SOLUTIONS TO THE PROBLEMS OF THE AG EXAM OF JANUARY 9 2016

1 *Let X and Y be affine varieties and let $f : X \to Y$ be a morphism.*

1a *Prove that a morphism $f : X \to Y$ is a closed embedding if and only if $f^* : k[Y] \to k[X]$ is surjective.*

A closed subset of Y is given by a radical ideal in $k[Y]$ and conversely every radical ideal in $k[Y]$ defines a closed subset of Y. So f^* is a closed embedding if and only if for some radical ideal $I \subset k[Y]$, f^* factors through an isomorphism $k[Y]/I \cong k[X]$. This is equivalent to: f^* is surjective, for since $k[X]$ is reduced, $\ker(f)$ is then automatically a radical ideal.

1b *Prove that if f^* is injective and $f^*k[Y] \subset k[X]$ is a localization, then f is an isomorphism onto a principal open subset.*

Without loss of generality we may assume that $k[X] = S^{-1}k[Y]$, with $S \subset k[Y]$ a multiplicative subset. Then every $f \in k[X]$ can be written as g/h with $g \in k[Y]$ and $h \in S$. If we let f run over a finite collection of k-algebra generators $f_1, \dots, f_n$, and write $f_i = g_i/h_i$, then we see that we can take for S the powers of $h := h_1h_2 \cdots h_n$ so that $k[X] = k[Y][h^{-1}]$. This means that $X = Y_h$ is a principal open subset.

1c *Is it true that every open embedding of X into Y is thus obtained? If yes, prove this, if no, give a counterexample.*

No, a counterexample is described in (5.5) of the notes.

2a *Prove that $t \in \mathbb{A}^1 \mapsto [1 : s : \cdots : s^n] \in \mathbb{P}^n$ extends to a closed embedding $f : \mathbb{P}^1 \to \mathbb{P}^n$.*

The extension is given by $[S_0 : S_1] \mapsto [S_0^n : S_0^{n-1}S_1 : \cdots : S_1^n]$ and this is in fact the d-fold Veronese embedding. We proved in the notes that it is closed.

2b *Prove that the Hilbert polynomial of the image of f is $nz+1$. What is the degree of the image of f?*

If we denote the homogeneous coordinates of $\mathbb{P}^n$ by $[T_0 : \cdots : T_n]$, then every homogeneous polynomial of degree d in $T_0, \dots, T_n$ restricts to $\mathbb{A}^2$ as a homogeneous polynomial of degree dn in S_0 and S_1. All such polynomials are thus obtained, because any monomial of degree dn in S_0 and S_1 can be written as a product of d monomials of degree n in S_0 and S_1. So the graded coordinate ring of $f(C)$ has in degree d dimension $dn + 1$. Hence its Hilbert function is $z \mapsto nz + 1$. This is a polynomial of degree 1 with leading coeffcient n and so $f(C)$ has degree n.

3 *Let be given integers $n > m \geq 0$ and regard $\mathbb{A}^m$ as the closed subset of $\mathbb{A}^n$ by putting the last $n - m$ coordinates equal to zero. Prove that $\mathbb{A}^n - \mathbb{A}^m$ is affine if and only if $n = m + 1$.*

For $n = m + 1$, observe that $\mathbb{A}^n - \mathbb{A}^{n-1}$ is just the principal open subset $\mathbb{A}^n_{z_n}$ and hence affine. We next show that for $n > m + 1$, the inclusion $j : \mathbb{A}^n - \mathbb{A}^m \subset \mathbb{A}^n$ induces an isomorphism $j^* : k[z_1, \dots, z_n] = \mathcal{O}(\mathbb{A}^n) \cong \mathcal{O}(\mathbb{A}^n - \mathbb{A}^m)$ on the algebras of regular functions. So if $\mathbb{A}^n - \mathbb{A}^m$ were affine, then this would imply that j is an isomorphism, which is obviously not the case. To see this, let $f \in \mathcal{O}(\mathbb{A}^n - \mathbb{A}^m)$. Then for $i = m + 1, \dots, n$, f is regular on the principal open $\mathbb{A}^n_{z_i}$ and so can be written as $z_i^{-r_i} g_i(z)$ with $r_i \geq 0$ and $g_i \in k[z_1, \dots, z_n]$ not divisible by z_i. From the identity $z_j^{r_j} g_i = z_i^{r_i} g_j$, we see that g_i is divisible by $z_i^{r_i}$. This can only be if $r_i = 0$. Hence $f \in k[z_1, \dots, z_n] = k[\mathbb{A}^n]$ as asserted.

