Convex Polygon Intersection Graphs

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Generic results

- Main result
  - A (new) look at the combinatorics of geometric intersection graphs.
  - Representation issues for "*P*-intersection graphs".
- Concrete general case
  - All convex polygon intersection graphs have polynomial representations (in integers).
- Consequently:
  - Recognition problem for classes of P-intersection graphs NP-complete when proven NP-hard, answers (completes) several open questions in classifying known classes.
  - Framework for further combinatorial study.

Intersection graphs?

- *Relevant model for overlap patterns and graph presentation*
  - With/without constraints on objects and/or patterns.
  - Optimisation problems on geometric intersection graphs e.g. domination and set cover problems (on disk graphs, intersection graphs of rectangles with bounded aspect ratio, etc).

- *Kratochvíl & Pergel (2008): initiate the study of intersection graphs of homothetic polygons in the plane.*
  - "[…] which properties of these [two] classes of graphs translate (or at least have a chance to) to intersection graphs of [general] homothetic polygons."
  - P_{hom}-intersection graphs, P_{hom}-contact graphs.
  - "Given a convex polygon P, what is the complexity of recognition."

*Les **informaticiens** n'étudient pas des objets, mais des relations entre les objets ; il leur est donc indifférent de remplacer ces objets par d'autres, pourvu que les relations ne changent pas. La matière ne leur importe pas, la forme seule les intéresse.*

After J. Henri Poincaré(1854-1912).

(Some) classes of geometric intersection graphs

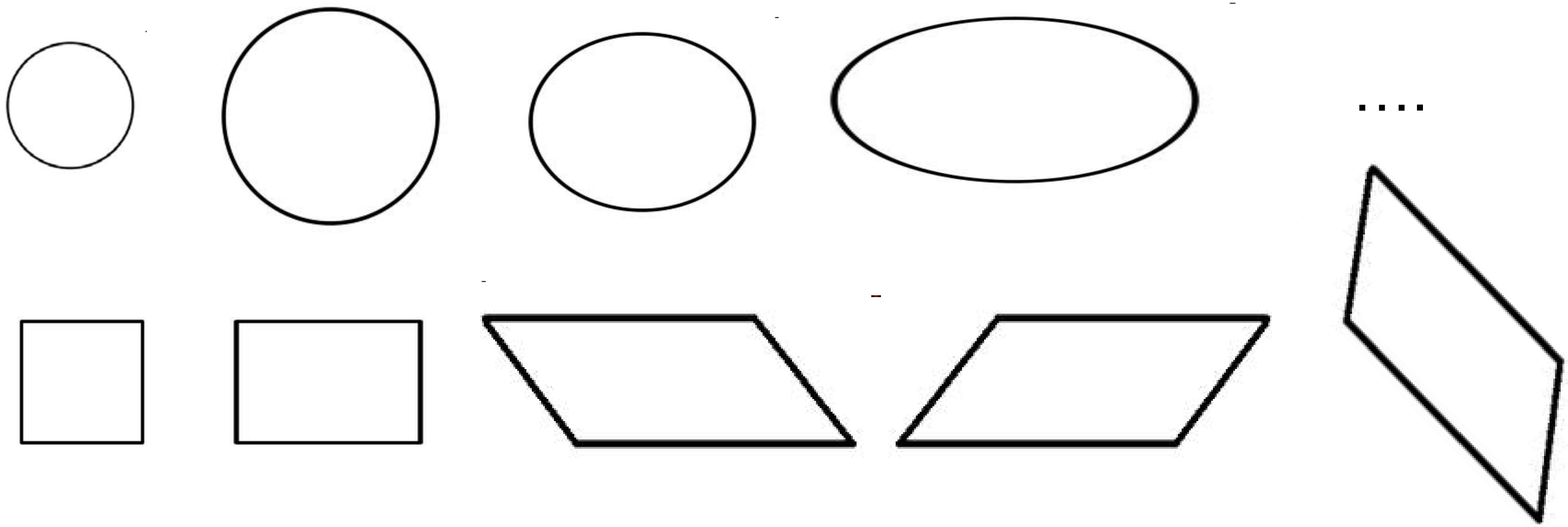
Graph Class	Objects -	Recognition	Repres.
interval	intervals	linear	poly
unit interval	unit intervals	linear	poly
circular-arc	arcs	linear	poly
(unit) disk	disks	NP-hard, $\in$ PSPACE	exp.
string	simple curves in R^2	NP-complete	exp.
tolerance	intervals w/tolerances	$\in$ NP	poly
segment	segments in R^2	NP-hard	exp.
box (rectangle)	rectangles in R^2	NP-complete	poly
unit square	unit squares	NP-complete	poly
square	*squares*	*NP-hard, $\in$ NP*	*poly*
max-tolerance	*semi-squares*	*NP-hard, $\in$ NP*	*poly*
polygon intersect.	*rat.repr. convex poly's*	*$\in$ NP*	*poly*
convex intersect	convex sets $\subset R^2$	NP-hard, $\in$ PSPACE	exp.
(planar	segments in R^2	linear	poly)

