COURSE NOTES INFINITE DIMENSIONAL LIE ALGEBRAS OCCURRING IN ALGEBRAIC GEOMETRY

EDUARD LOOIJENGA

We find it convenient to work over an arbitrary closed field k of characteristic zero. Where we discuss the link with topological quantum field theory, we assume that $k = \mathbb{C}$. As an intermediate base we use a smooth k-algebra R. Usually R is a k-algebra of finite type (so that $\mathrm{Spec}(R)$ is a nonsingular affine variety) or a local ring thereof.

1. Flat and projectively flat connections

Let S be a nonsingular k-variety. The sheaf of 0th order differential operators $\mathcal{D}_0(\mathcal{O}_S)$ of $\mathcal{O}_S$ is $\mathcal{O}_S$ (acting $\mathcal{O}_S$ by scalar multiplication) and the sheaf $\mathcal{D}_1(\mathcal{O}_S)$ of first order differential operators is its direct sum with θ_S, the sheaf of vector fields on S (acting $\mathcal{O}_S$ by derivation). Both are $\mathcal{O}_S$-modules and closed under the Lie bracket. As a Lie algebra it is a nontrivial extension of θ_S by $\mathcal{O}_S$. Indeed, the projection $\mathcal{D}_1(\mathcal{O}_S) \to \theta_S$ is also the map which assigns to $D \in \mathcal{D}_1(\mathcal{O}_S)$ the derivation $f \in \mathcal{O}_S \mapsto [D, f] \in \mathcal{O}_S$.

Now let $\mathcal{H}$ be a rank r vector bundle over S (= a locally a free $\mathcal{O}_S$-module of rank r). Then the Lie algebra $\mathcal{D}_1(\mathcal{H})$ of first order differential operators $\mathcal{H} \to \mathcal{H}$ is given in terms of a local basis of $\mathcal{H}$ by a $r \times r$ matrix of first order differential operators of $\mathcal{O}_S$. More intrinsically, it is the $\mathcal{O}_S$-submodule $\mathcal{D}_1(\mathcal{H})$ of $\mathcal{E}nd_k(\mathcal{H})$ characterized by the property that $[\mathcal{D}_1(\mathcal{H}), \mathcal{O}_S] \subset \theta_S \otimes_{\mathcal{O}_S} \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{H})$. This property yields an exact sequence of coherent sheaves of Lie algebras

$$0 \to \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{H}) \to \mathcal{D}_1(\mathcal{H}) \xrightarrow{\mathrm{symb}} \theta_S \otimes_{\mathcal{O}_S} \mathcal{E}nd_{\mathcal{O}_S}(\mathcal{H}) \to 0,$$

where symb is the symbol map which assigns to $D \in \mathcal{D}_1(\mathcal{H})$ the k-derivation $\phi \in \mathcal{O}_S \mapsto [D, \phi] \in \mathcal{O}_S$. The local sections of $\mathcal{D}_1(\mathcal{H})$ whose symbol land in $\theta_S \otimes 1_{\mathcal{H}} \cong \theta_S$ make up a coherent subsheaf of Lie subalgebras $\mathrm{symb}^{-1}(\theta_S) \subset \mathcal{D}_1(\mathcal{H})$ so that we have an exact sequence of coherent sheaves of Lie algebras

$$0 \to \mathcal{E}nd(\mathcal{H}) \to \mathrm{symb}^{-1}(\theta_S) \to \theta_S \to 0.$$

A connection on $\mathcal{H}$ is then simply a section ∇ of $\mathrm{symb}^{-1}(\theta_S) \to \theta_S$ (the image of $X \in \theta_S$ is denoted $\nabla_X \in \mathrm{symb}^{-1}(\theta_S)$). Its curvature $R(\nabla)$ is the $\mathcal{E}nd(\mathcal{H})$-valued 2-form given by $X \wedge Y \in \wedge^2_{\mathcal{O}_S} \theta_S \mapsto [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. So ∇ is flat precisely if it is a Lie homomorphism. This suggests the following definition.

Definition 1.1. A flat *projective connection* on $\mathcal{H}$ is a Lie subalgebra $\mathcal{D} \subset \mathcal{D}_1(\mathcal{H})$ with the property that $\mathrm{symb}(\mathcal{D}) = \theta_S$ and $\mathcal{D} \cap \mathcal{E}nd(\mathcal{H}) = \mathcal{O}_S \otimes 1_{\mathcal{H}} \cong \mathcal{O}_S$ so that we have an exact subsequence

$$0 \to \mathcal{O}_S \to \mathcal{D} \to \theta_S \to 0.$$

This indeed defines a flat connection on the associated projective space bundle $\mathbb{P}_S(\mathcal{H})$, for an $\mathcal{O}_S$-linear section ∇ of $\mathcal{D} \to \theta_S$ defines a connection on $\mathcal{H}$ whose curvature form $R(\nabla)$ is an ordinary 2-form on S. So this gives rise to a flat connection in $\mathbb{P}_S(\mathcal{H})$. Any other section

∇' differs from ∇ by an $\mathcal{O}_S$-linear map $\theta_S \to \mathcal{O}_S$, in other words, by a differential α, and on readily checks that $R(\nabla') = R(\nabla) + d\alpha$. This implies that the flat connection on $\mathbb{P}_S(\mathcal{H})$ is independent of the choice of the section. The curvature form $R(\nabla)$ is closed and hence locally exact, and so we can always choose ∇ to be flat as a connection. Any other such local section that is flat is necessarily of the form $\nabla + d\phi$ with $\phi \in \mathcal{O}$ and conversely, any such local section has that property. Beware that the Lie algebra sheaf $\mathcal{D}$ itself does not determine a connection on $\mathcal{H}$; this is most evident when $\mathcal{H}$ is a line bundle, for then we must have $\mathcal{D}(\mathcal{H}) = \mathcal{D}_1(\mathcal{H})$.

A flat projective connection $\mathcal{D}$ on $\mathcal{H}$ acts on the determinant bundle $\det(\mathcal{H}) = \wedge^r_{\mathcal{O}_S}\mathcal{H}$ by means of the formula

$$D(e_1 \wedge \cdots \wedge e_r) := \sum_{i=1}^{r} e_1 \wedge \cdots \wedge D(e_i) \wedge \cdots \wedge e_r, \quad (D \in \mathcal{D}).$$

This is indeed well-defined, and identifies $\mathcal{D}$ as a Lie algebra with the Lie algebra of first order differential operators $\mathcal{D}_1(\det(\mathcal{H}))$. Notice however that this identification makes $f \in \mathcal{O}_S \subset \mathcal{D}$ act on $\det(\mathcal{H})$ as multiplication by rf.

We can identify $\mathcal{D}$ also with the sheaf of first order differential operators of other line bundles on S, for if λ is a line bundle on S and N is a positive integer, then a similar formula identifies $\mathcal{D}_1(\lambda)$ with $\mathcal{D}_1(\lambda^{\otimes N})$, both as $\mathcal{O}_S$-modules and as k-Lie algebras, although this induces multiplication by N on $\mathcal{O}_S$. This leads us to:

Definition 1.2. Let λ be an invertible $\mathcal{O}_S$-module and $\mathcal{H}$ a locally free $\mathcal{O}_S$-module of finite rank. A λ-*flat connection* on $\mathcal{H}$ is homomorphism of $\mathcal{O}_S$-modules $u : \mathcal{D}_1(\lambda) \to \mathcal{D}_1(\mathcal{H})$ that is also a Lie homomorphism over k, commutes with the symbol maps (so these must land in θ_S) and takes scalars to scalars: $\mathcal{O}_S \subset \mathcal{D}_1(\lambda)$ is mapped to $\mathcal{O}_S \subset \mathcal{D}_1(\mathcal{H})$.

It follows from the preceding that such a homomorphism u determines a flat connection on the projectivization of $\mathcal{H}$. The map u preserves $\mathcal{O}_S$ and since this restriction is $\mathcal{O}_S$-linear, it is given by multiplication by some regular function w on S. If $D \in \theta_S$ is lifted to $\hat{D} \in \mathcal{D}_1(\lambda)$, then $u(\hat{D}) \in \mathcal{D}_1(\mathcal{H})$ is also a lift of D and so $D(w) = [u(\hat{D}), u(1)] = u([\hat{D}, 1]) = 0$. This shows that w must be locally constant (and hence constant, since a variety is connected); we call this the *weight* of u. So in the above discussion, $\mathcal{D}$ comes with $\det(\mathcal{H})$-flat connection of weight r^{-1}.

It is clear that if the weight of u is constant zero, then u factors through θ_S, so that we get a flat connection in $\mathcal{H}$. This is also the case when λ comes with a global nonzero section s, for then $\mathcal{D}_1(\mathcal{O}_S)$ splits both as $\mathcal{O}_S$-module and as a sheaf of k-Lie algebras, with θ_S identified as the annihilator of s. The flat connection is then given by the action of θ_S via this direct summand. This has an interesting consequence: if $\pi : \mathbb{L}^\times(\lambda) \to S$ is the geometric realization of the $\mathbb{G}_m$-bundle defined by λ, then $\pi^*\lambda$ has a 'tautological' generating section and thus gets identified with $\mathcal{O}_{\mathbb{L}^\times(\lambda)}$. Hence a λ-flat connection on $\mathcal{H}$ defines an ordinary flat connection on $\pi^*\mathcal{H}$. One checks that if w is the weight of u, then the connection is homogeneous of degree w along the fibers. So in case $k = \mathbb{C}$, $s \in S$ and $\tilde{s} \in \mathbb{L}^\times(\lambda(s))$ lies over $s \in S$, then the multi-valued map $(z, h) \in \mathbb{C}^\times \times H_s \mapsto (z\tilde{s}, z^w h) \in \mathbb{L}^\times(\lambda(s)) \times H_s$ is flat, and so the monodromy of the connection in $\mathbb{L}^\times(\lambda(s))$ is scalar multiplication by $e^{2\pi\sqrt{-1}w}$.

There is an obvious generalization involving several line bundles $\boldsymbol{\lambda} = \{\lambda_i\}_{i\in I}$ on S (here I is a finite nonempty set), a situation that we will indeed encounter. We then take for $\mathcal{D}_1(\boldsymbol{\lambda})$ the Lie subalgebra of $\oplus_{i\in I}\mathcal{D}_1(\lambda_i)$ of I-tuples of operators with the same symbol, so that we have an exact sequence

$$0 \to \mathcal{O}_S^I \to \mathcal{D}_1(\boldsymbol{\lambda}) \to \theta_S \to 0.$$

We define a *$\boldsymbol{\lambda}$-flat connection* on $\mathcal{H}$ is homomorphism of $\mathcal{O}_S$-modules $u : \mathcal{D}_1(\boldsymbol{\lambda}) \to \mathcal{D}_1(\mathcal{H})$ that is also a Lie homomorphism over k, commutes with the symbol maps and takes $\mathcal{O}_S^I$ to $\mathcal{O}_S \subset \mathcal{D}_1(\mathcal{H})$. Such a structure come with a multi-weight $\mathbf{w} = (w_i)_i \in k^I$. There is a geometric interpretation as before, if we let $\pi : \mathbb{L}^\times(\boldsymbol{\lambda}) \to S$ be the $\mathbb{G}_m^I$-bundle associated to $\boldsymbol{\lambda}$, then $\pi^*\mathcal{H}$ comes with a natural flat connection whose holonomy in a fiber is prescribed by $\mathbf{w}$.

Remark 1.3. We will also encounter a logarithmic version. Here we are given a closed subvariety $\Delta \subset S$ of lower dimension (usually a normal crossing hypersurface). Then the θ_S-stabilizer of the ideal defining Δ, denoted $\theta_S(\log\Delta)$, is a coherent $\mathcal{O}_S$-submodule of θ_S closed under the Lie bracket. If in Definition 1.2 we have u only defined on the preimage of $\theta_S(\log\Delta) \subset \theta_S$ in $\mathcal{D}_1(\boldsymbol{\lambda})$ (which we denote here by $\mathcal{D}_1(\boldsymbol{\lambda})(\log\Delta)$), then we say that we have a *logarithmic $\boldsymbol{\lambda}$-flat connection relative to* Δ on $\mathcal{H}$.

2. The Virasoro algebra and its basic representation

Much of the material exposed in this section is a conversion of certain standard constructions (as can be found for instance in [10]) into a coordinate invariant and relative setting. But the way we introduce the Virasoro algebra is perhaps less standard. A similar remark applies to part of the last subsection (in particular, Corollary 3.6), which is devoted to the Fock module attached to a symplectic local system.

We fix an R-algebra $\mathcal{O}$ isomorphic to the formal power series ring $R[[t]]$. In other words, $\mathcal{O}$ comes with a principal ideal $\mathfrak{m}$ so that $\mathcal{O}$ is complete for the $\mathfrak{m}$-adic topology and the associated graded R-algebra $\oplus_{j=0}^\infty \mathfrak{m}^j/\mathfrak{m}^{j+1}$ is a polynomial ring over R in one variable. The choice of a generator t of the ideal $\mathfrak{m}$ identifies $\mathcal{O}$ with $R[[t]]$. We put $L := \mathcal{O}[\mathfrak{m}^{-1}]$ so that L gets identified with $R((t))$. The $\mathfrak{m}$-*adic topology* on L is the topology that has the collection of cosets $\{f + \mathfrak{m}^N\}_{f\in L, N\geq 1}$ as a basis of open subsets. We often write $F^N L$ for $\mathfrak{m}^N$.

A continuous R-derivation D of L into a topological L-module M has the property that $D(\sum_{i\geq c} r_i t^i) = \sum_{i\geq c} i r_i t^{i-1} Dt$ and so is determined by its value on t. It is then easily seen that there is a universal one: $d : L \to \omega$ for which ω is the free L-module of rank one generated dt. We put $F^N\omega := F^{N-1}Ld\mathcal{O}$. A topological basis for ω is $(\omega_i := t^{i-1}\frac{dt}{t})_{i\in\mathbb{Z}}$. We write θ for the R-module of continuous R-derivations from L into L, which is just $\mathrm{Hom}_L(\omega, L)$. This is also a free L-module of rank one, a generator being the homomorphism $\frac{d}{dt}$ which takes dt to 1. The derivations that take F^sL to $F^{s+N}L$ form the $\mathcal{O}$-submodule $F^N\theta \subset \theta$ generated by $D_N := t^{N+1}\frac{d}{dt}$. Note that $\omega_i(D_j) = t^{i+j}$.

The residue map $\mathrm{Res} : \omega \to R$ which assigns to an element of $R((t))dt$ the coefficient of $t^{-1}dt$ is canonical, i.e., is independent of the choice of t (an elegant proof is due to Tate [21]). Its kernel is just the image of the universal derivation so that we have an exact sequence of R-modules $0 \to L \xrightarrow{d} \omega \xrightarrow{\mathrm{Res}} R \to 0$. The R-bilinear map

$$r : \omega \times L \to R, \quad (\alpha, f) \mapsto \mathrm{Res}(f\alpha)$$

is a topologically perfect pairing of filtered R-modules: we have $r(t^k, D_l) = \delta_{k+l,0}$ and so any R-linear $\phi : L \to R$ which is continuous (i.e., ϕ zero on $\mathfrak{m}^N$ for some N) is definable by an element of ω (namely by $\sum_{k>-N} \phi(t^{-k})\omega_k$) and likewise for an R-linear continuous map $\omega \to R$. More generally, if for $n \in \mathbb{Z}$, ω^n denotes the completed n-fold tensor power of ω

(this is the free L-module generated by $(dt)^n$, where it is understood that $(dt)^{-1} = \frac{d}{dt}$ so that $\omega^{-1} = \theta$), then besides the obvious duality $\omega^n \times \omega^{-n} \to \omega^0 = L$ of L-modules we have a perfect duality

$$(\alpha, \beta) \in \omega^n \times \omega^{1-n} \mapsto \mathrm{Res}(\alpha\beta)$$

of topological R-modules.

Observe that

$$(f, g) \in L \times L \mapsto \langle f, g\rangle := r(g, df) = \mathrm{Res}(g\, df) \in R$$

is an antisymmetric R-bilinear pairing that sends to (t^k, t^l) to $k\delta_{k+l,0}$. The kernel of his pairing is clearly $R \subset L$. In fact, this pairing factors through the universal derivation, for the result only depends on df and dg: if $dg = 0$, then g is a constant and $\mathrm{Res}(g\, df)$ is zero. So we have an antisymmetric R-bilinear nondegenerate pairing on dL (which is also the kernel of $\mathrm{Res} : \omega \to R$). This pairing naturally extends to $(dL \times \omega) \cup (\omega \times dL)$ (its value on (ω_k, ω_l) with $k \neq 0$ is $k^{-1}\delta_{k+l,0}$), but not to all of $\omega \times \omega$.

On the other hand, any continuous derivation $D \in \theta$ defines a R-bilinear pairing $\omega\times\omega \to R$, $(\alpha, \beta) \mapsto r(\alpha(D), \beta) = \mathrm{Res}(\alpha(D)\beta)$. This pairing is symmetric, for it is so on generators $(D_n, \omega_k, \omega_l)$, where it follows from the identity $\omega_k(D_n)\omega_l = \omega_{n+k+l}$.

2.1. **A trivial Lie algebra.** If we think of the multiplicative group of L as an algebraic group over R (or rather, as a group object in a category of ind schemes over R), then its Lie algebra, denoted here by $\mathfrak{l}$, is just L, regarded as a R-module with trivial Lie bracket. By writing $\mathfrak{l}$ we ignore the multiplication of L, but we retain the filtration $F^N\mathfrak{l} = \mathfrak{m}^N$. The universal enveloping algebra $U\mathfrak{l}$ is clearly the symmetric algebra of $\mathfrak{l}$ as an R-module, $\mathrm{Sym}^\bullet_R(\mathfrak{l})$. We complete it $\mathfrak{m}$-adically: given an integer N, then an R-basis of the truncation $U\mathfrak{l}/(U\mathfrak{l}\circ F^N\mathfrak{l})$ is indexed by the set of nondecreasing sequences of integers $\mathbf{k} := (k_1 \leq k_2 \leq \cdots \leq k_r)$ including the empty sequence: the associated basis element is $t^{\mathbf{k}} := t^{k_1} \circ \cdots \circ t^{k_r}$ (read 1 for the empty sequence). So elements of the completion

$$U\mathfrak{l} \to \overline{U}\mathfrak{l} := \varprojlim_N U\mathfrak{l}/U\mathfrak{l} \circ F^N\mathfrak{l}$$

are series of the form $\sum_{\mathbf{k}} r_{\mathbf{k}} t^{\mathbf{k}}$ with $r_{\mathbf{k}} \in R$ with the property that for every c the subsum over the sequences $\mathbf{k}$ whose last term is less than c is finite. We will refer to this construction as the $\mathfrak{m}$-*adic completion on the right*, although in the present case there is no difference with the analogously defined $\mathfrak{m}$-adic completion on the left, as $\mathfrak{l}$ is commutative. For example, if $\mathfrak{l}_2$ denotes the image of $\mathfrak{l} \otimes \mathfrak{l} \to U\mathfrak{l} \to \overline{U}\mathfrak{l}$, then its closure $\overline{\mathfrak{l}_2}$ in $\overline{U}\mathfrak{l}$ consists of the expressions $\sum_{i_0 \leq i \leq j} a_{i,j} t^i \circ t^j$ for some i_0.

Remark 2.1. An R-linear map $U\mathfrak{l} \to R$ is continuous precisely when it is so in every variable. It extends continuously to $\overline{U}\mathfrak{l} \to R$ precisely when it vanishes on some $U\mathfrak{l} \circ F^N\mathfrak{l}$. It follows that the continuous R-dual of $U\mathfrak{l}$ resp. $\overline{U}\mathfrak{l}$ can be identified with $\prod_{n\geq 0} \mathrm{Sym}^n_R\, \omega$ resp. $\oplus_{n\geq 0} \mathrm{Sym}^n_R\, \omega = \mathrm{Sym}^\bullet_R\, \omega$.

Given a $D \in \theta$, then we observed that the R-linear map $\omega \to L$ is self-adjoint with respect to the residue pairing. Since ω is the continuous R-dual of L this yields an element of $\overline{\mathfrak{l}_2}$. Let $C(D)$ be half this element, so that in terms of the above topological basis,

$$C(D) = \tfrac{1}{2} \sum_{i,j\in\mathbb{Z}} \mathrm{Res}(\omega_{-i}(D)\omega_{-j}) t^i \circ t^j.$$

In particular for $D = D_n = t^{n+1}\frac{d}{dt}$, $C(D_k) = \frac{1}{2}\sum_{i+j=n} t^i \circ t^j$. Observe that the map $C : \theta \to \overline{U}\mathfrak{l}$ is continuous.

2.2. **Oscillator and Virasoro algebra.** The residue map defines a central extension of $\mathfrak{l}$ by a copy of the trivial Lie algebra R. This is particular kind of Heisenberg algebra, called the *oscillator algebra* $\hat{\mathfrak{l}}$. As an R-module it is given as a direct sum $\mathfrak{l} \oplus R$ and if we denote the generator of the second summand by $\hbar$, then the Lie bracket is given by

$$[f + \hbar r, g + \hbar s] := \mathrm{Res}(g\, df)\hbar.$$

So this is a Lie algebra extension together with an R-linear splitting. Observe that $[t^k, t^l] = k\delta_{k+l,0}\hbar$ and that the center of $\hat{\mathfrak{l}}$ is $Re \oplus R\hbar$, where $e = t^0$ denotes the unit element of L viewed as an element of $\mathfrak{l}$. It follows that $U\hat{\mathfrak{l}}$ is an $R[t^0, \hbar]$-algebra. As an $R[\hbar]$-algebra it is obtained as follows: take the tensor algebra of $\mathfrak{l}$ (over R) tensored with $R[\hbar]$, $\otimes^{\bullet}_R \mathfrak{l} \otimes_R R[\hbar]$, and divide that out by the two-sided ideal generated by the elements $f \otimes g - g \otimes f - \mathrm{Res}(gdf)\hbar$. This makes it also plain that $U\hat{\mathfrak{l}}$ inherits from the $\mathbb{Z}_{\geq 0}$-grading of the tensor algebra a $\mathbb{Z}/2$-grading: $U\hat{\mathfrak{l}} = U\hat{\mathfrak{l}}^{\mathrm{even}} \oplus U\hat{\mathfrak{l}}^{\mathrm{odd}}$. The obvious surjection $\pi : U\hat{\mathfrak{l}} \to U\mathfrak{l} = \mathrm{Sym}^{\bullet}_R(\mathfrak{l})$ is the reduction modulo $\hbar$.

We filter $\hat{\mathfrak{l}}$ by letting $F^N\hat{\mathfrak{l}}$ be $F^N\mathfrak{l}$ for $N > 0$ and $F^N\mathfrak{l} + R\hbar$ for $N \leq 0$. This filtration is used to complete $U\hat{\mathfrak{l}}$ $\mathfrak{m}$-adically on the right:

$$U\hat{\mathfrak{l}} \to \overline{U}\hat{\mathfrak{l}} := \varprojlim_i U\hat{\mathfrak{l}}/U\hat{\mathfrak{l}} \circ F^i\mathfrak{l}.$$

In terms of the coordinate t, this has the same description as $\overline{U}\mathfrak{l}$, the difference being that the coefficients now lie in $R[\hbar]$. Since $\hat{\mathfrak{l}}$ is not abelian, the left and right $\mathfrak{m}$-adic topologies now differ. For instance, $\sum_{k\geq 1} t^k \circ t^{-k}$ does not converge in $\overline{U}\hat{\mathfrak{l}}$, whereas $\sum_{k\geq 1} t^{-k} \circ t^k$ does. The obvious surjection $\pi : \overline{U}\hat{\mathfrak{l}} \to \overline{U}\mathfrak{l}$ is of course still given by reduction modulo $\hbar$. We also observe that the filtrations of $\mathfrak{l}$ and $\hat{\mathfrak{l}}$ determine decreasing filtrations of their (completed) universal enveloping algebras, e.g., $F^N U\hat{\mathfrak{l}} = \sum_{r\geq 0}\sum_{n_1+\cdots+n_r\geq N} F^{n_1}\hat{\mathfrak{l}} \circ \cdots \circ F^{n_r}\hat{\mathfrak{l}}$.