4 *Let k and l be positive integers an let P be a projective space of dimension $n := k+l+1$.*

4a *Prove that the set $I_{k,l} \subset \mathrm{Gr}_k(P) \times \mathrm{Gr}_l(P)$ of pairs $([Q],[Q']) \in \mathrm{Gr}_k(P) \times \mathrm{Gr}_l(P)$ for which $Q \cap Q' \neq \emptyset$ is a hypersurface in $\mathrm{Gr}_k(P) \times \mathrm{Gr}_l(P)$.*

We may assume that $P = \mathbb{P}(V)$ with V a k-vector space of dimension $n+1 = k+l+2$ so that we can identify $I_{k,l}$ with the subset of $([W],[W']) \in \mathrm{Gr}_{k+1}(V) \times \mathrm{Gr}_{l+1}(V)$ for which $\dim(W \cap W') > 0$. This last condition is equivalent to $(\wedge^{k+1}W) \wedge (\wedge^{l+1}W') = 0$. The exterior product defines a bilinear map $\wedge^{k+1}V \times \wedge^{l+1}V \to \wedge^{n+1}V \cong k$ and hence defines a hypersurface $D \subset \mathbb{P}(\wedge^{k+1}V) \times \mathbb{P}(\wedge^{l+1}V)$ of bidegree $(1,1)$. Then $I_{k,l}$ is just the preimage of D under the product of Plücker embeddings $\mathrm{Gr}_{k+1}(V) \times \mathrm{Gr}_{l+1}(V) \to \mathbb{P}(\wedge^{k+1}V) \times \mathbb{P}(\wedge^{l+1}V)$ and so is closed in $\mathrm{Gr}_{k+1}(V) \times \mathrm{Gr}_{l+1}(V)$. As it is given by a single equation, it is a hypersurface or empty or all of $\mathrm{Gr}_{k+1}(V) \times \mathrm{Gr}_{l+1}(V)$ (for the latter space is irreducible). The last two cases are easily excluded.

4b *Let $\tilde{I}_{k,l} \subset \mathrm{Gr}_k(P) \times P \times \mathrm{Gr}_l(P)$ be the set of triples $(([Q],p,[R])$ with $p \in Q \cap R$. Prove that $\tilde{I}_{k,l}$ is closed in $\mathrm{Gr}_k(P) \times P \times \mathrm{Gr}_l(P)$.*

The proof is similar: we have a (trilinear) map

$$\wedge^{k+1}V \times V \times \wedge^{l+1}V \to \wedge^{k+2}V \oplus \wedge^{l+2}V, \quad (a,v,b) \mapsto (a \wedge v, v \wedge b),$$

and for every linear form on $\wedge^{k+2}V \oplus \wedge^{l+2}V$ we obtain closed subset of $\mathbb{P}(\wedge^{k+2}V) \times \mathbb{P}(V) \times \mathbb{P}(\wedge^{l+2}V)$. Denote by $\tilde{D}$ the intersection of these closed subsets (a closed subset of $\mathbb{P}(\wedge^{k+2}V) \times \mathbb{P}(V) \times \mathbb{P}(\wedge^{l+2}V)$). Then $\tilde{I}_{k,l}$ is the preimage of $\tilde{D}$ under the natural map $\mathrm{Gr}_{k+1}(V) \times \mathbb{P}(V) \times \mathrm{Gr}_{l+1}(V) \to \mathbb{P}(\wedge^{k+1}V) \times \mathbb{P}(V) \times \mathbb{P}(\wedge^{l+1}V)$ and hence closed in $\mathrm{Gr}_{k+1}(V) \times \mathbb{P}(V) \times \mathrm{Gr}_{l+1}(V)$.

