THE STABLE COHOMOLOGY OF THE SATAKE COMPACTIFICATION OF $\mathcal{A}_g$

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ABSTRACT. Charney and Lee [7] have shown that the rational cohomology of the Satake-Baily-Borel compactification $\mathcal{A}_g^{bb}$ of $\mathcal{A}_g$ stabilizes as $g \to \infty$ and they computed this stable cohomology as a Hopf algebra. We give a relatively simple algebro-geometric proof of their theorem that also takes into account the mixed Hodge structure that is present here. We find the latter to be impure.

1. THE THEOREM

Let $\mathcal{A}_g = \mathcal{A}_g(\mathbb{C})$ denote the coarse moduli space of principally polarized complex abelian varieties of genus g endowed with the analytic (Hausdorff) topology. Recall that the Satake-Baily-Borel compactification $j_g : \mathcal{A}_g \subset \mathcal{A}_g^{bb}$ realizes $\mathcal{A}_g$ as a Zariski open-dense subset in a normal projective variety $\mathcal{A}_g^{bb}$. Forming the product of two principally polarized abelian varieties defines a morphism of moduli spaces $\mathcal{A}_g \times \mathcal{A}_{g'} \to \mathcal{A}_{g+g'}$ which extends to these compactifications: we have a commutative diagram

$$\begin{array}{ccc} \mathcal{A}_g \times \mathcal{A}_{g'} & \longrightarrow & \mathcal{A}_{g+g'} \\ {\scriptstyle j_g \times j_{g'}}\Big\downarrow & & \Big\downarrow{\scriptstyle j_{g+g'}} \\ \mathcal{A}_g^{bb} \times \mathcal{A}_{g'}^{bb} & \longrightarrow & \mathcal{A}_{g+g'}^{bb} \end{array} \tag{1}$$

By taking $g' = 1$ and choosing a point of $\mathcal{A}_1$, we get the 'stabilization maps'

$$\begin{array}{ccc} \mathcal{A}_g & \longrightarrow & \mathcal{A}_{g+1} \\ {\scriptstyle j_g}\Big\downarrow & & \Big\downarrow{\scriptstyle j_{g+1}} \\ \mathcal{A}_g^{bb} & \longrightarrow & \mathcal{A}_{g+1}^{bb} \end{array} \tag{2}$$

whose homotopy type does not depend on the point we choose, for $\mathcal{A}_1$ is isomorphic to the affine line and hence connected. Since we are only concerned with homotopy classes and commutativity up to homotopy, we can for the definition of the map $\mathcal{A}_g^{bb} \to \mathcal{A}_{g+1}^{bb}$ even choose this point to be represented by the singleton $\mathcal{A}_0$. Then this map is a homeomorphism onto the Satake boundary (since $\mathcal{A}_1^{bb} \cong \mathbb{P}^1$ the maps are not just homotopic, but even induce the same map on Chow groups). This gives rise to two Hopf algebras with a mixed Hodge structure. We begin with the following lemma, which in essence is merely quoting work of Borel and Borel-Serre.

Lemma 1.1. *The stabilization maps defined above $\mathcal{A}_g \hookrightarrow \mathcal{A}_{g+1}$ (multiplication by a fixed elliptic curve) and $\mathcal{A}_g^{bb} \to \mathcal{A}_{g+1}^{bb}$ (mapping onto the boundary) induce on rational cohomology an isomorphism in degree $< g$ and are injective in degree g.*

Proof. Since the $\mathbb{Q}$-rank of $\mathcal{S}p_g$ is g, this first assertion follows from a theorem of Borel [3]. The second stability assertion is equivalent to the vanishing of the relative cohomology $H^k(\mathcal{A}_{g+1}^{bb}, \mathcal{A}_g^{bb}; \mathbb{Q})$ for $k \le g$. But $H^k(\mathcal{A}_{g+1}^{bb}, \mathcal{A}_g^{bb}; \mathbb{Q})$ can be identified with $H_c^k(\mathcal{A}_{g+1}; \mathbb{Q})$ and according to Borel-Serre [5] the latter space is zero in degrees $\le g$. □

We then form the stable rational cohomology spaces

$$H^k(\mathcal{A}_\infty; \mathbb{Q}) := \varinjlim_g H^k(\mathcal{A}_g; \mathbb{Q}), \quad H^k(\mathcal{A}_\infty^{bb}; \mathbb{Q}) := \varinjlim_g H^k(\mathcal{A}_g^{bb}; \mathbb{Q}),$$

where the notation is only suggestive, for there is here no pretense of introducing spaces $\mathcal{A}_\infty$ and $\mathcal{A}_\infty^{bb}$. If we take the direct sum over k we get a $\mathbb{Q}$-algebra in either case. It follows from the homotopy commutativity of the diagram (2) above that the inclusions j_g define a graded $\mathbb{Q}$-algebra homomorphism

$$j_\infty^* : H^\bullet(\mathcal{A}_\infty^{bb}; \mathbb{Q}) \to H^\bullet(\mathcal{A}_\infty; \mathbb{Q}).$$