Let us denote by $\hat{\mathfrak{l}}_2$ the image of $\mathfrak{l} \otimes_R \mathfrak{l} \subset \hat{\mathfrak{l}} \otimes_R \hat{\mathfrak{l}} \to U\hat{\mathfrak{l}}$. Its closure $\overline{\hat{\mathfrak{l}}_2}$ in $\overline{U}\hat{\mathfrak{l}}$ consists of the expressions $r\hbar + \sum_{i\leq j} r_{ij} t^i \circ t^j$ with for any given i, $r_{ij} \in R$ is nonzero for only finitely many j's. Under the reduction modulo $\hbar$, $\overline{\hat{\mathfrak{l}}_2}$ maps onto $\overline{\mathfrak{l}}_2$ in $\overline{U}\mathfrak{l}$ with kernel $R\hbar$. The generator t determines a continuous R-linear map $\hat{C} : \theta \to \overline{\hat{\mathfrak{l}}_2} \subset \overline{U}\hat{\mathfrak{l}}$ characterized by

$$\hat{C}(D_k) := \tfrac{1}{2}\sum_{i+j=k} : t^i \circ t^j : \ .$$

We here adhered to the *normal ordering convention*, which prescribes that the factor with the highest index comes last and hence acts first (here the exponent serves as index). So $: t^i \circ t^j :$ equals $t^i \circ t^j$ if $i \leq j$ and $t^j \circ t^i$ if $i > j$. This map is clearly a lift of $C : \theta \to \overline{U}\mathfrak{l}$, but is otherwise non-canonical.

Lemma 2.2. *We have*

(i) $[\hat{C}(D), f] = -\hbar D(f)$ *as an identity in* $\overline{U}\hat{\mathfrak{l}}$ *(where* $f \in \mathfrak{l} \subset \hat{\mathfrak{l}}$*) and*

(ii) $[\hat{C}(D_k), \hat{C}(D_l)] = -\hbar(l-k)\hat{C}(D_{k+l}) + \hbar^2\frac{1}{12}(k^3 - k)\delta_{k+l,0}$.

Proof. For (i) we compute $[\hat{C}(D_k), t^l]$. If we substitute $\hat{C}(D_k) = \frac{1}{2}\sum_{i+j=k} : t^i \circ t^j :$, then we see that only terms of the form $[t^{k+l} \circ t^{-l}, t^l]$ or $[t^{-l} \circ t^{k+l}, t^l]$ (depending on whether $k + 2l \leq 0$

or $k+2l \geq 0$) can make a contribution and then have coefficient $\frac{1}{2}$ if $k+2l=0$ and 1 otherwise. In all cases the result is $-\hbar l t^{k+l} = -\hbar D_k(t^l)$.

Formula (i) implies that

$$[\hat{C}(D_k), \hat{C}(D_l)] = \lim_{N\to\infty} \sum_{|i|\leq N} \tfrac{1}{2}\left(D_k(t^i)\circ t^{l-i} + t^i \circ D_k(t^{l-i})\right)$$
$$= -\hbar \lim_{N\to\infty} \sum_{|i|\leq N} \left(it^{k+i}\circ t^{l-i} + t^i \circ (l-i)t^{k+l-i}\right).$$

This is up to a reordering equal to $-\hbar(l-k)\hat{C}(D_{k+l})$. The terms which do not commute and are in the wrong order are those for which $0 < k+i = -(l-i)$ (with coefficient i) and for which $0 < i = -(k+l-i)$ (with coefficient $(l-i)$). This accounts for the extra term $\hbar^2 \frac{1}{12}(k^3-k)\delta_{k+l,0}$. □

This lemma shows that $-\hbar^{-1}\hat{C}$ behaves better than $\hat{C}$ (but requires us of course to assume that $\hbar$ be invertible). In fact, it suggests to consider the R-module $\hat{\theta}$ of pairs $(D,u)\in\theta\times\hbar^{-1}\bar{\mathfrak{l}}_2$ for which $C(D)\in\overline{U}\mathfrak{l}$ is the mod $\hbar$ reduction of $-\hbar u$. This R-module is independent of the choice of t and we have an exact sequence

$$0\to R\to\hat{\theta}\to\theta\to 0$$

of R-modules. Here R is identified with the R-module generated by the unit of $\overline{U}\hat{\mathfrak{l}}[\frac{1}{\hbar}]$. In order to avoid confusion, we shall denote that generator by c_0. Note that a non-canonical section of $\hat{\theta}\to\theta$ is defined by $D\mapsto\hat{D} := (D, -\hbar^{-1}\hat{C}(D))$.

Corollary-Definition 2.3. *This defines a central extension of Lie algebras, called the* Virasoro algebra *(of the R-algebra L). Precisely, if $T:\hat{\theta}\to\overline{U}\hat{\mathfrak{l}}[\frac{1}{\hbar}]$ is given by the second component, then T is an injective R-Lie algebra homomorphism which sends c_0 to 1. If we transfer the Lie bracket on $\overline{U}\hat{\mathfrak{l}}[\frac{1}{\hbar}]$ to $\hat{\theta}$, then in terms of our non-canonical section,*

$$[\hat{D}_k,\hat{D}_l] = (l-k)\hat{D}_{k+l} + \frac{k^3-k}{12}\delta_{k+l,0}c_0.$$

Moreover, $\mathrm{ad}_{T(\hat{D})}$ *leaves $\mathfrak{l}$ invariant (as a subspace of $\overline{U}\hat{\mathfrak{l}}$) and acts on that subspace by derivation with respect to $D\in\theta$.*

An alternative coordinate free definition of the Virasoro algebra, based on the algebra of pseudo-differential operators on L, can be found in [3].

The continuous dual of the Virasoro algebra has an interesting description in terms of polydifferentials. Put $L^{(2)} := L\hat{\otimes}_R L$, where the symbol ˆ refers to completion relative to the $(\mathfrak{m}\otimes 1 + 1\otimes\mathfrak{m})$-adic filtration. In terms of coordinates, $L^{(2)}$ is simply $R[[t_1,t_2]][t_1^{-1},t_2^{-1}]$. Then the analogously defined $\omega^{(2)} := \omega\hat{\otimes}_R\omega$ is a module over $L^{(2)}$, it is in fact the free $L^{(2)}$-module generated by dt_1dt_2 (we regard dt_1dt_2 as a symmetric tensor). We have a reduction to the diagonal $L^{(2)}\to L$ (defined by putting $t_1=t_2=t$). This is covered by a module homomorphism $\omega^{(2)}\to\omega^2$, where we recall that ω^2 denotes the module of quadratic differentials. Let $I_\Delta\subset L^{(2)}$ the reduced ideal defining the diagonal, that is, the ideal generated by (t_1-t_2). There is a canonically defined 'biresidue map' $\mathrm{Res}^{(2)}: I_\Delta^{-2}\omega^{(2)}\to L$ which assigns to $(t_1-t_2)^{-2}f(t_1,t_2)dt_1dt_2$ the reduction of f to the diagonal. Observe that the kernel of this map is $I_\Delta^{-1}\omega^{(2)}$.

Exchanging the factors defines an involution ι of $I_\Delta^{-2}\omega^{(2)}$ (it takes dt_1dt_2 to $dt_2dt_1 = dt_1dt_2$). We will only be concerned with the ι-invariants in $I_\Delta^{-1}\omega^{(2)}$. Now note that $(t_1 - t_2)^{-1}f(t_1,t_2)dt_1dt_2$ is ι-invariant if and only if f is anti-invariant. This means that f is divisible by t_1-t_2 and so this shows that $(I_\Delta^{-1}\omega^{(2)})^\iota = (\omega^{(2)})^\iota$. So if $\hat\omega^2 \subset (I_\Delta^{-2}\omega^{(2)}/I_\Delta\omega^{(2)})^\iota$ denotes the R-submodule of elements that have biresidue in R, then the preceding discussion shows that this R-module fits naturally in an exact sequence of R-modules:

$$0 \to \omega^2 \to \hat\omega^2 \to R \to 0.$$

Lemma 2.4. *The sequence* $0 \to \omega^2 \to \hat\omega^2 \to R \to 0$ *is topologically dual to the Virasoro extension sequence* $0 \to R\hbar \to \hat{\overline{\mathfrak{l}_2}} \to \overline{\mathfrak{l}_2} \to 0$*, where the duality arises from the natural pairing* $(R + \hbar^{-1}\mathfrak{l}\otimes_R \mathfrak{l}) \times \hat\omega^2 \to R$ *defined by*

$$\langle \lambda + \hbar^{-1}f\otimes_R g|\zeta\rangle := \lambda \operatorname{Res}^{(2)}\zeta + \operatorname{Res}_{t_1=0}\operatorname{Res}_{t_2=0} f(t_1)g(t_2)\zeta.$$

Proof. We only verify that $\langle \hbar^{-1}f\otimes_R g - \hbar^{-1}g\otimes_R f - \operatorname{Res}(gdf)\,|\,\zeta\rangle = 0$ and leave the rest of the argument to the reader.

We check this on generators and choose $f = t^p$ and $g = t^q$. As we have seen, we can write $\zeta = r(t_1-t_2)^{-2}dt_1dt_2 + \sum_{i,j} r_{ij}t_1^{i-1}t_2^{j-1}dt_1dt_2$ with $r_{ij} = r_{ji}$. We first take $\zeta = (t_1^{i-1}t_2^{j-1} + t_1^{j-1}t_2^{i-1})dt_1dt_2$. Since $\operatorname{Res}_{t_1=0}\operatorname{Res}_{t_2=0} t_1^{p+i-1}t_2^{q+j-1}dt_1dt_2 = \delta_{p+i,0}\delta_{q+j,0}$, we find that

$$\langle \hbar^{-1}t^p\otimes t^q\,|\,(t_1^{i-1}t_2^{j-1} + t_1^{j-1}t_2^{i-1})dt_1dt_2\rangle = \delta_{p+i,0}\delta_{q+j,0} + \delta_{p+j,0}\delta_{q+i,0}$$

which this is symmetric in (p,q). This in fact establishes a topologically perfect duality between $\overline{\mathfrak{l}_2}$ and $\omega^{(2)}$. Next we consider the case when $\zeta = (t_1-t_2)^{-2}dt_1dt_2$. Then

$$\begin{aligned}\langle \hbar^{-1}t^p\otimes t^q\,|\,\zeta\rangle &= \operatorname{Res}_{t_1=0}\operatorname{Res}_{t_2=0} t_1^pt_2^q(t_1-t_2)^{-2}dt_1dt_2 = \\ &\operatorname{Res}_{t_1=0}\operatorname{Res}_{t_2=0} t_1^{p-2}t_2^q(1-t_2/t_1)^{-2}dt_1dt_2 = \\ &\operatorname{Res}_{t_1=0} t_1^{p-2}dt_1 \operatorname{Res}_{t_2=0} t_2^q\sum_{i\geq 1} i(t_2/t_1)^{i-1}dt_2.\end{aligned}$$

Now $\operatorname{Res}_{t_2=0} t_2^q\sum_{i\geq 1} i(t_2/t_1)^{i-1}dt_2$ is zero unless $q<0$ and is then equal to $-qt_1^{q+1}$. (Note that here the order in which we a take the residues has become important.) We find that $\langle t^p\otimes t^q\,|\,\zeta\rangle$ is zero unless $-q = p > 0$ and then we get p. If we interchange the roles of p and q, the result is $-p$ when $-q = p < 0$ and zero otherwise. It follows that $\langle t^p\otimes t^q - t^q\otimes t^p|\,\zeta\rangle = p\delta_{p+q,0}$. But this is also the value of $\operatorname{Res} t^q d(t^p)\langle \hbar\,|\,\zeta\rangle = p\delta_{p+q,0}\operatorname{Res}^{(2)}(\zeta) = p\delta_{p+q,0}$. □

2.3. **A digression: Virasoro Lie algebra and the Schwarz derivative.** Let C be a Riemann surface and consider on C^2 the sheaf $\pi_1^*\Omega_C\otimes_{\mathcal{O}_{C^2}}\pi_2^*\Omega_C$, where $\pi_i : C\times C \to C$ is the projection on the ith factor. We could identify this with the sheaf of holomorphic 2-forms on C^2, but that has of course the effect that the involution $\iota : (p_1,p_2) \in C^2 \mapsto (p_2,p_1) \in C^2$ introduces an unwanted sign. We therefore prefer to regard this as a subsheaf of the sheaf of quadratic differentials on C^2 and denote it by $\Omega_C^{(2)}$.

Denote by $\Delta : C \to C^2$ the diagonal embedding and let $\mathcal{I}_\Delta \subset \mathcal{O}_{C^2}$ be the ideal defined by its image. The differential $d : \mathcal{I}_\Delta \to \pi_i^*\Omega_C$ takes $\mathcal{I}_\Delta^2$ to $\mathcal{I}_\Delta\pi_i^*\Omega_C$ and the resulting map $\Delta^*\mathcal{I}_\Delta \to \Delta^*\pi_i^*\Omega_C (\cong \Omega_C)$ is an isomorphism of $\mathcal{O}_C$-modules. Hence we have a natural isomorphism $\Delta^*\mathcal{I}_\Delta^2 \cong \Delta^*\Omega_C^{(2)}$, or equivalently, a natural isomorphism $\Delta^*\mathcal{I}_\Delta^{-2}\Omega_C^{(2)} \cong \mathcal{O}_C$. This defines the *biresidue* map $\operatorname{Res}^{(2)} : \Delta^{-1}\mathcal{I}_\Delta^{-2}\Omega_C^{(2)} \to \mathcal{O}_C$. We then define $\hat\Omega_C^2$ as the sheaf on C that is the subsheaf of $\Delta^{-1}\mathcal{I}_\Delta^{-2}\Omega_C^{(2)}/\mathcal{I}_\Delta\Omega_C^{(2)}$ consisting of forms that are invariant under the involution ι and have constant biresidue on C. This is an extension of the constant sheaf $\mathbb{C}_C$ by the module of quadratic differentials Ω^2_C on C. The subsheaf $\Omega^2_C(1) \subset \hat\Omega^2_C$ for which this biresidue equals 1 is a torsor over Ω^2_C on C.

Let us now do a local computation. Suppose we have a local coordinate z of C at some point $p \in C$. Then any element of $\Omega^2_{C,p}(1)$ can be written as a sum $\Delta^*((z_1-z_2)^{-2}dz_1dz_2) + f(z)dz^2$, with $f \in \mathcal{O}_{C,p}$. Let us see what the effect is of a coordinate change is on the first term: So let w be another local coordinate at p so that we can write $z = h(w)$, with $h(0) = 0$ and $h'(0) \neq 0$. We have $(z_1-z_2)^{-2}dz_1dz_2 = (h(w_1)-h(w_2))^{-2}h'(w_1)h'(w_2)dw_1dw_2$. Next put $(w, w+\varepsilon) : (w_1, w_2)$. So $\varepsilon = w_2 - w_1$ defines the diagonal and we can compute modulo ε:

$$\begin{aligned}
h^*\Delta^*\frac{dz_1dz_2}{(z_1-z_2)^2} &= \Delta^*(h\times h)^*\frac{dz_1dz_2}{(z_1-z_2)^2} = \Delta^*\Big(\frac{h'(w_1)h'(w_2)dw_1dw_2}{(h(w_1)-h(w_2))^2}\Big)\\
&= \frac{h'(w)h'(w+\varepsilon)dw^2}{(h(w)-h(w+\varepsilon))^2} \equiv \frac{h'(w)(h'(w)+\varepsilon h''(w)+\varepsilon^2.h'''(w)/2+\cdots)dw^2}{(\varepsilon h'(w)+\varepsilon^2h''(w)/2+\varepsilon^3h'''(w)/6+\cdots)^2}\\
&\equiv \frac{h'(w)^2+\varepsilon h'(w)h''(w)+\varepsilon^2h'(w)h'''(w)/2+\cdots)dw^2}{\varepsilon^2(h'(w)^2+\varepsilon h'(w).h''(w)+\varepsilon^2(h''(w)^2/4+h'(w)h'''(w)/3)+\cdots)}\\
&\equiv \frac{1}{\varepsilon^2}\Big(1+\varepsilon^2.\frac{h'(w)h'''(w)/6-h''(w)^2/4+\cdots}{\varepsilon^2h'(w)^2+\cdots}\Big)dw^2\\
&\equiv \Delta^*\Big(\frac{dw_1dw_2}{(w_1-w_2)^2}\Big)+\frac{S(h)(w)dw^2}{6}
\end{aligned}$$

where

$$S(h) := \frac{h'h'''-3/2.h''^2}{h'^2}$$

is known as the *Schwarz derivative*. You can check that for every $f \in \mathbb{C}\{w\}$ there exists a $h \in \mathbb{C}\{w\}$ with $h(0) = 0 \neq h'(0)$ such that $S(h) = f$ and so the group $\mathrm{Aut}(\mathcal{O}_{C,p})$ acts transitively on $\omega^2_{C,p}(1)$. It is clear that $S(h) = 0$ if and only if $h'h''' = \frac{3}{2}h''^2$. This last property turns out to be equivalent with h being locally a fractional linear transformation: $h(w) = (aw+b)/(cw+d)$ (with $ad-bc = 1$ and $d \neq 0$). It follows that an element $\zeta \in \omega^2_{C,p}(1)$ can be understood as defining a local projective structure at p, for all the coordinates z in which ζ takes the simple form $(z_1-z_2)^{-2}dz_1dz_2$ lie in the same orbit of the group $\mathrm{PGL}(2,\mathbb{C})$. A global section of $\Omega^2_{C,p}(1)$ amounts to a global projective structure on C and such a structure is just given by it holonomy: a homomorphism $\pi_1(C,P) \to \mathrm{PGL}(2,\mathbb{C})$ given up to conjugacy in $\mathrm{PGL}(2,\mathbb{C})$. This establishes a embedding

$$H^0(C, \Omega^2_{C,p}(1)) \hookrightarrow \mathrm{Hom}(\pi_1(C,P), \mathrm{PGL}(2,\mathbb{C})/\,\mathrm{PGL}(2,\mathbb{C}),$$

which in fact is an isomorphism (see [8]). There is a distinguished projective structure on C: assuming, as we may, that C connected, then by the uniformization theorem C is universally covered by $\mathbb{P}^1(\mathbb{C})$, $\mathbb{C}$ or the upper half plane. The covering group (which is of course isomorphic to $\pi_1(C,P)$) will be a discrete torsion free subgroup of the automorphism group of the covering space. So in the first case it is trivial, in the second case a discrete translation group and in the third case a discrete subgroup of $\mathrm{PGL}(2,\mathbb{C})$. As these are all subgroups of $\mathrm{PGL}(2,\mathbb{C})$, we have thus defined a projective structure on C.

More on this can be found in the book of Gunning [8] and a series of papers by Biswas and Raina [4].

2.4. **Another digression: the Bott-Virasoro group.** For $k = R = \mathbb{C}$, we can think of $\mathcal{O} = \mathbb{C}[[t]]$ as the algebra of formal functions on the germ of $\mathbb{C}$ at 0 and likewise $L = \mathbb{C}((t))$ as the algebra of formal functions on the punctured germ of $\mathbb{C}$ at 0. A real analogue is to take for L the algebra of $\mathbb{R}$-valued C^∞-functions of the unit circle S^1 (for simplicity we assume this to be the unit circle in $\mathbb{C}$, but actually one could take for S^1 any manifold diffeomorphic to S^1: for what follows no preferred coordinate is needed). Fourier expanding a $\mathbb{C}$-valued C^∞-function on S^1 makes it indeed look like an element of L.

In what follows, Ω^n will stand for the space of C^∞-sections of the kth tensor power of the cotangent bundle $T^*(S^1)$ so that for $k = 1, 0, -1$, we get respectively 1-forms, functions and vector fields. This is the real analogue of ω^n. The residue map is replaced by the map $\mathrm{Res} : \Omega^n \to \mathbb{R}$ given by $\alpha \mapsto \int_{S^1} \alpha$. So for each $n \in \mathbb{Z}$ we have a duality pairing

$$\Omega^n \times \Omega^{1-n} \to \mathbb{R}, \quad (\alpha, \beta) \mapsto \int_{S^1} \alpha\beta.$$

which is nondegenerate in a suitable sense. This space is acted on the right by the group of orientation preserving diffeomorphisms $\mathrm{Diff}^+(S^1)$ (acting by substitutions). The Lie algebra of that $\mathrm{Diff}^+(S^1)$ may be regarded as the Lie algebra of vector fields Ω^{-1}. We proceed as before and with the help of a residue pairing we obtain an oscillator algebra: a Lie algebra extension $\widehat{\Omega}^0$ of Ω^0 by $\mathbb{R}\hbar$ with underlying vector space $\Omega^0 \oplus \mathbb{R}$. This leads to a Virasoro extension: a central extension $\hat{\Omega}^{-1}$ of Ω^{-1} by $\mathbb{R}$. The formula is the same as for the formal case except that we must replace Res by $\int_{S^1}$. Its predual is an exact sequence of the form

$$0 \to \Omega^2 \to \hat{\Omega}^2 \to \mathbb{R} \to 0,$$

where $\hat{\Omega}^2$ is defined in a similar manner as $\hat{\omega}^2$: it is the space of symmetric bidifferentials ζ on $S^1 \times S^1$ with a pole of order 2 along the diagonal with constant double residu on the diagonal modulo those that vanish on the diagonal.

The subspace $\Omega^2(1) \subset \hat{\Omega}^2$ of forms with biresidue 1 on the diagonal is an affine space over the space Ω^2 of quadratic differentials; it has the interpretation as the space of real projective structures on S^1.