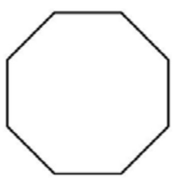
Ref. Breu, Kratochvil, Matousek, Pergel, … (see paper)

Generic notion of intersection graphs

- Base objects
  - One object (disk, square, convex polygon,…)
  - Finite non-empty set $S = \{o_1, \ldots, o_m\}$ of geometric objects in the plane
- Similarity transformations
  - Homothetic transformations (translating, scaling)
  - Template: parametric families of bi-continuous functions that are shape-preserving
  - With every object an finite set of allowed similarity templates
  - Assignment function $T: S \rightarrow$ "finite set of templates"
- Signature (or `type')
  - $P = \langle S,T \rangle$
- *P*-intersection graph
  - Intersection graph of plane objects $O_1, \ldots, O_n$
  - where every O_i $(1 \leq j \leq n)$ is similar to an $o \in S$ using a similarity transformation that conforms to a template in $T(o)$.
- Representation
  - By the concrete (description of a) set of objects that realize the graphs, by their specifying parameters.

Similarity





···· $\rightarrow$ ragged ranges in intersection models

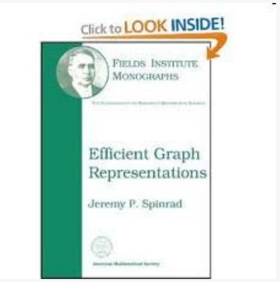
Similarity templates are preferably *smooth* families: if $t \in T(o)$, then images of O under $t(u_1,\ldots,u_k)$ and $t(v_1,\ldots,v_k)$ \`almost equal' if $(u_1,\ldots,u_k)$ and $(v_1,\ldots,v_k)$ are.

Measuring complexity (size)..) of representations

Let *P* = <S,T> be a signature

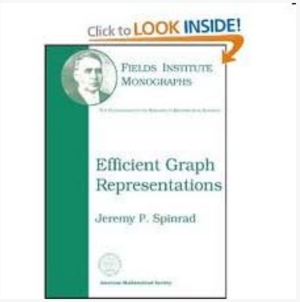
- *A P*-intersection graph with *n* vertices is said to be *polynomially represented* (using *p*),
  - if it is the intersection graph of a finite set of objects $O_1, \ldots, O_n$, where every O_i $(1 \leq i \leq n)$ is similar to an object $o \in S$ according to an allowed template of T(o), and has all its *specifying parameters* equal to rationals a/b with $|a|,|b| \leq 2^{p(n)}$
- A class *C* of *P*-intersection graphs is said to be *polynomially represented*
  - if there is a polynomial p=p(n) such that every graph in *C* is polynomially represented using p.

Polynomially represented $\rightarrow$ polynomial space



Generic problems for *P*-Intersection Graphs

- *P*-Intersection Graph Recognition
  - Given a graph G, decide whether G is a P-intersection graph.
- *P*-Intersection Graph Construction (`Drawing')
  - Given a graph G that is known to be a *P*-intersection graph, construct (`draw') a representation of G as the intersection graph of a set of objects in the plane according to signature *P*
- Complexity (size, …) of the above:
  - do representations in polynomial space (`bits') always exist and can these be found `efficiently'...?



Convex polygons

- Signatures from now on
  - **S**: a finite, non-∅ set of convex polygons $o_1,\ldots,o_m$
  - **T** assigns *linear* similarity templates t: every t is a family of affine transformations $x \to u + vQx$, where:
    - $u = (u_1,u_2)$ is any 2-dimensional vector,
    - v>o any scalar
    - Q a non-singular 2x2-matrix with rational coeffficients (fixed for t)
- Properties
  - Linear templates are smooth
  - S can be arbitrary and varying, T conceptually independent of S (although equiv to homothetic transforms of $Q(o_1),\ldots,Q(o_m)$)
  - Given S and T, similarity of convex O to base polygon in S under T is decidable.
  - Enlarging/shrinking by polynomially bounded margin achievable while preserving polynomial representation.
- Behaviour
  - Preserve convexity, sign of det(Q) determines orientation
  - Defining inequalities of base objects map faithfully

Open or closed?

- No difference for: (unit) disk graphs, … , intersection graphs of all `scalable' objects.