The multiplication maps exhibited in diagram (1) are (almost by definition) compatible with the stabilization maps and hence induce a graded coproduct on either algebra so that j_∞^* becomes a homomorphism of (graded bicommutative) Hopf algebras. Since the multiplication maps and the stability maps are morphisms in the category of complex algebraic varieties, these Hopf algebras come with a natural mixed Hodge structure such that j_∞^* is also a morphism in the mixed Hodge category. The Hopf algebra $H^\bullet(\mathcal{A}_\infty; \mathbb{Q})$ is well-known and due to Borel [3]: it has as its primitive elements classes $ch_{2r+1} \in H^{4r+2}(\mathcal{A}_\infty; \mathbb{Q})$, $r \ge 0$, where ch_{2r+1} restricts to $\mathcal{A}_g$ as the rational $(2r+1)$th Chern character of the Hodge bundle on $\mathcal{A}_g$, and so $H^\bullet(\mathcal{A}_\infty; \mathbb{Q}) = \mathbb{Q}[ch_1, ch_3, ch_5, \dots]$ with ch_{2r+1} of type $(2r+1, 2r+1)$ (if we are happy with multiplicative generators, we can just as well replace ch_{2r+1} by the corresponding Chern class c_{2r+1}). The principal and essentially only result of this paper is Theorem 1.2 below. Its first assertion is due Charney and Lee [7], who derive this from a determination of a limit of homotopy types. We shall obtain this in a relatively elementary manner from the algebraic geometry and the classical vanishing results of Borel and Borel-Serre. Our approach has the advantage that it helps us to understand the new classes that appear here geometrically, to the extent that this enables us to determine their Hodge type. We address the homotopy discussion of Charney and Lee and a generalization thereof in another paper [6] that will not be used here.

Theorem 1.2. *The graded Hopf algebra $H^\bullet(\mathcal{A}_\infty^{bb}; \mathbb{Q})$ has for every integer $r \ge 1$ a primitive generator y_r of degree $4r+2$ and for every integer $r \ge 0$ a primitive generator $\widetilde{ch}_{2r+1}$ of degree $4r+2$ such that the map $j_\infty^* : H^\bullet(\mathcal{A}_\infty^{bb}; \mathbb{Q}) \to H^\bullet(\mathcal{A}_\infty; \mathbb{Q})$ sends $\widetilde{ch}_{2r+1}$ to ch_{2r+1} and is zero on y_r when $r \ge 1$. In particular, if $\tilde{c}_{2r+1} \in H^{4r+2}(\mathcal{A}_\infty^{bb}; \mathbb{Q})$ denotes the Chern lift of $c_{2r+1} \in H^{4r+2}(\mathcal{A}_\infty; \mathbb{Q})$ defined by $\widetilde{ch}_1, \dots, \widetilde{ch}_{2r+1}$, then $H^\bullet(\mathcal{A}_\infty^{bb}; \mathbb{Q}) = \mathbb{Q}[y_1, y_2, y_3 \dots, \tilde{c}_1, \tilde{c}_3, \tilde{c}_5, \dots]$ is as a commutative $\mathbb{Q}$-algebra.*

The mixed Hodge structure on $H^\bullet(\mathcal{A}_\infty^{bb}; \mathbb{Q})$ is such that y_r is of bidegree $(0,0)$ and $\widetilde{ch}_{2r+1}$ (or equivalently, $\tilde{c}_{2r+1}$) is of bidegree $(2r+1, 2r+1)$.

Remark 1.3. So for $r \ge 1$, the primitive part $H_{\mathrm{pr}}^{4r+2}(\mathcal{A}_\infty^{bb}; \mathbb{Q})$ of the Hopf algebra $H^\bullet(\mathcal{A}_\infty^{bb}; \mathbb{Q})$ is two-dimensional in degree $4r+2$ and defines a Tate extension

$$0 \to \mathbb{Q} \to H_{\mathrm{pr}}^{4r+2}(\mathcal{A}_\infty^{bb}; \mathbb{Q}) \to \mathbb{Q}(-2r-1) \to 0,$$

with $\mathbb{Q}$ spanned by y_r and $\mathbb{Q}(-2r-1)$ spanned by ch_{2r+1}. We discuss the nature of this extension briefly in Remark 2.6 below.

2. The proof

The topological identification of the Satake boundary $\mathcal{A}_g^{bb} \smallsetminus \mathcal{A}_g$ with $\mathcal{A}_{g-1}^{bb}$ gives rise to the decomposition $\mathcal{A}_g^{bb} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \cdots \sqcup \mathcal{A}_0$ so that for $r \leq g$, the natural morphism $\mathcal{A}_r^{bb} \to \mathcal{A}_g^{bb}$ is injective and a normalization of its image $\cup_{k\leq r}\mathcal{A}_k$. Since we are only concerned with topological issues, we often identify $\mathcal{A}_r^{bb}$ with its image in $\mathcal{A}_g^{bb}$. [1]