If we think of Ω^{-1} as the Lie algebra of $\mathrm{Diff}^+(S^1)$, then we may hope for a central extension of the group $\mathrm{Diff}^+(S^1)$:

$$0 \to \mathbb{R} \to \widehat{\mathrm{Diff}}^+(S^1) \to \mathrm{Diff}^+(S^1) \to 1$$

whose Lie algebra yields the Virasoro extension. Such an extension exists: any such extension admits a section (as any $\mathbb{R}$-bundle has this property and so we can assume the underlying space is $\mathbb{R} \times \mathrm{Diff}^+(S^1)$. Then the product will have the form $(\lambda_1, h_1) \cdot (\lambda_2, h_2) = (\lambda_1 + \lambda_2 + c(h_1, h_2), h_1 \circ h_2)$, where c must be a cocycle: $c(h_1, h_2) + c(h_1 \circ h_2, h_3) = c(h_1, h_2 \circ h_3) + c(h_2, h_3)$. We take this c to be in terms of an angular coordinate ϕ,

$$c(h_1, h_2) = \int_{S^1} h_1'(h_2(\phi)).h_2''(\phi) d\phi.$$

2.5. **Fock representation.** It is clear that $F^0\hat{\mathfrak{l}} = R\hbar \oplus \mathcal{O}$ is an abelian subalgebra of $\hat{\mathfrak{l}}$. We let it act on a free rank one module Rv_o by via the projection $F^0\hat{\mathfrak{l}} \to R\hbar$ and by letting $\hbar$ act as the identity. The induced representation of $\hat{\mathfrak{l}}$ over R,

$$\mathbb{F} := U\hat{\mathfrak{l}} \otimes_{UF^0\hat{\mathfrak{l}}} Rv_o,$$

will be regarded as a $U\mathfrak{l}[\hbar^{-1}]$-module. It comes with an increasing PBW (Poincaré-Birkhoff-Witt) filtration $PBW_\bullet\mathbb{F}$ by R-submodules, with $PBW_r\mathbb{F}$ being the image of $\oplus_{s\le r}\hat{\mathfrak{l}}^{\otimes s}\otimes Rv_o$. Since $t^0\in\mathfrak{l}$ is central in $\hat{\mathfrak{l}}$ and contained in $\mathcal{O}$ it is in the kernel of this representation. As an R-module, $\mathbb{F}$ is free with basis the collection $t^{\mathbf{k}}\otimes v_o$, where $\mathbf{k}=(k_r\le\cdots\le k_1)$ is such that $k_1<0$. (In fact, $\mathrm{Gr}^{PBW}_\bullet\mathbb{F}$ can be identified as a graded R-module with the symmetric algebra $\mathrm{Sym}^\bullet(\mathfrak{l}/F^0\mathfrak{l})$.) This also shows that $\mathbb{F}$ is even a $\overline{U}\hat{\mathfrak{l}}[\hbar^{-1}]$-module. Thus $\mathbb{F}$ affords a representation of $\hat{\theta}$ over R, called its *Fock representation*.

It follows from Lemma 2.2 that for any $D\in\theta$ with lift $\hat{D}\in\hat{\theta}$,

$$T(\hat{D})t^{\mathbf{k}}\otimes v_o-t^{\mathbf{k}}\circ T(\hat{D})v_o=\left(\sum_{i=1}^r t^{k_r}\circ\cdots\circ D(t^{k_i})\circ\cdots\circ t^{k_1}\right)v_o.$$

By definition $T(\hat{D}_n)v_0=-\frac{1}{2\hbar}\sum_k:t^k\circ t^{n-k}:v_o$. This (finite) sum is over all k with $\max\{k,n-k\}<0$ and so gives zero when $n\ge 0$. In other words, $T(\hat{D})v_o=0$ when $D\in F^0\theta$. Equivalently, $F^0\theta$ acts on $\mathbb{F}$ by coefficient-wise derivation. This observation has an interesting consequence. Consider the module $\theta_{R/k}$ of k-derivations $R\to R$, denoted here simply by θ_R instead of the more accurate $\theta_{R/k}$ (the module of vector fields on $\mathrm{Spec}(R)$). Denote by $\theta_{L,R}$ the module of k-derivations of L that are continuous for the $\mathfrak{m}$-adic topology and preserve $R\subset L$ (the vector fields on the formal scheme defined by $(\mathcal{O},\mathfrak{m})$ that are lifts of a vector fields on $\mathrm{Spec}(R)$; it contains θ as the lifts of the zero vector field). Since $L\cong R((t))$ as an R-algebra, every k-derivation $R\to R$ extends to one from L to L and so we have an exact sequence

$$0\to\theta\to\theta_{L,R}\to\theta_R\to 0.$$

The following corollary essentially says that we have defined in the L-module $\mathbb{F}$ a Lie algebra $\hat{\theta}_{L,R}$ of first order (k-linear) differential operators which contains R as the degree zero operators and for which the symbol map (which is just the formation of the degree one quotient) has image $\theta_{L,R}$.

Corollary 2.5. *The actions on $\mathbb{F}$ of $F^0\theta_{L,R}\subset\theta_{L,R}$ (given by coefficient-wise derivation, killing the generator v_o) and $\hat{\theta}$ coincide on their intersection $F^0\theta$ and generate an extension of Lie algebras*

$$0\to Rc_o\to\mathcal{D}_{\mathbb{F}}\to\theta_{L,R}\to 0.$$

Its defining representation on $\mathbb{F}$ (still denoted T) is faithful and has the property that for every lift $\hat{D}\in\mathcal{D}_{\mathbb{F}}$ of $D\in\theta_{L,R}$ and $f\in\mathfrak{l}$ we have $[T(\hat{D}),f]=Df$ (in particular, it preserves every $U\hat{\mathfrak{l}}$-submodule of $\mathbb{F}$).

Proof. The first assertion has already been observed. The generator t can be used to define a section of $\theta_{L,R}\to\theta_R$: the set of elements of $\theta_{L,R}$ which kill t is a k-Lie subalgebra of $\theta_{L,R}$ which projects isomorphically onto θ_R. Now if $D\in\theta_{L,R}$, write $D=D_{\mathrm{vert}}+D_{\mathrm{hor}}$ with $D_{\mathrm{vert}}\in\theta$ and $D_{\mathrm{hor}}(t)=0$ and define an R-linear operator $\hat{D}$ in $\mathbb{F}$ as the sum of $T(\hat{D}_{\mathrm{vert}})$ and coefficient-wise derivation by D_{hor}. This map clearly has the properties mentioned.

As to its dependence on t: another choice yields a decomposition of the form $D=(D_{\mathrm{hor}}+D_0)+(D_{\mathrm{vert}}-D_0)$ with $D_0\in F^0\theta$ and in view of the above $\hat{D}_0$ acts in $\mathbb{F}$ by coefficient-wise derivation. ∎

3. The WZW connection in the abelian case

3.1. **The Fock representation for a symplectic local system.** We shall run into a particular type of finite rank subquotient of the Fock representation and it seems best to discuss the resulting structure here. We start out with the set data described in the following definition.

Definition 3.1. A *Fock system of genus g* consists of

(i) a free R-module H of rank $2g$ endowed with a symplectic form $\langle\ ,\ \rangle : H\otimes_R H\to R$ which is nondegenerate in the sense that the induced map $H\to H^*$, $a\mapsto \langle\ ,a\rangle$ is an isomorphism of R-modules,

(ii) a Lie action of θ_R on H on H by k-derivations, denoted by $D\mapsto \nabla_D$ (so it is k-linear and obeys the Leibniz rule: $\nabla_D(ra)=r\nabla_D(a)+D(r)a$), which preserves the symplectic form in the sense that $D\langle a,b\rangle=\langle\nabla_D(a),b\rangle+\langle a,\nabla_D(b)\rangle$,

(iii) a maximal isotropic R-submodule $F\subset H$ (of rank g).

One might think of ∇ as a flat meromorphic connection on the symplectic bundle represented by H.

Example 3.2. We encounter the complex-analytic analogue when we are dealing with a proper submersion $f:\mathcal{C}\to S$ of complex manifolds whose fibers are compact connected Riemann surfaces of fixed genus g. Then $\mathbb{H}=R^1f_*\mathbb{C}_{\mathcal{C}}$ is a local system on S of rank $2g$. Hence $\mathcal{H}:=\mathcal{O}_S^{\rm an}\otimes\mathcal{H}\cong R^1f_*f^{-1}\mathcal{O}_S^{\rm an}$ is a holomorphic symplectic vector bundle over S of rank $2g$, endowed with a flat connection that has $\mathbb{H}$ as its subsheaf of flat sections (the Gauss-Manin connection). We have a natural embedding of $f_*\Omega^{\rm an}_{\mathcal{C}/S}$ in $\mathcal{H}$ whose image $\mathcal{F}$ is a coherent $\mathcal{O}_S^{\rm an}$-submodule that is maximal isotropic. If we happen to be in a complex-algebraic setting, with R the local ring of an smooth point of an affine variety, then $\mathcal{H}$ and its Gauss-Manin connection are algebraically defined and we could take for H resp. F the R-module of regular sections of $\mathcal{H}$ resp. $\Omega_{\mathcal{C}/S}$ (both will be free).

A somewhat more universal example is that of family of polarized abelian varieties over S. Indeed, the data above are an abstract version of a polarised variation of Hodge structure of weight 1.

In this setting a Heisenberg algebra $\hat H$ is defined in an obvious manner: an R-module it is $H\oplus R\hbar$ endowed with the bracket $[a+R\hbar,b+R\hbar]=\langle a,b\rangle\hbar$. We also have defined a Fock representation $\mathbb{F}(H,F)$ of $\hat H$ as the induced module of the rank one representation of $\hat F=F+R\hbar$ on R given by the coefficient of $\hbar$: it is the quotient of $U\hat H$ by the right ideal generated by $\hbar-1$ and F. Notice that if we grade $\mathbb{F}(H,F)$ with respect to the PBW filtration, we get a copy of the symmetric algebra of H/F over R. Our goal is to define a flat projective connection on $\mathbb{F}(H,F)$.

Remark 3.3. In case our data come form a polarized family of complex abelian varieties $f:\mathscr{A}\to S$ (with $S=\operatorname{Spec}(R)$, where R is regular local $\mathbb{C}$-algebra), then the polarization defines a line bundle $\mathscr{L}$ over $\mathscr{A}$. The universal cover of the associated $\mathbb{C}^\times$-bundle is a Heisenberg group whose Lie algebra is as above. The sections of the tensor powers of $\mathscr{L}$ make up an algebra (of theta functions) and $\mathbb{F}(H,F)$ can be understood as the space of formal expansions these along the zero section. Mumford [15] has shown that the space of theta functions relative to a given polarization admits a projectively flat connection. In that algebraic setting this is a consequence of the action a finite analogue of the Heisenberg group acting on such a space and the fact this group has essentially only one interesting faithful representation. In the analytic setting is related to the fact that such theta functions are solutions to a certain heat operator. The projectively flat connection that we are going to define is the formal counterpart of this phenomenon.

We begin with extending the θ_R-action to $\hat{H}$ by stipulating that it kills $\hbar$. This action clearly preserves the Lie bracket and hence determines one of θ_R on the universal enveloping algebra $U\hat{H}$. This does not however induce one in $\mathbb{F}(H,F)$, for ∇_D will not respect the right ideal in $U\hat{H}$ generated by $\hbar-1$ and F. We will remedy this by means of a 'twist'.

We shall use the isomorphism $\sigma : H\otimes_R H \cong \operatorname{End}_R(H)$ of R-modules defined by associating to $a\otimes b$ the endomorphism $\sigma(a\otimes b) : x\in H \mapsto a\langle b,x\rangle \in H$. If we agree to identify an element in the tensor algebra of H, in particular, an element of H, as the operator in $U\hat{H}$ or $\mathbb{F}(H,F)$ given by left multiplication, then it is readily checked that for $x\in H$,

$$[a\circ b, x] = \sigma(a\otimes b + b\otimes a)(x).$$

We choose a Lagrangian supplement of F in H, i.e., a maximal isotropic R-submodule $F^-\subset H$ that is also a section of $H\to H/F$ and we write sometimes F^+ for F. Since F^- is an abelian Lie subalgebra of $\hat{H}$, we have a natural map $\operatorname{Sym}^\bullet_R(F^-)\to\mathbb{F}(H,F)$. This is clearly an isomorphism of $\operatorname{Sym}^\bullet_R(F^-)$-modules. We write ∇_D in matrix form according to the Lagrangian decomposition $H = F^-\oplus F^+$:

$$\nabla_D = \begin{pmatrix} \nabla^-_D & \sigma^-_D \\ \sigma^+_D & \nabla^+_D \end{pmatrix}.$$

Here the diagonal entries represent the induced connections on $F^\pm$, whereas $\sigma^\pm_D\in\operatorname{Hom}_R(F^\mp,F^\pm)$. Since σ identifies $F^\pm\otimes_R F^\pm$ with $\operatorname{Hom}_R(F^\mp,F^\pm)$, we can write $\sigma^\pm_D=\sigma(s^\pm_D)$ with $s^\pm_D\in F^\pm\otimes_R F^\pm$. The tensor $s^\pm_D$ is symmetric. For example, if $a,b\in F^+$, then $\langle a,b\rangle=0$ and hence

$$0 = D(\langle a,b\rangle) = \langle\nabla_D(a),b\rangle + \langle a,\nabla_D b\rangle = \langle\sigma^-_D(a),b\rangle + \langle a,\sigma^-_D(b)\rangle = \langle\sigma^-_D(a),b\rangle - \langle\sigma^-_D(b),a\rangle$$

which implies that σ^-_D is symmetric. We may regard $\sigma^\mp_D$ as the second fundamental form of $F^\pm\subset H$. Moreover, we have in $U\hat{H}$, $[s^\pm_D,a]=2\sigma^\pm(a)$. So if $a\in F^\mp$, then

$$[\nabla_D,a] = \nabla_D(a) = \nabla^\mp_D(a)+\sigma^\pm_D(a) = \nabla^\mp_D(a) + \tfrac{1}{2}[s^\pm_D,a].$$

This suggests we should assign to $D\in\theta_R$ the first order differential operator $T_{F^-}(D)$ in $\operatorname{Sym}^\bullet F^-\cong\mathbb{F}(H,F)$ defined by

$$T_{F^-}(D) := \nabla^-_D + \tfrac{1}{2}s^+_D + \tfrac{1}{2}s^-_D.$$

Proposition 3.4. *The map $T_{F^-} : \theta_R\to\operatorname{End}_k(\mathbb{F}(H,F))$ is k-linear and has the property that $[T_{F^-}(D),a]=\nabla_D(a)$ for every $D\in\theta_R$ and $a\in\hat{H}$. Any other map $\theta_R\to\operatorname{End}_k(\mathbb{F}(H,F))$ enjoying these properties differs from T_{F^-} by a multiple of the identity operator, in other words, is of the form $D\mapsto T_{F^-}(D)+\eta(D)$ for some $\eta\in\operatorname{Hom}_R(\theta_R,R)=\Omega_{R/k}$.*

Proof. That $T_{F^-}(D)$ has the stated property follows from the preceding. Let $\eta:\theta_R\to\operatorname{End}_k(\operatorname{Sym}^\bullet F^-)$ be the difference of two such maps. Then for every $D\in\theta_R$, $\eta(D)\in\operatorname{End}_R(\mathbb{F}(H,F))$ commutes with all elements of $\hat{H}$. Since $\mathbb{F}(H,F)$ is irreducible as a representation of $\hat{H}$, it follows that $\eta(D)$ is a scalar in R. □

Notice that if $a_1, \dots, a_n \in F^-$ and $r \in R$, then

$$\begin{aligned} &T_{F^-}(D)(a_n \circ \cdots \circ a_1 \circ rv_o) \\ &\quad = \Big(\sum_{i=1}^{n} a_n \circ \cdots \circ (\nabla^-_D + \sigma^+_D)(a_i) \circ \cdots \circ a_1\Big) rv_0 + a_n \circ \cdots \circ a_1 \circ (Drv_0 + \tfrac{1}{2} s^-_D \circ rv_o) \\ &\qquad\qquad\qquad\qquad = \nabla_D(a_n \circ \cdots \circ a_1 \circ r)v_0 + a_n \circ \cdots \circ a_1 \circ r \tfrac{1}{2} s^-_D \circ v_0. \end{aligned}$$

So this looks like the operator $T(\hat{D})$ acting in $\mathbb{F}$ with s^-_D playing the role of $-\hbar^{-1}\hat{C}(D)$. We observe that $T(\hat{D})$ is in fact a connection on $\mathrm{Sym}^\bullet F^-$. Here is the key result about its curvature. We again use the identification $\mathbb{F}(H, F) \cong \mathrm{Sym}^\bullet F^-$.

Proposition 3.5. *The curvature of the connection $T_{F^-} : \theta_R \to \mathrm{End}_k(\mathbb{F}(H, F))$ is scalar and $1/2$ times the curvature of ∇^+ on $\det(F^+)$: for $D, E \in \theta_R$, then $[T_{F^-}(D), T_{F^-}(E)] - T_{F^-}([D, E]) \in \mathrm{End}_k(\mathbb{F}(H, F))$ is scalar multiplication by $\frac{1}{2}\nabla^{\det F^+}(D, E)$. In particular, the operators $T_{F^-}(D)$ and the scalars R generate in $\mathrm{End}_k(\mathbb{F}(H, F))$ a Lie subalgebra $\mathcal{D}_{\mathbb{F}(H,F)}$ that is independent of the choice of F^- and which endows $\mathbb{F}(H, F)$ with the structure of a flat projective connection.*

Proof. The fact that ∇ preserves the Lie bracket is expressed by the following identities:

$$\begin{aligned} \nabla^+_D \nabla^+_E - \nabla^+_E \nabla^+_D - \nabla^+_{[D,E]} &= \sigma^+_E \sigma^-_D - \sigma^+_D \sigma^-_E, \\ \nabla^-_D \nabla^-_E - \nabla^-_E \nabla^-_D - \nabla^-_{[D,E]} &= \sigma^-_E \sigma^+_D - \sigma^-_D \sigma^+_E, \\ \nabla^{\mathrm{Hom}(F^-,F^+)}_D(\sigma^+_E) - \nabla^{\mathrm{Hom}(F^-,F^+)}_E(\sigma^+_D) &= \sigma^+_{[D,E]}, \\ \nabla^{\mathrm{Hom}(F^+,F^-)}_D(\sigma^-_E) - \nabla^{\mathrm{Hom}(F^+,F^-)}_E(\sigma^-_D) &= \sigma^-_{[D,E]}. \end{aligned}$$

The first two give the curvature of F^+ and F^- on the pair (D, E). The last two can also be written as operator identities in $\mathrm{Sym}^\bullet F^-$:

$$[\nabla^\pm_D, s^\pm_E] - [\nabla^\pm_E, s^\pm_D] = s^\pm_{[D,E]}.$$

We now expand

$$\begin{aligned} &[T_{F^-}(D), T_{F^-}(E)] - T_{F^-}([D, E]) = \\ &\quad = [\nabla^-_D + \tfrac{1}{2}s^+_D + \tfrac{1}{2}s^-_D, \nabla^-_E + \tfrac{1}{2}s^+_E + \tfrac{1}{2}s^-_E] - (\nabla^-_{[D,E]} + \tfrac{1}{2}s^+_{[D,E]} + \tfrac{1}{2}s^-_{[D,E]}) = \\ &\qquad = \big([\nabla^-_D, \nabla^-_E] - \nabla^-_{[D,E]}\big) + \tfrac{1}{2}\big([\nabla^-_D, s_E] - [\nabla^-_E, s_D] - s^+_{[D,E]}\big)\big) \\ &\qquad\qquad + \tfrac{1}{2}\big([\nabla^-_D, s^-_E] - [\nabla^-_E, s^-_D] - s^-_{[D,E]}\big)\big) + \tfrac{1}{4}\big([s^+_D, s^-_E] - [s^+_E, s^-_D]\big). \end{aligned}$$

If we feed the above identities in the last expression, we obtain

$$[T_{F^-}(D), T_{F^-}(E)] - T_{F^-}([D, E]) = \big(\sigma^-_E \sigma^+_D + \tfrac{1}{4}[s^+_D, s^-_E]\big) - \big(\sigma^-_D \sigma^+_E + \tfrac{1}{4}[s^-_D, s^+_E]\big).$$

We must show that the right hand side is equal to $\frac{1}{2}\mathrm{Tr}(\sigma^+_E \sigma^-_D - \sigma^+_D \sigma^-_E)$, or perhaps more specifically, that $\sigma^-_E \sigma^+_D + \frac{1}{4}[s^+_D, s^-_E] = \frac{1}{2}\mathrm{Tr}(\sigma^+_E \sigma^-_D)$ (and similarly if we exchange D and E). This reduces to the following identity in linear algebra: if $a \in F^+$ and $b \in F^-$, then in $\mathrm{Sym}^\bullet F^-$ we have

$$\sigma(b \otimes b)\sigma(a \otimes a) + \tfrac{1}{4}[a \circ a, b \circ b] = \tfrac{1}{2}\mathrm{Tr}_{F^-}(\sigma(b \otimes b)\sigma(a \otimes a)),$$

Indeed, a straightforward computation shows that

$$[a \circ a, b \circ b] = 2\langle a, b\rangle(a \circ b + b \circ a) = 4\langle a, b\rangle b \circ a + 2\langle a, b\rangle^2.$$

If we interpret $\langle a,b\rangle b\circ a$ as an operator in $\operatorname{Sym}^\bullet F^-$, then applying it to $x\in F^-$ yields $\langle a,b\rangle b\langle a,x\rangle = -b\langle b,a\rangle\langle a,x\rangle = -\sigma(b\otimes b)\sigma(a\otimes a)(x)$. By taking $x=b$, we get $\operatorname{Tr}_{F^+}(\sigma(b\otimes b)\sigma(a\otimes a)) = \langle a,b\rangle b\langle a,b\rangle = \langle a,b\rangle^2$ and so the desired identity follows.

The last assertion follows from this curvature computation and Proposition 3.4. □

Corollary 3.6. *The R-module $\mathbb{F}(H,F)$ is canonically endowed with a $\det_R(F)$-connection of weight $1/2$.*

Proof. Let F^- be as above so that we can identify $\mathbb{F}(H,F)$ with $\operatorname{Sym}^\bullet F^-$ and we endow $\det_R(F)$ with the connection $\nabla^{\det F^+}$. We define $u_{F^-}:\mathcal{D}_1(\det F)\to\mathcal{D}_1(\operatorname{Sym}^\bullet F^-)$ as multiplication by $1/2$ on R and for every $D\in\theta_R$ we assign to $\nabla_D^{\det F^+}$ the operator $\sqrt{1/2}T(D)$. It follows from Proposition 3.5 that u_{F^-} is a Lie homomorphism over R.

It remains to show that u_{F^-} is independent of F^-. This can be verified by a computation, but rather than carrying this out, we give an abstract argument that avoids this. It is based on the well-known fact that if H_o is a fixed symplectic k-vector space of finite dimension, and $F_o\subset H_o$ is Lagrangian, then the set of Lagrangian supplements of F_o in H_o form in the Grassmannian of H_o an affine space over $\operatorname{Sym}_k^2 F_o$ (and hence is simply connected). Now by doing the preceding construction universally over the corresponding affine space over $\operatorname{Sym}_R^2 F$, we see that the flatness on the universal example immediately gives the independence. □

Remark 3.7. In the context of a family of polarized abelian varieties, the quadratic terms that enter in the definition of T_{F^-} can be understood as a relict of the heat operator, of which the associated theta functions are solutions (see Welters [29]).

3.2. **The semi-local setting.** This refers to the situation where we allow the R-algebra L to be a finite direct sum of R-algebras isomorphic to $R((t))$: $L=\oplus_{i\in I}L_i$, where I is a nonempty finite index set and L_i as before. (The motivation is in the next subsection.) We then extend the notation employed earlier in the most natural fashion. For instance, $\mathcal{O}$, $\mathfrak{m}$, ω, $\mathfrak{l}$ are now the direct sums over I (as filtered modules) of the items suggested by the notation. If $r:L\times\omega\to R$ denotes the sum of the residue pairings of the summands, then r is still topologically perfect. The associated antisymmetric bilinear map on L, $(f,g)\in L\times L\mapsto\langle f,g\rangle := r(df,g)$, has kernel is R^I.

However, we take for the oscillator algebra $\hat{\mathfrak{l}}$ not the direct sum of the $\hat{\mathfrak{l}}_i$, but rather the quotient of $\oplus_i\hat{\mathfrak{l}}_i$ that identifies the central generators of the summands with a single $\hbar$. We thus get a Virasoro extension $\hat\theta$ of θ by c_0R and a (faithful) oscillator representation of $\hat\theta$ in $\overline{U\hat{\mathfrak{l}}}$. The decreasing filtrations are the obvious ones. We shall denote by $\mathbb{F}$ the Fock representation $\mathbb{F}$ of $\hat{\mathfrak{l}}$ that ensures that every summand $\mathcal{O}_i$ acts as zero; it is then the induced representation of the rank one representation of $F^0\hat{\mathfrak{l}}=\mathcal{O}\oplus R\hbar$ in Rv_o.

