Goresky-Pardon extensions of Chern classes and associated Tate extensions

EDUARD LOOIJENGA

ABSTRACT. Let $X$ be an irreducible complex-analytic variety, $S$ a stratification of $X$ and $F$ a holomorphic vector bundle on the open statum $X^\circ$. We give geometric conditions on $S$ and $F$ that produce a natural extension of the Chern class $c_k(F) \in H^{2k}(X^\circ; \mathbb{C})$ to $H^{2k}(X; \mathbb{C})$, which, in the algebraic setting, is of Hodge level $\geq k$. When applied to the Baily-Borel compactification $X$ of a locally symmetric variety $X^\circ$ and $F$ an automorphic vector bundle, this refines a theorem of Goresky-Pardon. In passing we define a class of simplicial resolutions of the Baily-Borel compactification that can be used to define its mixed Hodge structure. We use this to show that the stable cohomology of the Satake (=Baily-Borel) compactification of $A_g$ contains nontrivial Tate extensions.

1. Introduction

Let $X$ be an irreducible complex-analytic variety, $X^\circ$ a nonsingular Zariski open-dense subset of $X$ and $F$ a holomorphic vector bundle on $X^\circ$. In this paper we give conditions under which the rational Chern classes $c_k(F) \in H^{2k}(X^\circ; \mathbb{Q})$ extend in a canonical manner as complex classes to $X$, even (and especially) in situations where $F$ is known not to extend to $X$ as a complex vector bundle. The passage to complex cohomology is not just an artefact of our method, for we find examples for which the imaginary part of such a extension is nonzero. Before we say more about what is in this paper, we mention the situation that is both the origin and the motivation for addressing this question. This is when $X^\circ$ is a locally symmetric variety, $X = (X^\circ)_{bb}$ its Baily-Borel compactification, and $F$ an automorphic vector bundle on $X^\circ$. Mumford defined in 1977 [9] Chern numbers for an automorphic bundle $F$ as integrals of Chern forms relative to some metric on $X^\circ$ (using his toroidal compactifications to prove their absolute convergence) and proved them to have properties that Hirzebruch had earlier established in case $X^\circ$ is compact. A quarter of a century later Goresky and Pardon [7] proved that the Chern classes of such an $F$ can be naturally extended to $X$ in such a manner that the associated Chern numbers (i.e., polynomials in these classes evaluated on the fundamental class of $X$) yield those of Mumford.

Returning to the content of this article, it has four principal results. The first one may be characterized as putting the result of Goresky and Pardon in (what we feel is) its natural setting. This has in any case the effect of making statements more transparent and proofs shorter. Key to this approach are the rather simple concepts formulated in Section 2. Our point of departure is not just $X$ with its Zariski open-dense subset, but rather an analytic stratification $S$ of $X$ for which $X^\circ$ is the union of the open strata. We introduce (in 2.2) certain analytic control data on $(X, S)$ embodied in the notion of a system of local retractions. For a stratification $(X, S)$ thus endowed, we define (in 2.4) a corresponding notion for a holomorphic vector
bundle on $X^\circ$, namely that of an isoholonomic flat structure. This structure may also be regarded as a set of control data (in the sense of stratification theory), but now on the vector bundle and compatible with the local retractions. Both notions are analytic in character and have algebraic counterparts. Proposition 2.5 states that this last structure suffices to produce a natural extension to $X$ of the complex Chern classes. We then show that such structures are present on the Baily-Borel stratification resp. an automorphic vector bundle, so that this recovers the result of Goresky-Pardon. (We work this out in the case of the symplectic group.)

The second main result pertains to the complete, complex-algebraic setting, where we prove (Theorem 2.8) that these Chern class extensions have the expected Hodge level, provided that $(X, S)$ admits (what we have called) a stratified resolution (Definition 2.10). This leads to a simplicial resolution of $X$ by complete nonsingular varieties which satisfies cohomological descent so that it can be used to describe the mixed Hodge structure on the cohomology of $X$.

The third part applies this to Baily-Borel compactifications: Theorem 4.4 states that some of Mumford’s toroidal resolutions of a Baily-Borel compactification give rise to a stratified resolution. Since these can be used to identify the mixed Hodge structure on the cohomology of a Baily-Borel compactification, we hope that this will find other applications as well.

Our fourth contribution is an application of the preceding to the stable cohomology of the Satake (=Baily-Borel) compactification $A_{gb}^\text{bb}$ of $A_g$. Charney and Lee [4] have shown that for a fixed $k$, $H^k(A_g; \mathbb{Q})$ stabilizes as $g \to \infty$ and that the resulting stable cohomology $H^\bullet$ has the structure of a $\mathbb{Q}$-Hopf algebra. This is in fact a polynomial algebra with primitive basis $\tilde{c}h_{2r+1} \in H^{4r+2}$ ($r \geq 0$) and $y_r \in H^{4r+2}$ ($r > 0$), although the $\tilde{c}h_{2r+1}$’s are not canonically defined and the $y_r$’s are only defined up to sign. So $H_{pr}^{4r+2}$ is of dimension 2 when $r > 0$. Jiaming Chen and the author [5] have recently shown that $H^\bullet$ has a natural mixed Hodge structure that gives $H_{pr}^{4r+2}$ the structure of a Tate extension: it is an extension of $\mathbb{Q}(-2r-1)$ (which has the image of $\tilde{c}h_{2r+1}$ as generator) by $\mathbb{Q}(0)$ (which has $y_r$ as generator). With the help of the results described above, we find that the one-dimensional space $F^{2r+1}H_{pr}^{4r+2}$ is in fact spanned by the Goresky-Pardon Chern character of the Hodge bundle on $A_g$ ($g \gg r$). We then use the theory that underlies the construction of the Beilinson regulator to compute the class of this Tate extension (Theorem 5.1) and find it to be nonzero. At the same time we show that the Goresky-Pardon Chern character in question has a real part that is rational (so lies in $H_{pr}^{4r+2}$), but that its imaginary part is nonzero. This answers (negatively) the question asked by Goresky-Pardon ((1.6) of [7]) whether their extension always lives in rational cohomology. Our examples leave open the possibility that this is so for the real part of this class (say, in the setting of an automorphic vector bundle over the interior of a Baily-Borel compactification).

We close this introduction with a brief discussion of how this is connected with other work in this area. Goresky and Tai proved in [8] that an automorphic vector bundle on a locally symmetric variety $X^\circ$ extends naturally to what is called the reductive Borel-Serre compactification of $X^\circ$. This compactification, which we shall denote for the purpose of this introduction by $\hat{X}$, has a real-analytic structure and dominates $X$ in the sense that the latter is naturally a quotient of $\hat{X}$, but lives by no means in the complex-analytic category. Goresky and Tai predicted that the Chern
classes of their extension are simply pull-backs of the Goresky-Pardon Chern classes to $\hat{X}$ and this was later proved by Zucker [13] (with some corrections supplied by Ayoub and Zucker [2], see also [10]). Shortly afterwards Zucker [14] showed that the quotient map $\hat{X} \to X$, despite not being in any sense a morphism of complex-algebraic varieties, behaves from a cohomological point of view as if it were, for he proved that $H^\bullet(\hat{X})$ carries a natural mixed Hodge structure such that the induced map $H^\bullet(X) \to H^\bullet(\hat{X})$ is a morphism in this category. Very recently Arvind Nair showed in [11] that the Chern classes of the Goresky-Tai extension have the expected Hodge level and he there formulated our second main result as a conjecture ((4.2) of op. cit.), something we had not been aware of while working on this project. In light of Zucker’s result, our theorem implies the property proved by Nair, but is not equivalent to it, as the map $H^\bullet(X) \to H^\bullet(\hat{X})$ may not be injective.

In correspondence with Klaus Hulek and others in connection with [5] we had wondered about the possibly nontrivial nature of the above Tate extension. Via audience feedback to a talk of his at the IAS (that apparently had made mention of this question), we learned that the work of Nair might shed light on this and indeed, when we wrote Nair, he informed us (in April 2015) that his techniques—which involve among other things local Hecke operators and analytic results due to Franke—enable him to determine the class of this extension (which turned out to be nonzero). The proof given here was found thereafter (September 2015), but is, we understand, quite different from his.

It is a pleasure to acknowledge the numerous conversations with Spencer Bloch on this material. He drew my attention to the Chern class extensions defined by Goresky-Pardon, and also suggested (at a time when neither of us was aware that Nair had in fact conjectured this) that these classes might have the Hodge level property that is established here. I am also indebted to Mark Goresky, who pointed out to me a subtlety regarding the partial flat structures on automorphic bundles that I had overlooked.

2. CHERN CLASSES IN A STRATIFIED SETTING

**Isoholonomic relative connections.** Let $\rho : M \to S$ be a submersion of complex manifolds and let $\mathcal{F}$ be a holomorphic vector bundle on $M$ of rank $r$. We need the following three notions relative to $\rho$.

**Definition 2.1.** We say that a $C^\infty$-differential form on $M$ is $\rho$-basic if it is locally the pull-back along $\rho$ of a form on $S$.

A flat $\rho$-structure on $\mathcal{F}$ is a flat holomorphic connection along the fibers of $\rho$, i.e., is given by a $\rho^{-1}\mathcal{O}_S$-linear map $\nabla_\rho : \mathcal{F} \to \Omega_\rho \otimes \mathcal{F}$ satisfying the Leibniz property:

$$\nabla_\rho(\phi s) = \phi \nabla_\rho(s) + d_\rho(\phi) \otimes s$$

whose relative curvature (an $\mathcal{O}_M$-homomorphism $\mathcal{F} \to \Omega^2_\rho \otimes \mathcal{F}$) is identically zero.

We say that such a flat $\rho$-structure on $\mathcal{F}$ is isoholonomic if we can cover $S$ by open subsets $V$ such that $\nabla_\rho|_{\rho^{-1}V}$ can be lifted to a flat holomorphic connection on $\mathcal{F}|_{\rho^{-1}V}$.

Let us comment on these definitions. We begin with observing that if $\rho$ factors through a submersion $\rho' : M \to S'$ and one of the three properties above holds for $\rho$, then that property also holds for $\rho'$.
Next we note that we can drop the adjective ‘locally’ in the definition of a $\rho$-basic form if the fibers of $\rho$ are connected: it is then just the pull-back of a form on $S$. This is still true if the set of $v \in S$ for which $\rho^{-1}(v)$ is connected contains an open-dense subset of $S$ (we then say that a general fiber of $\rho$ is connected).

For a flat $\rho$-structure $\nabla_\rho$ on $\mathcal{F}$, the $\rho$-flat local sections make up a subsheaf $\mathcal{F} \subset \mathcal{F}$. This is a locally free $\rho^{-1}\mathcal{O}_S$-submodule of rank $r$ with the property that the natural map $\mathcal{O}_M \otimes_{\rho^{-1}\mathcal{O}_S} \mathcal{F} \rightarrow \mathcal{F}$ is an isomorphism. The converse also holds: a subsheaf $\mathcal{F} \subset \mathcal{F}$ with these properties defines a flat $\rho$-structure on $\mathcal{F}$. This also amounts to giving a (maximal) atlas of local holomorphic trivializations of $\mathcal{F}$ whose transition functions factor through $\rho$. In the situations that we shall consider, $\rho$ will be topologically locally trivial with connected fibers, and then a given flat $\rho$-structure is isoholonomic precisely if its holonomy along $\rho^{-1}(v)$ (given as a $GL(\mathcal{F})$-orbit in $\text{Hom}(\pi_1(\rho^{-1}(v)), GL(\mathcal{F}, \mathbb{C}))$) is locally constant in $v \in S$ in an evident sense. Whence the terminology.

Let be given an isoholonomic flat $\rho$-structure $\nabla_\rho$ on $\mathcal{F}$. We then have an open covering $\{V_\alpha\}_\alpha$ of $S$ and for every $\alpha$ a flat holomorphic connection $\nabla^\alpha : \mathcal{F}|_{\rho^{-1}V_\alpha} \rightarrow \Omega_M \otimes \mathcal{F}|_{\rho^{-1}V_\alpha}$ which extends $\nabla_\rho|_{\rho^{-1}V_\alpha}$ if $\{\phi_\alpha\}_\alpha$ is a $C^\infty$ partition of unity on $S$ with $\sup(\phi_\alpha) \subset V_\alpha$, then $\nabla := \sum_\alpha \rho^*(\phi_\alpha)\nabla^\alpha$ is a $C^\infty$-connection on $\mathcal{F}$ which globally lifts $\nabla_\rho$ in a particular way: in terms of a local trivialization in the atlas described above, this connection is given by a matrix of $\rho$-basic forms of type $(1, 0)$. Its curvature is therefore given by a matrix of $\rho$-basic 2-forms of Hodge level $\geq 1$ (i.e., is a sum of forms of type $(2, 0)$ and $(1, 1)$). Hence the Chern form $C_k(\nabla)$ is a $\rho$-basic $2k$-form of Hodge level $\geq k$. Note that this remains so if we alter the connection $\nabla_\rho$ by adding to it a nilpotent relative differential $\eta$ (i.e., a section of $\Omega_\rho \otimes \mathcal{E}_{nil}(\mathcal{F})$, that takes values in the nilpotent endomorphisms), for then the curvature form of $\nabla$ will be nilpotent along the fibers of $\rho$ and so $C_k(\nabla + \eta)$ is also closed, it is then still $\rho$-basic. So when the general fiber of $\rho$ is connected, $C_k(\nabla + \eta)$ is the pull-back of one on $S$. In particular, the complex $k$th Chern class of $\mathcal{F}$ lies in the image of $\rho^* : H^{2k}(S; \mathbb{C}) \rightarrow H^{2k}(M; \mathbb{C})$.

But as we will see in our main application, it is possible for $F$ to have nontrivial holonomy along the fibers of $\rho$ and so $\mathcal{F}$ need not be a sheaf pull-back of a holomorphic vector bundle on $S$.