We will use the fact that the higher direct images $R^\bullet j_{g*}\mathbb{Q}_{\mathcal{A}_g}$ are locally constant on each stratum $\mathcal{A}_r$. The stalk of this restriction can be identified with the rational cohomology algebra of the arithmetic group $P_g(r)$ defined as follows. Let H stand for $\mathbb{Z}^2$ (with basis denoted (e, e')) and endowed with symplectic form characterized by $\langle e, e'\rangle = 1$. We also put $I := \mathbb{Z}e$. We regard H^g as a direct sum of symplectic lattices with g summands. Then $P_g(r)$ is in terms of the decomposition $H^g = H^r \oplus H^{g-r}$, the group of symplectic transformations in H^g that are the identity on $H^r \oplus 0$ and preserve $H^r \oplus I^{g-r}$. The orbifold fundamental group of $\mathcal{A}_r$ is isomorphic to $\mathrm{Sp}(H^r)$ (the isomorphism is of course given up to conjugacy) and its representation on a stalk of $R^\bullet j_{g*}\mathbb{Q}_{\mathcal{A}_g}|\mathcal{A}_r$ corresponds to its obvious action (given by conjugation) on $P_g(r)$. Note that this action is algebraic in the sense that it extends to a representation of the underlying affine algebraic group (which assigns to a commutative ring R the group $\mathrm{Sp}(H^r \otimes R)$). If $p \in \mathcal{A}_r$ and U_p is a regular neighborhood of p in $\mathcal{A}_g^{bb}$ such that the natural map $H^\bullet(U_p \cap \mathcal{A}_g; \mathbb{Q}) \to (R^\bullet j_{g*}\mathbb{Q}_{\mathcal{A}_g})_p$ is an isomorphism, then for every $r \leq s \leq g$ and $q \in U_p \cap \mathcal{A}_s$ the restriction map yields a map of $\mathbb{Q}$-algebras $(R^\bullet j_{g*}\mathbb{Q}_{\mathcal{A}_g})_p \to (R^\bullet j_{g*}\mathbb{Q}_{\mathcal{A}_g})_q$. Under the above identification this is represented by the $\mathrm{Sp}(H^r)$-orbit of the obvious inclusion $P_g(s) \hookrightarrow P_g(r)$. Similarly, the natural sheaf homomorphism

$$R^\bullet j_{g+g'*}\mathbb{Q}_{\mathcal{A}_{g+g'}}|\mathcal{A}_g^{bb} \times \mathcal{A}_{g'}^{bb} \to R^\bullet(j_g \times j_{g'})_*\mathbb{Q}_{\mathcal{A}_g \times \mathcal{A}_{g'}} \cong R^\bullet j_{g*}\mathbb{Q}_{\mathcal{A}_g} \boxtimes R^\bullet j_{g'*}\mathbb{Q}_{\mathcal{A}_{g'}}$$

(we invoked the Künneth isomorphism) is at $\mathcal{A}_r \times \mathcal{A}_{r'} \subset \mathcal{A}_g \times \mathcal{A}_{g'}$ induced by the obvious embedding $P_g(r) \times P_{g'}(r') \hookrightarrow P_{g+g'}(r + r')$, or rather its $\mathrm{Sp}(H^{r+r'})$-orbit.

The proof of the first assertion of our main theorem rests on careful study of the Leray spectral sequence for the inclusion $j_g : \mathcal{A}_g \subset \mathcal{A}_g^{bb}$,

$$E_2^{p,q} = H^p(\mathcal{A}_g^{bb}, R^q j_{g*}\mathbb{Q}) \Rightarrow H^{p+q}(\mathcal{A}_g; \mathbb{Q}). \tag{3}$$

Lemma 2.1. *Let $r \leq g$. Then the natural map $H^p(\mathcal{A}_g^{bb}, R^\bullet j_{g*}\mathbb{Q}) \to H^p(\mathcal{A}_r^{bb}, R^\bullet j_{g*}\mathbb{Q})$ is an isomorphism for $p < r$ and is injective for $p = r$.*

Proof. It suffices to show that when $r < g$, the natural map $H^p(\mathcal{A}_{r+1}^{bb}, R^\bullet j_{g*}\mathbb{Q}) \to H^p(\mathcal{A}_r^{bb}, R^\bullet j_{g*}\mathbb{Q})$ has this property. For this we consider the exact sequence

$$\begin{aligned}\cdots \to H_c^p(\mathcal{A}_{r+1}, R^\bullet j_{g*}\mathbb{Q}) \to H^p(\mathcal{A}_{r+1}^{bb}, R^\bullet j_{g*}\mathbb{Q}) \to H^p(\mathcal{A}_r^{bb}, R^\bullet j_{g*}\mathbb{Q}) \to \\ \to H_c^{p+1}(\mathcal{A}_{r+1}, R^\bullet j_{g*}\mathbb{Q}) \to \cdots\end{aligned}$$

The restriction $R^q j_{g*}\mathbb{Q}|\mathcal{A}_{r+1}$ is a local system whose monomdromy comes from an action of the algebraic group $\mathcal{S}p(H^r)$. Following Borel [4], $H_c^i(\mathcal{A}_{r+1}, R^\bullet j_{g*}\mathbb{Q})$ then vanishes in degree $\leq r$. The lemma follows. $\square$

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[1] We learned from Riccardo Silvati Manni that the Satake boundary of $\mathcal{A}_g$ is normal so that we have here in fact an isomorphism.