From now on, unless otherwise stated, we place ourselves in the semi-local setting, so that $L=\oplus_{i\in I}L_i$ with I nonempty and finite and $L_i\cong R((t))$.

3.3. **Computations on a pointed projective curve.** The following example motivates what follows. Assume k algebraically closed. Suppose we are given a connected nonsingular projective curve C of genus g and a nonempty finite set I and an embedding $i\in I\mapsto x_i\in C$. Denote its image by X so that $C^\circ := C-X$ is affine. This brings us in the semilocal case considered above with $R=k$: we let $\mathcal{O}_i$ be is the formal completion of $\mathcal{O}_{C,x_i}$, L_i its field of fractions, $d:L_i\to\omega_i$ the universal continuous derivation, $\mathcal{O}:=\oplus_i\mathcal{O}_i$, $L:=\oplus_iL_i$ and $\omega:=\oplus_i\omega_i$. The residue pairing $r : \omega \times L \to k$ which sends $(\alpha = (\alpha_i)_i, f = (f_i)_i)$ to $\sum_i \mathrm{Res}(f_i \alpha_i)$ is a topologically perfect. The associated pairing $(f, g) \in L \times L \mapsto \langle f, g\rangle := r(df, g) = \sum_i \mathrm{Res}(g_i df_i)$ is degenerate with kernel the locally constant functions, i.e., k^I. Indeed, $r(df, g)$ only depends on the pair (df, dg) and defines a nondegenerate pairing on dL (the differentials with all resides zero).

Let A be the image in L of the k-algebra of regular functions on $C^\circ := C - X$ and likewise $B \subset \omega$ the image of the A-module of regular differentials on C°. The two determine each other via the residue pairing $r : \omega \times L \to k$: the Riemann-Roch theorem can be interpreted as saying that they are each others annihilator: $B = \mathrm{ann}(A)$ and $A = \mathrm{ann}(B)$. So we find that $d : L \to \omega$ takes A to $\mathrm{ann}(A)$ and that the resulting map $\Omega_{A/k} \to \mathrm{ann}(A)$ is an isomorphism of R-modules. Observe that $AdA \subset B = \mathrm{ann}(A)$. We also note that $A \cap \mathcal{O}$ is the space of regular functions on C and hence equal to k.

Following Weil, $H^1(C, \mathcal{O}_C)$ can be described as follows. We have an exact sequence

$$0 \to \mathcal{O}_C \to j_*\mathcal{O}_{C^\circ} \to \oplus_{i\in I} L_i/\mathcal{O}_i \to 0$$

where $j : C^\circ \subset C$ is the inclusion and $L_i/\mathcal{O}_i$ is understood as a skyscraper sheaf at x_i. Since $H^1(C, j_*\mathcal{O}_{C^\circ}) = H^1(C, \varinjlim_n \mathcal{O}_C(nX)) = \varinjlim_n H^1(C, \mathcal{O}_C(nX))$ is trivial (Serre duality and Riemann-Roch), the exact cohomology sequence gives an isomorphism

$$L/(\mathcal{O} + A) \cong H^1(C, \mathcal{O}_C).$$

Via this isomorphism, the Serre duality pairing $H^0(C, \Omega_C) \times H^1(C, \mathcal{O}_C) \to k$ is given by the residue pairing: $(\alpha, f + \mathcal{O} + A) \in H^0(C, \Omega_C) \times L/(\mathcal{O} + A) \mapsto \mathrm{Res}(f\alpha)$. The latter is a perfect pairing and so we find that the image of $H^0(C, \Omega_C)$ in ω is the annihilator of $\mathcal{O} + A$. The annihilator of $\mathcal{O}$ is $F^1\omega$ and so $H^0(C, \Omega_C)$ is identified with $F^1 \mathrm{ann}(A) := F^1\omega \cap \mathrm{ann}(A)$.

We can rework $L/(\mathcal{O} + A) \cong H^1(C, \mathcal{O}_C)$ in terms of differentials. We first note that since $k^I \subset \mathcal{O}$, it then follows that d induces an isomorphism of $L/(\mathcal{O} + A)$ onto $dL/(F^1\omega + dA)$. Now $dL/F^1\omega$ describes the polar parts of meromorphic differentials at X whose residues are all zero. It is a classical fact that such a polar part is always the polar part of a meromorphic differential on C which is regular on C°, is of an element of $\mathrm{ann}(A)$: $F^1\omega + \mathrm{ann}(A) = dL$ (but we will see that it is also a formal consequence of the identity $A \cap L = k$). So $dL/(F^1\omega + dA) \cong \mathrm{ann}(A)/F^1\omega \cap \mathrm{ann}(A)$. So we can now identify the short exact sequence for the De Rham cohomology

(Hodge) $$0 \to H^0(C, \Omega_C) \to H^1_{DR}(C, k) \to H^1(C, \mathcal{O}_C) \to 0$$

with

(ω-Hodge) $$0 \to F^1 \mathrm{ann}(A) \to \mathrm{ann}(A)/dA \to \mathrm{ann}(A)/F^1 \mathrm{ann}(A) \to 0,$$

which is nothing but the classical representation of $H^1_{DR}(C, k)$ as the quotient of the space of differentials of the second kind that are regular on C° modulo the exact ones. If we apply d^{-1}, then after noting that $d^{-1}(F^1 \mathrm{ann}(A)) = F^0A^\perp$, we get the sequence

(L-Hodge) $$0 \to F^1A^\perp \to A^\perp/(A + k^I) \to A^\perp/F^0A^\perp \to 0.$$

In the last case, the De Rham pairing $H^1_{DR}(C, k) \times H^1_{DR}(C, k) \to k$ (which is $2\pi\sqrt{-1}$ times the Betti intersection pairing) is induced by $\langle\ ,\ \rangle : L \times L \to R$. Furthermore, $A^\perp/F^0A^\perp \xrightarrow{\cong} L/(A + \mathcal{O})$.

3.4. **The case of a family of curves.** The preceding can also be carried over a smooth k-variety S (for instance $S = \operatorname{Spec}(R)$). We then have a smooth proper morphism $\pi : \mathcal{C} \to S$ whose geometric fibers are connected curves of genus g over S, endowed with pairwise disjoint sections $(x_i)_{i\in I}$. If we denote by $j : \mathcal{C}^\circ \subset \mathcal{C}$ the complement of their union, then $\pi^\circ := \pi j : \mathcal{C}^\circ \to S$ is an affine morphism and $\mathcal{A} := \pi^\circ_* \mathcal{O}_{\mathcal{C}^\circ}$ is a sheaf of $\mathcal{O}_S$-algebras. Via the relative residue pairing, we can identify $\operatorname{ann}(\mathcal{A})$ with $\pi^\circ_*\Omega_{\mathcal{C}^\circ/S}$. The L-Hodge sequence

$$0 \to F^1\mathcal{A}^\perp \to \mathcal{A}^\perp/(\mathcal{A} + R^I) \to \mathcal{A}^\perp/F^0\mathcal{A}^\perp \to 0.$$

is naturally isomorphic to the Hodge exact sequence

$$0 \to \pi_*\Omega_{\mathcal{C}/S} \to R^1\pi_*\pi^{-1}\mathcal{O}_S \to R^1\pi_*\mathcal{O}_\mathcal{C} \to 0.$$

Every local vector field $D \in \theta_S$ can be lifted to a vector field $\tilde D$ on $\mathcal{C}^\circ$. This will necessarily preserve $\mathcal{A}^\perp$. Since $\tilde D$ also preserves $\mathcal{O}_S^I$, it will act on $\mathcal{A}^\perp/(\mathcal{A} + \mathcal{O}_S^I)$. This action only depends on D, for any other lift differs by an element of $\theta_{\mathcal{C}^\circ/S} \cong \operatorname{Hom}_{\mathcal{O}_S}(\Omega_{\mathcal{C}^\circ/S}, \mathcal{O}_S) \cong \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{A}^\perp/\mathcal{O}_S^I, \mathcal{A})$ and such an element will clearly induce the zero map in $\mathcal{A}^\perp/(\mathcal{A} + \mathcal{O}_S^I)$. The resulting (Lie-)action of θ_S on $\mathcal{A}^\perp/(\mathcal{A}+\mathcal{O}_S^I)$ (denoted $D \mapsto \nabla_D$) is via the latter's identification with $R^1\pi_*\pi^{-1}\mathcal{O}_S$ just the Gauss-Manin connection. It preserves the symplectic pairing on $\mathcal{A}^\perp/(\mathcal{A} + \mathcal{O}_S^I)$. We thus end up with the data needed to define a Fock system attached to a symplectic local system.

3.5. **Allowing curves with nodal singularities.** Eventually we also want to allow in our family some singular fibers. Most of the discussion will go through if these have complete intersection singularities, but our interest is in the much more restricted case of ordinary double points. A nodal singularity is in fact a plane curve singularity which has two branches crossing normally: the formal completion of its local ring is isomorphic to that of the union of the coordinate axes in $\mathbb{A}^2$ at the origin. A curve C with nodal singularities only has an invertible dualizing sheaf ω_C that is easy to describe explicitly: it is the sheaf of rational differentials on C that are regular on its smooth part and have on each branch of a node a pole of order at most one at the singular point with the property that the two residues of the branches add up to zero. If a node at $p \in C$ is given by means of a local planar equation $f \in k[z_1, z_2]$ with $f = az_1z_2+$ higher order terms and $a \neq 0$, then a generator of $\omega_{C,o}$ is represented by what we could regard as the restriction to C of the quotient of $dz_1 \wedge dz_2$ by df: on the branch tangent to $z_2 = 0$ resp. $z_1 = 0$ it is given by $(\frac{\partial f}{\partial z_1})^{-1}dz_2$ resp. $-(\frac{\partial f}{\partial z_2})^{-1}dz_1$. Its residue at p is a^{-1} resp. $-a^{-1}$. If C is connected and complete and $X \subset C_{\mathrm{reg}}$ is a finite set which meets every irreducible component of C, then the residue pairing establishes a perfect pairing between $A = k[C^\circ]$ and $H^0(C^\circ, \Omega_{C^\circ})$ and between $H^1(C, \mathcal{O}_C)$ and $H^0(C, \omega_C)$. In particular, the dimension of $H^0(C, \omega_C)$ is that of the arithmetic genus of C.

This works just as well, if not better, in families with a nonsingular total space: suppose given a proper and flat morphism between k-varieties $\pi : \mathcal{C} \to S$ whose base S and domain $\mathcal{C}$ are nonsingular and connected and whose fibers are reduced connected curves with at worst ordinary double point singularities. Since the family is flat, the arithmetic genus of the fibers is locally constant and hence constant, say equal to g: $R^1\pi_*\mathcal{O}_\mathcal{C}$ is locally free of rank g. We denote by $\Delta \subset S$ the discriminant of π so that we have over $S - \Delta$ a smooth family. In our particular case, the locus where π fails be smooth is itself nonsingular and of codimension 2 (it is locally given by the vanishing to two partial derivatives) and the restriction of π is finite with image Δ. In the generic (versal) case, Δ has only normal crossing singularities with the local branches at $s \in S$ in an evident manner indexed by the nodes of C and *for simplicity we shall henceforth assume this to be the case.* Then the critical locus $(\mathcal{C}/S)_{\rm sing}$ of π is the normalization of Δ and that the normal bundle of the map $(\mathcal{C}/S)_{\rm sing} \to S$ is naturally identified with the tensor product of the two normal bundles of $(\mathcal{C}/S)_{\rm sing}$ in each of the local branches defined along it. To see this, assume (for simplicity) that S is of dimension one and that over $o \in S$, the fiber has just one double point, and that in terms of local coordinates at that point and at o, $f = \pi^*s$ with f as above. Then our local discussion then indeed shows that the derivative of f sends $\frac{\partial}{\partial z_1} \otimes \frac{\partial}{\partial z_2}$ to $a\frac{\partial}{\partial s}$. (Since this is the intrinsic description of the hessian of f, we call this the *hessian isomorphism*.)

The vector fields on S that are locally liftable to a vector field on $\mathcal{C}$ are precisely those that belong to $\theta_S(\log \Delta)$, i.e., those that are tangent to Δ, or equivalently, respect the principal ideal sheaf defining Δ. (We see this also illustrated in the local description above, for then $f\sum_{i=1}^2 (\frac{\partial f}{\partial z_i})^{-1}\frac{\partial}{\partial z_i}$ is a lift over π of the Euler vector field $s\frac{\partial}{\partial s} \in \theta_S(\Delta)$.)

The family $\mathcal{C}/S$ carries a relative dualizing sheaf $\omega_{\mathcal{C}/S}$ which is invertible (=locally free of rank one) and may be characterized simply as follows: the locus $\mathcal{U} \subset \mathcal{C}$ where π is smooth has a complement on $\mathcal{C}$ of codimension two (or is empty) and then $\omega_{\mathcal{C}^\circ/S}$ is simply the direct image of $\Omega_{\mathcal{U}/S}$ on $\mathcal{C}$. We have a natural isomorphism $\Omega_{\mathcal{U}/S}\otimes_{\mathcal{O}_\mathcal{U}}\pi^*\omega_S \cong \omega_\mathcal{U}$ (where $\omega_S = \Omega^{\dim S}$ and similarly for $\mathcal{U}$). This continues to hold on $\mathcal{C}$ if we replace $\Omega_{\mathcal{U}/S}$ by $\omega_{\mathcal{C}/S}$. (In the local description above, a local generator of $\omega_{\mathcal{C}/S}$ is represented by the quotient of $dz_1 \wedge dz_2$ by df.) Note that we have a natural homomorphism $\mathcal{H}om_{\mathcal{O}_\mathcal{C}}(\omega_{\mathcal{C}/S}, \mathcal{O}_\mathcal{C}) \to \mathcal{H}om_{\mathcal{O}_\mathcal{C}}(\Omega_{\mathcal{C}/S}, \mathcal{O}_\mathcal{C}) \cong \theta_{\mathcal{C}/S}$. This is in fact an isomorphism: it is clearly so outside the codimension 2 locus $(\mathcal{C}/S)_{\rm sing}$ and as $\omega_{\mathcal{C}/S}$ is invertible and $\theta_{\mathcal{C}/S}$ torsion free, this must be so on all of $\mathcal{C}$.

The characterizing property of a relative dualizing sheaf entails that we may identify $R^1\pi_*\omega_{\mathcal{C}/S}$ with $\mathcal{O}_S$ and that for every locally free $\mathcal{O}_\mathcal{C}$-module $\mathcal{F}$, the pairing $R^k\pi_*\mathcal{F} \otimes_{\mathcal{O}_S} R^{1-k}\pi_*\,\mathcal{H}om(\mathcal{F}, \omega_{\mathcal{C}/S}) \to R^1\pi_*\omega_{\mathcal{C}/S} \cong \mathcal{O}_S$ is perfect. In particular, we have a perfect pairing $\pi_*\omega_{\mathcal{C}/S} \otimes_{\mathcal{O}_S} R^1\pi_*\mathcal{O}_\mathcal{C} \to \mathcal{O}_S$ (and so $\pi_*\omega_{\mathcal{C}/S}$ is locally free of rank g). These pairings can be obtained in terms of a residue map as we will explain below.

Suppose given pairwise disjoint sections x_i of π, indexed by the finite nonempty set I whose union $\cup_{i\in I}x_i(S)$ lies in the smooth part of $\mathcal{C}$ and meets every irreducible component of a fiber. The last condition ensures that if $j : \mathcal{C}^\circ := \mathcal{C} - \cup_{i\in I}x_i(S) \subset \mathcal{C}$ is the inclusion, then $\pi^\circ := \pi j$ is an affine morphism. Then $R^1\pi_*\mathcal{O}_\mathcal{C}$ has a description à la Weil as before: let $(\mathcal{O}_i, \mathfrak{m}_i)$ be the formal completion of $\mathcal{O}_\mathcal{C}$ along $x_i(S)$, $\mathcal{L}_i$ the subsheaf of fractions of $\mathcal{O}_i$ with denominator a local generator of $\mathfrak{m}_i$ and denote by $\mathcal{O}$, $\mathfrak{m}$ and $\mathcal{L}$ the corresponding direct sums (= the direct image on S). We also keep on using ω, θ, $\hat\theta$ etc. for their sheafified counterparts. So these are now all $\mathcal{O}_S$-modules and the residue map is also one of $\mathcal{O}_S$-modules: $\mathrm{Res} : \omega \to \mathcal{O}_S$. If we write $\mathcal{A}$ for the sheaf of $\mathcal{O}_S$-algebras $\pi^\circ_*\mathcal{O}_{\mathcal{C}^\circ}$ and identify it with its image in $\mathcal{L}$, then $R^1\pi_*\mathcal{O}_\mathcal{C}$ can be identified with $\mathcal{L}/(\mathcal{A} + \mathcal{O})$. The same reasoning shows that for any locally free $\mathcal{O}_S$-module of finite rank $\mathcal{F}$, $R^1\pi_*\mathcal{F}$ can be identified with $\mathcal{F} \otimes_{\mathcal{O}_x} \mathcal{L}/(\pi^\circ_* j^*\mathcal{F}) + \mathcal{F} \otimes_{\mathcal{O}_x} \mathcal{O})$ (where $\mathcal{O}_x := \oplus_i\mathcal{O}_{\mathcal{C},x_i}$). Now note that we have a natural map of $\mathcal{O}_S$-modules

$$(\mathcal{F} \otimes_{\mathcal{O}_x} \mathcal{L}) \otimes_{\mathcal{O}_S} \pi_*\,\mathcal{H}om(\mathcal{F}, \omega_{\mathcal{C}/S}) \to \omega \xrightarrow{\mathrm{Res}} \mathcal{O}_S,$$

where the first map is evaluation along the sections x_i. This factors through a pairing $R^1\pi_*\mathcal{F} \otimes_{\mathcal{O}_S} \pi_*\,\mathcal{H}om(\mathcal{F}, \omega_{\mathcal{C}/S})$ which accomplishes relative Serre duality.

We may regard

$$\mathcal{DR}^\bullet_{\mathcal{C}/S} : 0 \to \mathcal{O}_\mathcal{C} \xrightarrow{d} \omega_{\mathcal{C}/S} \to 0$$

as a kind of 'relative De Rham complex', which fits in the exact sequence of complexes $0 \to \omega_{\mathcal{C}/S}[-1] \to \mathcal{DR}^\bullet_{\mathcal{C}/S} \to \mathcal{O}_\mathcal{C} \to 0$. We put $\mathcal{H}^k_{\mathcal{C}/S} := R^k\pi_*\mathcal{DR}^\bullet_{\mathcal{C}/S}$ (a relative hypercohomology module). Then the connecting homomorphisms $R^j\pi_*\mathcal{O}_{\mathcal{C}/S} \to R^j\pi_*\omega_{\mathcal{C}/S}$ (induced by d) in the long exact sequence of $\mathcal{O}_S$-modules

$$\cdots \to R^{k-1}\pi_*\mathcal{O}_\mathcal{C} \to R^{k-1}\pi_*\omega_{\mathcal{C}/S} \to \mathcal{H}^k_{\mathcal{C}/S} \to R^k\pi_*\mathcal{O}_\mathcal{C} \to R^k\pi_*\omega_{\mathcal{C}/S} \to \cdots$$

are trivial (for $j=0$ this is clear and for $j=1$ we get its Serre adjoint) and so we have a short exact sequence

$$0 \to \pi_*\omega_{\mathcal{C}/S} \to \mathcal{H}^1_{\mathcal{C}/S} \to R^1\pi_*\mathcal{O}_\mathcal{C} \to 0$$

In particular, the middle term is free of rank $2g$. The fiber of $\mathcal{H}^1_{\mathcal{C}/S}$ over $s \in S - \Delta$ is the De Rham cohomology space $H^1_{DR}(C_s)$ and so $\mathcal{H}^1_{\mathcal{C}/S}$ furnishes a natural extension of the Hodge bundle over $S - \Delta$.

We will see that the Gauss-Manin connection ∇ manifests itself here as a connection with logarithmic singularities so that we have covariant derivation in $\mathcal{H}^1_{\mathcal{C}/S}$ with respect to $\theta(\log\Delta)$.

Since we want to take a bit of an axiomatic approach, we collect the properties that we consider relevant for this purpose. We denote the sheaf of $\mathcal{O}_S$-derivations $\mathcal{A} \to \mathcal{A}$ by $\theta_{\mathcal{A}/S}$ and the sheaf $\pi^\circ_*\omega_{\mathcal{C}^\circ/S}$ by $\omega_{\mathcal{A}/S}$. The final clause of the proposition below generalizes the classical fact that the first De Rham cohomology group of a compact Riemann surface is its space of the differentials of the second kind modulo the exact ones.

Proposition 3.8. *The sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ satisfies the following properties.*

($\mathcal{A}_1$) *$\mathcal{A}$ is as a sheaf of $\mathcal{O}_S$-algebras flat and of finite type and $\mathcal{A} \cap \mathcal{O} = \mathcal{O}_S$,*
($\mathcal{A}_2$) *$\mathcal{L}/(\mathcal{A}+\mathcal{O})$ and $F^1\omega_{\mathcal{A}/S}$ locally free of rank g and are perfectly paired with respect to the residue pairing,*
($\mathcal{A}_3$) *$\omega_{\mathcal{A}/S}$ is the annihilator of $\mathcal{A}$ with respect to the residue pairing and we have $\theta_{\mathcal{A}/S} = \mathcal{H}om_\mathcal{A}(\omega_{\mathcal{A}/S}, \mathcal{A})$,*
($\mathcal{A}_4$) *The local vector fields on S that can be lifted to a vector field on $\mathcal{C}^\circ$ make up the Lie subalgebra $\theta_S(\log\Delta)$.*

Moreover, $\mathcal{H}^1_{\mathcal{C}/S}$ may be identified with $\ker(\omega_{\mathcal{A}/S} \xrightarrow{(\mathrm{Res}_i)_i} \mathcal{O}^I_S)/d\mathcal{A}$. Via this identification the Lie algebra $\theta_S(\log\Delta)$ acts naturally in $\mathcal{H}^1_{\mathcal{C}/S}$ and induces over $S-\Delta$ the Gauss-Manin connection.

Proof. By relative Serre duality and Riemann-Roch, the $\mathcal{O}_S$-module $\pi_*\mathcal{O}_\mathcal{C}(nX)$ is free of rank $1-g+n|I|$ when $n|I| > 2g-2$. Hence $\mathcal{A} = \varinjlim_n(\pi_*\mathcal{O}_\mathcal{C}(nX))$ is a direct limit of free $\mathcal{O}_S$-modules and therefore flat. It is clear that $A \cap \mathcal{O} = \pi_*\mathcal{O}_\mathcal{C}$ and since π is projective with connected fibers and S is normal, the latter can be identified with $\mathcal{O}_S$. This proves ($\mathcal{A}_1$).

Property ($\mathcal{A}_2$) is essentially relative Serre duality: the residue pairing establishes a perfect $\mathcal{O}_S$-duality between $R^1\pi_*\mathcal{O}_\mathcal{C}$ and $\pi_*\omega_{\mathcal{C}/S}$. The Weil isomorphism identifies $R^1\pi_*\mathcal{O}_\mathcal{C}$ with $\mathcal{L}/(\mathcal{A}+\mathcal{O})$ and it is clear that $\pi_*\omega_{\mathcal{C}/S} = \omega_{\mathcal{A}/S} \cap F^1\omega = F^1\omega_{\mathcal{A}/S}$.