- **Lemma**: *Every polynomially represented P-intersection graph with a base set of closed convex polygons can be obtained as a polynomially represented P-intersection graph with a base set of open convex polygons, and vice versa.*

- Ingredients:
  - Let O_1 and O_2 be two disjoint convex polygons in the plane, both having nonempty interior. The (shortest) distance between O_1 and O_2 is realized as the distance between a vertex of one polygon and an edge of the other.
  - Let a,b,c,v_1,v_2 be rationals with numerator and denominator bounded in absolute value by q for some $q > 0$. If the following fraction is $\neq 0$, then distance of (v_1,v_2) to $ax+by+c=0$: $|a v_1 + b v_2 + c|/\sqrt{(a^2 + b^2)} \geq 1/(2q^5)$.
  - Topological argument using enlarging/shrinking while preserving polynomial representation. Eliminate touchings without altering intersection pattern.

- Thus assume closed objects
  - disjoint $\Leftrightarrow$ "distance > 0", …

Modeling (a) representation

- Let *G* be a *P*-intersection graph, assume it has a geometric representation using $O_1, \ldots, O_n$ (assumed given). Describe the pattern of intersections and non-intersections $\rightarrow$ LP model in polynomially represented rationals.
- Issues:
  - *Which inequalities to use*?
    - Use suitable, allowed templates u+vQ with u,v as unknowns
    - Map defining inequalities, linear and homogeneous in u's and v's etc
  - *How to use them*?
    - Case: intersecting
      - Inequalities to assert existence of common point
    - Case: nonintersecting
      - **Theorem:** *Two closed convex polygons in the plane are disjoint iff they can be separated by a line that precisely coincides with an edge of one of them.*
      - Use appropriate inequality to assert separation (…<0 or …>0)
  - *How to fill them in*?
    - Scale the variables (cf the given representation)
      - appropriate inequality to assert separation (…$\leq$ 1 resp … $\geq$ 1)
    - Translate
      - Assert all u's are $\geq$ o

Model

- All constraints of the affine transformations that are used (taking the scaling into account): $u_{i,1}$, $u_{i,2} \geq 0$ and $v_i \geq 1$.
- For any pair O_i, O_j we have k_{ij} or l_{ij} inequalities respectively of the following form:
  - if O_i, O_j must be disjoint: $k_{ij} \leq \max\{k_i, k_j\}$ inequalities
    - **DISJ**$_{i,j}(u_{i,1}, u_{i,2}, v_i, u_{j,1}, u_{j,2}, v_j) \leq -1 \,/\, \geq 1$
  - if O_i, O_j must intersect: $l_{ij} = k_i + k_j$ inequalities
    - **IN**$_i(x_{i,j}, y_{i,j}, u_{i,1}, u_{i,2}, v_i) \leq / \geq 0$, $IN_j(x_{i,j}, y_{i,j}, u_{j,1}, u_{j,2}, v_j) \leq / \geq 0$.
- To be written as "**Ax** =**b** with **x**≥ 0" where
  - **A** a N-by-N+3n+2m all-rational matrix (with $N=n+\Sigma k_{ij}+\Sigma l_{ij} = O(n^2)$), Numerators and denominators bounded by $2^{O(q)}$.
  - **b** a vector of 0's and $\pm$ 1's,
  - with feasible solution!
- **Lemma:** *The model has an all-rational solution for u's and v's with numerators and denominators polynomially bounded (i.e. in bits).*

Combining

- **Theorem** (Existence): *Let G be a P-intersection graph. Then G has a polynomial representation, even fully in integers.*

- **Theorem** (Visualization): *The construction (`drawing') problem of P-intersection graphs can be solved algorithmically, in exponential time.*
  - Assumption to know an embedding `ahead of time' unnecessary
  - Existence and drawing problems algorithmic, although `exponential'.

- Applications
  - Max-tolerance graphs
    - Kaufmann et al: recognition problem NP-hard
    - Max-tolerance graphs have polynomial-size (integer) representations ($\rightarrow$ recognition problem NP-complete).
  - P_{hom}-intersection graphs
    - Kratochvíl, Pergel (2008): P_{hom}-intersection graph recognition NP-hard
    - P_{hom}-intersection graphs have polynomial-size (integer) representations ($\rightarrow$ recognition problem NP-complete).
  - P_{hom}-contact graphs

- Further (linear homogeneous) constraints can be included in the model.

Finally

- Kratochvíl, Pergel (2008)
  - Study of (convex) polygon intersection graphs

- Our contribution: generic perspective and (first) results on the representation of convex polygon intersection graphs

Recent work jointly with Tobias Müller (*CWI,* Amsterdam)

- Upper bounds
  - Concrete and better (upper) bounds using the combinatorics of linear template mappings

- Lower bounds e.g.
  - **Theorem**: There are *P*-intersection graphs whose representation in integers requires a $2^{\Omega(n)}$x $2^{\Omega(n)}$ grid, i.e. Ω(n) bits in the coordinates.


THANK YOU