By viewing I^{g-r} as the subquotient $(H^r \oplus I^{g-r})/(H^r \oplus 0)$ of H^g, we see that there is a natural homomorphism of arithmetic groups $P_g(r) \to \mathrm{GL}(I^{g-r}) = \mathrm{GL}(g-r, \mathbb{Z})$.

Lemma 2.2. *The homomorphism $P_g(r) \to \mathrm{GL}(g-r, \mathbb{Z})$ induces an isomorphism on rational cohomology in degrees $< \frac{1}{2}(g-r-1)$. In that range the rational cohomology of $\mathrm{GL}(g-r, \mathbb{Z})$ is stable and is canonically isomorphic to the stable cohomology of $\mathrm{GL}(\mathbb{Z}) := \cup_r \mathrm{GL}(r, \mathbb{Z})$. The inclusion $P_g(r) \times P_{g'}(r') \subset P_{g+g'}(r+r')$ induces on rational cohomology in the stable range (relative to both factors) the coproduct in the Hopf algebra $H^\bullet(\mathrm{GL}(\mathbb{Z}); \mathbb{Q})$.*

Proof. According to Borel [4], the cohomology of the arithmetic group $\mathrm{GL}(r, \mathbb{Z})$ with values in an irreducible representation of the underlying algebraic group $\mathcal{SL}_r^{\pm}$ (the group of invertible matrices of determinant ± 1) is zero in degree $< \frac{1}{2}(r-1)$, unless the representation is trivial. Let $N_g(r)$ be the kernel of $P_g(r) \to \mathrm{GL}(g-r, \mathbb{Z})$. This is a nilpotent subgroup whose center, when written additively, may be identified with the symmetric quotient $\mathrm{Sym}_2(I^{g-r})$ of $I^{g-r} \otimes I^{g-r}$. The quotient of $N_g(r)$ by this center is abelian, and when written additively, naturally identified with the lattice $H^r \otimes I^{g-r}$. So in view of the Leray spectral sequence

$$H^p(\mathrm{GL}(g-r, \mathbb{Z}), H^q(N_g(r), \mathbb{R})) \Rightarrow H^{p+q}(P_g(r), \mathbb{R}),$$

it suffices to show that $H^q(N_g(r); \mathbb{R})$ does not contain the trivial representation of $\mathrm{GL}^{\pm 1}(g-r, \mathbb{R})$ in positive degree $< \frac{1}{2}(g-r-1)$. This follows from another Leray spectral sequence

$$H^s(I^{g-r} \otimes H^r, H^t(\mathrm{Sym}_2 I^{g-r}, \mathbb{R})) \Rightarrow H^{s+t}(N_g(r), \mathbb{R}).$$

The left hand side is isomorphic to

$$\wedge^s \mathrm{Hom}(I^{g-r} \otimes H^r, \mathbb{R}) \otimes \wedge^t \mathrm{Hom}(\mathrm{Sym}_2 I^{g-r}, \mathbb{R})$$

as a representation of $\mathrm{SL}^{\pm 1}(g-r, \mathbb{R})$. The classical invariant theory of $\mathrm{SL}(g-r; \mathbb{R})$ tells us that this representation does not contain the trivial representation when $0 < s + 2t < g - r$ and so the first part lemma follows. The second assertion merely quotes a theorem of Borel [3] and the last assertion is easy. □

Corollary 2.3. *For $q < \frac{1}{2}(g-r-1)$, $R^q j_{g*}\mathbb{Q})|\mathcal{A}_r^{bb}$ is a constant local system whose stalk is canonically isomorphic to $H^q(\mathrm{GL}(\mathbb{Z}), \mathbb{Q})$. This identification is compatible with the multiplicative structure. It is also compatible with the coproduct in the sense that when $0 \leq r' \leq g'$, then in degree $< \frac{1}{2}\min\{g-r-1, g'-r'-1\}$, the natural map $R^\bullet j_{g+g'*}\mathbb{Q}_{\mathcal{A}_{g+g'}}|\mathcal{A}_r^{bb} \times \mathcal{A}_{r'}^{bb} \to (R^\bullet j_{g*}\mathbb{Q}_{\mathcal{A}_g}|\mathcal{A}_r^{bb}) \boxtimes (R^\bullet j_{g'*}\mathbb{Q}_{\mathcal{A}_{g'}}|\mathcal{A}_{r'}^{bb})$ is stalkwise identified with the coproduct on $H^\bullet(\mathrm{GL}(\mathbb{Z}); \mathbb{Q})$.*

Proof of the first assertion of Theorem 1.2. We have shown (Lemma 2.1 and Corollary 2.3) that when $p < r$ and $q < \frac{1}{2}(g-r-1)$ we have

$$E_2^{p,q} = H^p(\mathcal{A}_g^{bb}, R^q j_{g*}\mathbb{Q}) = H^p(\mathcal{A}_r^{bb}, \mathbb{Q}) \otimes H^q(\mathrm{GL}(\mathbb{Z}); \mathbb{Q})$$