We next extend this to a stratified setting. This naturally leads us to consider ‘germ versions’ of the notions we just introduced. Let $X$ be a complex-analytic variety endowed with a stratification $\mathcal{S}$, by which we mean a finite partition of $X$ into connected nonsingular locally closed subvarieties, called strata, such that the closure of a stratum is a subvariety that is a union of strata. We partially order the collection of strata by letting $S' \leq S$ mean that $S' \subset \overline{S}$.

**Definition 2.2.** A system of retractions $\rho = (\rho_S)_S$ for $(X, \mathcal{S})$ assigns to each $S \in \mathcal{S}$ an analytic retraction $\rho_S : X_S \rightarrow S$ with the property that when $S' \leq S$, then $\rho_S|_{X_{S'}} = \rho_{S'}$. We then say that that $(X, \mathcal{S}, \rho)$ is a rigidified stratified variety\(^1\).

Here $X_S$ denotes the germ of $X$ at $S$ and so this means that $\rho_S$ is represented by an analytic retraction whose domain is unspecified neighborhood $U_S$ of $S$ in

\(^1\)There is of course also a $C^\infty$-variant of this notion, but we will here be only interested in the holomorphic version.
$X$. If the stratification $S$ satisfies Whitney’s (a) condition, then we may take $U_S$ so small such that for every $S' \in S$, $\rho_S|U_S \cap S'$ is a submersion. Note that for every $S \in S$, the collection $\{\rho_S|\bar{S}\}_{S' \leq S}$ is a system of retractions for $(\bar{S}, S, \bar{S})$. A complex submanifold of a complex manifold need not be a holomorphic retract of some neighborhood of it and so the mere existence of such a system indicates that the stratification is quite special. A standard example is the natural stratification of a torus embedding. We will see that the Satake and toric compactifications of $X$.

We make the rigidified stratified varieties objects of a category: a morphism $(\bar{X}, \bar{S}, \bar{\rho}) \to (X, S, \rho)$ is given by a complex-analytic morphism $\pi: \bar{X} \to X$ that takes any stratum $\bar{S}$ of $\bar{\rho}$ submersively to a stratum $S$ of $S$ in such a manner that on a neighborhood of $\bar{S}$ we have $\pi\bar{\rho}_\bar{S} = \rho_S\pi$ and we demand that the preimage of the union of the open strata in $X$ is equal to the union of the open strata in $\bar{X}$.

From now on $(X, S, \rho)$ is a rigidified stratified variety with $X$ topologically normal (in the sense that normalization is a homeomorphism). We denote by $X^0 \subset X$ the union of the open strata and by $j: X^0 \subset X$ and $i_S: S \subset X$ ($S \in S$) the inclusions.

We write $\rho^0_S$ for the restriction of $\rho_S$ to $X^0$. The assumption of topological normality guarantees that a general fiber of $\rho^0_S$ is connected.

**Definition 2.3.** We say that a $C^\infty$-differential form on $X^0$ is $\rho^0$-basic if for every stratum $S \in S$, its germ at $S$ is $\rho^0_S$-basic.

The $\rho^0$-basic $C^\infty$-differential forms in $j_*A^*_X$ make up a differential bigraded sub-algebra $A^*_X, \rho$ that is a fine resolution of the constant sheaf $\mathbb{C}_X$ on $X$ (see Verona [12] and Theorem 4.2 in [7]). It has the property that for all $S \in S$, $i_S^{-1}A^*_X, \rho = A^*_S$. Its holomorphic part defines a subcomplex $\Omega^p_X, \rho \subset j_*\Omega^p_X$ with a similar property: $i_S^{-1}\Omega^p_X, \rho = \Omega^p_S$. This is also a resolution of the constant sheaf $\mathbb{C}_X$ and we can regard $A^*_X, \rho$ as a double complex which resolves it. So $(A^p_{X, \rho}^\bullet, \bar{\partial})$ resolves $\Omega^p_X, \rho$ and we have a Hodge-De Rham spectral sequence

$$E^q_r = H^q(X, \Omega^p_X, \rho) \Rightarrow H^{p+q}(X, \mathbb{C}).$$

If we are in the complex projective setting, then one may wonder whether this spectral sequence degenerates and yields the Hodge filtration of the mixed Hodge structure on $H^*(X)$. This is not so in general (there exist examples for which $H^0(X, \Omega^p_X, \rho) = 0 \neq F^pH^p(X)$), but if it is at least true that the limit filtration of this spectral sequence refines the Hodge filtration, then we would have a generalization of Theorem 2.8 below and probably also end up with a simpler proof of it.

Note that a morphism $\pi: (\bar{X}, \bar{S}, \bar{\rho}) \to (X, S, \rho)$ determines a map of sheaf complexes $\pi^{-1}A^*_X, \rho \to A^*_X, \rho$, which induces the usual map $\pi^*: H^*(X; \mathbb{C}) \to H^*(\bar{X}; \mathbb{C})$ on cohomology.

**Definition 2.4.** Let $F$ be a holomorphic vector bundle on $X^0$. A $\rho^0$-flat structure $\nabla_\rho = (\nabla_{\rho_S})_{S \in S}$ on $F$ assigns to every $S \in S$ a $\rho_S$-flat structure $\nabla_{\rho_S}$ on $i_S^{-1}j_*F$ and is subject to a compatibility condition up to nilpotents: noting that for any pair of incident strata $S \geq S'$, the connection $\nabla_{\rho_{S'}}$ induces a flat connection along the fibers of $\rho_S'$ on the common domain of $\rho_S$ and $\rho_{S'}$ (for the submersion $\rho_S'$ there factors through $\rho^0_S$) so that we may write this connection there as $\nabla_{\rho_S} + \eta^S_{S'}$ with $\eta^S_{S'}$ a section of $\Omega^p_{\rho_S} \otimes \text{End}(F)$, then we require that $\eta^S_{S'}$ is a nilpotent relative differential. More generally, whenever we have a chain of strata $S > S_1 > \cdots > S_n$,
we ask that the $\eta^S_1, \ldots, \eta^S_n$ span on their common domain of definition a complex vector space of nilpotent relative differentials.

We say that the $\rho$-flat structure on $\mathcal{F}$ is isoholonomic if for every $S \in \mathcal{S}$, $\nabla_{\rho_S}$ is so on $i^{-1}_S j_* \mathcal{F}$.

Given a $\rho$-flat structure on $\mathcal{F}$, then its $\rho$-flat local sections define a subsheaf of $j_* \mathcal{F}$, but this subsheaf can be zero on certain strata and is probably of little interest unless the holonomies are trivial (as is the case discussed in Remark ??). More relevant to us is the subsheaf of $\mathcal{O}_{X, \rho}$-algebras $\mathcal{End}(\mathcal{F}, \nabla_{\rho}) \subset j_* \mathcal{End}(\mathcal{F})$ of $\rho$-flat local endomorphisms of $j_* \mathcal{F}$, at least when the $\rho$-flat structure on $\mathcal{F}$ is isoholonomic, for then $i^{-1}_S \mathcal{End}(\mathcal{F}, \nabla_{\rho})$ is locally like $\mathcal{End}_{\mathcal{O}}(\mathcal{O}_S)$ (it is a sheaf of Azumaya $\mathcal{O}_S$-algebras).

Note that if $\pi : (\hat{X}, \hat{S}, \hat{\rho}) \to (X, S, \rho)$ is a morphism of rigidified stratified, topological normal varieties, then a $\rho$-flat structure on $\mathcal{F}$ determines one on the pullback of $\mathcal{F}$ along $\pi^\circ : \hat{X}^\circ \to X^\circ$ (that we simply denote by $\pi^* \nabla_{\rho}$): if $\pi$ maps $\hat{S} \in \hat{S}$ to $S \in S$, then $\nabla_{\rho_S}$ determines in an obvious manner a flat connection along the fibers of $\rho^\circ_S$ and the resulting system has the required properties. It is isoholonomic when $\nabla_{\rho}$ is.

**Proposition 2.5.** With a holomorphic vector bundle $\mathcal{F}$ on $X^\circ$ that is endowed with an isoholonomic $\rho$-flat structure $\nabla_{\rho} = (\nabla_{\rho_S})_{S \in \mathcal{S}}$ is associated a complex Chern class extension $c_k(\mathcal{F}, \nabla_{\rho}) \in H^{2k}(X; \mathbb{C})$ of $c_k(\mathcal{F}) \subset H^{2k}(X^\circ; \mathbb{C})$ ($k \geq 0$). This is functorial in the following sense: if $\pi : (\hat{X}, \hat{S}, \hat{\rho}) \to (X, S, \rho)$ is a morphism of rigidified stratified spaces, then $\pi^* c_k(\mathcal{F}, \nabla_{\rho}) = c_k(\pi^* \mathcal{F}, \nabla_{\hat{\rho}})$. It also has the property that if the relative holonomy of $\nabla_{\rho_S}$ is trivial at every point of $S$ and for any pair of incident strata $S \geq S'$, $\eta^S_S = 0$, then $(\mathcal{F}, \rho)$ extends naturally to $(\hat{\mathcal{F}}, \hat{\rho})$ on $X$ (as a holomorphic vector bundle with flat connections along the retractions) and we have $c_k(\mathcal{F}, \nabla_{\hat{\rho}}) = c_k(\hat{\mathcal{F}}) \subset \mathbb{C}$.

**Proof.** By assumption $X$ admits a covering by open subsets $U_\alpha$ with the property that there is unique $S_\alpha \in \mathcal{S}$ such that $S_\alpha \cap U_\alpha$ is closed, $\rho_{S_\alpha}$ and $\nabla_{\rho_{S_\alpha}}$ are defined on $U_\alpha$ and $\nabla_{\rho_{S_\alpha}}$ lifts to a flat holomorphic connection $\nabla^\alpha$ on $\mathcal{F}|_{U_\alpha}$. We can choose a partition of unity $\{\phi_\alpha : X \to [0, 1]\}_{\alpha}$ with $\sup(\phi_\alpha) \subset U_\alpha$ and with $\phi_\alpha|_{U_\alpha}$ factoring through $\rho_{S_\alpha}$. Then $\nabla := \sum \phi_\alpha \nabla^\alpha$ is a $C^\infty$-connection on $\mathcal{F}$ with the property that any $S \in \mathcal{S}$ admits a neighborhood $U_S$ in $X$ such that for any chain $S > S_1 > \cdots > S_n$ in $\mathcal{S}$, the relative connection $\nabla$ induces along $\rho_S|_{U_S} \cap U_S \cap \cdots \cap U_{S_n}$ is a convex linear combination of $\nabla_{\rho_S}$ and $\eta^S_1, \ldots, \eta^S_n$. This implies that the Chern form $C_k(\mathcal{F}, \nabla)$ is a $\rho$-basic closed form of Hodge level $\geq k$. By the fine resolution property cited above, it therefore defines a cohomology class $c_k(\mathcal{F}, \nabla)$.

The proof that this class is independent of our choices is a straightforward generalization of the standard proof and is based on the observation that the $C^\infty$-connections on $\mathcal{F}$ satisfying the above property are an affine space. Indeed, if $\nabla$ is another such connection, then we define on the pull-back $\hat{\mathcal{F}} := \rho^{-1}_X \mathcal{F}$ of $\mathcal{F}$ along $\rho_X : \mathbb{C} \times X^\circ \to X^\circ$ a connection $\hat{\nabla}$ given on a pulled back section $\rho^{-1}_X s$ as $(1 - t)\nabla(s) + t \cdot \nabla(s)$. Then $C_k(\mathcal{F}, \nabla)$ defines (by the result above) a class $c_k(\mathcal{F}, \nabla) \in H^{2k}(\mathbb{C} \times X)$. This class evidently restricts to $c_k(\mathcal{F}, \nabla)$ resp. $c_k(\mathcal{F}, \nabla)$ if we take the first coordinate 0 resp. 1 and so $c_k(\mathcal{F}, \nabla) = c_k(\mathcal{F}, \nabla)$. It is straightforward to verify that these Chern classes have the asserted naturality behaviour.

Assume now that the relative holonomy of $\nabla_{\rho_S}$ is trivial at every point of $S$ and for any pair of incident strata $S \geq S'$, $\eta^S_{S'} = 0$. Choose for every stratum $S$ a
neighborhood $U_S$ of $S$ in $X$ contained in the domain of $\rho_S$ and $\nabla_{\rho_S}$ such that $\nabla_{\rho_S}$ has no holonomy on $U^\circ$. Then the subsheaf of $\mathcal{F}|U_S^\circ$ of $\nabla_{\rho_S}$-flat sections has a direct image on $U_S$, whose restriction to $S$ is a holomorphic vector bundle. Its pullback as a vector bundle along $\rho_S$ can on a neighborhood of $S$ in $X^\circ$ be identified with $\mathcal{F}$ and so this defines an extension of $\mathcal{F}$ across $U_S$. Since the $\eta^S_{\hat{\omega}}$ vanish, such extensions agree on overlaps. It follows that $\mathcal{F}$ extends to a holomorphic bundle $\hat{\mathcal{F}}$ on $X$. Although $X$ may be singular, a connection as constructed above extends to a connection $\hat{\nabla}$ on $\hat{\mathcal{F}}$ in the sense that it is locally given by a matrix with entries in $\Omega_X$ (so restrictions of holomorphic differentials on an ambient complex manifold).

Then $C_k(\hat{\mathcal{F}}, \hat{\nabla})$ is a $C^\infty$ 2k-form (i.e., locally the restriction to $X$ of a form defined on an ambient $C^\infty$-manifold) and therefore defines a class in $H^{2k}(X; \mathbb{C})$. Since its restriction to $X^\circ$ is $C_k(\mathcal{F}, \nabla)$, this class is in fact $c_k(\mathcal{F}, \nabla_{\rho})$.