4c *Prove that the projection onto the middle factor, $\pi_P : \tilde{I}_{k,l} \to P$, is Zariski locally trivial with fiber isomorphic to $\mathrm{Gr}_{k-1}(\mathbb{P}^{n-1}) \times \mathrm{Gr}_{l-1}(\mathbb{P}^{n-1})$. Conclude that $\tilde{I}_{k,l}$ is a nonsingular subvariety of $\mathrm{Gr}_k(P) \times P \times \mathrm{Gr}_l(P)$ of codimension $n+1$.*

Fix a hyperplane $V' \subset V$ so that we have defined a hyperplane $P' := \mathbb{P}(V')$ in P. Its complement $U := P - P'$ is an affine open subset of P such that every $u \in U$ defines a one dimensional subspace $L_u \subset V$ not contained V'. Every k-plane in P passing through u corresponds to a $(k+1)$-plane in V passing through L_u which will meet V' in a k-plane. Conversely, a k-plane in V' spanes with L_u a $(k+1)$-plane in V which defines a k-plane in P passing through u. Similarly for a l-plane in P passing through u. We thus find an isomorphism $\mathrm{Gr}_k(P') \times U \times \mathrm{Gr}_l(P') \cong \pi_P^{-1}U$ compatible with the projection onto U. It follows that $\tilde{I}_{k,l}$ is nonsingular and irreducible. Its codimension is equal to $\dim \mathrm{Gr}_k(P) + \dim \mathrm{Gr}_l(P) - \dim \mathrm{Gr}_k(P') - \dim \mathrm{Gr}_l(P') = (k+1)(n-k)+(l+1)(n-l)-k(n-k)-l(n-l) = (n-k)+(n-l) = 2n-(k+l) = n+1$.

4d *Prove that the projection $\pi : \tilde{I}_{k,l} \to \mathrm{Gr}_k(P) \times \mathrm{Gr}_l(P)$ has image $I_{k,l}$ and that there exists an open subset of $I_{k,l}$ over which π is a bijection. Conclude that $I_{k,l}$ is irreducible.*

It is clear that $\pi(\tilde{I}_{k,l}) = I_{k,l}$ and so $I_{k,l}$ is irreducible. Denote by $J_{k,l} \subset I_{k,l}$ the set of pairs $([Q],[Q']) \in \mathrm{Gr}_k(P) \times \mathrm{Gr}_l(P)$ for which $Q \cap Q'$ is of dimension > 1. Clearly $\pi^{-1}([Q],[Q'])$ is a singleton if $([Q],[Q']) \in I_{k,l} - J_{k,l}$. So it suffices to show that $I_{k,l} - J_{k,l}$ is open-dense in $I_{k,l}$.

For the trivialization over $U \subset P$ as defined above, $\pi^{-1}(J_{k,l}) \cap \pi_P^{-1}U$ is identified with the subset of $([R],u,[R']) \in \mathrm{Gr}_k(P) \times U \times \mathrm{Gr}_l(P')$ for which $R \cap R' \neq \emptyset$. Since this is a closed subset of $(\mathrm{Gr}_k(P) \times U \times \mathrm{Gr}_l(P))$, it follows that $\pi^{-1}(J_{k,l})$ is closed in $\tilde{I}_{k,l}$ (let U run over a standard open covering of P). It is clearly not all of $\tilde{I}_{k,l}$ and so its complement is open-dense in $\tilde{I}_{k,l}$. Since P is a projective space, π is closed and hence $\pi\pi^{-1}(J_{k,l})) = J_{k,l}$ is closed in $I_{k,l}$. So the complement of $J_{k,l}$ is as required.