Homework of Serre Duality

Chen Jiaming

1. Let X be an integral scheme of finite type over a field k(not necessarily algebraic closed), then for any closed point $p \in X$, $\dim X = \dim \mathcal{O}_{X,p}$.

2. If there is some property $\mathcal{P}$ of points of a scheme that is 'open'(If a point x has $\mathcal{P}$, then there is some neighborhood U of x such that all the points in U have $\mathcal{P}$), then to check if all points of a quasi-compact scheme have $\mathcal{P}$, it suffices to check only the closed points.

3. Let X be a noetherian scheme, and let $\mathscr{F}, \mathscr{G} \in \mathfrak{Mod}(X)$.

(a) If $\mathscr{F}, \mathscr{G}$ are both coherent, then $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is coherent, for all $i \geqslant 0$.

(b) If $\mathscr{F}$ is coherent and $\mathscr{G}$ is quasi-coherent, then $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is quasi-coherent, for all $i \geqslant 0$.

4. Let A be a regular local ring, and let M be a finitely generated A-module.

(a) M is projective if and only if $\mathrm{Ext}^i(M, A) = 0$.

(b) Use (a) to show that for any n, $\mathrm{pd}\, M \leqslant n$ if and only if $\mathrm{Ext}^i(M, A) = 0$ for all $i > n$.

5. Suppose that Y is a projective scheme of dimension n satisfying strong Serre duality. If $X \hookrightarrow Y$ is a closed subscheme of codimension r, show that $\mathscr{E}xt^i_Y(\mathcal{O}_X, \omega_Y^\circ)$ is the dualizing sheaf for X.

**6.* Suppose X is a complete intersection in $\mathbb{P}^n$, of hypersurfaces of degrees $d_1, \cdots, d_r$. Then X satisfies strong Serre duality, with dualizing sheaf $\omega_X^\circ \cong \mathcal{O}_X(-n-1-d_1-\cdots-d_r)$.