In the situation of the last clause of Proposition 2.5 we find that $c_k(\mathcal{F}, \nabla_{\rho})$ lifts to an integral class. But we will see that this is not so in general.

Remark 2.6. We shall later want to work with Chern characters $ch_k$ rather than with Chern classes $c_k$. Since we always use $\mathbb{Q}$-vector spaces as coefficients, there is no loss of information here: $ch_k$ is a universal polynomial of weighted degree $k$ with rational coefficients in $c_1, \ldots, c_k$ and vice versa. These Chern characters can also be obtained via an Atiyah class, which is perhaps closer to the spirit of algebraic geometry, albeit that they then come to be realized as De Rham classes. An isoholonomic $\rho$-flat structure defines a natural lift of the Atiyah class of $\mathcal{F}$, $\text{At}(\mathcal{F}) \in H^1(X^\circ, \Omega_{X^\circ}^1 \otimes \text{End}(\mathcal{F}))$ to an element $\text{At}(\mathcal{F}, \nabla_{\rho}) \in H^1(X, \Omega_{X,\rho}^1 \otimes \text{End}(\mathcal{F}, \nabla_{\rho}))$. A representative as a 1-Čech cocycle is obtained from the collection $(U_\alpha, \nabla^\alpha)_\alpha$ in the proof above: it is given by $U_{\alpha\beta} = U_\alpha \cap U_\beta \mapsto \nabla^\beta - \nabla^\alpha$. We then define the twisted Goresky-Pardon Chern character as the image of $\text{At}(\mathcal{F}, \nabla_{\rho})$ under the map

$$H^1(X, \Omega_{X,\rho}^1 \otimes \text{End}(\mathcal{F}, \nabla_{\rho})) \to \bigoplus_{k=0}^\infty H^k(X, \Omega_{X,\rho}^k),$$

$$A \mapsto \text{Tr}(\exp(-A)) = \sum_{k=0}^\infty \frac{\text{Tr}((-A)^{\lfloor k/2 \rfloor})}{k!}.$$ 

This class is closed for all the differentials in the Hodge-De Rham spectral sequence and then yields $(2\pi \sqrt{-1})^k ch_k(\mathcal{F}, \nabla_{\rho})$. This observation leads us to:

Corollary 2.7. Suppose that in the situation of Proposition 2.5, the setting is algebraic over $\mathbb{R}$, that is, $X$, its stratification and the retractions appearing there and the vector bundle $\mathcal{F}$ are defined over $\mathbb{R}$. Then the twisted Goresky-Pardon Chern character $(2\pi \sqrt{-1})^k ch_k(\mathcal{F}, \nabla_{\rho})$ is fixed under full complex conjugation (acting on both $X$ and the coefficient field $\mathbb{C}$).

Proof. We must verify that $(2\pi \sqrt{-1})^k ch_k(\mathcal{F}, \nabla_{\rho})$ is fixed under the anti-linear map $z \in H^*(X; \mathbb{C}) \mapsto \overline{z} \in H^*(X; \mathbb{C})$, where $\iota : X \to X$ is complex conjugation. In the Čech description of the Atiyah class above we can choose the collection $(U_\alpha, \nabla^\alpha)_\alpha$ in such a manner that $\iota$ acts compatibly on our index set: $\iota(U_\alpha) = U_{\iota \alpha}$ and $\iota^* \nabla^\alpha = \nabla^{\iota \alpha}$. Then it is clear from the definition that $(2\pi \sqrt{-1})^k ch_k(\mathcal{F}, \nabla_{\rho})$ has the asserted property. \hfill \Box

Theorem 2.8. Suppose that in the situation of Proposition 2.5, the setting is algebraic and that $X$ is complete. Suppose moreover that the resolution $\pi : \tilde{X} \to X$ that satisfies
the holonomy property with respect to \((F, \nabla_F)\) extends to a stratified resolution in the sense below. Then \(c_k(F, \nabla_F)\) is of Hodge level \(\geq k\), i.e., lies in \(F^kH^{2k}(X; \mathbb{C})\).

**Remark 2.9.** Since the cup product is compatible with the Hodge filtration, it then follows that the corresponding class \(c_k(F, \nabla_F)\) is also of Hodge level \(\geq k\).

The notion of a resolution of a stratified variety that appears in the formulation of the theorem above expresses the fact that such a variety is equisingular along strata in a rather strong sense. Among other things, it can be shown to imply Whitney’s \((a)\) condition. We define this notion and prove the theorem in the next subsection.

**Stratified resolutions.** We begin with noting that if on a complex manifold \(Y\) is given a normal crossing divisor \(D\), then \(Y\) acquires a natural stratification, where a stratum is a connected component of the locus where for some integer \(l \geq 0\) exactly \(l\) local branches of \(D\) meet. With \(D\) given, we will often write \(Y^{(l)}\) for the normalization of the locus where at least \(l\) branches of \(D\) meet (so that \(Y^{(0)} = Y\)). This is clearly a complex manifold. If \(E\) is a connected component of \(Y^{(l)}\), then the locus where \(> l\) branches of \(D\) meet traces out on \(E\) a normal crossing divisor, which is simple when \(D\) is and whose normalization is contained in \(Y^{(l+1)}\). When \(l > 0\), then for the same reason, \(E\) naturally maps to a number of connected components of \(Y^{(l-1)}\). When \(D\) is simple, this number is \(l\) and the maps are embeddings.

Let \((X, S)\) be a stratified analytic variety and assume that the normalization of \(X\) is a homeomorphism.

**Definition 2.10.** An \(S\)-resolution of \((X, S)\) consists of giving for every stratum \(S \in \mathcal{S}\) a resolution of its closure \(\pi_S : \tilde{S} \to \overline{S}\) such that

1. \(\pi_S : \tilde{S} \to \overline{S}\) is an isomorphism over \(S\) and the preimage of \(\partial S\) is a simple normal crossing divisor \(\tilde{D}_{\tilde{S}}\) (so that \(\tilde{S}\) comes with a natural stratification),
2. when \(S' < S\), then \(\tilde{S}[S'] := \overline{\pi_S^{-1}S'}\) is a union of irreducible components of \(\tilde{D}_{\tilde{S}}\) and we have a factorization

\[
\pi_S : \tilde{S}[S'] \xrightarrow{\pi_S'} \tilde{S}' \xrightarrow{\pi_S''} \overline{S}'
\]

that maps every stratum of \(\tilde{S}[S']\) onto a stratum of \(\tilde{S}'\) and
3. when \(S'' < S'\), then \(\pi_S'''[\tilde{S}[S''] \cap \tilde{S}[S']]\) factors as

\[
\pi_S''' : \tilde{S}[S''] \cap \tilde{S}[S'] \xrightarrow{\pi_S''} \tilde{S}[S''] \xrightarrow{\pi_S'} \overline{S}'
\]

Given such an \(S\)-resolution, we denote by \(\tilde{X}\) the disjoint union of the \(\tilde{S}\) with \(S \subset X^0\). Then the projection \(\pi : \tilde{X} \to X\) is a normal crossing resolution of \(X\). Also note that for any \(S \in \mathcal{S}\), the collection \(\\{\pi_{S'}\}_{S' < S}\) defines a \(S[\tilde{S}]\)-resolution of \(\overline{S}\).

In order to prove Theorem 2.8 we first show how the above notion gives rise to a simplicial resolution that can be used compute the cohomology of \(X\) and its mixed Hodge structure, when that makes sense.

Obviously, the collection of \(\pi_S : \tilde{S} \to \overline{S}\), where \(S \in \mathcal{S}\) runs over the open strata, defines a resolution \(\pi : \tilde{X} \to X\) of \(X\) whose exceptional set is a normal crossing divisor. So \(X\) can be regarded as a quotient space of \(\tilde{X}\) with the identifications taking place over the strata of depth \(\geq 1\). Let \(S > S'\) be a pair of incident strata whose depths differ by 1. When we regard \(\overline{S}\) as a quotient of \(\tilde{S}\), then the identification over \(S'\) is exhibited by \(\pi_{S'} : \tilde{S}[S'] \to \tilde{S}'\). In order to let all such identifications take place by means of morphisms between smooth varieties, it is best to replace
components are glued to each other in a normalization. It is then wise to remember that these connected components are glued to each other in $\tilde{S}[S']$. We may continue this process with any stratum of depth 2 and finally end up with a small category $\mathcal{S}$ of compact complex manifolds over $X$ that has $X$ as a direct limit in the category of topological spaces. Here is more precise description of $\mathcal{S}$.

An object of $\mathcal{S}$ is a connected component $E$ of $\tilde{S}(l)$ for some $S \in \mathcal{S}$ and some $l \geq 0$. So when we regard $E$ as a subvariety of $\tilde{S}$, it is the closure of a stratum. We describe two types of basic morphisms $E \rightarrow E'$ between two such objects and stipulate that these generate the $\mathcal{S}$-morphisms. The first one is when $E'$ is obtained from $E$ by forgetting one of the $l$ irreducible components of $D_S$ which contains $E$ and $E \rightarrow E'$ is the obvious embedding. The other is defined only when $E$ is contained in $\tilde{S}[S']$ for some $S' \subset S$. Then $E' := \pi^S_S(E)$ is a connected component of $\tilde{S}(l')$ for some $l'$ and the resulting map $E \rightarrow E'$ is the other type of basic morphism.

Now recall that the nerve of the small category $\mathcal{S}$ is a simplicial set whose $n$-simplices are chains of length $n$: $E_* = (E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_n)$ in $\mathcal{S}$. We then obtain a simplicial space $X_n$ by taking for $X_n$ the disjoint union of the objects $\mathrm{in}(E_n)$ where $E_n$ runs over the chains of length $n$ in $\mathcal{S}$ and $\mathrm{in}(E_n)$ stands for the first term of such a chain. (So the connected components of $X_n$ are objects of $\mathcal{S}$, but the indexing is by the $n$-simplices, so that several copies of the same $\mathcal{S}$-object may appear.) Its geometric realization $|X_*|$ (which is obtained in a standard fashion as a quotient of the disjoint union of the products $|\Delta^n| \times X_i$) is a space over $X$ and this structural map is a homotopy equivalence. It can be used for cohomological descent: the face maps $\partial_i : X_n \rightarrow X_{n-1}$, $0 \leq i \leq n$, are used to define a double cochain complex

$$C^*(X_*) : 0 \rightarrow C^*(X_0) \rightarrow C^*(X_1) \rightarrow \cdots$$

and the obvious chain homomorphism from $C^*(X)$ to the associated simple complex $sC^*(X_*)$ induces an isomorphism on (integral) cohomology. In particular, we have spectral sequence

$$E_1^{r,s} = H^s(X_r) \Rightarrow H^{r+s}(X).$$

We note that the edge homomorphism $H^*(X) \cong H^*(sC^*(X_*)) \rightarrow H^*(X_0) = \bigoplus_{E \in \mathcal{E}} H^*(E)$ is induced by the obvious map $\sqcup_{E \in \mathcal{E}} E = X_0 \rightarrow X$.

Suppose that we are in the algebraic setting so that varieties and morphisms are complex-algebraic. Then $H^*(X)$ carries a mixed Hodge structure and we can use this construction to identify that structure: the above spectral sequence is one of mixed Hodge structures. This implies that it degenerates at $E_2$ (all higher differentials are zero since their source and target when nonzero have different weight) and yields the weight filtration:

$$\text{gr}_s^W H^{r+s}(X;\mathbb{Q}) = E_2^{r,s} = H(H^s(X_{r-1};\mathbb{Q}) \rightarrow H^s(X_r;\mathbb{Q}) \rightarrow H^s(X_{r+1};\mathbb{Q})).$$

Moreover, if we use $A^\bullet(M)$ to denote the $\mathbb{C}$-valued De Rham complex of a complex algebraic manifold $M$ and $F^\bullet A^\bullet(M)$ its Hodge filtration, then the Hodge filtration of $sA^\bullet(X_*)$ defines the Hodge filtration of $H^\bullet(X;\mathbb{C})$.

Perhaps the simplest nontrivial example is when $X$ has only two strata: $S$ and $X - S$ and $\pi : \tilde{X} \rightarrow X$ is a resolution with $\pi^{-1}S$ nonsingular. Then the complex $C^\bullet(X_*)$ is just $0 \rightarrow C^\bullet(\tilde{X}) \oplus C^\bullet(S) \rightarrow C^\bullet(\pi^{-1}S) \rightarrow 0$ and the associated exact
sequence
\[ \cdots \to H^{s-1}(\pi^{-1}S) \to H^s(X) \to H^s(\tilde{X}) \oplus H^s(\pi^{-1}S) \to H^s(\pi^{-1}S) \to \cdots \]
yields the weight filtration: \( W_s H^s(X) = 0 \), \( W_{s-1} H^s(X) \) is the image of the map \( H^{s-1}(\pi^{-1}S; \mathbb{Q}) \to H^s(X; \mathbb{Q}) \) and \( W_s H^s(X) = H^s(X; \mathbb{Q}) \).