The Leray spectral sequences (3) for j_{g*} and j_{g+1*} are compatible and so we may form a limit: we fix p and q, but we let r and $g - r$ tend to infinity. This then yields a spectral sequence

$$E_2^{p,q} = H^p(\mathcal{A}_\infty^{bb}; \mathbb{Q}) \otimes H^q(\mathrm{GL}(\mathbb{Z}); \mathbb{Q}) \Rightarrow H^{p+q}(\mathcal{A}_\infty; \mathbb{Q}). \tag{4}$$

This spectral sequence is not just multiplicative, but also compatible with the coproduct. So the differentials take primitive elements to primitive elements (or zero) and the spectral sequence restricts to one of graded vector spaces by restricting to the primitive parts. The primitive part of $E_2^{p,q}$ is zero unless $p=0$ or $q=0$. A theorem of Borel [3] tells us that $H^\bullet(\mathrm{GL}(\mathbb{Z});\mathbb{Q})_{\mathrm{pr}}$ has for every positive integer r a generator a_r in degree $4r+1$ (and is zero in all other positive degrees) and that $H^\bullet(\mathcal{A}_\infty;\mathbb{Q})_{\mathrm{pr}}$ has for every odd integer s a primitive generator ch_s in degree $2s$ (and is zero in all other positive degrees). This implies that $d^k(1\otimes a_r)=0$ for $k=2,3,\dots 4r+1$, but that $y_r := d^{4r+2}(1\otimes a_r)$ will be nonzero and primitive. We also see that for $s>0$ odd, $H^{2s}(\mathcal{A}_g^{bb};\mathbb{Q})$ must contain a lift $\widetilde{ch}_s$ of c_s. Since $H^\bullet(\mathcal{A}_\infty^{bb};\mathbb{Q})$ is a Hopf algebra, it then follows that the Hopf algebra $H^\bullet(\mathcal{A}_\infty^{bb};\mathbb{Q})$ is primitively generated by $y_1, y_2, \dots, \widetilde{ch}_1, \widetilde{ch}_3, \widetilde{ch}_5, \dots$. So as a commutative $\mathbb{Q}$-algebra it is freely generated by $y_1, y_2, \dots, \tilde{c}_1, \tilde{c}_3, \tilde{c}_5, \dots$. □

The spectral sequence (4) suggests that the space $\mathcal{A}_\infty$ (which we did not define) has the homotopy type of a $B\mathrm{GL}(\mathbb{Z})$-bundle over $\mathcal{A}_\infty^{bb}$ (which we did not define either). Indeed, the contents of [7] amounts to proving this in some form.

Remark 2.4. The long exact sequence for the pair $(\mathcal{A}_g^{bb}, \mathcal{A}_g)$ shows that the cohomology $H^\bullet(\mathcal{A}_g^{bb}, \mathcal{A}_g;\mathbb{Q})$ stabilizes as well with g and is equal to the ideal in $\mathbb{Q}[y_1, y_2, \dots, \tilde{c}_1, \tilde{c}_3, \tilde{c}_5, \dots]$ generated by the y_r's. We shall therefore denote this ideal by $H^\bullet(\mathcal{A}_\infty^{bb}, \mathcal{A}_\infty;\mathbb{Q})$. We use the occasion to point out that the y-classes are canonically defined, but that this is not at all clear for the $\tilde{c}$-classes (for more on this, see Remark 2.6).

Remark 2.5. We can account geometrically for the classes y_r as follows. Denote the single point of $\mathcal{A}_0 \subset \mathcal{A}_g^{bb}$ by ∞ (the worst cusp of $\mathcal{A}_g^{bb}$), and take g so large that the natural maps $H^{4r+1}(\mathrm{GL}(\mathbb{Z});\mathbb{Q}) \to H^{4r+1}(\mathrm{GL}(g,\mathbb{Z});\mathbb{Q}) \to (R^{4r+1}j_{g*}\mathbb{Q})_\infty$ and $H^{4r+2}(\mathcal{A}_\infty^{bb}, \mathcal{A}_\infty;\mathbb{Q}) \to H^{4r+2}(\mathcal{A}_g^{bb}, \mathcal{A}_g;\mathbb{Q})$ are isomorphisms. Choose a regular neighborhood U_∞ of ∞ in $\mathcal{A}_g^{bb}$ so that if we put $\mathring{U}_\infty := U_\infty \cap \mathcal{A}_g$, the natural maps

$$(R^{4r+1}j_{g*}\mathbb{Q})_\infty \leftarrow H^{4r+1}(\mathring{U}_\infty;\mathbb{Q}) \xrightarrow{\delta} H^{4r+2}(U_\infty, \mathring{U}_\infty;\mathbb{Q})$$

are also isomorphisms. If we identify $a_r \in H^{4r+1}(\mathrm{GL}(\mathbb{Z});\mathbb{Q})$ with its image in $H^{4r+1}(\mathring{U}_\infty;\mathbb{Q})$, then $\delta(a_r) \in H^{4r+2}(U_\infty, \mathring{U}_\infty;\mathbb{Q})$ is precisely the image of y_r under the restriction map $H^{4r+2}(\mathcal{A}_\infty^{bb}, \mathcal{A}_\infty;\mathbb{Q}) \cong H^{4r+2}(\mathcal{A}_g^{bb}, \mathcal{A}_g;\mathbb{Q}) \to H^{4r+2}(U_\infty, \mathring{U}_\infty;\mathbb{Q})$.