We next do ($\mathcal{A}_3$). We have the Weil isomorphism $R^1\pi_*\mathcal{O}(-nx) \cong \mathcal{L}/(\mathfrak{m}^{-n}+\mathcal{A})$. The Serre-dual of $R^1\pi_*\mathcal{O}_\mathcal{C}(-nx)$ is $\pi_*\omega_{\mathcal{C}/S}(nx)$ and so $\pi_*\omega_{\mathcal{C}/S}(nx)$ and $\mathfrak{m}^{-n}+\mathcal{A}$ are each others annihilator. Hence the same is true for $\omega_\mathcal{A} = \cup_n\pi_*\omega_{\mathcal{C}/S}(nx)$ in relation to $\cap_n(\mathfrak{m}^{-n}+\mathcal{A}) = \mathcal{A}$. For the rest of ($\mathcal{A}_3$) we note that $\theta_{\mathcal{C}/S}$ and $\omega_{\mathcal{C}/S}$ are line bundles on $\mathcal{C}$ that are each others $\mathcal{O}_\mathcal{C}$-dual. Since $\mathcal{C}^\circ$ is affine over S, the natural map

$$\theta_{\mathcal{A}/S} = \pi^\circ_*\theta_{\mathcal{C}^\circ/S} \to \mathcal{H}om_{\pi^\circ_*\mathcal{O}^\circ_\mathcal{C}}(\pi^\circ_*\omega_{\mathcal{C}^\circ/S}, \pi^\circ_*\mathcal{O}^\circ_\mathcal{C}) = \mathcal{H}om_\mathcal{A}(\omega_{\mathcal{A}/S}, \mathcal{A})$$

is an isomorphism.

For $(\mathcal{A}_4)$ we note that the elements of $\theta_S(\log\Delta)$ admit local lifts in $\mathcal{C}$. On $\mathcal{C}^\circ$, these local lifts make up a subsheaf of $\theta_{\mathcal{C}^\circ}$ that is an extension of $(\pi^\circ)^{-1}\theta_S(\log\Delta)$ by $\theta_{\mathcal{C}^\circ/S}$. Since π° is affine, $R^1\pi^\circ_*\theta_{\mathcal{C}^\circ/S}=0$ and so any element of $\theta_S(\log\Delta)$ lifts to $\theta_{\mathcal{C}^\circ}$.

Finally, d maps $\mathcal{L}/\mathcal{O}_S^I$ isomorphically onto $\ker(\omega \xrightarrow{(\mathrm{Res}_i)_i} \mathcal{O}_S^I)$. This maps $\mathcal{O}$ onto $F^1\omega$ and so identifies $\mathcal{L}/(\mathcal{A}+\mathcal{O})$ with $\ker(\omega_{\mathcal{A}/S} \xrightarrow{(\mathrm{Res}_i)_i} \mathcal{O}_S^I)/F^1\omega + d\mathcal{A}$. The asserted identification of $\mathcal{H}^1_{\mathcal{C}/S}$ then follows.

Now let be given sections D of $\theta_S(\log\Delta))$ and α of $\omega_{\mathcal{A}/S}$ on an an affine open subset U of S which is small enough for there to exist generator μ of ω_U. Then $\pi^*\mu\wedge : \omega_{\mathcal{C}/S}|\pi^{-1}U \to \omega_{\mathcal{C}}|\pi^{-1}U$ is an isomorphism. Choose a lift $\tilde{D}\in H^0(U,\pi_*\theta_{\mathcal{C}^\circ})$ of D. We then define the relative Lie derivative $\mathcal{L}_{\tilde{D}}(\alpha)\in H^0(U,\omega_{\mathcal{A}/S})$ by the property

$$\mu\wedge\mathcal{L}_{\tilde{D}}(\alpha)=\mathcal{L}_{\tilde{D}}(\mu\wedge\alpha)-\pi^{\circ *}\mathcal{L}_D(\mu)\wedge\alpha = d\iota_{\tilde{D}}(\mu\wedge\alpha)-\pi^{\circ *}(d\iota_D\mu)\wedge\alpha.$$

This formula shows that $\mathcal{L}_{\tilde{D}}(\alpha)$ is exact near x_i, so that it has zero residue along x_i. In other words, $\mathcal{L}_{\tilde{D}}(\alpha)$ is a section over U of $\ker(\omega_{\mathcal{A}/S} \xrightarrow{(\mathrm{Res}_i)_i} \mathcal{O}_S^I)$. Another choice of $\tilde{D}$ differs from this element by some $\tilde{D}_o\in H^0(U,\theta_{\mathcal{A}/S})$. Since $\mu\wedge\mathcal{L}_{\tilde{D}_o}(\alpha)=d\iota_{\tilde{D}_o}(\mu\wedge\alpha)=\mu\wedge d\iota_{\tilde{D}_o}\alpha\in \mu\wedge dH^0(U,\mathcal{A})$, it follows that $\mathcal{L}_{\tilde{D}_o}(\alpha)\in H^0(U,d\mathcal{A})$. This shows that we have well-defined Lie action of $\theta_S(\log\Delta)$ in $\mathcal{H}^1(\mathcal{C}/S)$. It is well-known (and not hard to prove) that this is over $S-\Delta$ covariant derivativation with respect to the Gauss-Manin connection. □

3.6. **Covacua spaces for Fock data.** Proposition 3.8 shows that in the situation to which it pertains, we can cover S be affines whose coordinate ring satisfies the properties below.

Definition 3.9. We say that an R-subalgebra $A\subset L$ is of *Fock type* if

- (A_1) A is a flat R-algebra of finite type and $A\cap\mathcal{O}=R$,
- (A_2) the R-modules $L/(A+\mathcal{O})$ and $F^1\operatorname{ann}(A)$ are free of finite rank which are perfectly paired over R by the residue pairing,
- (A_3) the universal continuous R-derivation $d : L\to\omega$ maps A to $\operatorname{ann}(A)$ and the A-dual of the resulting A-homomorphism $\Omega_{A/R}\to\operatorname{ann}(A)$ yields an R-isomorphism $\operatorname{Hom}_A(\operatorname{ann}(A),A)\xrightarrow{\cong}\theta_{A/R}$,
- (A_4) there exists a reduced nonzero ideal $\mathscr{D}_A\subset R$ such that the Lie algebra $\theta_{A,R}$ of k-derivations of R which preserve A map in θ_A onto the subalgebra $\theta_R(\log\mathscr{D}_A)$ of $\theta_R)$ of derivations of A which preserve Δ.

This ideal $\mathscr{D}_A\subset R$ (which must be unique) is then called the *discriminant ideal* of A and the closed subset $\Delta\subset\operatorname{Spec}(R)$ defined by it is called the *discriminant* of A.

Let $A\subset L=\mathfrak{l}$ be an R-subalgebra of Fock type as above and put $H:=A^\perp/(A+R^I)\cong A^\perp/dA$. Then the residue pairing on $\mathfrak{l}$ induces one on H so that $\hat{H}$ becomes a subquotient of $\hat{\mathfrak{l}}$ (namely $A^\perp+R\hbar/A+R^I$). Observe that the maximal isotropic subspace $F^0\mathfrak{l}$ then corresponds to $F^0A^\perp/F^0(A+R^I)\cong F^1A^\perp\cong F^1\operatorname{ann}(A)$ (use (A_1)) and the latter is maximal isotropic by (A_3). This property also shows that the form on H is nondegenerate. The following theorem links the two Fock systems $\mathbb{F}$ and $\mathbb{F}(H,F^1A^\perp)$ and can be regarded as an abelian version of the WZW-connection.

Theorem 3.10. *Let $A \subset L$ be an R-subalgebra of Fock type. Denote by $\mathcal{D}_{A,\mathbb{F}} \subset \mathcal{D}_{\mathbb{F}}$ resp. $\mathcal{D}_{\mathbb{F}}(\log \mathscr{D}_A) \subset \mathcal{D}_{\mathbb{F}_A}$ the preimage of $\theta_{A,R}$ resp. $\theta_R(\log \mathscr{D}_A)$ (these are Lie algebras of first order differential operators acting on $\mathbb{F}$ resp. $\mathbb{F}_A$). Then:*

(i) *The space of covariants $\mathbb{F}_A := \mathbb{F}/A\mathbb{F}$ is naturally identified (in a $\hat{A}^\perp$-equivariant manner) with the Fock representation $\mathbb{F}(H, F^1A^\perp)$,*

(ii) *every $D \in \theta_{A/R}$ admits a unique lift $\hat{D} \in \hat{\theta}_{A/R}$ such that $T(\hat{D})$ lies in the $\overline{U}\hat{\mathfrak{l}}$-closure of $A \circ \mathfrak{l}$,*

(iii) *the Lie action of $\mathcal{D}_{A,\mathbb{F}}$ on $\mathbb{F}$ preserves the submodule $A\mathbb{F}$ and acts on $\mathbb{F}_A$ through a Lie-epimorphism $\mathcal{D}_{A,\mathbb{F}} \to \mathcal{D}_{\mathbb{F}}(\log \mathscr{D}_A)$ (with c_0 acting as the identity).*

In particular, $\mathcal{D}_{A,\mathbb{F}}$ acts on $\mathbb{F}_A$ naturally via its $\det_R(F^1A^\perp)$-connection of weight $\frac{1}{2}$.

Proof. For (i) we note that the kernel of the composite $A^\perp \subset L = \mathfrak{l} \subset \hat{\mathfrak{l}} \to \mathbb{F} \to \mathbb{F}_A$ contains $A + F^0A^\perp$. The fact that $A + R^I$ is an abelian subalgebra of $\hat{\mathfrak{l}}$ then implies that there is an induced R-linear map $\mathbb{F}(H, F^1A^\perp) \to \mathbb{F}_A$. It is straightforward to verify that this is an isomorphism of R-modules and then (i) follows.

To prove (ii), let $D \in \theta_{A/R}$. According to (A_3), we may view D as a L-linear map $\omega \to L$ which maps $\operatorname{ann}(A)$ to A. This implies that $\hat{C}(\hat{D})$ lies in the closure of the image of $A \otimes_R \hat{\mathfrak{l}} + \hat{\mathfrak{l}} \otimes_R A$ in $\overline{U}\hat{\mathfrak{l}}$. It follows that $\hat{C}(\hat{D})$ has the form $\hbar r + \sum_{n\geq 1} f_n \circ g_n$ with $r \in R$, one of $f_n, g_n \in L$ being in A and the order of f_n smaller than that of g_n for almost all n. In view of the fact that the nonzero elements of A are of lower order than those of $\mathcal{O}$ and $f_n \circ g_n \equiv g_n \circ f_n \pmod{\hbar R}$, we can assume that all f_n lie in A. We then modify our lift by taking the coefficient of $\hbar$ to be zero. It then follows that $T(\hat{D})$ lies in the $\overline{U}\hat{\mathfrak{l}}$-closure of $A \circ \mathfrak{l}$. A lift with this property is unique: any other lift differs from this one by an element of $R\hbar$. But $A \subset \mathfrak{l}$ is isotropic for $\langle\ ,\ \rangle$ and so $\overline{A \circ \mathfrak{l}} \cap R\hbar = 0$.

For (iii) we observe that if $D \in \theta_{A,R}$ and $f \in A$, then $[D, f] = Df$ lies in A. This shows that if $\hat{D} \in \mathcal{D}_{A,\mathbb{F}}$, then $T(\hat{D})$ preserves $A\mathbb{F}$ and hence acts in $\mathbb{F}_A$. When $D \in \theta_{A/R}$ and we choose $\hat{D} \in \hat{\theta}_{A/R}$ as in (ii), then $T(\hat{D})$ is clearly zero in $\mathbb{F}_A$. Thus $\mathcal{D}_{A,\mathbb{F}}$ acts in $\mathbb{F}_A$ through its quotient $\mathcal{D}_{A,\mathbb{F}}/\theta_{A/R}$. This quotient can be identified $\mathcal{D}_{\mathbb{F}}(\log \mathscr{D}_A)$. $\square$

4. Loop algebras and the Sugawara construction

Here we show how the Virasoro algebra acts in the standard represenions of a centrally extended loop algebra. This construction goes back to the physicist H. Sugawara (in 1968), but it was probably Graeme Segal who first noticed its relevance for the present context. Most of this material can be found for instance in [10] (Lecture 10) and [9] (Ch. 12), except that we approach the Sugawara construction via the construction discussed in the previous section and put it in the (coordinate free) setting that makes it appropriate for the present application.

In this section, we fix a simple Lie algebra $\mathfrak{g}$ over k of finite dimension. We retain the data and the notation of Section 2.

4.1. **Canonical invariant symmetric form and dual Coxeter number.** We identify $\mathfrak{g} \otimes \mathfrak{g}$ with the space of bilinear forms $\mathfrak{g}^* \times \mathfrak{g}^* \to k$, where $\mathfrak{g}^*$ denotes the k-dual of $\mathfrak{g}$, as usual. We form its space of $\mathfrak{g}$-invariants (relative to the adjoint action on both factors) $(\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$. This space is known to be of dimension one and to consist of symmetric tensors. So a generator c is represented by a nondegenerate symmetric bilinear form $c : \mathfrak{g}^* \times \mathfrak{g}^* \to k$. Since c is nondegenerate, it defines an isomorphism $\mathfrak{g}^* \cong \mathfrak{g}$ as $\mathfrak{g}$-representations and hence defines a nondegenerate $\mathfrak{g}$-invariant symmetric bilinear form $\check{c} : \mathfrak{g} \times \mathfrak{g} \to k$. Its $\mathfrak{g}$-invariance amounts to $\check{c}([X,Y],Z]) + \check{c}(y,[X,Z]) = 0$ for all $X,Y,Z \in \mathfrak{g}$. If $(X_\kappa)_{\kappa=1}^{\dim \mathfrak{g}}$ is an orthonormal basis with respect to $\check{c}$, then $c = \sum_\kappa X_\kappa \otimes X_\kappa$, and its $\mathfrak{g}$-invariance says that $\sum_\kappa([Z,X_\kappa] \otimes Y_\kappa + X_\kappa \otimes [Z,Y_\kappa]) = 0$ for all $Z \in \mathfrak{g}$. In other words, the image of c in $U\mathfrak{g}$ lies in the centre of $U\mathfrak{g}$. This means that c acts in every irreducible representation of $\mathfrak{g}$ as a scalar. Up to know, c is determined up to a nonzero scalar, but we will fix our c by requiring an additional property.

For this we briefly recall the basic structure theory of semisimple Lie algebras. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ yields a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha$, where $\Phi \subset \mathfrak{h}^* - \{0\}$ and $X \in \mathfrak{g}_\alpha$ means that $[H,X] = \alpha(H)X$ for all $H \in \mathfrak{h}$. The finite set Φ is a root system and its elements are called the roots of the pair $(\mathfrak{g},\mathfrak{h})$. We also choose a root basis $(\alpha_1,\dots,\alpha_r)$ in $\mathfrak{h}^*$ so that Φ decomposes into positive and negative roots: $\Phi = \Phi_+ \sqcup \Phi_-$ with $\Phi_- = -\Phi_+$; the elements of Φ_+ are the nonnegative linear combinations of $(\alpha_1,\dots,\alpha_r)$.

For every root $\alpha \in \Phi$, the root space $\mathfrak{g}_\alpha$ is of dimension one and we also have $-\alpha \in \Phi$. Moreover $\mathfrak{h}_\alpha := [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is 1-dimensional subspace of $\mathfrak{h}$ and has a generator H_α characterized by the property that $\alpha(H_\alpha) = 2$. . It is then clear that $\mathfrak{g}_\alpha + \mathfrak{h}_\alpha + \mathfrak{g}_{-\alpha}$ closed under the Lie bracket, hence makes up a Lie subalgebra of $\mathfrak{g}$. This Lie algebra is isomorphic to $\mathfrak{sl}(2)$: for if we choose $X_{\pm\alpha} \in \mathfrak{h}_{\pm\alpha}$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$, then $X_\alpha \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $X_{-\alpha} \mapsto \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $H_\alpha \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ defines a Lie isomorphism. The roots come in at most two types: if we regard $\mathfrak{h}^*$ as subspace of $\mathfrak{g}^*$ via the decomposition $\mathfrak{g}^* = \mathfrak{h}^* \oplus \oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha^*$, then these are distinguished by the value of $c(\alpha,\alpha)$. We now normalize c by requiring that $\alpha \in \Phi \mapsto c(\alpha,\alpha)$ is $\mathbb{Q}$-valued and takes 2 as its maximal value[1].

With this definition, c acts in the adjoint representation $\mathfrak{g}$ as multiplication by an even number. Half of this is called the *dual Coxeter number* of $\mathfrak{g}$ and denoted by $\check{h}$. So in terms of a basis $\{X_\kappa\}_\kappa$ of $\mathfrak{g}$ that is orthonormal for $\check{c}$ (so that c takes the form $\sum_\kappa X_\kappa \otimes X_\kappa$), we have

$$\sum_{\kappa=1}^{\dim \mathfrak{g}} [X_\kappa,[X_\kappa,Y]] = 2\check{h}Y \quad \text{for all } Y \in \mathfrak{g}.$$

We will write $C \in U\mathfrak{g}$ for the image of $c \in \mathfrak{g} \otimes_k \mathfrak{g}$ so that $C = \sum_\kappa X_\kappa \circ X_\kappa$. (This notational distinction becomes important in the next subsection, where both c and C appear as elements of the same algebra.)

The action of C on a finite dimensional irreducible representation of $\mathfrak{g}$ V commutes with the $\mathfrak{g}$-action and hence must be scalar (by Schur's Lemma), denoted C_V. This scalar can be given in terms of the highest weight $\lambda \in \mathfrak{h}^*$ of V. To explain, we choose a root basis $(\alpha_1,\dots,\alpha_r)$ in $\mathfrak{h}^*$. Since $\mathfrak{h}$ acts semisimply in V we can decompose V according to the characters in $\mathfrak{h}$: $V = \oplus_{\lambda\in\mathfrak{h}^*}V(\lambda)$. If we denote by $\Lambda(V)$ the set of $\lambda \in \mathfrak{h}^*$ for which $V_\lambda \neq 0$, then the *highest weight* of V is by definition the $\lambda_V \in \Lambda(V)$ where $\lambda \in \Lambda(V) \mapsto \lambda(H_{\alpha_1} + \cdots + H_{\alpha_r})$ takes its maximum (this is indeed unique). Then

$$C_V = c(\lambda_V, \lambda_V + 2\rho) = c(\lambda_V + \rho, \lambda_V + \rho) - c(\rho,\rho),$$

where $\rho := \frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha$ (see [9]). In case $V = \mathfrak{g}$ is the adjoint representation, this highest weight is spanned by the *highest root*, denoted here by $\tilde{\alpha}$ and we have $c(\tilde{\alpha},\tilde{\alpha}) = 2$.

[1] Then $c(\alpha,\alpha) \in \{1,2\}$ for all roots unless $\mathfrak{g}$ is of type G_2, in which case we get $\{\frac{2}{3}, 2\}$.

4.2. **Centrally extended loop algebras.** Let $L\mathfrak{g}$ stand for $\mathfrak{g}\otimes_k L$, but considered as a filtered R-Lie algebra (so we restrict the scalars to R) with $F^N L\mathfrak{g} = \mathfrak{g}\otimes_k \mathfrak{m}^N$. The quotient $UL\mathfrak{g}/UL\mathfrak{g}\circ F^N L\mathfrak{g}$ is a free R-module: a set of generators is $X_{\kappa_r}(t_i^{k_r})\cdots X_{\kappa_1}(t_i^{k_1})$, $k_r \leq \cdots \leq k_1 < N$, $i\in I$, where we adopt the custom to write $X(f)$ for $X\otimes f$. We complete $UL\mathfrak{g}$ $\mathfrak{m}$-adically on the right:

$$UL\mathfrak{g}\circ F^N L\mathfrak{g}.$$

Denote by $q:\mathfrak{g}\otimes\mathfrak{g}\to(\mathfrak{g}\otimes\mathfrak{g})^{\mathfrak{g}}$ the $\mathfrak{g}$-invariant projection (given by $X\otimes Y\mapsto \check{c}(X,Y)c$). Then an argument similar to the one we used to prove that the pairing r is topologically perfect shows that the pairing

$$r_{\mathfrak{g}}:(\mathfrak{g}\otimes_k\omega)\times(\mathfrak{g}\otimes_k L)\to cR,\quad (X(\alpha),Y(f))\mapsto q(X\otimes Y)\operatorname{Res}(f\alpha)$$

is topologically perfect (the basis dual to $(X_\kappa(t^l))_{\kappa,l}$ is $(X_\kappa(t^{-l-1}dt))_{\kappa,l}$).

A central extension of Lie algebras

$$0\to cR\to\widehat{L\mathfrak{g}}\to L\mathfrak{g}\to 0$$

is defined by endowing the sum $L\mathfrak{g}\oplus cR$ with the Lie bracket

$$[X(f)+cr, Y(g)+cs] := [X,Y](fg)+r_{\mathfrak{g}}(Xdf,Yg),$$

Since the residue is zero on $\mathcal{O}$, the inclusion $\mathcal{O}\mathfrak{g}\subset\widehat{L\mathfrak{g}}$ is a homomorphism of Lie algebras. In fact, this is a canonical (and even unique) Lie section of the central extension over $\mathcal{O}\mathfrak{g}$, for it is just the derived Lie algebra of the preimage of $\mathcal{O}\mathfrak{g}$ in $\widehat{L\mathfrak{g}}$. The $\operatorname{Aut}(\mathfrak{g})$-invariance of c implies that the tautological action of $\operatorname{Aut}(\mathfrak{g})$ on $\mathfrak{g}$ extends to $\widehat{L\mathfrak{g}}$.

We filter $\widehat{L\mathfrak{g}}$ by setting $F^N\widehat{L\mathfrak{g}} = F^N L\mathfrak{g}$ for $N>0$ and by letting for $N\leq 0$, $F^N\widehat{L\mathfrak{g}}$ be the preimage of $F^N L\mathfrak{g}+\mathfrak{c}_R$. Then $U\widehat{L\mathfrak{g}}$ is a filtered $R[c]$-algebra whose reduction modulo c is $UL\mathfrak{g}$. The $\mathfrak{m}$-adic completion on the right

$$\overline{U}\widehat{L\mathfrak{g}} := \varprojlim_N U\widehat{L\mathfrak{g}}/(U\widehat{L\mathfrak{g}}\circ F^N L\mathfrak{g})$$

is still an $R[c]$-algebra and the obvious surjection $\overline{U}\widehat{L\mathfrak{g}}\to\overline{U}L\mathfrak{g}$ is the reduction modulo c. These (completed) enveloping algebras not only come with the (increasing) Poincaré-Birkhoff-Witt filtration, but also inherit a (decreasing) filtration from L.

4.3. **A digression: the loop space analogue.** Suppose $R=k$ so that L is a complete local field. Let $\mathscr{G}$ be an algebraic group defined over k whose Lie algebra is $\mathfrak{g}$. Then the *algebraic loop group* $\mathscr{L}\mathscr{G}$ of $\mathscr{G}$ is a limit of algebraic groups defined over k and is characterized by the property that $\mathscr{G}(L)=\mathscr{L}\mathscr{G}(k)$. Its Lie algebra is $L\mathfrak{g}$. The central extension $\widehat{L\mathfrak{g}}$ of $L\mathfrak{g}$ by k can often be integrated to one of $\mathscr{L}\mathscr{G}$ by $\mathbb{G}_m$ by means of a procedure that was initiated by Steinberg [18]. We here describe the analogue for the loop group of a compact form of $\mathscr{G}$.