**Proof of Theorem 2.8.** For \( \nabla \) as constructed in the proof of Proposition 2.5, the Chern form \( C_k(F, \nabla) \) defines a closed \( 2k \)-form on \( S \) for every \( S \in S \). This form is of Hodge level \( \geq k \). It restricts to a \( 2k \)-form \( C_k(F_E, \nabla_E) \) on every \( S \)-object \( E \) with the same property and for every \( S_p \)-morphism \( \phi : E \to E' \) we have \( \phi^* C_k(F_{E'}, \nabla_{E'}) = C_k(F_E, \nabla_E) \). This means that \( (C_k(F_E, \nabla_E))_E \) defines a cocycle of degree \( 2k \) in \( A^*(X_\ast) \) and thus defines a class \( c_k(F, \nabla) \in F^k H^{2k}(X) \). \( \Box \)

3. A first application to Baily-Borel compactifications

**Review of the Baily-Borel compactification.** Let \( \mathcal{G} \) be a connected reductive complex algebraic group that is defined over \( \mathbb{R} \). Write \( G \) resp. \( G_\mathbb{C} \) for \( \mathcal{G}(\mathbb{R}) \) resp. \( \mathcal{G}(\mathbb{C}) \) endowed with the Hausdorff topology. We assume that \( G \) has compact center and that the symmetric space \( X \) of \( G \) is endowed with a \( G \)-invariant complex structure. To say that \( X \) is the symmetric space \( \mathcal{X} \) of \( G \) means that for every \( x \in X \) the stabilizer \( G_x \) is a maximal compact subgroup of \( G \) and to say that \( \mathcal{X} \) comes with a \( G \)-invariant complex structure amounts to the property that \( G_x \) contains an embedded a copy of the circle group \( U(1) \) in its center whose action on \( T_x X \) defines its complex structure (it is a nontrivial action by scalars of unit norm). This makes \( X \) a bounded symmetric domain. It appears naturally as an open \( G \)-orbit in a complex projective manifold \( \tilde{X} \), called the compact dual of \( X \), on which \( G_\mathbb{C} \) acts transitively. It has the property that for \( x \in X \), the \( G_\mathbb{C} \)-stabilizer \( G_{\mathbb{C}, x} \) is simply the complexification of \( G_x \).

We now assume that \( \mathcal{G} \) is defined over \( \mathbb{Q} \) and let \( \Gamma \subset G(\mathbb{Q}) \) be an arithmetic subgroup. For what follows the passage to a subgroup of \( \Gamma \) of finite index will be harmless, and so we will assume from the outset that \( \Gamma \) is neat. This means that for every finite dimensional representation \( \rho : G_\mathbb{C} \to GL(n, \mathbb{C}) \) the subgroup of \( \mathbb{C}^\times \) generated by the eigenvalues of elements of \( \rho(\Gamma) \) has no torsion (actually it suffices to verify this for just one faithful representation). This implies that the arithmetically defined subquotients of \( \Gamma \) are torsion free. The action of \( \Gamma \) on \( X \) is then proper and free so that the orbit space \( \mathcal{X}_\Gamma \) is complex manifold. The Baily-Borel compactification, which we will presently recall, shows that \( \mathcal{X}_\Gamma \) has even the structure of nonsingular quasi-projective variety.

A central role in the Baily-Borel theory is played by the collection \( \mathcal{P}_{\text{max}} = \mathcal{P}_{\text{max}}(G) \) of maximal proper parabolic subgroups of \( \mathcal{G} \) defined over \( \mathbb{Q} \) and so let us fix some \( P \in \mathcal{P}_{\text{max}} \). We review the structure of \( P \) and the way it acts on \( \mathcal{X} \). Its unipotent radical \( R_u(P) \subset P \) is at most 2-step unipotent: if \( U_P \subset R_u(P) \) denotes its center (a nontrivial vector group), then \( V_P := R_u(P)/U_P \) is also a (possibly trivial) vector group. Adopting the convention to denote the associated Lie algebras by the corresponding Fraktur font, then the Lie bracket defines an antisymmetric bilinear map \( v_P \times v_P \to u_P \). This map is equivariant with respect to the adjoint action of \( P \) on these vector spaces. Note that \( P \) acts on \( u_P \) and \( v_P \) through its Levi quotient \( L_P := P/R_u(P) \). The reductive group \( L_P \) has in \( u_P \) a distinguished open orbit that is a strictly convex cone \( C_P \) with the property that if we exponentiate \( \sqrt{-1}C_P \) to a semigroup in \( G_\mathbb{C} \), then this semigroup leaves \( X \) invariant (think of the upper half plane in \( \mathbb{C} \) that is invariant under the semigroup of translations in...
The $G$-stabilizer of $u_P$ is $P$ and so $u_P$ determines $P$. This gives rise to a partial order $\leq$ on $\mathcal{P}_{\text{max}}$ by stipulating that $Q \leq P$ in case $u_Q \subset u_P$ (in $g$). This last property is equivalent to $C_Q \subset U_P$. The other (nonmaximal) $\mathbb{Q}$-parabolic subgroups of $G$ are obtained from chains in $\mathcal{P}_{\text{max}}$: for a chain $P_0 < P_1 < \cdots < P_n$ in $\mathcal{P}_{\text{max}}$, $P := P_0 \cap \cdots \cap P_n$ is a $\mathbb{Q}$-parabolic subgroup. It has the property that its unipotent radical contains the unipotent radical of each $P_i$: $\bigcup_{i=1}^n R_u(P_i) \subset R_u(P)$.

For $P \in \mathcal{P}_{\text{max}}$, the $\mathbb{Q}$-split center $A_P$ of $L_P$ is isomorphic as such to the multiplicative group (and so $A_P \cong \mathbb{R}^\times$). It acts on $u_P$ by a faithful character (multiplication by scalars) and on $u_P$ by the square of that character (so that it indeed preserves $C_P$). The horizontal subgroup $M_P^h \subset L_P$ (but for some if not most authors the superscript stands for hermitian) is the kernel of the action of $L_P$ on $u_P$. This is a reductive subgroup defined over $\mathbb{Q}$ with compact center. The centralizer of $M_P^h$ in $L_P$ is a reductive $\mathbb{Q}$-subgroup whose commutator subgroup we denote by $M_P'$ (we like to think that the symbol $\ell$ should refer to link rather than linear—the explanation for this terminology will become clear below). This group acts in such a manner on the real projectivization of $C_P$ (in the projective space of $u_P$) that the latter is the symmetric space of $M_P^h$. So we may regard $C_P$ as the symmetric space of $L_P := M_P \cdot A_P$. The group $L_P$ supplements $M_P^h$ in $L_P$ up to a finite central subgroup. We denote the preimage of $L_P' \subset P$ by $P'$. The action of $P$ on $X$ is still transitive. Important for what follows is that the formation of the $P'$-orbit space of $X$ remains in the holomorphic category: it defines a holomorphic submersion of complex manifolds $X \to X(P)$, with $X(P)$ appearing as the symmetric domain of $M_P^h$. This is called a rational boundary component of $X$. The $P'$-orbits in $X$ (so the fibers of $X \to X(P)$) are also orbits of the semi-subgroup $R_u(P) + \exp(\sqrt{-1}C_P) \subset G_C$ in $X$ and this description is essentially an abstract way of realizing $X$ as a Siegel domain of the third kind. To be precise, we have a natural factorization of $\rho_P : X \to X(P)$:

$$
\rho_P : X \xrightarrow{\rho_P'} X(P)' \xrightarrow{\rho_P''} X(P),
$$

where is the first map $\rho_P'$ is a bundle of tube domains (a ‘torsor’ over $X(P)'$ for the semigroup $\exp(u + \sqrt{-1}C_P)$) and the second map $\rho_P''$ is a principal bundle of the vector group $V_P = \exp(v_P)$ (so a bundle of affine spaces). The latter has also the structure of a complex affine space bundle, but beware that this complex structure on a fibre (which can be given as a complex structure on its translation space $v$) will in general vary with the base point. The map $M_P^h \to L_P/L_P' = P/P'$ is an isogeny: it is onto and has finite kernel. We write $G_P$ for $P/P'$. The action of $M_P^h$ on $X(P)$ is through this quotient and we prefer to regard $X(P)$ as the symmetric space of the quotient $G_P$ of $P$ rather than of the subquotient $M_P^h$ of $P$ (see the example of the symplectic group below).

Every $Q \in \mathcal{P}_{\text{max}}(G)$ with $Q \supset P$ has by definition the property that $u_Q \subset u_P$. But it is then even true that $Q^t \supset P^t$ and so the projection $\rho_Q : X \to X(Q)$ factors through $\rho_P : X \to X(P)$ via a morphism that we shall denote by $\rho_{Q/P} : X(P) \to X(Q)$. The latter can be understood as the formation of a rational boundary component of $X(P)$. Indeed, $Q$ defines a maximal proper parabolic subgroup of $G_P$, namely the image of $Q \cap L_P$ in $L_P/L_P' = G_P$ (that we shall denote by $Q_{/P}$). This identifies $\mathcal{P}_{\text{max}}(G_P)$ as a PSet with $\mathcal{P}_{\text{max}}(G)_{>P}$. The unipotent radical of $Q_{/P}$ is the image of $R_u(P) \cap R_u(Q)$ in $Q_{/P}$. Its center $U_{Q_{/P}}$ is the image of $U_Q$, $U_Q/U_Q \cap R_u(P)$. Similarly, the cone $C_{Q_{/P}} \subset u_{Q_{/P}}$ is the image of $C_Q \subset u_Q$ under
the projection $u_Q \to u_Q/\Gamma \cap R_u(p) \cong u_{Q/P}$:

$$
\begin{align*}
\cup C_P < C_Q & \longrightarrow C_{Q/P}.
\end{align*}
$$

We define the Satake extension of $X$ as a ringed space. As a set it is the disjoint union

$$
\mathcal{X}_{\text{bb}} := X \sqcup \bigcup_{P \in \mathcal{P}_{\text{max}}} X(P).
$$

It is endowed with the horocyclic topology: the topology generated by the open subsets of $X$ and the subsets $\Omega^{bb}_{Q/P}$, where $P \in \mathcal{P}_{\text{max}}$ and $\Omega \subset X$ is open and invariant under both the semigroup $\sqrt{1-\mathcal{C}}_P$ and the group $\Gamma \cap P^G$, and

$$
\Omega^{bb}_{Q/P} := \Omega \sqcup \bigcup_{Q \in \mathcal{P}_{\text{max}}, Q \leq P} \rho_Q(\Omega).
$$

Since $\Gamma \cap R_u(P)$ is cocompact in $R_u(P)$, we may replace here invariance under $\Gamma \cap P^G$ by invariance under $R_u(P)_G. (\Gamma \cap P^G)$ (but not in general by invariance under $P^G$). Yet this topology is independent of $\Gamma$: it only depends on the $Q$-structure on $G$. This construction is natural in the sense that the closure of any rational boundary component in $X_{\text{bb}}$ can be identified with its Satake extension. The structure sheaf $O_{\mathcal{X}_{\text{bb}}}$ is the sheaf of complex-valued continuous functions that are holomorphic on every stratum $X(P)$. It is clear that $\Gamma$ acts on this ringed space. The main theorem of Baily-Borel asserts (among other things) that the orbit space $(\mathcal{X}_{\Gamma}, O_{\mathcal{X}_{\Gamma}})$ is as a ringed space a normal compact analytic space that underlies the structure of normal projective variety (by a theorem of Chow, this structure is then unique). Moreover, the decomposition of $X_{\text{bb}}$ into $X$ and its rational boundary components defines a decomposition of $X_{\Gamma}^{bb}$ into nonsingular subvarieties (strata) such that the closure of any such is a union thereof. Any stratum is of the same type as $X_{\Gamma}$: it is of the form $X(P)_G \cap \Gamma$ and hence has its own Baily-Borel compactification. The preceding shows that the Baily-Borel compactification of a stratum maps homeomorphically onto its closure in $X_{\Gamma}^{bb}$. This map is also morphism of varieties (that could be an isomorphism, but it is conceivable that this closure is not normal). This shows among other things:

**Corollary 3.1.** The retractions $\{\rho_P : X \to X(P)\}$ $P \in \mathcal{P}_{\text{max}}$ endow the Baily-Borel stratification of $X_{\Gamma}^{bb}$ with a natural system $\rho^{bb}_{\Gamma}$ of retractions, thus making it a rigidified stratified space.

**Satake extension of automorphic bundles.** Let $F$ be an automorphic vector bundle on $X$, that is, a complex vector bundle on $X$ endowed with a $G$-action lifting the one on $X$ in such a manner that for some (and hence for any) $x \in X$ the copy of $U(1)$ in the stabilizer $G_x$ acts also complex linearly on the fiber $F(x)$. Such a vector bundle is completely given by the action of $G_x$ on $F(x)$ and conversely, any finite dimensional complex representation of $G_x$ defines such a vector bundle. The bundle $F$ with its $G$-action extends to the compact dual $\bar{X}$ as a vector bundle with $G_C$-action and this extension (that we denote by $\bar{F}$) is unique. This is because the $G_x$-action on the complex vector space $F(x)$ extends to one of the complexification $G_{x,C}$ of $G_x$, and $G_{x,C}$ is just the $G_C$-stabilizer of $x$. Since the $G_C$-bundle $\bar{F}$
is defined in the holomorphic category, it follows that $\mathcal{F}$ comes with a $G$-invariant holomorphic structure.

Given $x \in X$, the compactness of $G_x$ implies that $G_x$ leaves invariant an inner product in the fiber $\mathcal{F}(x)$. This inner product then extends in a unique manner to a $G$-invariant inner product $h$ on $\mathcal{F}$. As is well-known, we then have a unique hermitian connection $\nabla$ on $(\mathcal{F}, h)$ whose $(0,1)$-part is zero on local holomorphic sections. This connection is of course also $G$-invariant. It is in fact independent of $h$. This is clear when $\mathcal{F}(x)$ is irreducible as a representation of $G_x$, for then the inner product is unique up to scalar and the general case then follows from this by decomposing $\mathcal{F}(x)$ into irreducible subrepresentations. So we have canonically associated Chern forms $C_n(\mathcal{F}) = C_n(\mathcal{F}, \nabla)$ on $X$. Such a form is harmonic relative to a $G$-invariant metric on $X$ is $G$-invariant and of Hodge bidegree $(n, n)$. The $G$-equivariance allows us to descend all of this to $X$, so that we get a holomorphic bundle $\mathcal{F}_\Gamma$ with connection on $X$ whose Chern forms pull back to the ones of $(\mathcal{F}, \nabla)$. The $G$-invariant connection $\nabla$ will in general not extend to $\mathcal{F}$ and neither will the associated Chern forms.