We may also get a homology class this way: the Hopf algebra $H_\bullet(\mathrm{GL}(\mathbb{Z});\mathbb{Q})$ has a primitive generator in $H_{4r+1}(\mathrm{GL}(\mathbb{Z});\mathbb{Q})$ that is dual to a_r and if we represent this generator as $(4r+1)$-cycle B_r in $\mathring{U}_\infty$, then B_r bounds both in U_∞ (almost canonically) and in $\mathcal{A}_g$ (not canonically). The two bounding $(4r+2)$-chains make up a $(4r+2)$-cycle in $\mathcal{A}_g^{bb}$ whose class $z_r \in H_{4r+2}(\mathcal{A}_g^{bb};\mathbb{Q})$ pairs nontrivially with the image of y_r in $H^{4r+2}(\mathcal{A}_g^{bb};\mathbb{Q})$.

Proof that the y-classes are of weight zero. In view of Remark 2.5 it is enough to show that the image of $H^\bullet(\mathrm{GL}(\mathbb{Z});\mathbb{Q})$ in the stalk $(R^\bullet j_{g*}\mathbb{Q})_\infty$ has weight zero. For this we will need a toroidal resolution of U_∞ as described in [1], but we will try to get by with the minimal input necessary (for a somewhat more detailed review of this construction one may consult [6]).

Consider the symmetric quotient $\mathrm{Sym}_2\,\mathbb{Z}^g$ of $\mathbb{Z}^g\otimes\mathbb{Z}^g$ and regard it as a lattice in the space $\mathrm{Sym}_2\,\mathbb{R}^g$ of quadratic forms on $\mathrm{Hom}(\mathbb{Z}^g,\mathbb{R})$. The positive definite quadratic forms make up a cone $C_g \subset \mathrm{Sym}_2\,\mathbb{R}^g$ that is open and convex and is as such spanned by its intersection with $\mathrm{Sym}_2\,\mathbb{Z}^g$. Let $C_g^+ \supset C_g$ be the convex cone spanned by $\overline{C}_g \cap \mathrm{Sym}_2\,\mathbb{Z}^g$; this is just the set of semipositive quadratic forms on $\mathrm{Hom}(\mathbb{Z}^g,\mathbb{R})$

whose kernel is spanned by its intersection with $\mathrm{Hom}(\mathbb{Z}^g,\mathbb{Z})$. The obvious action of $\mathrm{GL}(g,\mathbb{Z})$ on $\mathrm{Sym}_2\,\mathbb{Z}^g$ preserves both cones and is proper on C_g.

Consider the algebraic torus $T_g := \mathbb{C}^\times \otimes_{\mathbb{Z}} \mathrm{Sym}_2\,\mathbb{Z}^g$. If we apply the 'log norm' $\mathrm{lgnm} : z \in \mathbb{C}^\times \mapsto \log|z| \in \mathbb{R}$ to the first tensor factor, we get a $\mathrm{GL}(g,\mathbb{Z})$-equivariant homomorphism $\mathrm{lgnm}_{T_g} : T_g \to \mathrm{Sym}_2\,\mathbb{R}^g$ with kernel the compact torus $U(1) \otimes_{\mathbb{Z}} \mathrm{Sym}_2\,\mathbb{Z}^g$. We denote by $\mathcal{T}_g \subset T_g$ the preimage of C_g so that we have defined a proper $\mathrm{GL}(g,\mathbb{Z})$-equivariant homomorphism of semigroups $\mathrm{lgnm}_{\mathcal{T}_g} : \mathcal{T}_g \to C_g$. Since $\mathrm{GL}(g,\mathbb{Z})$ acts properly on C_g it does so on $\mathcal{T}_g$ and hence the orbit space $\mathring{V} := \mathrm{GL}(g,\mathbb{Z})\backslash\mathcal{T}_g$ has the structure of a complex-analytic orbifold. There is an natural extension of $V \supset \mathring{V}$ in the complex analytic category (it is in fact the Stein hull of $\mathring{V}$ in case $g > 1$) that comes with a distinguished point that we will (for good reasons) also denote by ∞ such that $\mathring{V}$ is open-dense in V and $(V, V \smallsetminus \mathring{V})$ is topologically the open cone over a pair of spaces with vertex ∞. It has the property that there exists an open embedding of U_∞ in V that takes ∞ to ∞ and identifies U_∞ with a regular neighborhood of ∞ in V in such a way that $\mathring{U}_\infty = U_\infty \cap \mathring{V}$. This justifies that we focus on the triple $(V, \mathring{V}; \infty)$. All else we need to now about V is that the toroidal extension of $\mathring{V}$ that we are about to consider provides a resolution of V as an orbifold.