So let G be a compact simply connected real Lie group with (real) Lie algebra is $\mathfrak{g}$. It is known that then G is in fact 2-connected (every continuous map $S^2\to G$ is null-homotopic), but not 3-connected, for $\pi_3(G)$ is infinite cyclic. The simplest example, $\mathrm{SU}(2)$, illustrates this: when we identify this group as the group of unit quaternions, we see that it is diffeomorphic to S^3.

These homotopy properties are reflected by the bi-invariant De Rham complex of G: note that the group $G\times G$ acts on G by left-right multiplication with $(g_1,g_2)\in G\times G$ sending $h\in G$ to $g_1hg_2^{-1}$. This group then also acts on the De Rham complex of $\mathscr{E}^\bullet(G)$ of G and an averaging argument shows that $\mathscr{E}^\bullet(G)$ is homotopy equivalent to its subcomplex $\mathscr{E}^\bullet(G)^{G\times G}$.

The complex $\mathscr{E}^\bullet(G)^{\{1\}\times G}$ is of course the Lie algebra of right-invariant forms and so can be identified with the exterior algebra on $\mathfrak{g}^*$, the differential being expressed in terms of the Lie bracket. Then $\mathscr{E}^\bullet(G)^{G\times G}$ is the subcomplex of $\mathfrak{g}$-invariants in $\wedge^\bullet\mathfrak{g}^*$ (relative to the adjoint representation). On this subcomplex the differential is zero, so that we can identify it with $H^\bullet(G;\mathbb{R})$ (this is in fact the space of harmonic forms on G relative a bi-invariant metric). There are clearly no nonzero $\mathfrak{g}$-invariants in $\mathfrak{g}^*$ and neither are there in $\wedge^2\mathfrak{g}^*$ ($\mathfrak{g}$ admits no nonzero invariant antisymmetric bilinear form), but $(\wedge^3\mathfrak{g}^*)^G$ is generated by the form $(X,Y,Z)\mapsto\langle[X,Y],Z\rangle$, where $\langle\ ,\ \rangle$ is a nonzero G-invariant symmetric form on $\mathfrak{g}$. We normalize this 3-form in such a manner that the corresponding bi-invariant form α on G has the property that integration over α defines an isomorphism $\pi_3(G)\to\mathbb{Z}$. It has the property that under the multiplication map $m: G\times G\to G$, $m^*\alpha=\pi_1^*\alpha+\pi_2^*\alpha$.

Now let Σ be a connected closed oriented surface and consider the space $\mathrm{Map}(\Sigma,G)$ of differentiable maps $S^2\to G$ regarded as a group for pointwise multiplication. We show that the preceding gives rise to a continuous character $\chi:\mathrm{Map}(\Sigma,G)\to\mathrm{U}(1)$ as follows. Choose a connected compact oriented manifold M with boundary together with an orientation preserving identification of ∂M with Σ. We suppose M equipped with a smooth retraction r of neighbourhood of ∂M in M onto ∂M Since G is 2-connected, any $F\in\mathrm{Map}(\Sigma,G)$ extends to a differentiable map $\tilde F: M\to G$. By a small homotopy, we can arrange that on a neighbourhood of $\partial M\cong\Sigma$ this map factors through r. We claim that then $\exp(2\pi\sqrt{-1}\int_M\tilde F^*\mu$ does not depend on $\tilde F$: if $\tilde F': M\to G$ is another such extension, then we can glue these maps to produce a differentiable map: $H: M\cup_\Sigma(-M)\to G$ by letting H be on M resp. $-M$ be $\tilde F$ resp. $\tilde F'$. Now $\int_M H^*\mu$ is an integer by construction. But this is easily seen to be equal to $\int_M\tilde F^*\mu-\int_M\tilde F'^*\mu$. It follows that we have a well-defined map $\chi:\mathrm{Map}(\Sigma,G)\to\mathrm{U}(1)$.

To see that χ is a homomorphism, let be given $F_1,F_2:\Sigma\to G$. If $\tilde F_i: M\to G$ is an extension of F_i as above, then $\tilde F_1\cdot\tilde F_2: M\to G$ is an extension of $F_1\cdot F_2$ with the same regular boundary behaviour. Since $\tilde F_1\cdot\tilde F_2: M\to G$ is the composite of $(\tilde F_1,\tilde F_2): M\to G\times G$ and $m: G\times G\to G$, we have

$$\int_M(\tilde F_1\cdot\tilde F_2)^*\alpha=\int_M(\tilde F_1,\tilde F_2)^*m^*\alpha=\int_M(\tilde F_1,\tilde F_2)^*(\pi_1^*\alpha+\pi_2^*\alpha)=\int_M\tilde F_1^*\alpha+\int_M\tilde F_2^*\alpha,$$

so that after exponentiating we find that $\chi(F_1\cdot F_2)=\chi(F_1)\chi(F_2)$.

We have several 'applications' of this character χ, one of which involves $\Sigma=S^2$ only. For this we consider the space $\mathrm{Map}(S^1,G)$ of differentiable maps $S^1\to G$. It is a group for pointwise multiplication whose complexified Lie algebra may be viewed as an analogue of the our $L\mathfrak{g}$. The central extension of $\widehat{L\mathfrak{g}}$ has as its counterpart a central extension of $\mathrm{Map}(S^1,G)$ by $\mathrm{U}(1)$ defined as follows. Since G is simply connected, every loop $S^1\to G$ extends to a differentiable map $D^2\to G$. We consider only those maps $D^2\to G$ that are near the boundary factor through radial projection onto S^1. So if $\mathrm{Map}^+(D^2,G)$ is the group of such maps, then restriction defines a surjective homomorphism of groups $\mathrm{Map}^+(D^2,G)\to\mathrm{Map}(S^1,G)$. Notice that $F\in\mathrm{Map}^+(D^2,G)$ is in the kernel K of this homomorphism precisely when it is constant 1 near the boundary circle. So via a stereographic projection such an F may be thought of as an element of $\mathrm{Map}(S^2,G)$ that is constant 1 on a neighborhood of the north pole ∞. We thus get an embedding $K\subset\mathrm{Map}(S^2,G)$. We now obtain the central extension as a push-out: $K\cap\ker(\chi)$, where $\chi:\mathrm{Map}(S^2,G)\to\mathrm{U}(1)$ is as defined above, is normal in $\mathrm{Map}^+(D^2,G)$ (this is clear) and so if we put $\widehat{\mathrm{Map}}(S^1,G):=\mathrm{Map}^+(D^2,G)/K\cap\ker(A)$, then we find a central extension

$$1 \to \mathrm{U}(1) \to \widehat{\mathrm{Map}}(S^1, G) \to \mathrm{Map}(S^1, G) \to 1.$$

This is known to be the universal central extension of $\mathrm{Map}(S^1, G)$. In a way a topological justification of this extension is that we have $\pi_2(\mathrm{Map}(S^1, G)) \cong \pi_3(G) \cong \mathbb{Z}$ and that the central extension 'restores' the vanishing of π_2 in the same way as the Hopf bundle $S^3 \to \mathbb{P}^1(\mathbb{C}) = S^2$ (which is indeed a $\mathrm{U}(1)$-bundle) does this for S^2. But the main motivation for this central extension is that it has a representation theory which is much better behaved than the one for $\mathrm{Map}(S^1, G)$. The same is true on the level Lie algebras (which is an extension of $\mathrm{Map}(S^1, \mathfrak{g})$ by a one-dimensional Lie algebra). This Lie algebra is a real analogue of our $\widehat{L\mathfrak{g}}$.

4.4. **Another digression: a $\mathrm{U}(1)$-bundle over a moduli space.** The second use of χ involves a principal G-bundle P/Σ (where G acts on the right as is usual for such bundles). The gauge group $\mathrm{Aut}(P/\Sigma)$, the group of automorphisms of P/Σ, can be identified with $\mathrm{Map}(\Sigma, G)$: since P/Σ is trivial, it admits a section $s : \Sigma \to P$, which then defines the Σ-isomorphism $(x, g) \in \Sigma \times G \mapsto s(x).g \in P$. Any other section is of the form $x \in \Sigma \mapsto s(x)f(x)$ for some $f \in \mathrm{Map}(\Sigma, G)$. Since $\mathrm{Aut}(P/\Sigma)$ permutes these sections, it is identified with $\mathrm{Map}(\Sigma, G)$. Another choice of s gives an identification which differs form this one by a conjugacy with an element of $\mathrm{Map}(\Sigma, G)$. Hence we have a well-defined character (independent of choices) $\chi : \mathrm{Aut}(P/\Sigma) \to \mathrm{U}(1)$.

Now consider the space of flat connections on P/Σ. This is acted on by $\mathrm{Aut}(P/\Sigma)$ and so the orbit space may be regarded as the moduli space $\mathrm{Flat}(G, \Sigma)$ of principal G-bundles with flat connection. If we take the orbit space with respect to the subgroup $\ker(\chi) \subset \mathrm{Aut}(P/\Sigma)$ instead, we get a $\mathrm{U}(1)$-bundle over $\mathrm{Flat}(G, \Sigma)$. There is a description of $\mathrm{Flat}(G, \Sigma)$ in simpler terms, which makes it immediate that it is a finite dimensional space: if we choose a base point $x_o \in \Sigma$, then any flat connection on P/Σ defines via its holonomy a group homomorphism $\rho : \pi_1(\Sigma, x_o) \to \mathrm{Aut}(P_{x_o}) \cong G$, where the last identification involves a choice of a point in P_{x_o}. If we change base point x_o or the isomorphism of the right G spaces $P_{x_o} \cong G$, changes ρ by post-composition with an inner automorphism of G. Conversely, any homomorphism $\rho : \pi_1(\Sigma, x_o) \to G$ determines a flat connection on the trivial bundle $\Sigma \times G \to \Sigma$, and any two such differ by an automorphism of this bundle if and only if the associated maps are in the same G-conjugacy class. In other words, the space $\mathrm{Flat}(G, \Sigma)$ is naturally identified with $\mathrm{Hom}(\pi_1(\Sigma, x_o), G)/G$. The preceding construction defines a $\mathrm{U}(1)$-bundle over $\mathrm{Flat}(G, \Sigma)$. A heuristic computation (which can be made rigorous) shows that $\dim(\mathrm{Hom}(\pi_1(\Sigma, x_o), G)/G) = (2g(\Sigma) - 2) \dim G$ (use that $\pi_1(\Sigma, x_o)$ has $2g(\Sigma)$ generators subject to one relation).

This has a complex-analytic counterpart: let Σ be endowed with a conformal structure so that it becomes a Riemann surface C. Then any element of $\mathrm{Hom}(\pi_1(\Sigma, x_o), G)$ can be complexified to an element of $\mathrm{Hom}(\pi_1(\Sigma, x_o), \mathscr{G}(\mathbb{C}))$ to produce a flat (hence complex-analytic) $\mathscr{G}(\mathbb{C})$-principal bundle over C. This bundle is semistable and 'semisimple'. Conversely, any such $\mathscr{G}(\mathbb{C})$-principal bundle so arises. In fact, a theorem of Narasimhan-Sheshadri and its subsequent generalization by Donaldson implies that $\mathrm{Flat}(G, \Sigma)$ can be identified with the coarse moduli space $M(C, \mathscr{G}(\mathbb{C}))$ of 'semisimplified' semistable $\mathscr{G}(\mathbb{C})$-principal bundles. The Picard group of this bundle has a canonical generator that corresponds to the $\mathrm{U}(1)$-bundle over $\mathrm{Flat}(G, \Sigma)$ constructed above.

4.5. **Segal-Sugawara representation.** Tensoring with $c = \sum_\kappa X_\kappa \otimes X_\kappa \in \mathfrak{g} \otimes_k \mathfrak{g}$ defines the R-linear map

$$\mathfrak{l} \otimes_R \mathfrak{l} \to L\mathfrak{g} \otimes_R L\mathfrak{g}, \quad f \otimes g \mapsto \sum_\kappa X_\kappa(f) \otimes X_\kappa(g),$$

which, when composed with $L\mathfrak{g} \otimes_R L\mathfrak{g} \subset \widehat{L\mathfrak{g}} \otimes_R \widehat{L\mathfrak{g}} \to U\widehat{L\mathfrak{g}}$, yields a map $\gamma : \mathfrak{l} \otimes_R \mathfrak{l} \to U\widehat{L\mathfrak{g}}$. Since $\gamma(f \otimes g - g \otimes f) = \sum_\kappa [X_\kappa f, X_\kappa g] = c \dim \mathfrak{g}. \operatorname{Res}(gdf)$, this factors through a map $\hat{\gamma} : \hat{\mathfrak{l}}_2 \to UL\mathfrak{g}\otimes_R$ if we put $\gamma(\hbar) := c \dim \mathfrak{g}$. This R-module homomorphism is continuous and so extends to the completions $\hat{\gamma} : \overline{\hat{\mathfrak{l}}_2} \to \overline{U}L\mathfrak{g}\otimes_R$. Let us see how this induces an action of the Virasoro algebra. Let us first consider $\hat{C}_\mathfrak{g} := \hat{\gamma}\hat{C} : \theta \to (\overline{U}\widehat{L\mathfrak{g}})^{\operatorname{Aut}(\mathfrak{g})}$ (where we recall that $\hat{C} : \theta \to \overline{U}\hat{\mathfrak{l}})$ and describe it in the spirit of Section 2: given $D \in \theta$, then the R-linear map

$$1 \otimes D : \mathfrak{g} \otimes_k \omega \to \mathfrak{g} \otimes_k L$$

is continuous and self-adjoint relative to $r_\mathfrak{g}$ and the perfect pairing $r_\mathfrak{g}$ allows us to identify it with an element of $\overline{U}\widehat{L\mathfrak{g}}$; half this element produces our $\hat{C}_\mathfrak{g}(D)$. Thus the choice of the parameter t yields

$$\hat{C}_\mathfrak{g}(D_n) = \tfrac{1}{2} \sum_{i \in I} \sum_{\kappa, k+l=n} : X_\kappa(t_i^k) \circ X_\kappa(t_i^l) : \quad .$$

This formula could have been used to define $\hat{C}_\mathfrak{g}$ but that approach would make its naturality less apparent.

Lemma 4.1. *For $X \in \mathfrak{g}$, $f \in L$ and $D \in \theta$, we have*

$$[\hat{C}_\mathfrak{g}(D), X(f)] = -(c + \check{h})X(Df)$$

(an identity in $\overline{U}\widehat{L\mathfrak{g}}$) and upon a choice of a parameter t, then with the preceding notation

$$[\hat{C}_\mathfrak{g}(D_k), \hat{C}_\mathfrak{g}(D_l)] = (c + \check{h})(k - l)\hat{C}_\mathfrak{g}(D_{k+l}) + c(c + \check{h})\delta_{k+l,0}\frac{k^3 - k}{12} \dim \mathfrak{g}.$$

For the proof (which is a bit tricky, but not very deep), we refer to Lecture 10 of [10] (our $C_\mathfrak{g}(\hat{D}_k)$ is their T_k). This formula suggests that we make the central element $c + \check{h}$ of $\overline{U}\widehat{L\mathfrak{g}}$ invertible (its inverse might be viewed as a rational function on $(\operatorname{Sym}^2 \mathfrak{g}^2)^\mathfrak{g}$), so that we can state this lemma in a more natural manner:

Corollary 4.2 (Sugawara representation). *The map $D \mapsto \frac{-1}{c+\check{h}}\hat{C}_\mathfrak{g}(D)$ induces a natural homomorphism of R-Lie algebras*

$$T_\mathfrak{g} : \hat{\theta} \to \left(\overline{U}\widehat{L\mathfrak{g}}[\tfrac{1}{c+\check{h}}]\right)^{\operatorname{Aut}(\mathfrak{g})}$$

which sends the central element $c_0 \in \hat{\theta}$ to $c(c + \check{h})^{-1} \dim \mathfrak{g}$. Moreover, if $\hat{D} \in \hat{\theta}$, then $\operatorname{ad}_{T_\mathfrak{g}(\hat{D})}$ leaves $L\mathfrak{g}$ invariant (as a subspace of $\overline{U}\widehat{L\mathfrak{g}}$) and acts on that subspace by derivation with respect to the image of $\hat{D}$ in θ.

5. Algebraic description of the WZW connection

5.1. **Review of the theory of highest weight modules of a simple Lie algebra.** For what follows we briefly review from [9] the theory of highest weight representations of a loop algebra such as $\widehat{L\mathfrak{g}}$. According to that theory, the natural analogues for $\widehat{L\mathfrak{g}}$ of the finite dimensional irreducible representations of the finite dimensional semi-simple Lie algebras are obtained as follows, *assuming for the moment that I is a singleton*. We fix a Cartan subalgebra

$\mathfrak{h} \subset \mathfrak{g}$ and a system of positive roots $(\alpha_1, \dots, \alpha_r)$ in $\mathfrak{h}^*$. We decompose $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$ with $\mathfrak{g}_0 = \mathfrak{h}$ and $\mathfrak{g}_+$ resp. $\mathfrak{g}_-$ the direct sum of the root spaces associated to the positive resp. negative roots so that $\mathfrak{g}_{\geq 0} := \mathfrak{g}_0 \oplus \mathfrak{g}_+$ is a Borel subalgebra and $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are nilpotent subalgebras of $\mathfrak{g}$.

Let V be a finite dimensional irreducible representation of $\mathfrak{g}$ and denote by $\lambda = \lambda_V \in \Lambda(V) \subset \mathfrak{h}^*$ the highest weight of V. Then the weight space $\dim V(\lambda) = 1$ and $\mathfrak{g}_{\geq 0}$ leaves $V(\lambda)$ invariant, acting on it via $\mathfrak{g}_0 = \mathfrak{h}$ with character λ. Thus V appears as the irreducible quotient of the induced (Verma) module $U\mathfrak{g} \otimes_{U\mathfrak{g}_{\geq 0}} V(\lambda)$. This Verma module is already generated by $V(\lambda)$ as a $U\mathfrak{g}_-$-module.

The weight λ_V is *integral dominant* in the sense that if $(\varpi_1, \dots, \varpi_r)$ denotes the basis of $\mathfrak{h}^*$ dual to $(H_{\alpha_1}, \dots, H_{\alpha_r})$, then $\lambda_V \in \mathbb{Z}_{\geq 0}\varpi_1 + \cdots + \mathbb{Z}_{\geq 0}\varpi_r$. The map $V \mapsto \lambda_V \in \mathfrak{h}^*$ defines a bijection between the set of equivalence classes of finite dimensional irreducible representation of $\mathfrak{g}$ and the integral dominant weights of $(\mathfrak{h}, \Phi_+)$. Given an integral dominant weight λ, we often denote by V_λ a representative of the equivalence class of irreducible representations defined by λ.

Definition 5.1. The *level* of the finite dimensional irreducible $\mathfrak{g}$-representation V is $\ell_V := c(\tilde{\alpha}, \lambda_V)$.

This level is a nonnegative integer. Each $c(\tilde{\alpha}, \varpi_i)$ is a positive integer and so for any $\ell \geq 0$, the set of dominant integral weights λ with $c(\tilde{\alpha}, \lambda) \leq \ell$ is finite. In other words, the collection of equivalence classes of irreducible representations of level ℓ is finite. We denote that collection by P_ℓ.

5.2. **Review of the theory of highest weight modules of $\widehat{L\mathfrak{g}}$.** There is a similar construction for the irreducible highest weight modules of $\widehat{L\mathfrak{g}}$. Let t be a generator $\mathfrak{m}$ and put $L^- := t^{-1}R[t^{-1}]$. Then we have a decomposition

$$\widehat{L\mathfrak{g}} = \widehat{L\mathfrak{g}}_- \oplus \widehat{L\mathfrak{g}}_0 \oplus \widehat{L\mathfrak{g}}_+,$$

where

$$\widehat{L\mathfrak{g}}_0 := R \otimes_k \mathfrak{h} \oplus Rc, \ \widehat{L\mathfrak{g}}_+ := F^1 L\mathfrak{g} + F^0 L\mathfrak{g}_+ \text{ and } \widehat{L\mathfrak{g}}_- := L^-\mathfrak{g} \oplus R\mathfrak{g}_-.$$

(note that only $\widehat{L\mathfrak{g}}_-$ involves the choice of t).

We now fix a nonnegative integer ℓ and make $R \otimes_k V$ an R-representation of $F^0 L\mathfrak{g}$ by letting c act as multiplication by ℓ and by letting $\mathfrak{g} \otimes_k \mathcal{O}$ act via the reduction modulo $\mathfrak{m}$, $\mathfrak{g} \otimes_k \mathcal{O} \to R \otimes_k \mathfrak{g}$. If we induce this up to $\widehat{L\mathfrak{g}}$ we get a representation $\widetilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}}, V)$ of $\widehat{L\mathfrak{g}}$. Clearly, the definition of $\widetilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}}, V)$ does not involve the choice of t.

By the PBW-theorem, the natural map $\mathrm{Sym}^\bullet_R(\mathfrak{g} \otimes_k L^-) \otimes_k V \to \tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}}$ is an isomorphism of R-modules (here $\mathrm{Sym}^\bullet_R(\mathfrak{g} \otimes_k L^-)$ is understood as the subalgebra of the tensor algebra on $\mathfrak{g} \otimes_k L^-$ which consists of symmetric tensors).

Let $X_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ be generators such that $[X_\alpha, X_{-\alpha}] = H_\alpha$. We put $\tilde{X} := X_{\tilde{\alpha}}(t^{-1})$.

Lemma 5.2. *For every nonnegative integer p, $X_{-\tilde{\alpha}}(t)\tilde{X}^{p+1}$ acts on the highest weight space $V(\lambda)$ as $(p+1)(p-\ell+\ell_V)\tilde{X}^p$. Moreover, a supplement of $X_{-\tilde{\alpha}}(t)$ in $\widehat{L\mathfrak{g}}_+$ annihilates $\tilde{X}^{p+1}V(\lambda)$. If $\ell \geq \ell_V$, then $\tilde{X}^{\ell+1-\ell_V}V(\lambda)$ generates a $\widehat{L\mathfrak{g}}$-submodule of $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}}, V)$ that is a quotient of $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}}, V_{\lambda+2(\ell+1-\ell_V)\tilde{\alpha}})$.*

Proof. We put $\tilde{H} := [\tilde{X}, X_{-\tilde{\alpha}}t^{-1}]$. Note that $\tilde{H} = H_{\tilde{\alpha}} - c$. Then $[\tilde{H}, \tilde{X}] = 2\tilde{X}$ and so $\tilde{H}\tilde{X}^r = \tilde{X}^r\tilde{H} + r.2\tilde{X}^r = \tilde{X}^r.(H_{\tilde{\alpha}} - c + 2r)$. This acts on $V(\lambda)$ as $(\ell_V - \ell + 2r)\tilde{X}^r$. Since $X_{-\tilde{\alpha}}(t)$ annihilates

$V(\lambda)$, it follows that $X_{-\tilde\alpha}(t)\circ \tilde X^{p+1}$ acts on $V(\lambda)$ as $-\sum_{r=0}^p \tilde X^r \tilde H \tilde X^{p-r} = \sum_{r=0}^p(\ell_V-\ell+2r)\tilde X^p = (p+\ell_V-\ell)(p+1)\tilde X^p$. This proves the first assertion.