**Lemma 3.2.** The action of the semigroup $R_\alpha(P) \exp(\sqrt{-1}C_P)$ on $X$ defines a natural $\rho_P$-flat structure on $\mathcal{F}$. This identifies $\mathcal{F}$ with the $\rho_P$-pull-back of a vector bundle $\mathcal{F}(P)$ on $X(P)$ with $P/R_\alpha(P)$-action (lifting the obvious $P/R_\alpha(P)$-action on $X(P)$). In particular, $\mathcal{F}(P)$ is automorphic relative the $M_P^0$-action on $X(P)$ with $L_P^0$ acting (possibly nontrivially) as a group of bundle automorphisms over $X(P)$.

Finally, for any chain $P < P_1 < \cdots < P_n$ in $P_{\max}$, the $\rho_P$-flat structure on $\mathcal{F}$ restricted to $\rho_P$ differs from the $\rho_P$-flat structure by a differential that takes its values in the nilpotent Lie algebra $R_n(p \cap p_1 \cdots \cap p_n)$.

**Proof.** Recall that the morphism $\rho_P : X \to X(P)$ is a principal bundle for the semisubgroup $R_\alpha(P) \exp(\sqrt{-1}C_P)$ of $G_C$ and so the restriction of $\mathcal{F}$ is canonically trivialized as a complex vector bundle along the fibers of $\rho_P$. This trivialization can be made holomorphic. We show this by means of the factorization (3.1): the fibers of the first factor $\rho_P' : X \to X(P)'$ are orbits of the semisubgroup $\exp(u_P + \sqrt{-1}C_P) \subset U_P(C)$ and so over such orbits we get a holomorphic trivialization: we end up with a vector bundle $\mathcal{F}'$ on $X(P)'$ such that $\mathcal{F}$ is identified with the pull-back of $\mathcal{F}'$ along $\rho_P'$.

The second factor $\rho_P'' : X(P)' \to X(P)$ is a torsor for the vector group $V_P = \exp(u_P)$. Thus the trivial vector bundle over $X(P)$ with fiber $V_P$ has a holomorphic structure yielding a holomorphic vector bundle over $X(P)$ (whose total space we denote by $V_P$) such that for every holomorphic local section $\sigma$ of $X(P)' \to X(P)$ with domain $N$, the map $V_P|_N \to X(P)'|_N, v_2 \mapsto v_2 + \sigma(z)$ is biholomorphic. Any trivialization of $V_P|_N$ then yields a holomorphic trivialization of $X(P)'|_N \to N$ that gets covered by a trivialization of $\mathcal{F}'$ over $X(P)'|_N$. This defines the natural $\rho_P$-flat structure on $\mathcal{F}$.

It is clear that the $\rho_P$-flat structure induced by $P_k$ differs by the one defined by $P$ by a differential that takes values in the complexification of $R_n(p) + R_n(p_k)$. But this last space is contained in the nilpotent Lie algebra $R_n(p \cap p_1 \cdots \cap p_n)$. ⊓⊔

The quotient of $\mathcal{F}$ by the $\Gamma$-action gives a holomorphic vector bundle $\mathcal{F}_\Gamma$ on $X$. The following corollary somewhat sharpens the main result of [7].

**Corollary-definition 3.3.** The bundle $\mathcal{F}_\Gamma$ on $X$ admits a natural $\rho_{\Gamma_\bb}^h$-flat structure $\nabla_{\rho_{\Gamma_\bb}^h}$. This structure is isoholonomic so that we have defined the Goresky-Pardon
Chern class extension $c^\bb_k(F_T) := c_k(F_T, \nabla_{\rho^\bb_T}) \in H^{2k}(\bb; \mathbb{R})$, $k = 0, 1, \ldots$. The restriction of such a class to the closure of a stratum is of the same type (it is the Chern class of an automorphic bundle on that stratum).

**Proof.** It is clear that Lemma 3.2 produces $\rho^\bb_T$-flat structure $\nabla_{\rho^\bb_T}$ on $F_T$. What remains to see is that this structure is isoholonomic. The lemma in question identifies $\mathcal{F}$ in a $P$-equivariant manner with the $\rho_P$-pull-back of a vector bundle $\mathcal{F}(P)$ on $\mathbb{X}(P)$. We may cover $\mathbb{X}(P)$ by open subsets $V \subset \mathbb{X}(P)$ such that $\mathcal{F}(P)|V$ is trivial and the $\Gamma$-stabilizer of $V$ is in fact its $\Gamma_P$-stabilizer. Such a $V$ then embeds as an open subset in a Baily-Borel stratum $\mathbb{X}(P)|\Gamma(P)$ of $\bb$ and a trivialization of $\mathcal{F}(P)|V$ yields the flat connection that is being asked for. □

**Remark 3.4.** The argument used in the proof of Lemma 3.2 can probably be extended to prove that the automorphic bundles $\mathcal{F}$ and $\{\mathcal{F}(P)\}_{P \in \mathbb{P}_{\max}}$ define a bundle $\mathcal{F}^\bb$ over the Satake extension $\bb$ in the sense that it becomes a locally free module over the structure sheaf of $\bb$. This has some interest, as it may somewhat simplify the proof in [15] of Conjecture 9.5 of [8]. But since $\Gamma$ defines an arithmetic subgroup of $L^+_\ell$, which then may act nontrivially in the fibers of $\mathcal{F}(P)$, $\mathcal{F}^\bb$ will in general not be a locally free module over the structure sheaf of $\bb$.

**Example 3.5. A symplectic group and its Hodge bundle.** Let be given a finite dimensional real vector space $V$ endowed with a nondegenerate symplectic form $\omega : V \times V \to \mathbb{R}$. The automorphism group of $(V, \omega)$ defines an almost simple algebraic group defined over $\mathbb{R}$ whose group of real resp. complex points (endowed with the Hausdorff topology) is $\text{Sp}(V)$ resp. $\text{Sp}(V_{\mathbb{C}})$. Let $h : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ be the Hermitian form defined by $h(v, v') := \sqrt{-1}\omega(v, v')$. It has signature $(g, g)$. The associated symmetric space of $\text{Sp}(V)$ and its compact dual are obtained as follows: $\mathbb{X} = \mathbb{H}(V)$ is the locus in the Grassmannian $\text{Gr}_g(V_{\mathbb{C}})$ which parametrizes the $g$-dimensional subspaces $F \subset V_{\mathbb{C}}$ that are totally isotropic relative to $\omega_{\mathbb{C}}$ and $\mathbb{X} = \mathbb{H}(V)$ is the open subset of $\mathbb{H}(V)$ parametrizing those $F$ on which in addition $h$ is positive definite. The group $\text{Sp}(V)$ indeed acts transitively on $\mathbb{H}(V)$ and the stabilizer of any $[F] \in \mathbb{H}(V)$ restricts isomorphically to the unitary group $U(F)$, which is a maximal compact subgroup of $\text{Sp}(V)$. The restriction of the tautological rank $g$ bundle on $\text{Gr}_g(V_{\mathbb{C}})$ to $\mathbb{H}(V)$ resp. $\mathbb{H}(V)$ is an automorphic bundle $\mathcal{F} = \mathcal{F}_V$ resp. its natural extension $\overline{\mathcal{F}}$ to $\mathbb{H}(V)$. If $[F] \in \mathbb{H}(V)$, then $V_{\mathbb{C}} = F \oplus \overline{F}$ and so $F$ is the $(1, 0)$-part of a Hodge structure on $V$ of weight 1 polarized by $\omega$. Thus $\mathbb{H}(V)$ also parametrizes polarized Hodge structures on $V$ of this type. For this reason, $\mathcal{F}$ is often called the *Hodge bundle*. Notice that $h$ defines on $\mathcal{F}$ an inner product (that we continue to denote by $h$).

Now assume $V$ and $\omega$ defined over $\mathbb{Q}$ so that our group is also defined over $\mathbb{Q}$. A maximal proper $\mathbb{Q}$-parabolic subgroup $P \subset \text{Sp}(V)$ is the $\text{Sp}(V)$-stabilizer of a nonzero isotropic subspace $I \subset V$ defined over $\mathbb{Q}$ and vice versa. So $\mathbb{P}_{\max}$ may be identified with the set $\mathbb{J}(V)$ of nonzero $\mathbb{Q}$-isotropic subspaces of $V$. We will therefore index our objects accordingly.

Let $I \in \mathbb{J}(V)$ and write $V'_I$ for $V/I$ and $V_I \subset V'_I$ for $I^\perp/I$. Note that the symplectic form identifies $V'_I$ with the dual of $I^\perp$ and induces on $V_I$ a nondegenerate symplectic form. Then the unipotent radical $R_u(P_I)$ of $P_I$ is the subgroup that acts trivially on $I$ and $V_I$ (this group then automatically acts trivially on $V/I^\perp$). This identifies the Levi quotient $L_I$ of $P_I$ with $\text{GL}(I) \times \text{Sp}(V_I)$. The center $U_I$ of $R_u(P_I)$ is the subgroup that acts trivially on $I^\perp$ (or equivalently, on $V'_I$). Its (abelian) Lie
algebra \( u_I \) can be identified with \( \text{Sym}_2 I \subset \text{Sym}_2 V \cong g \) and \( C_I \subset \text{Sym}_2 I \) is the cone of positive definite elements. This identifies \( R_u(P_I)/U_I \) with a group of elements in \( \text{GL}(I^+) \) that act trivially on both \( I \) and \( V_I \); this group is abelian with Lie algebra \( \text{Hom}(V_I, I) \) (which we shall identify with \( I \otimes V_I \) by means of the non-degenerate symplectic form on \( V_I \)). The central subgroup \( A_I \subset P_I \) appears here as the group of scalars in \( \text{GL}(I) \) (so this is a copy of \( \mathbb{R}^x \)). The adjoint action of \( L_I = \text{GL}(I) \times \text{Sp}(V_I) \) on this Lie algebra is the obvious one. In terms of these isomorphisms, \( M^I = \{ \pm I \} \times \text{Sp}(V_I) \), \( M^I = \text{SL}(I) \), \( L^I = \text{GL}(I) \), \( G_I = \text{Sp}(V_I) \) and \( P_I \subset G_I \) is the group that acts as \( \pm 1 \) on \( V_I \).

We next describe the maps \( \mathbb{X} \to \mathbb{X}(P_I)' \to \mathbb{X}(P_I) \). For this we note that for \( [F] \in \mathbb{H}(V) \), the projection \( F \to V_{I,C} \) is into and the projection \( F \to V_C/I_C \cong I_C \) is onto with kernel \( F \cap I_C \) projecting isomorphically onto a subspace of \( V_{I,C} \) that defines an element of \( \mathbb{H}(V_I) \). If we denote by \( \mathbb{H}(V_I) \) the subspace of Grassmannian of \( V_{I,C} \) parameterizing the subspaces whose intersection with \( V_{I,C} \) defines an element of \( \mathbb{H}(V_I) \), then we obtain a diagram

\[
\rho_I : \mathbb{X} = \mathbb{H}(V) \xrightarrow{\rho_I'} \mathbb{X}(P_I)' = \mathbb{H}(V_I') \xrightarrow{\rho_I''} \mathbb{X}(P_I) = \mathbb{H}(V_I),
\]

where the maps are the obvious ones. It is clear that \( F \) is the \( \rho_I \)-pull-back of the tautological bundle \( F_{I,V_I} \) over \( \mathbb{H}(V_I') \). We note that \( \mathbb{H}(V_I') \to \mathbb{H}(V_I) \) is a principal \( \text{Hom}(V/I, V_I) \cong V_I \otimes I \)-action lifts to \( F_{I,V_I} \).

Let us determine the \( \rho_I \)-flat structure on \( F_{I,V_I} \). First observe that any subspace \( F'' \subset V_I \) that defines an element of \( \mathbb{H}(V_I) \) determines a complex structure on \( V_I \) characterized by the property that the \( \mathbb{R} \)-linear isomorphism \( V_I \subset V_{I,C} \to V_{I,C}/F'' \cong \text{Hom}_C(F''/C) \) is in fact \( C \)-linear. This gives the constant vector bundle \( \mathbb{H}(V_I) \) with fiber \( V_I \) a holomorphic structure (which can be identified with the total space of the dual \( F_{I,V_I} \) of \( F_{I,V_I} \)). Hence the constant vector bundle on \( \mathbb{H}(V_I) \) with fiber \( V_I \otimes I \) also acquires a holomorphic structure (namely as the total space of \( F_{I,V_I} \times I \); let us denote that total space by \( V_I \). Then \( \rho_I'' : \mathbb{H}(V_I') \to \mathbb{H}(V_I) \) is a \( V_I \)-torsor in the complex-analytic category. A local section of \( \rho_I'' \) identifies \( F_{I,V_I} \) with the pull-back along \( \rho_I'' \) of \( F_{I,V_I} \) with \( \mathbb{H}(V_I') \) with \( \text{Sp}(V_I) \times \text{GL}(I) \) acting in the obvious way. This gives the \( \rho_I \)-flat structure on \( F_{I,V_I} \).

The preceding also makes it clear that \( P_I \leq P_J \) is equivalent to \( I \subset J \). In other words, \( (\mathcal{F}_I)_{\mathbb{H}^C} \) is identified with \( (\mathcal{F}(V_I), C) \). Note that for an inclusion \( I \subset J \) of \( \mathbb{Q} \)-isotropic subspaces, the diagram involving the associated cones is

\[
\begin{array}{ccc}
\text{Sym}^2 I \subset \text{Sym}^2 J & \longrightarrow & \text{Sym}^2 (J/I) \\
\cup & \cup & \\
C_I \leq C_J & \longrightarrow & C_{J/I}.
\end{array}
\]

4. Resolving a Baily-Borel compactification as a stratified space

In this section we describe the data needed for Mumford’s toroidal compactifications introduced in [1] and explain how this compares with the Baily-Borel construction. We will then show:

**Theorem 4.1.** Every toroidal resolution \( \pi : \tilde{X} \to X_{\mathbb{H}}^{\text{hb}} \) of the Baily-Borel compactification has the property that for every Baily-Borel stratum, the holonomy of \( F_I \) relative to
its local retraction is trivial over the preimage of that stratum in $\tilde{X}$ so that (by Proposition 2.5) $F_T$ extends naturally to a holomorphic vector bundle $\tilde{F}_T$ on $\tilde{X}$ and we have $\pi^* c_k^{sp}(F_T) = c_k(\tilde{F}_T)$. We can choose such a resolution to be part of a stratified resolution of $X^{\text{hbl}}$ so that (by Theorem 2.8) $c_k^{sp}(F_T) \in F^k H^k(X^{\text{hbl}}; \mathbb{C})$.