The universal cover of $\mathcal{T}_r$ is contractible (with covering group $\mathrm{Sym}_2\,\mathbb{Z}^g$) and hence the universal cover of $\mathring{V}$ as an orbifold is also contractible and has covering group $\mathrm{GL}(g,\mathbb{Z}) \ltimes \mathrm{Sym}_2\,\mathbb{Z}^g$ (it is in fact a virtual classifying space for this group). Similarly, the orbit space, $\mathcal{I} := \mathrm{GL}(g,\mathbb{Z})\backslash C_g$ exists as a real-analytic orbifold and is a virtual classifying space for $\mathrm{GL}(g,\mathbb{Z})$. The map lgnm_{T_g} induces a projection $\nu : \mathring{V} \to \mathcal{I}_g$ and the classes that concern us lie in the image of

$$H^\bullet(\mathrm{GL}(\mathbb{Z});\mathbb{Q}) \to H^\bullet(\mathrm{GL}(g,\mathbb{Z});\mathbb{Q}) \to H^\bullet(\mathcal{I}_g;\mathbb{Q}) \xrightarrow{\nu^*} H^\bullet(\mathring{V};\mathbb{Q}). \tag{5}$$

A *nonsingular admissible decomposition* of C_g^+ is a collection $\{\sigma\}_{\sigma\in\Sigma}$ of closed cones in C_g^+, each of which is spanned by a partial basis of $\mathrm{Sym}_2\,\mathbb{Z}^g$, such that the collection is closed under 'taking faces' and 'taking intersections' and whose relative interiors are pairwise disjoint with union C_g^+. Let Σ be such a decomposition that is also $\mathrm{GL}(g,\mathbb{Z})$-invariant and is fine enough in the sense that every $\mathrm{GL}(g,\mathbb{Z})$-orbit in C_+^g meets every member of Σ in at most one point. Such decompositions exist [1]. The associated torus embedding $T_g \subset T_g^\Sigma$ is then nonsingular and comes with an action of $\mathrm{GL}(g,\mathbb{Z})$. We denote by $\mathcal{T}_g^\Sigma$ the interior of the closure of $\mathcal{T}_g$ in T_g^Σ. This is an open $\mathrm{GL}(g,\mathbb{Z})$-invariant subset of T_g^Σ on which $\mathrm{GL}(g,\mathbb{Z})$ acts properly so that $V^\Sigma := \mathrm{GL}(g,\mathbb{Z})\backslash\mathcal{T}_g^\Sigma$ exists as an analytic orbifold. It is of the type alluded to above: we have a natural proper morphism $f : V^\Sigma \to V$ that is complex-algebraic over V and is an isomorphism over $\mathring{V}$. Moreover, the exceptional set is a simple normal crossing divisor in the orbifold sense.

As for every torus embedding there is also a real counterpart in the sense that $\mathrm{lgnm}_{\mathcal{T}_g}$ extends in a $\mathrm{GL}(g,\mathbb{Z})$-equivariant manner to a proper and surjective map $\mathrm{lgnm}_{\mathcal{T}_g^\Sigma} : \mathcal{T}_g^\Sigma \to C_g^\Sigma$, where C_g^Σ is a certain stratified locally compact Hausdorff space which contains C_g as an open dense subset. In the present case C_g^Σ is simply a manifold with corners, because Σ is nonsingular. The strata of C_g^Σ are indexed by Σ, with the stratum defined by σ being the image of C_g under the projection along the real subspace of $\mathrm{Sym}_2\,\mathbb{R}^g$ spanned by σ. So each stratum of C_g^Σ appears as a convex open subset of some vector space and it is all of this vector space precisely when the relative interior of σ is contained C_g. This is also equivalent to the stratum having compact closure in C_g^Σ.

Let us call a *wall* of C_g^Σ, the closure of a stratum defined by a ray (= a one-dimensional member) of Σ. So a wall is compact if and only if the associated ray lies in $C_g \cup \{0\}$. We denote by $\partial_{\mathrm{pr}} C_g^\Sigma$ the union of these compact walls. This is a closed subset of C_g^Σ and its covering by such compact walls is a *Leray covering*: the covering is locally finite and each nonempty intersection is contractible (and is in fact the closure of a stratum). Its nerve is easily expressed in terms of Σ. Let us say that a member of Σ is *proper* if it is contained in $C_g \cup \{0\}$. The proper members of Σ make up a subset $\Sigma_{\mathrm{pr}} \subset \Sigma$ that is also closed under 'taking faces' and 'taking intersections' and their union makes up a $\mathrm{GL}(g,\mathbb{Z})$-invariant cone contained in $C_g \cup \{0\}$. If we projectivize that cone we get a simplicial complex in the real projective space of $\mathrm{Sym}_2 \mathbb{R}^g$ that we denote by $P(\Sigma_{\mathrm{pr}})$. A vertex of $P(\Sigma_{\mathrm{pr}})$ corresponds of course to a ray of Σ_{pr} and this in turn defines a compact wall of C_g^Σ. In this way $P(\Sigma_{\mathrm{pr}})$ can be identified in a $\mathrm{GL}(g,\mathbb{Z})$-equivariant manner with the nerve complex of the covering of $\partial_{\mathrm{pr}} C_g^\Sigma$ by the compact walls of C_g^Σ. A standard argument shows that we have a $\mathrm{GL}(g,\mathbb{Z})$-equivariant homotopy equivalence between $\partial_{\mathrm{pr}} C_g^\Sigma$ and the nerve $P(\Sigma_{\mathrm{pr}})$ of this covering.