The second is proved similarly: Let β be a positive root and $f\in\mathcal{O}$. Then $[X_\beta(f),\tilde X]=0$ and $X_\beta(f)$ kills $V(\lambda)$. Hence $X_\beta(f)$ kills $\tilde X^{p+1}V(\lambda)$. Likewise, if $\beta\neq\tilde\alpha$, then $[X_{-\beta}(tf),\tilde X]$ is zero or $\tilde\alpha-\beta$ is a root (which is then necessarily positive) and $[X_{-\beta}(tf),\tilde X]$ is proportional to $X_{\tilde\alpha-\beta}(f)$. In the last case we note that $X_{\tilde\alpha-\beta}(f)$ commutes $\tilde X = X_{\tilde\alpha}(t^{-1})$ and so $X_{-\beta}(tf)\tilde X^{p+1}$ is a linear combination of $\tilde X^{p+1}X_{-\beta}(tf)$ and $\tilde X^p X_{\tilde\alpha-\beta}f$. Such an element kills V_λ. Similarly, $[X_{-\tilde\alpha}(t^2f),\tilde X] = -H_{\tilde\alpha}(tf)$, $[H_{\tilde\alpha}(tf),\tilde X]=2X_{\tilde\alpha}f$. Both $X_{\tilde\alpha}f$ and $H_{\tilde\alpha}(tf)$ kill $V(\lambda)$ and hence the same is true of $X_{-\tilde\alpha}(t^2f)\tilde X^{p+1}$. A similar argument shows that when $H\in\mathfrak{h}$, then $H(tf)$ kills $\tilde X^{p+1}V(\lambda)$. This proves the second assertion and combined with the first one, we find that when $\ell\geq\ell_V$, $\widehat{L\mathfrak{g}}_+$ annihilates $\tilde X^{\ell+1-\ell_V}V(\lambda)$.

Since $[H,\tilde X]=\tilde\alpha(H)\tilde X$, H acts on $\tilde X^{p+1}V(\lambda)$ as multiplication by $(\lambda+(p+1)\tilde\alpha)(H)$. In other words, $\mathfrak{h}$ acts on $\tilde X^{p+1}V(\lambda)$ with weight $\lambda+(p+1)\tilde\alpha$. This implies the last assertion. □

For future reference we mention:

Corollary 5.3. *Assume that $\ell\geq\ell_V$. Then for every $p>\ell-\ell_V$ and every $n\geq 0$, we have $X_{-\tilde\alpha}(t)^n\tilde X^{p+n}V(\lambda)=\tilde X^pV(\lambda)$.*

Proof. Let $v\in V(\lambda)$. Iterated application of Lemma 5.2 yields

$$X_{-\tilde\alpha}(t)^n\tilde X^{p+n}v = \binom{p+n}{n}\binom{p+n-\ell+\ell_V}{n}\tilde X^p v.$$ □

When $\ell\geq\ell_V$ we define $\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V)=\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V_\lambda)$ as the quotient $\widehat{L\mathfrak{g}}$-module of $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}},V_\lambda)$ by the submodule generated by $\tilde X^{\ell+1-\ell_V}V(\lambda)$, so that we have an exact sequence of $\widehat{L\mathfrak{g}}$-modules

$$\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}},V_{\lambda+2(\ell+1-\ell_V)\tilde\alpha})\to\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}},V_\lambda)\to\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V)\to 0.$$

For the proof of the following assertion we refer to Exercise (12.12 of [9]).

Proposition-Definition 5.4. *The $\widehat{L\mathfrak{g}}$-module $\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V)$ is irreducible and* integrable *in the sense that the elements of $\widehat{L\mathfrak{g}}_+\cup\widehat{L\mathfrak{g}}_-$ act locally nilpotently in $\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V)$ (which means that for each $u\in\widehat{L\mathfrak{g}}_+\cup\widehat{L\mathfrak{g}}_-$, $\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V)=\cup_N\ker(u^N)$). We call this the* integrable irreducible representation of $\widehat{L\mathfrak{g}}$ of highest weight (λ,ℓ)*. Any irreducible $\widehat{L\mathfrak{g}}$-module with this property is equivalent to such a representation.*

We now return to the general case where we allow I to be more than a singleton. We assume given for every $i\in I$ an irreducible representation V_i of $\mathfrak{g}$ of level $\leq\ell$. Applying the preceding to every pair $(L_i\mathfrak{g},V_i)$ yields a $\widehat{L_i\mathfrak{g}}$-representation $\mathbb{H}_\ell(V_i)$ so that we can form the $\widehat{L\mathfrak{g}}$-representation $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}},V):=\otimes_{i\in I}\tilde{\mathbb{H}}_\ell(V_i)$ and $\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V):=\otimes_{i\in I}\mathbb{H}_\ell(L\mathfrak{g},V_i)$. Proposition-definition 5.4 implies that $\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V)$ is irreducible and integrable as a $\widehat{L\mathfrak{g}}$-module. With our given R-subalgebra A of Fock type of genus g, we can form $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}},V)_{A\mathfrak{g}}$ and $\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V)_{A\mathfrak{g}}$. The following proposition says the latter is of finite rank.

Proposition 5.5 (Finiteness)**.** *The space $\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V)$ is finitely generated as a $UA\mathfrak{g}$-module (so that $\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V)_{A\mathfrak{g}}$ is a finitely generated R-module).*

Proof. Choose a subspace $F^- \subset L^-$ which maps isomorphically onto $L/(\mathcal{O}+A)$. Since $F^-\mathfrak{g}$ is a free R-module of finite rank, we may think of it as an affine space over R. If $K_N \subset F^-\mathfrak{g}$ denotes the set of $u \in F^-\mathfrak{g}$ for which u^N annihilates $\otimes_{i\in I} V_i = 0$, then K_N defines a Zariski closed subset of $F^-\mathfrak{g}$. Clearly, $K_N \subset K_{N+1}$. Since the elements of $F^-\mathfrak{g}$ act on $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)$ in a locally nilpotent manner, we have $\cup_N K_N = F^-\mathfrak{g}$. Such a countable union of Zariski closed subsets must be a finite one: $K_N = F^-\mathfrak{g}$ for some N (this is clear when the field k is uncountable and we can reduce to that case by embedding k in such a field). It then follows that $M := \sum_{r\geq 0}(F^-\mathfrak{g})^{\circ(r)} \otimes (\otimes_{i\in I} V_i) = \sum_{r=0}^{N-1}(F^-\mathfrak{g})^{\circ(r)} \otimes (\otimes_{i\in I} V_i)$ is a finitely generated R-submodule of $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)$ that is invariant under $F^-\mathfrak{g}$. The Poincaré-Birkhoff-Witt theorem then shows that M generates $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)$ as a $A\mathfrak{g}$-module.

The second assertion follows from Theorem 5.12. □

Remark 5.6. In the general context of Proposition 5.5 we expect the R-module $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}}$ to be locally free and this to be a consequence of a flatness property of the $UA\mathfrak{g}$-module $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)$. Such a result, or rather an algebraic proof of it, might simplify the argument in [24] (see Section **??** for our version).

Inspired by the physicists terminology, we make the following definition.

Definition 5.7. We call the finitely generated R-module $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}}$ the R-module of *covacua* attached to A and its R-dual $\mathbb{H}^\ell(\widehat{L\mathfrak{g}}, V^*)^{A\mathfrak{g}} := \mathrm{Hom}_R(\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}}, R)$ the *R-module of vacua*. We refer to $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}}$ and its topological dual $\tilde{\mathbb{H}}^\ell(\widehat{L\mathfrak{g}}, V^*)^{A\mathfrak{g}}$ (where we use the topology coming from the PBW-filtration) as the R-module of *precovacua* resp. *prevacua*.

Remark 5.8. If $F^- \subset A^\perp$ is a section of $A^\perp \to A^\perp/(A + F^0 A^\perp)$, then the PBW theorem shows that the natural map from the tensor algebra of $\mathfrak{g}\otimes F^-$ tensored with V to $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}}$ is onto. In other words, any element of $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}}$ can be represented by a linear combination of elements of the form $Y_1 f_1 \circ \cdots Y_n f_n \circ rv$ with $f_i \in F^-$ and $r \in R$. *In particular, we can always choose our coefficients f_i in $A^\perp$.* This last property is useful as it is preserved under the WZW-connection that we will define next.

5.3. **The extended Sugawara connection.** We fix a nonnegative integer ℓ. We first assume that I is a singleton and let V be an irreducible finite dimensional representation of $\mathfrak{g}$ of level $\leq \ell$. Recall that C_V denotes the scalar by which the Casimir element C acts on V. We let the Lie algebra $\hat{\theta}$ act on $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}}, V)$ and $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)$ over R via $T_\mathfrak{g}$. If we choose a generator t of $\mathfrak{m}$, then it follows from Corollary 4.2 that if $\hat{D} \in \hat{\theta}$ lifts $D \in \theta$, then for $v \in V$ and $r \in R$ we have

$$
\begin{aligned}
T_\mathfrak{g}(\hat{D})\big(Y_r f_r \circ \cdots \circ Y_1 f_1 \circ rv\big) =\\
= \sum_{i=1}^{r} Y_r f_r \circ \cdots \circ Y_i D(f_i) \circ \cdots \circ Y_1 f_1 \circ rv + Y_r f_r \circ \cdots \circ Y_1 f_1 \circ r T_\mathfrak{g}(\hat{D}) v
\end{aligned}
$$

.

If $D \in F^1\theta$, then clearly $T_\mathfrak{g}(\hat{D})v = 0$, but this is not so when $D \in F^0\theta$, for

$$T_\mathfrak{g}(\hat{D}_0)v = \frac{-1}{2(\ell+\check{h})}\sum_\kappa X_\kappa \circ X_\kappa \circ v = \frac{-C_V}{2(\ell+\check{h})} v.$$

Our generator defines a section $\sigma_t : \theta_R \to \theta_{L,R}$ of $\theta_{L,R} \to \theta_R$, characterized by the property that $\sigma_t(D)(t) = 0$ for every $D \in \theta_R$. We thus obtain a splitting $\theta_{L,R} = \theta \oplus \sigma_t(\theta_R)$ and we then define a representation $T_{\mathfrak{g},t}$ over k of $\hat{\mathcal{D}}_{L,R} = \hat{\theta}_R \oplus \sigma_t(\theta_{L,R})$ on $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}}, V)$ (and similarly on $(\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}}, V))$, by stipulating that on $\hat{\theta}_R$ the action is as above and on $\sigma_t(\theta_{L,R})$ as coefficientwise derivation. The following lemma explicates its dependence on t:

Lemma 5.9. *For any other generator t' of $\mathfrak{m}$, put $u := t'/t \in \mathcal{O}^\times$ and denote by $u_0 \in R^\times$ its reduction modulo $\mathfrak{m}$. Then for every $\hat{D} \in \hat{\mathcal{D}}_{L,R}$ which lifts $D \in \theta_R$ and for every $Z \in U\widehat{L\mathfrak{g}}$ we have*

$$T_{\mathfrak{g},t'}(\hat{D}) = T_{\mathfrak{g},t}(\hat{D}) - \frac{Du_0.C_V}{2(\ell + \check{h})u_0},$$

where the last term is an element of R that acts as a scalar. In particular, $T_{\mathfrak{g},t}$ and $T_{\mathfrak{g},t'}$ have the same image and $T_{\mathfrak{g},t}$ only depends on the image of t in $\mathfrak{m}/\mathfrak{m}^2$.

Proof. It suffices to check this for $\hat{D}$ of the form $\sigma_{t'}(D)$. It is straightforward to verify that $D - (t\dot{u} + u)^{-1}D(u)t\frac{\partial}{\partial t}$ is the lift of D to $\theta_{L,R}$ which kills $ut = t'$ (where $\dot{u}$ is the derivative with respect to t) and so

$$\sigma_{t'}(D) = \sigma_t(D) - \frac{D(u)}{t\dot{u} + u} t\frac{\partial}{\partial t}.$$

Notice that the coefficient of $D_0 = t\frac{\partial}{\partial t}$ in this expression is $-u_0^{-1}D(u_0)$. Now $T_{\mathfrak{g},t'}(\sigma_{t'}(D))$ is by definition given as coefficientwise derivation with respect to D. On the other hand $T_{\mathfrak{g},t}(\sigma_{t'}(D))$ is coefficientwise derivation plus multiplication by

$$-u_0^{-1}D(u_0)T_{\mathfrak{g}}(D_0) = \frac{D(u_0).C_V}{2(\ell + \check{h})u_0}. \qquad \square$$

Remark 5.10. The R-module $\mathfrak{m}/\mathfrak{m}^2$ is free of rank one and can be understood as a relative Zariski contangent space. The image $\bar{t}$ of t in $\mathfrak{m}/\mathfrak{m}^2$ defines a Lie splitting of $\mathcal{D}_1(\mathfrak{m}/\mathfrak{m}^2) \to \theta_R$: it assigns to $D \in \theta_R$ the lift $D_{\bar{t}} \in \mathcal{D}_1(\mathfrak{m}/\mathfrak{m}^2)$ characterized by $D_{\bar{t}}(\bar{t}) = 0$. It is easy to see to check that for $t' = u_0 t$ with $u_0 \in R^\times$, we have $D_{\bar{t}'} = D_{\bar{t}} - u_0^{-1}Du_0$.

Remark 5.11. Returning to the general case, where I as any finite set, we see that the preceding lemma takes the following form: if $t = (t_i)_{i\in I}$ is a generator of $\mathfrak{m} = \oplus_i \mathfrak{m}_i$, then we have defined an action $T_{\mathfrak{g},t}$ on $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}}, V)$ and $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)$. For any other generator t' of $\mathfrak{m}$, put $u_i := t_i'/t_i \in \mathcal{O}_i^\times$ and denote by $u_{i,0} \in R^\times$ its reduction modulo $\mathfrak{m}_i$. Then for every $\hat{D} \in \hat{\mathcal{D}}_{L,R}$ which lifts $D \in \theta_R$ and for every $Z \in U\widehat{L\mathfrak{g}}$ we have

$$T_{\mathfrak{g},t'}(\hat{D}) - T_{\mathfrak{g},t}(\hat{D}) = -\sum_{i\in I} \frac{Du_{i,0}.C_{V_i}}{2(\ell + \check{h})u_{i,0}},$$

where the right hand side is an element of R.

We have the following analogue of Theorem 3.10.

Theorem 5.12 (Algebraic version of the WZW connection)**.** *Let $A \subset R$ be a R-subalgebra of Fock type. Then:*

(i) *Every $D \in \theta_{A/R}$ has a unique lift $\hat{D} \in \hat{\theta}_{A/R}$ with the property that $T_{\mathfrak{g}}(\hat{D})$ lies in the $\overline{U}\widehat{L\mathfrak{g}}$-closure of $A\mathfrak{g} \circ L\mathfrak{g}$.*

(ii) *For any choice of a generator $t = (t_i)_{i\in I}$ of $\mathfrak{m}$, the associated Sugawara representation $T_{\mathfrak{g},t}$ of $\mathcal{D}_{\mathbb{F}}$ on $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)$ has the property that it makes $\mathcal{D}_{A,\mathbb{F}}$ preserve the submodule $A\mathfrak{g}\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V) \subset \mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)$ and acts in the space of $A\mathfrak{g}$-covariants in $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)$,*

$\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V)_{A\mathfrak{g}}$, via a Lie-epimorphism $\mathcal{D}_{A,\mathbb{F}}\to\mathcal{D}_{\mathbb{F}}(\log\mathscr{D}_A)$; this representation is one by differential operators of degree ≤ 1 (with c_0 acting as multiplication by $(c+\check{h})^{-1}c\dim\mathfrak{g}$).

(iii) *These data define a natural flat logarithmic connection (called the* WZW connection *and denoted ∇^{WZW}) on $\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V)_{A\mathfrak{g}}$ relative to the collection R-modules $\det(F^1\operatorname{ann}A)$, $\{\mathfrak{m}_i/\mathfrak{m}_i^2\}_{i\in I}$ with weights $\frac{1}{2}$ resp. $C_{V_i}/2(l+\check{h})$.*

The same properties hold for $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}},V)_{A\mathfrak{g}}$.

Proof. The proof is indeed similar to the arguments used to prove Theorem 3.10. Since D maps $A^\perp$ to $A\subset L$, $1\otimes D$ maps the submodule $\mathfrak{g}\otimes A^\perp$ of $\mathfrak{g}\otimes\omega$ to the submodule $\mathfrak{g}\otimes A=A\mathfrak{g}$ of $\mathfrak{g}\otimes L=L\mathfrak{g}$. It is clear that $\mathfrak{g}\otimes A^\perp$ and $A\mathfrak{g}$ are each others annihilator relative to the pairing $r_\mathfrak{g}$. This implies that $\hat{C}(\hat{D})$ lies in the closure of the image of $A\mathfrak{g}\otimes_k L\mathfrak{g}+L\mathfrak{g}\otimes_k A\mathfrak{g}$ in $\overline{U}\widehat{L\mathfrak{g}}$. It follows that $\hat{C}(\hat{D})$ has the form $cr+\sum_\kappa\sum_{n\ge 1}X_\kappa(f_{\kappa,n})X_\kappa(g_{\kappa,n})$ with $r\in R$, one of $f_{\kappa_n},g_{\kappa,n}\in L$ being in A and the order of f_{κ_n} smaller than that of $g_{\kappa,n}$ for almost all κ,n. Since the elements of A have order ≤ 0 and $X_\kappa(f_{\kappa,n})\circ X_\kappa(g_{\kappa,n})\equiv X_\kappa(g_{\kappa,n})X_\kappa(f_{\kappa,n}) \pmod{cR}$, we can assume that all $f_{\kappa,n}$ lie in A and so the first assertion follows.

Let $\hat{D}\in\mathcal{D}_{A,\mathbb{F}}$ have image $D\in\theta_{A,R}$. Then for $X\in\mathfrak{g}$ and $f\in A$, we have $[D,X(f)]=X(Df)$, which is an element of $A\mathfrak{g}$ (since $Df\in A$). This shows that $T_\mathfrak{g}(\hat{D})$ preserves $A\mathfrak{g}\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V)$. If $\hat{D}$ is as in (i), then we have seen that $T_\mathfrak{g}(\hat{D})$ maps $\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V)$ to $A\mathfrak{g}\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V)$ and hence induces the zero map in $\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V)_{A\mathfrak{g}}$. So $\mathcal{D}_{A,\mathbb{F}}$ acts on $\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V)_{A\mathfrak{g}}$ via a Lie-epimorphism $\mathcal{D}_{A,\mathbb{F}}\to\mathcal{D}_{\mathbb{F}}(\log\mathscr{D}_A)$.

The last observation follows from Theorem 3.10 , Lemma 5.9 and Remark 5.10.

The same proof applies to $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}},V)$. ☐

Remark 5.13. According to Remark 5.8 any element of $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}},V)_{A\mathfrak{g}}$ can be represented by a linear combination of expressions of the type $Y_rf_r\circ\cdots Y_1f_1\circ v$ with $f_i\in A^\perp$. The covariant derivative of such an element with respect to some $D\in\mathcal{D}_{\mathbb{F}}(\log\mathscr{D}_A)$ is given by implementing the receipe: choose a lift $\tilde{D}\in\mathcal{D}_{A,\mathbb{F}}$. Then $\tilde{D}$ preserves A and hence $A^\perp$ and so

$$\nabla_D^{\mathrm{wzw}}[Y_r(f_r)\cdots Y_1(f_1)v]=\sum_{k=1}^r[Y_r(f_r)\cdots Y(\tilde{D}f_k)\cdots Y_1(f_1)v]+[Y_r(f_r)\cdots Y_1(f_1)\tilde{D}(v)],$$

where we note that all these terms except the last have their coefficients in $A^\perp$. The last term is computed by decomposing $\tilde{D}$ for each $i\in I$ into a horizontal and a vertical part. The horizontal part has the effect of multiplication by some element of R, whereas the vertical part is given by the Sugawara representation. This too, has its coefficients in $A^\perp$.

Our discussion also suggests to 'twist' $\mathcal{O}_\mathcal{C}$ in the sense that

If we sheafify and apply this to the situation of Subsection 3.5 we find:

Corollary 5.14 (The WZW-connection). *Let $\pi:\mathcal{C}\to S$ be a family of curves endowed with sections $(x_i:X\to\mathcal{C})_{i\in I}$ as in Subsection 3.5: S and $\mathcal{C}$ are smooth, the geometric fibers have at worst ordinary double points and the discriminant Δ is a normal crossing divisor. Denote by $q:\mathbb{L}^\times(\det(\pi_*\omega_{\mathcal{C}/S},(x_i^*\omega_{\mathcal{C}/S})_{i\in I})\to S$ the $\mathbb{G}_m\times\mathbb{G}_m^I$-bundle over S defined by the $\mathbb{G}_m$-bundles associated to the line bundles mentioned as arguments. Let ℓ be a nonnegative integer and let for each $i\in I$ be given a finite dimensional irreducible representation V_i of $\mathfrak{g}$. Then $\mathcal{H}_\ell(\widehat{L\mathfrak{g}},V)$ is a coherent $\mathcal{O}_S$-module, locally free over $S-\Delta$ and the pull-back $q^*\mathcal{H}_\ell(\widehat{L\mathfrak{g}},V)$ admits a canonical flat connection with weights $(\frac{1}{2},(C_{V_i}/2(l+\check{h})_{i\in I})$ and with a logarithmic singularity along $q^*\Delta$.*

These properties follow from the preceding, except the assertion that $\mathcal{H}_\ell(\widehat{L\mathfrak{g}}, V)$ is locally free over $S-\Delta$. This is however a consequence of the following well-known

Lemma 5.15. *Let U be a nonsingular k-variety and $\mathcal{F}$ a coherent $\mathcal{O}_U$-module endowed with a flat connection. Then $\mathcal{F}$ is locally free.*

Indeed, this Lemma implies that $q^*\mathcal{H}_\ell(\widehat{L\mathfrak{g}}, V)$ is locally free over $q^{-1}(S-\Delta)$. Since q admits local sections, it then follows that $\mathcal{H}_\ell(\widehat{L\mathfrak{g}}, V)$ is locally free over $S-\Delta$.

5.4. **Propagation principle.** The following proposition is known as the *propagation of vacua*; it essentially shows that trivial representations may be ignored (as long as some representations remain: if all are trivial, then we can get rid of all but one of them). If we do not care about the WZW-connection, then this is even true for nontrivial representations (a fact that can be found in Beauville [1]) so that we then essentially reduce the discussion to the case where I is a singleton.

Proposition 5.16. *Let $I'\subset I$ be nonempty and put $I'':=I-I'$, $V':=\otimes_{i\in I'}V_i$ and $V'':=\otimes_{j\in I''}V_i$. Assume that the map $A\to\oplus_{j\in I''}L_j/\mathcal{O}_j$ is onto and denote by A' its kernel (so that the image of A' in L_j is contained in $\mathcal{O}_j$ for all $j\in I''$). Then the injections $(V_j\hookrightarrow H_\ell(V_j))_{j\in I''}$ define maps of $A'\mathfrak{g}$-modules $\tilde{\mathbb{H}}_\ell(V')\otimes_k V''\to\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}}, V)$ and $\mathbb{H}_\ell(V')\otimes_k V''\to\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)$ (where $A'\mathfrak{g}$ acts on the tensor factor $R\otimes_k V_j$, $j\in I''$, via the evaluation map $A'\mathfrak{g}\to R\otimes_k\mathfrak{g}$ given by reduction modulo $\mathfrak{m}_j$) which both induce an isomorphism on covariants:*

$$\big(\tilde{\mathbb{H}}_\ell(\widehat{L'\mathfrak{g}}, V')\otimes_k V''\big)_{A'\mathfrak{g}}\xrightarrow{\cong}\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}},\quad \big(\mathbb{H}_\ell(\widehat{L'\mathfrak{g}}, V')\otimes_k V''\big)_{A'\mathfrak{g}}\xrightarrow{\cong}\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}}.$$

In case the V_j, $j\in I''$ are all trivial, so that we have isomorphism $\mathbb{H}_\ell(V')_{A'\mathfrak{g}}\to\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}}$, then this isomorphism is equivariant for the flat logarithmic connections on either side relative to the collection of rank one R-modules $\det(F^1\operatorname{ann}A)$, $\{\mathfrak{m}_i/\mathfrak{m}_i^2\}_{i\in I'}$.