For what follows it is convenient to pretend that $G$ is also a maximal $Q$-parabolic subgroup: since $R_u(G)$ is trivial we have $C_G = U_G = V_G = 0$. We take $G^c = \{1\}$ and $X(G) = X$. We write $P^c = P \cup \{G\}$ for the corresponding collection. A partial order $\leq$ on $P^c$ is defined as before and has $G$ as its minimal element.

The cones $\{C_P\}_{P \in P^c}$ are pairwise disjoint as subsets of $G$. We denote their union by $C(g)$ and write $C_P$ for the union of all the $C_Q$ with $Q \leq P$ with $Q \in P^c$ (so $0 \in C_{P^c}$). Then $C_P^\circ$ is the closure of $C_P$ in $C(g)$ and is spanned by $\mathcal{U}_P \cap v_P(\mathbb{Q})$. Now $\Gamma_P := \Gamma \cap P$ is an arithmetic subgroup of $P$. In particular, $\Gamma \cap R_u(P)$ is an extension of a lattice (namely $\Gamma \cap U_P$) by a lattice (namely the image of $\Gamma_P$ in $U_P$). The image of $\Gamma_P$ in $GL(u_P)$ is a discrete subgroup which therefore acts on properly discretely on $C_P$. An important feature of this action is that it has in $C_P^\circ$ a fundamental domain that is a rational polyhedral cone (i.e., the convex cone spanned by a finite subset of $u_P(\mathbb{Q})$). The extra ingredient needed for a toroidal compactification is a $\Gamma$-admissible decomposition of $C(g)$, that is, a $\Gamma$-invariant collection of rational polyhedral cones that is closed under ‘taking faces’ and ‘taking intersections’, whose relative interiors are pairwise disjoint and whose union is $C(g)$. It is a basic fact [1] that $\Gamma$-admissible decompositions exist and that any two such have a common refinement.

Given $\Sigma$, then the restriction of $\Sigma$ to the open cone $C_P$, $\Sigma|C_P$, defines a relative torus embedding of $X_{\Gamma \cap U_P}$ over $X(P)'$. (Strictly speaking, there is not really a torus acting but rather an open semigroup in this torus, namely $\exp(u + \sqrt{-1}C_P)/(\Gamma \cap U_P) \subset U_P(\mathbb{C})/(\Gamma \cap U_P)$.) The result is a normal analytic variety with an action of the semigroup $\exp(u + \sqrt{-1}C_P)$ and which has toroidal singularities. The group $\Gamma_P/\Gamma_P \cap U_P$ acts on it properly discontinuously. When we divide out by the image $\Gamma(V_P)$ of $\Gamma \cap R_u(P)$ in $V_P$ (which is just a lattice) we get a family of toroidal embeddings over $X(P)_{\Gamma(V_P)}$. The latter is in fact a family of abelian varieties over $X(P)$ (or rather a torsor thereof). If we divide out by $\Gamma_P/\Gamma_P \cap U_P$ instead we get an abelian torsor with base the Baily-Borel stratum $X(P)_{\Gamma(G_P)}$ and $X_{\Gamma_P}$ appears as a toroidal embedding over this torsor.

It is perhaps more transparent, and also more in the Satake-Baily-Borel spirit, to do this construction before dividing out by $\Gamma_P \cap U_P$, that is, to first introduce a $\Gamma$-equivariant extension $X^{\Sigma}$ of $X$ of ringed spaces. This brings us, like the Satake extension, outside the realm of analytic spaces, but the advantage of this approach is that it allows us to concisely describe the maps that exist between various compactifications. Here is how to proceed. For every $\sigma \in \Sigma$ we can form a holomorphic quotient $\rho_\sigma : X \to X(\sigma)$. This map can be understood as the inclusion of $X$ in its $\exp((\sigma)C)$-orbit in the compact dual $\tilde{X}$ of $X$ (this is an open subset of $\tilde{X}$) followed by the formation of the $\exp((\sigma)C)$-orbit space. Alternatively, $\rho_\sigma$ is the formation of the quotient of $X$ with respect to the equivalence relation generated by the relation $z \sim z' \iff z' \in \exp((\sigma)C) + \sqrt{-1} \sigma z$. We let $X^{\Sigma}$ be the disjoint union of the $X(\sigma)$’s (this includes $X = X(\{\emptyset\})$) and equip this union with the topology generated by the open subsets of $X$ and those of the form $\Omega^{\text{hbl},\sigma}$: here $\sigma \in \Sigma$, $\Omega \subset X$ is an open
subset invariant under the semigroup $\exp(\langle \sigma \rangle_\mathbb{R} + \sqrt{-1}\sigma)$, and

$$\Omega^{\mathrm{bb}, \sigma} := \bigcup_{\tau \in \Sigma ; \tau \subset \Sigma} \rho_\tau(\Omega)$$

(note that $\Omega$ appears in this union for $\tau = \{0\}$). The structure sheaf is the sheaf of complex valued continuous functions that are holomorphic on each stratum. Note that when $\sigma \subset C_P$, the map $\rho_\sigma : \mathbb{X} \to \mathbb{X}(P')'$ factors through $\mathbb{X}(\sigma)$. It is then clear that the composite projections $\mathbb{X}(\sigma) \to \mathbb{X}(P')' \to \mathbb{X}(P)$ combine to define a continuous $\Gamma$-equivariant morphism $\pi^\Sigma : \mathbb{X}^\Sigma \to \mathbb{X}^{\mathrm{bb}}$ of locally ringed spaces, whose restriction over $\mathbb{X}(P)$ in fact factors over $\mathbb{X}(P)'$. This drops to a morphism $\pi^\Sigma : \mathbb{X}^\Sigma \to \mathbb{X}^{\mathrm{bb}}$ in the analytic category which has the property that it factors over a Baily-Borel stratum $\mathbb{X}(P)_{\Gamma(G_P)}$ through the abelian torsor $\mathbb{X}(P)'_{\Gamma_P/(\Gamma_P \cap U_P)}$ over $\mathbb{X}(P)_{\Gamma(G_P)}$. We can now prove part of Theorem 4.1.

**Lemma 4.2.** The retractions $\mathbb{X} \to \mathbb{X}(\sigma)$ turn $\mathbb{X}^\Sigma$ into a rigidified stratified space such that $\pi^\Sigma : \mathbb{X}^\Sigma \to \mathbb{X}^{\mathrm{bb}}$ is a morphism in this category. An automorphic bundle $\mathcal{F}_\Gamma$ on $\mathbb{X}_\Gamma$ satisfies the hypotheses of the last clause of Proposition 2.5 with respect to $\mathbb{X}^\Sigma$ and so the total Chern class of the resulting extension $\mathcal{F}^\Sigma_\Gamma$ to $\mathbb{X}^\Sigma_\Gamma$ (with complex coefficients) equals $(\pi^\Sigma)^* \exp(\mathcal{F}_\Gamma)$.

**Proof.** Let $\sigma \in \Sigma$ be such that its relative interior is contained in $C_P$. Then $\rho_\sigma$ (whose fibers are orbits of $R_\sigma(P)\exp(\sqrt{-1}C_P)$ factors through $\rho_\sigma$ (whose fibers are orbits $\exp(\langle \tau \rangle_\mathbb{R} + \sqrt{-1}\tau)$). This proves the first assertion. The resulting local flat connections $\nabla_{\rho_\sigma}$ on our automorphic bundle are compatible: if $\tau \in \Sigma$ is a face of $\sigma$, then $\exp(\langle \tau \rangle_\mathbb{R} + \sqrt{-1}\tau) \subset \exp(\langle \tau \rangle_\mathbb{R} + \sqrt{-1}\tau)$ and so the local flat connection associated to $\tau$ induces the one associated to $\sigma$. □

**Remark 4.3.** The extension $\mathcal{F}^\Sigma_\Gamma$ of $\mathcal{F}_\Gamma$ across $\mathbb{X}^\Sigma_\Gamma$ appears at various places in the literature; it is the canonical extension that appears in [9]. When $\mathcal{F}$ belongs to the Hodge filtration of a locally homogeneous variation of Hodge structure, then it is also the Deligne extension. Had we introduced the locally free $\mathcal{O}_\mathbb{X}^{\mathrm{bb}}$-module $\mathcal{F}^{\mathrm{bb}}$ as in Remark 3.4, then we could say that $\pi^{\Sigma,*} \mathcal{F}^{\mathrm{bb}}$ is a locally free $\mathcal{O}_\mathbb{X}^{\Sigma}$-module with $\Gamma$-action and $\mathcal{F}^\Sigma_\Gamma$ would simply be its $\Gamma$-quotient (the $\Gamma$-stabilizer of every $x \in \mathbb{X}^\Sigma$ acts trivially on the fiber $\mathcal{F}^{\mathrm{bb}}(x)$).

Let us say that the $\Gamma$-admissible decomposition $\Sigma$ is **smooth** if it is one into integral simplicial cones (i.e., each member of $\Sigma$ is the cone spanned by an integral partial basis of some $\log(\Gamma \cap U_P)$ for some $P$). This ensures that $\mathbb{X}^\Sigma$ is smooth. We will refer to $\mathbb{X}^\Sigma$ simply as a **toroidal resolution** of $\mathbb{X}^{\mathrm{bb}}$. Another basic fact is that any $\Gamma$-admissible decomposition admits a smooth refinement. The following proposition will complete the proof Theorem 4.1.

**Theorem 4.4.** The Baily-Borel compactification $\mathbb{X}^{\mathrm{bb}}$ admits a toroidal resolution relative to its natural stratification in the sense of Definition 2.10.

**Proof.** The first question we must address is the following. Let $P \in \mathcal{P}_{\max}$ and suppose we are given a $\Gamma$-admissible decomposition $\Sigma$ of $C(g)$ and a $\Gamma(G_P)$-admissible decomposition $\Sigma(P)$ of $C(g_P)$. The former defines $\pi^\Sigma : \mathbb{X}^\Sigma \to \mathbb{X}^{\mathrm{bb}}$ and the latter defines $\pi^{\Sigma(P)} : \mathbb{X}^{\Sigma(P)} \to \mathbb{X}(P)^{\mathrm{bb}}$ and we want to know when the closure of $(\pi^\Sigma)^{-1}(\mathbb{X}(P))$ naturally maps to $\mathbb{X}^{\Sigma(P)}$. For the strata of $\mathbb{X}^\Sigma$ that lie over $\mathbb{X}(P)$ there is no issue: they map onto $\mathbb{X}(P)$. 
A Satake stratum in the boundary of $X(P)^{bb}$ is of the form $X(Q)$, with $Q \in \mathcal{P}_{\text{max}}$ such that $Q > P$. A stratum of $X^\Sigma$ over $X(Q)$ that lies in the closure of a stratum over $X(P)$ is of the form $X(\sigma)$, with $\sigma \in \Sigma|C_Q^+$, and $\sigma \cap C_Q$ and $\sigma \cap C_P$ both nonempty (recall that we obtain $X(\sigma)$ as a quotient of the equivalence relation on $X$ generated by $z \sim z' \iff z' \in \exp(\langle \sigma \rangle_R + \sqrt{-1}\sigma)$). On the other hand, a stratum of $X(P)^{\Sigma(P)}$ over $X(Q)$ is of the form $X(P)(\tau)$, where $\tau \in \Sigma(P)$ is such that $\tau - \{0\} \subset C_{Q,P}$. We obtain it as a quotient of the equivalence relation on $X(P)$ generated by $z \sim z' \iff z' \in \exp(\langle \tau \rangle_R + \sqrt{-1}\tau)$. Now also recall that $C_{Q,P}$ is the image of $C_Q$ under the projection $u_Q \to u_Q/u_Q \cap R_u(p) \cong u_{Q,P}$. So $X(\sigma)$ maps onto $X(\tau)$ if and only if this projection maps $\sigma$ to the relative interior of $\tau$. In other words, we want that any member of $\Sigma$ in the star of $C_P$ in $C_Q^+$ is mapped by under this projection to a member of $\Sigma(P)$.

This reduces the proposition to a combinatorial issue: we must construct for every $P \in \mathcal{P}_{\text{max}}$ a $\Gamma(G_P)$-admissible decomposition $\Sigma(P)$ of $C(g_P)$ such that

(i) $\gamma \in \Gamma$ takes $\Sigma(P)$ to $\Sigma(\gamma P \gamma^{-1})$,

(ii) for every triple $Q \geq P \geq P_0$ in $\mathcal{P}_{\text{max}}$, the projection

$$u_{Q,P_0} \cong u_Q / u_Q \cap R_u(p_0) \to u_Q / u_Q \cap R_u(p) \cong u_{Q,P}$$

maps every member of $\Sigma(P_0)$ in the star of $C_{P,P_0}$ to a member of $\Sigma(P)$.

We begin with choosing a $\Sigma(Q)$ for every element of $\mathcal{P}_{\text{max}}$ that is maximal for $\leq$ and making sure that (i) is satisfied. We then proceed with downward induction on the POSet $(\mathcal{P}_{\text{max}}, \leq)$ and assume that we have constructed for every $P \in \mathcal{P}_{\text{max}}$ a $\Gamma(G_P)$-admissible decomposition $\Sigma(P)$ of $C(g_P)$ satisfying (i) and (ii) so that it remains to construct $\Sigma = \Sigma(G)$.