Each stratum closure in C_g^Σ can be retracted in a canonical manner onto its intersection with $\partial_{\mathrm{pr}} C_g^\Sigma$ and we thus find a $\mathrm{GL}(g,\mathbb{Z})$-equivariant deformation retraction $C_g^\Sigma \to \partial_{\mathrm{pr}} C_g^\Sigma$. This shows at the same time that the inclusion $C_g \subset C_g^\Sigma$ is a $\mathrm{GL}(g,\mathbb{Z})$-equivariant homotopy equivalence. So if we put $\mathcal{I}_g^\Sigma := \mathrm{GL}(g,\mathbb{Z})\backslash C_g^\Sigma$ and $\partial_{\mathrm{pr}}\mathcal{I}_g^\Sigma := \mathrm{GL}(g,\mathbb{Z})\backslash \partial_{\mathrm{pr}} C_g^\Sigma$, then we end up with homotopy equivalences $\mathcal{I}_g \subset \mathcal{I}_g^\Sigma \supset \partial_{\mathrm{pr}}\mathcal{I}_g^\Sigma$. We also have a homotopy equivalence $\partial_{\mathrm{pr}}\mathcal{I}_g^\Sigma \sim \mathrm{GL}(g,\mathbb{Z})\backslash P(\Sigma_{\mathrm{pr}})$.

Taking the preimage under lgnm makes walls of C_g^Σ correspond to irreducible components of the toric boundary $\mathcal{T}_g^\Sigma \smallsetminus \mathcal{T}_g$ and a wall of C_g^Σ is compact if and only if the associated irreducible component is. So the preimage $\partial_{\mathrm{pr}}\mathcal{T}_g^\Sigma$ of $\partial_{\mathrm{pr}} C_g^\Sigma$ is the union of the compact irreducible components of the toric boundary. It is clear that $P(\Sigma_{\mathrm{pr}})$ is also the nerve of the covering of $\partial_{\mathrm{pr}}\mathcal{T}_g^\Sigma$ by its irreducible components. The image of $\partial_{\mathrm{pr}}\mathcal{T}_g^\Sigma$ in V (in other words, its $\mathrm{GL}(g,\mathbb{Z})$-orbit space) is the normal crossing divisor $f^{-1}(\infty)$. The inclusion $f^{-1}(\infty) \subset V^\Sigma$ is also a deformation retract. So in the commutative diagram

$$\begin{array}{ccccc}
\mathring{V} & \hookrightarrow & V^\Sigma & \hookleftarrow & f^{-1}(\infty) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{I}_g & \hookrightarrow & \mathcal{I}_g^\Sigma & \hookleftarrow & \partial_{\mathrm{pr}}\mathcal{I}^\Sigma
\end{array}$$

the inclusion on the top right and those at the bottom are homotopy equivalences. It follows that the composite map in diagram (5) factors through the rational cohomology of $\mathcal{I}_g^\Sigma$ and hence also through the rational cohomology of V^Σ and that the nonzero classes in $H^\bullet(V^\Sigma;\mathbb{Q}) \cong H^\bullet(f^{-1}(\infty);\mathbb{Q})$ that we thus obtain come from the nerve of the covering of $f^{-1}(\infty)$ by its irreducible components. Such classes are known to be of weight zero [8]. □

Remark 2.6. Goresky and Pardon [9] have constructed a lift c_r^{bb} of the *real* Chern class $c_r \in H^{2r}(\mathcal{A}_g;\mathbb{R})$ to $H^{2r}(\mathcal{A}_g^{bb};\mathbb{R})$. The second author [2] recently proved that

c_r^{bb} (and hence also the corresponding Chern character ch_r^{bb}) lies in $F^rH^{2r}(\mathcal{A}_g^{bb};\mathbb{R})$. So the class of the Tate extension in Remark 1.3 is up to a rational number given by the value of c_{2r+1}^{bb} on the class $z_r \in H_{4r+2}(\mathcal{A}_g^{bb};\mathbb{Q})$ found in Remark 2.5 (two choices of z_r differ by a class of the form $j_{g*}(w)$ with $w \in H_{4r+2}(\mathcal{A}_g;\mathbb{Q})$ and c_{2r+1}^{bb} takes on such a class the rational value $c_{2r+1}(w)$). Arvind Nair, after learning of our theorem, informed us that his techniques enable him to show that this extension class is nonzero. Subsequently a different proof (based on the Beilinson regulator) was given in [2].

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