For every maximal element $P$ of $\mathcal{P}_{\text{max}}$ we choose a rationally polyhedral cone $\Pi_P \subset C^+_P$ that is a fundamental domain for the action of $\Gamma_P$ on $C^+_P$ in such a manner that $\Pi_P = \gamma(\Pi_P)$. For every face $Q \leq P$ such that $\Pi_P \cap C_Q \neq \emptyset$ the image of $\Pi_P$ in $C^+_{Q,P}$ is a rationally polyhedral cone and so meets only a finite number of members of $\Sigma(Q)$. Hence the pull-back of $\Sigma(Q)$ to $\Pi_P$ is a finite decomposition of $\Pi_P$ into rationally polyhedral cones. The set of $Q$ with $\Pi \cap C_Q \neq \emptyset$ is also finite and so the finitely intersections of these pull-backs make up a decomposition $\Sigma(\Pi_P)$ of $\Pi_P$ into finitely many rationally polyhedral cones.

Now let $P$ run over a system of representatives $\{P_i\}_{i=1}^r$ of the $\Gamma$-action in the collection of maximal elements of $\mathcal{P}_{\text{max}}$. So for each $i$ we have a rationally polyhedral cone $\Pi_i$ and a decomposition $\Sigma(\Pi_i)$ of that cone. It is easy to find a $\Gamma$-invariant admissible decomposition $\Sigma$ which refines each $\Sigma(\Pi_i)$. After possibly refining once more we can arrange $\Sigma$ that be smooth. It will then have the desired properties. □

5. Tate extensions in the stable cohomology of $A^{bb}_g$

The stable cohomology of $A^{bb}_g$. We here focus on what is perhaps the most ‘classical’ example and also is a special case of 3.5, namely the moduli stack $\mathcal{A}_g$ of principal polarized abelian varieties. We shall prove that the stable cohomology of its Baily-Borel compactification contains nontrivial Tate extensions and carries Goresky-Pardon Chern classes that have nonzero imaginary part (and hence are not defined over $\mathbb{Q}$).

Let $H$ stand for $\mathbb{Z}^2$ and endowed with standard symplectic form (characterized by $\langle e, e' \rangle = 1$ where $(e, e')$ is its standard basis) and regard $H^q(= \mathbb{Z}^{2q})$ as a direct
sum of symplectic lattices. In the notation of Example 3.5 we take for \( V \) the vector space \( \mathbb{R} \otimes H^q(= \mathbb{R}^{2g}) \) with its obvious rational symplectic structure so that we have defined the symmetric domain \( \mathbb{H}_g := \mathbb{H}(\mathbb{R} \otimes H^q) \) and we take for \( \Gamma \) the integral symplectic group \( \text{Sp}(H^q)(= \text{Sp}(2g, \mathbb{Z})) \). Then \( \mathcal{A}_g \) can be identified with \( \text{Sp}(H^q) \backslash \mathbb{H}_g \), when we think of the latter as a Deligne-Mumford stack. The Hodge bundle on \( \mathbb{H}_g \) descends to a rank \( g \) vector bundle \( \mathcal{F}_g \) on the stack \( \text{Sp}(H^q) \backslash \mathbb{H}_g \). As such it has integral Chern classes. In what follows we will work mostly with cohomology with coefficients in \( \mathbb{Q} \)-vector spaces. Then the distinction between the stack \( \mathcal{A}_g \) and underlying coarse moduli space (that we shall denote by \( \mathcal{A}_g \)) becomes moot, for the natural map from \( \mathcal{A}_g \) (which has the homotopy type of \( B \text{Sp}(2g, \mathbb{Z}) \)) to \( \mathcal{A}_g \) induces an isomorphism on rational (co)homology. The Hodge bundle on \( \mathbb{H}_g \) descends to a bundle \( \mathcal{F}_g \) on \( \mathcal{A}_g \) and thus we find \( \text{ch}_2(\mathcal{F}_g) \in H^{2k}(\mathcal{A}_g; \mathbb{Q}) \cong H^{2k}(\mathcal{A}_g; \mathbb{Q}) \). We will therefore pretend that \( \mathcal{F}_g \) is a vector bundle on \( \mathcal{A}_g \). According to Charney and Lee [4], \( H^k(\mathcal{A}_g; \mathbb{Q}) \) is independent of \( k \) for \( g \) sufficiently large. They prove that the direct sum of these stable cohomology spaces comes with the structure of a connected \( \mathbb{Q} \)-Hopf algebra \( H^\bullet \) whose primitive generators are classes \( \text{ch}_{2r+1} \in H^{4r+2} (\mathcal{A}_g; \mathbb{Q}) \) and classes \( y_r \in H^{4r+2} (\mathcal{A}_g; \mathbb{Q}) \). For \( g \gg r \), the image of \( \text{ch}_{2r+1} \in H^{4r+2}(\mathcal{A}_g; \mathbb{Q}) \) is \( \text{ch}_{2r+1}(\mathcal{F}_g) \) (which is known to be nonzero), whereas the image of \( y_r \) in \( H^{4r+2}(\mathcal{A}_g; \mathbb{Q}) \) is zero.

The class \( y_r \) is somewhat harder to describe: it comes from transgression of a primitive class in \( H^{4r+1}(B \text{GL}(\mathbb{Z}); \mathbb{Q}) \) about which we will say more below. Jiaming Chen and the author [5] have recently shown that the stability theorem holds if we take the mixed Hodge structure on \( H^\bullet(\mathcal{A}_g^{\text{bh}}; \mathbb{Q}) \) into account: \( H^\bullet \) inherits such a structure with \( H^k \) having weight \( k \) (for \( \mathcal{A}_g^{\text{bh}} \) is compact) and \( H^k/W_{k-1} H^k \) can be identified with \( H^k(\mathcal{A}_g; \mathbb{Q}) \) for \( g \) large. Moreover, for \( k = 4r + 2 \), the image of \( \text{ch}_{2r+1} \) in \( H^{4r+2}(\mathcal{A}_g; \mathbb{Q}) \) is \( \text{ch}_{2r+1}(\mathcal{F}_g) \), which is of bidegree \( (2r+1, 2r+1) \) (and nonzero for \( g \gg r \)), whereas \( y_r \) (\( r \geq 1 \)) has weight zero. So the primitive part \( H^{4r+2} \) of \( H^{4r+2} \) is a Tate extension:

\[
0 \to \mathbb{Q}(0) \to H^{4r+2}_{pr} \to \mathbb{Q}(-2r-1) \to 0,
\]

where \( \mathbb{Q}(-2r-1) \) is spanned by the image \( \text{ch}_{2r+1} \) of \( \text{ch}_{2r+1} \) and \( \mathbb{Q}(0) \) by the image of \( y_r \). The inclusion \( \mathbb{Q}(-2r-1) \subset \mathbb{C} \) comes about by regarding the twisted (De Rham) version \((2\pi \sqrt{-1})^{2r+1} \text{ch}_{2r+1} \) as the natural generator.

In what follows we take \( g \) large enough to be in the stable range, so that this sequence appears in \( H^{4r+2}(\mathcal{A}_g^{\text{bh}}; \mathbb{Q}) \). By Theorem 4.1 (in combination with Remarks 2.6 and 2.9), the Goresky-Pardon Chern character \( \text{ch}_{2r+1}^{\text{sp}}(\mathcal{F}_g) \) (being a universal polynomial with rational coefficients of weighted degree \( 2r+1 \) in the \( \mathcal{A}_g^{\text{sp}}(\mathcal{F}_g) \)) is then a generator of \( F^{2r+1} H^{4r+2} \). This will help us determine the class of this extension. For this purpose we also need to know a bit more about \( y_r \), when viewed as an element of \( H^{4r+2}(\mathcal{A}_g^{\text{bh}}; \mathbb{Q}) \). We will however not describe \( y_r \), but rather a stable primitive homology class on \( z_r \in H_{4r+2}(\mathcal{A}_g^{\text{bh}}; \mathbb{Q}) \) such that \( \langle y_r, z_r \rangle \neq 0 \). That will do, for then the map \( x \in H^{4r+2}_{pr} \mapsto \langle x, z_r \rangle/\langle y_r, z_r \rangle \in \mathbb{Q} = \mathbb{Q}(0) \) splits the above sequence and so the extension class is given by the image of \( \langle \text{ch}_{2r+1}^{\text{sp}}(\mathcal{F}_g), z_r \rangle \) in \( \mathbb{C}/\mathbb{Q} \). We prefer to replace \( \text{ch}_{2r+1}^{\text{sp}}(\mathcal{F}_g) \) by its De Rham variant \((2\pi \sqrt{-1})^{2r+1} \text{ch}_{2r+1}^{\text{sp}}(\mathcal{F}_g) \), so that the class of this Tate extension becomes more like a period; it is then the image of \( \langle (2\pi \sqrt{-1})^{2r+1} \text{ch}_{2r+1}^{\text{sp}}(\mathcal{F}_g), z_r \rangle \) in \( \mathbb{C}/\mathbb{Q}(2r+1) \). The following theorem
shows that this extension is nontrivial by showing that the Goresky-Pardon Chern character has a nonzero imaginary part.

**Theorem 5.1.** The class of the Tate extension (5.1) in \( \mathbb{C}/\mathbb{Q}(2r+1) \) (which is given by the image of \((2\pi \sqrt{-1})^{2r+1} \chi_{2r+1}(F_g), z_r) \) in \( \mathbb{C}/\mathbb{Q}(2r+1) \) is real and equal to a nonzero rational multiple of \( \pi^{-2r-1} \zeta(2r+1) \). In particular, \( \chi_{2r+1}(F_g) \) has a nonzero imaginary part, but its real part lies in \( H^{4r+2}(A_{\text{gh}}^g; \mathbb{Q}) \).

The computation uses Beilinson’s regulator for the field \( \mathbb{Q} \), which involves among other things Deligne cohomology and the Cheeger-Simons classes. We recall what we need below, referring to Burgos’ very accessible exposition [3] as a general reference for this topic.

**Refined Chern characters.** For a smooth complex variety \( X \) there is defined the Deligne cohomology group \( H^{2p}_D(X, \mathbb{Z}(p)) \) \( (p = 0, 1, 2, \ldots) \). It fits in an exact sequence

\[
0 \to J_p(X) \to H^{2p}_D(X, \mathbb{Z}(p)) \to F^p H^{2p}(X, \mathbb{Z}(p)) \to 0,
\]

where \( F^p H^{2p}(X, \mathbb{Z}(p)) \) denotes the intersection of the image of \( H^{2p}(X, \mathbb{Z}(p)) \to H^{2p}(\mathcal{J} \mathcal{E} D U) \) with \( F^p H^{2p}(X, \mathbb{C}) \) and \( J_p(X) \) is an abelian group that is the \( p \)-th intermediate Jacobian in case \( X \) is projective:

\[
J_p(X) := H^{2p-1}(X, \mathbb{C})/(F^p H^{2p-1}(X, \mathbb{C}) + H^{2p-1}(X, \mathbb{Z}(p))).
\]

We only need here the following somewhat informal description of this extension: when \( X \) is complete, an element of \( H^{2p}_D(X, \mathbb{Z}(p)) \) is representable by a pair \((b, \alpha)\), where \( b \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p)) \) and \( \alpha \) is closed 2\(p\)-form on \( X \) of Hodge level \( \geq p \) with periods in \( \mathbb{Z}(p) \) (we then write \( \alpha \in (F^p A_{\text{cl}}^{2p}(X, \mathbb{Z}(p))) \)), such that for every smooth singular \( \mathbb{Z} \)-valued 2\(p\)-chain \( Z \) on \( X \), the image of \( \int_Z \alpha \) in \( \mathbb{C}/\mathbb{Z}(p) \) is equal to \( b([-\partial]) \). In case \( X \) is not complete, we require that \( \alpha \) extends to a normal crossing compactification with logarithmic poles along \( D \) of \( X \) (so that it represents an element of \( F^p H^{2p}(X) \) with periods in \( \mathbb{Z}(p) \)). The equivalence relation is the one which produces the exact sequence and so \((b, \alpha)\) represents zero precisely when the cohomology class of \( \alpha \) is zero and \( b \) is in the image of \( F^p H^{2p-1}(X, \mathbb{C}) \to H^{2p-1}(X, \mathbb{C})/H^{2p-1}(X, \mathbb{Z}(p)) = H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p)) \). Beilinson and Gillet showed that for a vector bundle \( \mathcal{F} \) on \( X \) one has a natural lift of \((2\pi \sqrt{-1})^p \chi_{2p}(\mathcal{F}) \in F^p H^{2p}(X, \mathbb{Z}(p)) \) to \( H^{2p}_D(X, \mathbb{Z}(p)) \). It is called the Beilinson Chern character and—in order to come to terms with the fact that Beilinson and Betti have a common initial string—we denote it by \( \chi_{2p}^{B}(\mathcal{F}) \in H^{2p}_D(X, \mathbb{Z}(p)) \).

It was observed by Dupont, Hain and Zucker [6] that we can also get this class as a Cheeger-Simons differential character, which is defined in a \( C^{\infty} \)-setting. For a manifold \( M \) we have an extension that is similarly defined as \( H^{2p}_D(X, \mathbb{Z}(p)) \) above:

\[
0 \to H^{2p-1}(M, \mathbb{C}/\mathbb{Z}(p)) \to H^{2p}(M, \mathbb{C}/\mathbb{Z}(p)) \to A^{2p}_D(M, \mathbb{Z}(p)) \to 0,
\]

where \( A^{2p}_D(M, \mathbb{Z}(p)) \) denotes the space of closed 2\(p\)-forms on \( M \) with periods in \( \mathbb{Z}(p) \). A complex vector bundle \( \mathcal{F} \) on \( M \) endowed with a connection \( \nabla \) defines Cheeger-Simons Chern character \( \hat{\chi}_{2p}(\mathcal{F}, \nabla) \in H^{2p}(M, \mathbb{C}/\mathbb{Z}(p)) \), the closed 2\(p\)-form \( \text{Ch}_{2p}(\mathcal{F}, \nabla) \) then being given as \( \text{Tr}((-\text{R}(\nabla))^p)/p! \), where \( \text{R}(\nabla) \in A^{2p}_D(\text{End}(\mathcal{F})) \) denotes the curvature form of \( \nabla \). Dupont-Hain-Zucker [6] verified the compatibility with the Beilinson’s Chern character: if \( X \) is projective and \( \mathcal{F} \) is an algebraic vector bundle endowed with a connection \( \nabla \) of type \((1, 0)\), then the Chern character form \( \text{Ch}_{2p}(\mathcal{F}, \nabla) \) lands in \( (F^p A^{2p}_D(M, \mathbb{Z}(p)) \). This ensures that \( \hat{\chi}_{2p}(\mathcal{F}, \nabla) \) maps to
the corresponding subspace $F^p\dot{H}^{2p}(M;\mathbb{C}/\mathbb{Z}(p))$ of $\dot{H}^{2p}(M;\mathbb{C}/\mathbb{Z}(p))$ and the evident projection $F^p\dot{H}^{2p}(M;\mathbb{C}/\mathbb{Z}) \to H^{2p}_\mathbb{Z}(X;\mathbb{Z}(p))$ maps $\text{ch}_p(F,\nabla)$ to $\text{ch}^p_p(F)$. This is then also true when $X$ is quasiprojective, provided we know that $(F,\nabla)$ extends across a smooth normal crossing compactification, for both refinements of the Chern character behave functorially with respect to pull-backs.

The regulator map for $\mathbb{Q}$. The group homology of $\text{GL}(g,\mathbb{Z})$ stabilizes in $g$ and the resulting stable rational homology is a graded commutative $\mathbb{Q}$-Hopf algebra with a primitive generator for each degree $4r + 1$ (so it is an exterior algebra). This stable homology is in fact the homology of $B\text{GL}(\mathbb{Z})$, where $\text{GL}(\mathbb{Z})$ is the monotone union $\cdots \subset \text{GL}(g,\mathbb{Z}) \subset \text{GL}(g,\mathbb{Z}) \subset \cdots$. Applying Quillen’s plus construction does not affect the homology and hence this remains so for the homology of $B\text{GL}(\mathbb{Z})^+$. The latter is an $H$-space with distinguished generators up to sign for its primitive rational homology: following Quillen, the algebraic $K$-groups of $\mathbb{Z}$ are defined as $K_s(\mathbb{Z}) := \pi_s(B\text{GL}(\mathbb{Z})^+,*)$ and the Hurewicz map

$$K_s(\mathbb{Z}) = \pi_*(B\text{GL}(\mathbb{Z})^+,*) \to H_*(B\text{GL}(\mathbb{Z})^+) \cong H_*(B\text{GL}(\mathbb{Z}))$$

induces for $s > 0$ an isomorphism of $K_s(\mathbb{Z}) \otimes \mathbb{Q}$ onto $H^*_\mathbb{Q}(B\text{GL}(\mathbb{Z});\mathbb{Q})$. It is known for $s > 0$, $K_s(\mathbb{Z})$ is a torsion group unless $s = 4r + 1$ ($r = 0, 1, \ldots$) in which case it has rank one. We choose for $r > 0$ a generator $b_r$ of the image of $K_{4r+1}(\mathbb{Z}) \to K_{4r+1}(\mathbb{Z}) \otimes \mathbb{Q}$ and identify it with its image in $H^r_\mathbb{Q}(B\text{GL}(\mathbb{Z});\mathbb{Q})$. This element is of course defined up to sign. Over $B\text{GL}(g,\mathbb{Z})$ we have the universal local system $\mathcal{V}_g$ with fiber $\mathbb{Z}^g$. The inclusion $\text{GL}(g,\mathbb{Z}) \subset \text{GL}(g,\mathbb{C})$ induces a map $B\text{GL}(g,\mathbb{Z}) \to B\text{GL}(g,\mathbb{C})$. If we take direct limits, then the resulting map $B\text{GL}(\mathbb{Z}) \to B\text{GL}(\mathbb{C})$ is zero on rational homology in positive degree (being a homomorphism from an exterior algebra to a polynomial one), but the situation is different for Deligne cohomology. This of course requires that we are in an algebraic setting, which is kind of clear for $B\text{GL}(\mathbb{C})$, being an inductive limit of Grassmannians, but less so for $B\text{GL}(\mathbb{Z})$. Yet, as explained in [6] and [3], this can be given a sense by regarding $B\text{GL}(\mathbb{Z})$ as a simplicial projective manifold of dimension zero (and in order to get the map, we must then do the same for $B\text{GL}(\mathbb{Z}))$).

We are interested in the value $\text{ch}^B_{2r+1}(\mathcal{V}_g)(b_r) \in \mathbb{C}/\mathbb{Q}(2r+1)$, or rather its image in $\mathbb{C}/\mathbb{R}(2r+1)$. Since $\mathbb{R}(2r+1)$ is just the imaginary axis, we may identify $\mathbb{C}/\mathbb{R}(2r+1)$ with $\mathbb{R}$ so that we have a natural map $\mathbb{C}/\mathbb{Q}(2r+1) \to \mathbb{C}/\mathbb{R}(2r+1) \cong \mathbb{R}$. The image of $\text{ch}^B_{2r+1}(\mathcal{V}_g)(b_r) \in \mathbb{C}/\mathbb{Q}(2r+1)$ in $\mathbb{R}$ is according to Beilinson [3] given by a rational multiple of the corresponding regulator of $\mathcal{Q}$, which is $\zeta'(-2r)$, where $\zeta$ is the classical Riemann zeta function. (It is in fact known that $\text{ch}_{2r+1}^B(\mathcal{V}_g)(b_r)$ itself is represented by $\zeta'(-2r)$, but we will obtain this as an outcome of our computation.) If we then invoke the functional equation for $\zeta$, we find:

**Scholium 5.2.** The image of $\text{ch}^B_{2r+1}(\mathcal{V}_g)(b) \in \mathbb{C}/\mathbb{Q}(2r+1)$ under the natural map $\mathbb{C}/\mathbb{Q}(2r+1) \to \mathbb{R}$ is a nonzero rational multiple of $\pi^{-2r-1}\zeta(2r+1)$.

**Proof of Theorem 5.1.** Returning to the situation at hand, let us denote by $I^r$ the integral part of the first basis resp. second basis element of $H = \mathbb{Z}^2$, so that we have a decomposition $H^g = I^g \otimes I^g$ into maximal isotropic sublattices of $H^g$. The symplectic form identifies $I^r$ with $\text{Hom}(I^g,\mathbb{Z})$ and so we have an embedding $\text{GL}(g,\mathbb{Z}) = \text{GL}(I^g) \hookrightarrow \text{Sp}(H^g)$ defined by $\sigma \mapsto (\sigma, (\sigma^*)^{-1})$. This map commutes

---

\footnote{This can probably also be used to produce another proof that $y_r$ is of type $(0,0)$.}
with the stability maps on either side so that the map on rational homology also stabilizes, but this will yield the zero map as $H_*(B\text{GL}(\mathbb{Z});\mathbb{Q})$ is an exterior algebra and $H_*(B\text{Sp}(\mathbb{Z});\mathbb{Q})$ a polynomial algebra. However, as explained in [5], if $\infty \in A^\text{bb}_g$ is the worst cusp (the unique element of the zero-dimensional Satake stratum $A_0$ of $A^\text{bb}_g$), then we have a basis of regular neighborhoods $U_\infty$ of $\infty$ in $A^\text{bb}_g$ with the property that $U^\infty_\infty := U_\infty \cap A^\text{bb}_g$ is a virtual classifying space for the semidirect product $\text{GL}(g,\mathbb{Z}) \ltimes \text{Sym}_2(\mathbb{Z}^g)$ and so contains a virtual classifying space for $\text{GL}(g,\mathbb{Z})$. We will make use of the fact that this virtual classifying space can be chosen in the real locus. Here we note that the modular interpretation of $(A_g, F_g)$ endows this pair with a real structure. The Baily-Borel compactification $A^\text{bb}_g$ together with its stratification are defined over $\mathbb{R}$. In particular, $\infty$ is a real point so that we can take $U_\infty$ invariant under complex conjugation.

**Lemma 5.3.** The locus $U^\infty_\infty \cap A^\text{bb}_g(\mathbb{R})$ is a virtual classifying space for $\text{GL}(g,\mathbb{Z})$ and so we can represent $b_r$ by a cycle $B_r$ on $U^\infty_\infty \cap A^\text{bb}_g(\mathbb{R})$.

**Proof.** The real structure on $A^\text{bb}_g$ lifts to one on $\mathbb{H}_g$, which in relation to the cusp $\infty$ is best understood in terms of the Siegel upper half plane model. The symplectic form identifies the space of complex symmetric tensors $\text{Sym}_2(I_g^2)$ with the space of symmetric maps $I_g^2 \to I_g^2$. The graph of such a map lies in $\mathbb{H}_g$ if and only if the imaginary part of the symmetric tensor is positive. If $C_{I^g}$ denotes the locus $C_{I^g}$ of positive symmetric tensors, then $\sqrt{-1}C_{I^g}$ defines a real subset of $\mathbb{H}_g$. The $\text{Sp}(I^g)$-stabilizer of $\sqrt{-1}C_{I^g}$ is $\text{GL}(I^g)$ and the orbit space $\text{GL}(I^g)\backslash \sqrt{-1}C_{I^g}$ maps onto a connected component of the real locus of $A^\text{bb}_g$. Now $\text{GL}(I^g)\backslash C_{I^g}$ is a virtual classifying space for $\text{GL}(I^g) = \text{GL}(g,\mathbb{Z})$. This is still so if we replace $C_{I^g}$ by any $\text{GL}(I^g)$-invariant cocore $K \subset C_{I^g}$ [1]. In particular, $\text{GL}(I^g)\backslash (\sqrt{-1}K)$ supports a $(4r + 1)$-cycle $B_r(K)$ which represents the primitive element $b_r$ defined above. For an appropriate choice of $K$, $\text{GL}(I^g)\backslash (\sqrt{-1}K)$ embeds in $U^\infty_\infty$ and we then take $B_r$ to be the image of $B_r(K)$.

Since $H_{4r+1}(A_g;\mathbb{Q}) = 0$ (we are in the stable range), the cycle $B_r$ bounds a $\mathbb{Q}$-chain $Z_r$ in $A^\text{bb}_g$. As $U^\infty_\infty$ is contractible (even conical we make a careful choice for $U^\infty_\infty$), this cycle also bounds a chain $cB_r$ in $U^\infty_\infty \cap A^\text{bb}_g(\mathbb{R})$ so that we obtain a $(4r + 2)$-cycle $Z_r - cB_r$ on $A^\text{bb}_g$. It is shown in [5] that the stable cohomology class $y_r \in H^{4r+2}(A^\text{bb}_g;\mathbb{Q})$ takes a nonzero value on this class so that $[Z_r - cB_r]$ may serve as our $z_r \in H_{4r+2}(A^\text{bb}_g;\mathbb{Q})$. It remains to compute the value of $(2\pi\sqrt{-1})^{2r+1} \text{ch}^\text{sp}_{2r+1}(F_g) \cdot z_r$ on $[Z_r - cB_r]$.

Corollary 3.3 gives us a connection $\nabla$ on $F_g$ whose curvature form yields the twisted Goresky-Pardon Chern characters. According to Corollary 2.7 these are invariant under full complex conjugation. We assume that $U^\infty_\infty$ has been chosen so small that $\nabla$ is flat on $U^\infty_\infty$ and defines on $U^\infty_\infty \cap A^\text{bb}_g(\mathbb{R})$ a local system given by the obvious representation of degree $g$ of $\text{GL}(I^g)$. Then the form $\text{Ch}_{2r+1}(F_g, \nabla)$ vanishes on $cB_r$ and so we find that

$$\langle (2\pi\sqrt{-1})^{2r+1} \text{ch}^\text{sp}_{2r+1}(F_g), z_r \rangle = \int_{Z_r} \text{Ch}_{2r+1}(F_g, \nabla) \cdot \nabla(Z_r).$$

As $\text{Ch}_{2r+1}(F_g, \nabla)$ defines a class in $H^{4r+2}(A_g;\mathbb{Q}(2r + 1))$, the image of this integral in $\mathbb{C}/\mathbb{Q}(2r + 1)$ only depends on $\partial Z_r = B_r$ and is then given by the value $\text{ch}_{2r+1}(F_g)(b_r) \in \mathbb{C}/\mathbb{Q}(2r + 1)$. Since $\text{Ch}_{2r+1}(F_g, \nabla)$ and $b_r$ are invariant under full complex conjugation, this value lies in fact in the image of $\mathbb{R}$ in $\mathbb{C}/\mathbb{Q}(2r + 1)$. 


In other words, it is completely given by its image in $\mathbb{C}/\mathbb{R}(2r+1) \cong \mathbb{R}$. We have observed that $(\mathcal{F}, \nabla)$ extends as a holomorphic vector bundle with flat connection to a nonsingular toric compactification and so this is also equal to $\text{ch}_2^{b_r}(\mathcal{F}_g)(b_r) \in \mathbb{C}/\mathbb{Q}(2r+1)$. According to our Scholium 5.2 its image in $\mathbb{C}/\mathbb{R}(2r+1) \cong \mathbb{R}$ is a rational multiple of $\pi^{-2r-1} \zeta(2r + 1)$. This completes the proof.

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YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY BEIJING (CHINA) AND MATHEMATISCH INSTITUUT, UNIVERSITEIT UTRECHT (NEDERLAND)

E-mail address: eduard@math.tsinghua.edu.cn