Proof. It suffices to do the case when I'' is a singleton $\{o\}$. Let us abbreviate $\tilde{\mathbb{H}}':=\tilde{\mathbb{H}}_\ell(\widehat{L'\mathfrak{g}}, V')$, $\tilde{\mathbb{H}}_o:=\tilde{\mathbb{H}}_\ell(\widehat{L_o\mathfrak{g}}, V_o)$ and $\tilde{\mathbb{H}}:=\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}}, V)$ (and similarly without the tildes).

Our assumption says that $A\to L_o/\mathcal{O}_o$ is onto. So $\widehat{L_o\mathfrak{g}}=A\mathfrak{g}+F^0\widehat{L_o\mathfrak{g}}$ and hence $U\widehat{L_o\mathfrak{g}}=UA\mathfrak{g}.UF^0\widehat{L_o\mathfrak{g}}$ by the PBW-theorem. This implies that $\tilde{\mathbb{H}}_o$ is generated as an $A\mathfrak{g}$-module by V_o. In other words, the natural map $\tilde{\mathbb{H}}'\otimes V_o\to\tilde{\mathbb{H}}_{A\mathfrak{g}}$ is onto. The kernel is clearly $A'\mathfrak{g}(\tilde{\mathbb{H}}'\otimes V_o)$ and so this yields the first isomorphism.

The same reasoning shows that the map $\big(\mathbb{H}'\otimes_R\tilde{V}_o\big)_{A'\mathfrak{g}}\to\big(\mathbb{H}'\otimes_R\tilde{\mathbb{H}}_o\big)_{A\mathfrak{g}}$ is an isomorphism. So in order that the second map be an isomorphism, the natural map

$$\big(\mathbb{H}'\otimes_R\tilde{\mathbb{H}}_o\big)_{A\mathfrak{g}}\to\big(\mathbb{H}'\otimes_R\mathbb{H}_o\big)_{A\mathfrak{g}}$$

must be one. It is clearly onto. Let $f\in A'$ map to a generator of $\mathfrak{m}_o$. First note that $\mathbb{H}_o$ is the quotient of $\tilde{\mathbb{H}}_o$ by $U(\widehat{L_o\mathfrak{g}})X_{\tilde\alpha}(1/f)^{\ell+1-\ell_{V_o}}V_o(\lambda_o)=U(A\mathfrak{g})X_{\tilde\alpha}(1/f)^{\ell+1-\ell_{V_o}}V_o(\lambda_o)$. This implies that the kernel of $\big(\mathbb{H}'\otimes_R\tilde{\mathbb{H}}_o\big)_{A\mathfrak{g}}\to\big(\mathbb{H}'\otimes_R\mathbb{H}_o\big)_{A\mathfrak{g}}$ is generated by the image of $\mathbb{H}'\otimes_R X_{\tilde\alpha}(1/f)^{\ell+1-\ell_{V_o}}V_o(\lambda_o)$ in $\big(\mathbb{H}'\otimes_R\tilde{\mathbb{H}}_o\big)_{A\mathfrak{g}}$. So it suffices to prove that for every $w'\in\mathbb{H}'$, $w'\otimes X_{\tilde\alpha}(1/f)^{\ell+1-\ell_{V_o}}V_o(\lambda_o)\in A\mathfrak{g}\big(\mathbb{H}'\otimes_R\tilde{\mathbb{H}}_o\big)$. By Proposition-Definition 5.4, the action of $X_{-\tilde\alpha}(f)$ in $\mathbb{H}'$ is locally nilpotent and so $X_{-\tilde\alpha}(f)^n w'=0$ for some $n\geq 0$. Corollary 6.14 tells us that $X_{\tilde\alpha}(1/f)^{\ell+1-\ell_{V_o}}V_o(\lambda_o)=X_{-\tilde\alpha}(f)^n w_o$ for some $w_o\in\tilde{\mathbb{H}}_o$. Since $X_{-\tilde\alpha}(f)\in A\mathfrak{g}$ acts on

$\big(\mathbb{H}' \otimes_R \tilde{\mathbb{H}}_o\big)_{A\mathfrak{g}}$ as $X_{-\tilde\alpha}(f)\otimes 1 + 1\otimes X_{-\tilde\alpha}(f)$ it follows that

$$\begin{aligned} w'\otimes X_{\tilde\alpha}(1/f)^{\ell+1-\ell_{V_o}} V_o(\lambda_o) &= w'\otimes X_{-\tilde\alpha}(f)^n w_o \\ &\equiv (-1)^n X_{-\tilde\alpha}(f)^n w'\otimes w_o \equiv 0 \pmod{A'\mathfrak{g}\big(\mathbb{H}'\otimes_R \tilde{\mathbb{H}}_o\big)}, \end{aligned}$$

as desired.

The second assertion follows in a straightforward manner from our definitions: just observe that a derivation D of A' which preserves R and each $\mathfrak{m}_j$, $j\in I''$ will act in $\tilde{\mathbb{H}}_\ell(\widehat{L_j\mathfrak{g}}, V_j) = \tilde{\mathbb{H}}_\ell(\widehat{L_j\mathfrak{g}}, k)$ as coefficientwise derivation. □

Remark 5.17. Even when not all the V_j , $j\in I''$, are trivial, the isomorphism in Proposition 5.16 can be used to transfer the flat logarithmic connection on $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)$ relative to the collection $\{\det(F^1 \operatorname{ann} A), \{\mathfrak{m}_i/\mathfrak{m}_i^2\}_{i\in I}\}$ to $\big(\mathbb{H}_\ell(V')\otimes_k V''\big)_{A'\mathfrak{g}}$. However, $\widehat{L_j\mathfrak{g}}$ then enters for $j\in I''$ in a nontrivial manner: a derivation D of A' which preserves R may not preserve $\mathfrak{m}_j$ and so we must write the action of D in $\hat{L}_j$ as a sum of a derivation which acts as coefficientwise derivation and a vertical Sugarawa part of which only the coefficient of $\mathfrak{g}t_j^{-1}\otimes\mathfrak{g}$ matters. We will see this illustrated in the genus zero case discussed below.

Remark 5.18. Proposition 5.16 is sometimes used in the opposite direction: suppose $\mathfrak{m}_o\subset A$ is a principal ideal with the property that for a generator $t\in\mathfrak{m}_o$, the $\mathfrak{m}_o$-adic completion of A gets identified with $R((t))$, and $A[1/t]\to L/\mathcal{O}$ is onto. Denote by A_o the kernel of $A[1/t]\to L/\mathcal{O}$. If we associate to o the trivial $\mathfrak{g}$-representation k and let $(I\sqcup\{o\},\{o\})$ take the role of (I, I'), we then find an isomorphism $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}}\cong(\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, k)\otimes V)_{A_o\mathfrak{g}}$. We will see this illustrated by the following genus zero example.

5.5. **The genus zero case and the KZ-connection.** We here assume our curve C to be isomorphic to $\mathbb{P}^1$. Let $x_1,\dots,x_n\in C$ be pairwise distinct. Choose an affine coordinate z on C whose domain contains these points and write z_i for $z(x_i)$. Notice that $t_\infty := z^{-1}$ may serve as a parameter for the local field at $z=\infty$. So if $\mathbb{H}_\ell(k)$ denotes the representation of $\widehat{\mathfrak{g}((z^{-1}))}$ attached to the trivial representation k of $\mathfrak{g}$, then by the special case of the propagation principle formulated in Remark 5.18 we have $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}} = (\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, k)\otimes V_1\otimes\cdots\otimes V_n)_{A'[z]}$, where $\mathfrak{g}[z]$ acts on V_i via its evaluation at x_i and $A'=k[z]$. Now $\tilde{\mathbb{H}}_\ell(k)$ is generated by $\mathfrak{g}[z] = A'\mathfrak{g}$-module with kernel $\mathfrak{g}\subset\mathfrak{g}[z]$ and so $(\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}}, k)\otimes V_1\otimes\cdots\otimes V_n)_{\mathfrak{g}[z]}$ is just the space of $\mathfrak{g}$-covariants $(V_1\otimes\cdots\otimes V_n)_{\mathfrak{g}}$. According to Proposition 5.4 $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, k)$ is obtained from $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}}, k)$ by dividing out by $\mathfrak{g}[z]X(z)^{1+\ell}$, where X is a generator of the highest root space $\mathfrak{g}_{\tilde\alpha}$, and so $(\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, k)\otimes V_1\otimes\cdots\otimes V_n)_{\mathfrak{g}[z]}$ can be identified with the biggest quotient of $(V_1\otimes\cdots\otimes V_n)_{\mathfrak{g}}$ on which $(\sum_{i=1}^n z_i X^{(i)})^{1+\ell}$ acts trivially (where $X^{(i)}$ acts on V_i as X and on the other tensor factors V_j, $j\neq i$, as the identity).

Now regard $z_1,\dots,z_n$ as variables and consider the open subset $U_n\subset\mathbb{A}^n_k$ of pairwise distinct n-tuples parametrized by these and take $R=k[U_n]=k[z_1,\dots,z_n][\Delta^{-1}]$, where $\Delta=\prod_{i<j}(z_i-z_j)$. Then the projective line over U_n, $\mathbb{P}^1_{U_n}$, comes with $n+1$ 'tautological' sections (including the one at infinity). For such a trivial family, the relative Zariski cotangent space of these sections are automatically trivialized (e.g., $z=z_i$ is trivialized by dz and $z=\infty$ by $-z^{-2}dz$). We denote the complement of the union of the these sections by $\mathcal{C}^\circ\subset\mathbb{P}^1_{U_n}$ and by A the R-module $k[\mathcal{C}^\circ]=R[z][f^{-1}]$, where $f=\prod_i(z-z_i)$. The preceding discussion shows

that we then have an exact sequence

$$R \otimes_k (V_1 \otimes \cdots \otimes V_n)_{\mathfrak{g}} \to R \otimes_k (V_1 \otimes \cdots \otimes V_n)_{\mathfrak{g}} \to \mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}} \to 0,$$

where the first map is given by $(\sum_{i=1}^n z_i X^{(i)})^{1+\ell}$. We identify its WZW connection, or rather, a natural lift of that connection to $R \otimes_k (V_1 \otimes \cdots \otimes V_n)_{\mathfrak{g}}$. In order to compute the covariant derivative with respect to the vector field $\partial_i := \frac{\partial}{\partial z_i}$ on U_n, we follow our recipe and lift it to $\mathbb{P}^1_{U_n}$ in the obvious way (with zero vertical component). We continue to denote that lift by $\tilde{\partial}_i$ and determine its (Sugawara) action on $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)$. We first observe that ∂_i is tangent to all the sections, except the ith. Near that section we decompose it as $(\frac{\partial}{\partial z} + \tilde{\partial}_i) - \frac{\partial}{\partial z}$, where the first term is tangent to the ith section and the second term is vertical. The action of the former is easily understood: its lift to $R \otimes_k (V_1 \otimes \cdots \otimes V_n)_{\mathfrak{g}}$ acts as derivation with respect to z_i (in R). The vertical term, $-\frac{\partial}{\partial z}$, acts via the Sugawara representation, that is, it acts on the ith slot $R \otimes V_i$ as $\frac{1}{\ell+\check{h}} \sum_\kappa X_\kappa (z - z_i)^{-1} \circ X_\kappa$ and as the identity on V_j, $j \neq i$, in other words, it acts as $\frac{1}{\ell+\check{h}} \sum_\kappa X_\kappa^{(i)}((z-z_i)^{-1}) X_\kappa^{(i)}$. This action does not induce one in $R \otimes_k (V_1 \otimes \cdots \otimes V_n)_{\mathfrak{g}}$. To make it so, we subtract from this the action by the following element of $A\mathfrak{g} U\widehat{\mathcal{L}\mathfrak{g}}$ (which of course will act trivially in $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}}$):

$$\sum_\kappa X_\kappa \Big(\frac{1}{z - z_i}\Big) \circ \frac{1}{\ell + \check{h}} X_\kappa^{(i)} = \frac{1}{\ell + \check{h}} \sum_{j,\kappa} X_\kappa^{(j)} \Big(\frac{1}{z - z_i}\Big) X_\kappa^{(i)}.$$

This modification does indeed act in $R \otimes_k (V_1 \otimes \cdots \otimes V_n)_{\mathfrak{g}}$, namely as

$$\frac{-1}{\ell + \check{h}} \sum_{j \neq i} \frac{1}{z_j - z_i} \sum_\kappa X_\kappa^{(j)} X_\kappa^{(i)} = \frac{1}{\ell + \check{h}} \sum_{j \neq i} \frac{1}{z_i - z_j} \sum_\kappa X_\kappa^{(j)} X_\kappa^{(i)}.$$

We regard the Casimir element C as an element of $\mathrm{Sym}^2 \mathfrak{g}$, so that $C^{(i,j)} := \sum_\kappa X_\kappa^{(j)} X_\kappa^{(i)}$ is its action in $V_1 \otimes \cdots \otimes V_n$ on the ith and jth factor (note that $C^{(i,j)} = C^{(j,i)}$). We find that the WZW-connection is induced by the connection on $R \otimes_k (V_1 \otimes \cdots \otimes V_n)_{\mathfrak{g}}$ whose connection form is

$$\sum_i \frac{1}{\ell + \check{h}} \sum_{j \neq i} \frac{dz_i}{z_i - z_j} C^{(i,j)} = \frac{1}{\ell + \check{h}} \sum_{1 \le i < j \le n} \frac{d(z_i - z_j)}{z_i - z_j} C^{(i,j)}.$$

It commutes with the Lie action of $\mathfrak{g}$ on $V_1 \otimes \cdots \otimes V_n$ and so the connection passes to one on $R \otimes (V_1 \otimes \cdots \otimes V_n)_{\mathfrak{g}}$. This lift of the WZW-connection is known as the *Knizhnik-Zamolodchikov connection*. It is not difficult to verify that it is flat (see for instance [11]), so that we have not just a projectively flat connection, but a genuine one.

Proposition 5.19. *The map $R \otimes_k (V_1 \otimes \cdots \otimes V_n)_{\mathfrak{g}} \to \mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}}$ is an isomorphism for $n = 1, 2$. Hence for $n = 1$ (resp. $n = 2$), $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}}$ is zero unless V_1 is the trivial representation (resp. V_1 and V_2 are equivalent to each others dual), in which case it can be identified with R.*

Proof. For $n = 1$ this is clear. For $n = 2$, $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}}$ can be identified with the image in $(V_1 \otimes V_2)_{\mathfrak{g}}$ of the kernel of $(z_1 X^{(1)} + z_2 X^{(2)})^{1+\ell}$ acting in $V_1 \otimes V_2$. Since $X^{(1)} + X^{(2)}$ is zero in $(V_1 \otimes V_2)_{\mathfrak{g}}$ and $(X^{(1)})^{1+\ell}$ is zero in V_1, this $(V_1 \otimes V_2)_{\mathfrak{g}}$. □

Remark 5.20. The case $n = 3$ is also special, because a 3-pointed genus zero curve $(C \cong \mathbb{P}^1; x_1, x_2, x_3)$ has no moduli. Our identification of $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}}$ shows that $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)_{A\mathfrak{g}}$ is naturally identified with the biggest quotient of $V_1\otimes V_2\otimes V_3$ on which both $\mathfrak{g}$ and the endomorphisms $(z_1X^{(1)}+z_2X^{(2)}+z_3X^{(3)})^{1+\ell}$ act trivially. But this should be independent of (z_1, z_2, z_3). Indeed, as is shown in [1], if V_1, V_2, V_3 are the associated irreducible $\mathfrak{g}$-representations of level $\leq \ell$, then $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)_C$ is naturally identified with the biggest quotient of $V_1\otimes V_2\otimes V_3$ on which both $\mathfrak{g}$ and the endomorphisms $(z_1X^{(1)} + z_2X^{(2)} + z_3X^{(3)})^{1+\ell}$ act trivially for *all* values of (z_1, z_2, z_3). This last condition is of course equivalent to requiring that $X^p \otimes X^q \otimes X^r$ induces the zero map in $V_1 \otimes V_2 \otimes V_3$ whenever $p + q + r > \ell$.

When $\mathfrak{g} = \mathfrak{sl}(2)$, V_i will be equivalent to n_ith symmetric power of the defining representation for some n_i and then the Clebsch-Gordan rule says that $(V_1 \otimes V_2 \otimes V_3)_{\mathfrak{sl}(2)}$ is of dimension ≤ 1, with equality if and only if $n_1 + n_2 + n_3$ is even, say $2m$, and $n_i \leq m$ for all i. As is shown in [1], the biggest quotient of $V_1 \otimes V_2 \otimes V_3$ on which $\mathfrak{sl}(2)$ and $(z_1X^{(1)} + z_2X^{(2)} + z_3X^{(3)})^{1+\ell}$ act trivially is nonzero (and hence of dimension one) if and only if in addition $m \leq \ell$.

5.6. **Topological interpretation.** Let Σ be a closed oriented surface. Denote the genus of Σ by $g(\Sigma)$ and by $\mathcal{L}(\Sigma) \subset \wedge^{g(\Sigma)}H_1(\Sigma;\mathbb{R})$ the set of $L \in \wedge^g H_1(\Sigma;\mathbb{R})$ for which $K(L) := \ker(\wedge L : H_1(\Sigma;\mathbb{R}) \to \wedge^{g(\Sigma)+1}H_1(\Sigma;\mathbb{R}))$ is a Lagrangian subspace (so that L is a generator of $\wedge^{g(\Sigma)}K(L)$). Let us first assume that Σ is connected. It is known that if $g(\Sigma) > 0$, then $\mathcal{L}(\Sigma)$ is connected, has infinite cyclic fundamental group with a canonical generator and is an orbit of the symplectic group $\mathrm{Sp}(H_1(\Sigma;\mathbb{R}))$. For example, if $g(\Sigma) = 1$, then $\mathcal{L}(\Sigma) = H_1(\check{\Sigma};\mathbb{R}) - \{0\} \cong \mathbb{R}^2 - \{0\}$. (In the uninteresting case $g(\Sigma) = 0$, we have $\wedge^{g(\Sigma)}H_1(\Sigma;\mathbb{R}) = \wedge^0\{0\} = \mathbb{R}$ and so $\mathcal{L}(\Sigma)$ is then canonically identified with $\mathbb{R} - \{0\}$, in particular, $\mathcal{L}(\Sigma)$ contains a canonical element.) An element of $\mathcal{L}(\Sigma)$ may arise if Σ is given as the boundary of a compact oriented 3-manifold W: then the kernel of $H_1(\Sigma;\mathbb{Z}) \to H_1(W;\mathbb{Z})$ is a Lagrangian sublattice and so an orientation of it yields an element of $\mathcal{L}(\Sigma)$.

Given a 2-dimensional real vector space T, we denote by $S(T)$ its sphere of rays. Fix a finite set I.

Definition 5.21. An *I-marking* of Σ consists of the following data: an injection $x : I \hookrightarrow \Sigma$, $i \in I \mapsto x_i \in \Sigma$, for every $i \in I$ a universal cover $\tilde{S}_i \to S(T_{x(i)}\Sigma)$ of the sphere of rays in $T_{x(i)}\Sigma$ and the choice of a universal cover $\tilde{\mathcal{L}} \to \mathcal{L}(\Sigma)$. These are the objects of a groupoid for which a morphism $(\Sigma, \tilde{S}, \tilde{\mathcal{L}}) \to (\Sigma', \tilde{S}', \tilde{\mathcal{L}}')$ is given by a system $(h, (\tilde{h}_i)_i, \tilde{h}_L)$, where h is an orientation preserving diffeomorphism $h : \Sigma \cong \Sigma'$ which takes x_i to x'_i, and $\tilde{h}_i : \tilde{S}_i \to \tilde{S}'_i$ resp. $\tilde{h}_L : \tilde{\mathcal{L}} \to \tilde{\mathcal{L}}'$ are lifts of the natural maps induced by h, where it is understood that another such system defines the same morphism if it is isotopic to $(h, (\tilde{h}_i)_i, \tilde{h}_L)$ in the sense that it lies in the same arc component of the space such pairs.

Remark 5.22. Instead of choosing a universal cover $\tilde{S}_i \to S(T_{x(i)}\Sigma)$, we could fix a ray in $T_{x(i)}\Sigma$, for this ray can then be taken as a base point of $S(T_{x(i)}\Sigma)$ and thus define a canonical universal cover of it. While that may be more down to earth, the definition above is a bit more flexible.

Remark 5.23. The automorphism group of the I-marked surface $(\Sigma, \tilde{S}, \tilde{\mathcal{L}})$, denoted here by $\Gamma(\Sigma, \tilde{S}, \tilde{\mathcal{L}})$, is a central extension of the usual mapping class group $\Gamma(\Sigma, x)$ of the pair (Σ, x): the forgetful group homomorphism $\Gamma(\Sigma, \tilde{S}, \tilde{\mathcal{L}}(\Sigma)) \to \Gamma(\Sigma, x)$ is onto and fits in a short exact sequence

$$\mathbb{Z}^I \times \mathbb{Z} \to \Gamma(\Sigma, \tilde{S}, \tilde{\mathcal{L}}) \to \Gamma(\Sigma, S) \to \{1\},$$

where $\mathbb{Z}^I \times \mathbb{Z}$ is in a natural manner identified with the group of covering transformations of $(\prod_i \tilde{S}_i) \times \mathcal{L} \to \prod_i S(T_{x(i)}\Sigma) \times \mathcal{L}(\Sigma)$. The latter has image contained in the center of $\Gamma(\Sigma, \tilde{S})$

and is injective unless $g(\Sigma) = 0$: in that case the last $\mathbb{Z}$-summand maps to zero and the map $\mathbb{Z}^I \to \Gamma(\Sigma, \tilde{S})$ is injective when $|I| > 2$, has kernel the sum of the two generators when $|I| = 2$ and is zero when $|I| \leq 1$.

A conformal structure $\mathfrak{c}$ on Σ yields in combination with the given orientation a complex structure on Σ and thus endows it with the structure of a complex projective curve that we shall denote by $\Sigma(\mathfrak{c})$.

Lemma 5.24. *Let $L \in \mathcal{L}(\Sigma)$. Then L determines for every conformal structure $\mathfrak{c}$ on $\Sigma(\mathfrak{c})$ a generator of* $\det H^0(\Sigma(\mathfrak{c}), \omega_{\Sigma(\mathfrak{c})})$.

Proof. We note that every regular differential on $\Sigma(\mathfrak{c})$ defines by integration a linear map $K(L) \to \mathbb{C}$ and the basic theory of Riemann surfaces tells us that we thus obtain a complex-linear isomorphism $H^0(\Sigma(\mathfrak{c}), \omega_{\Sigma(\mathfrak{c})}) \cong \mathrm{Hom}(K(L), \mathbb{C})$. This induces an isomorphism between $\det H^0(\Sigma(\mathfrak{c}), \omega_{\Sigma(\mathfrak{c})})$ and $\mathrm{Hom}_{\mathbb{R}}(\det_{\mathbb{R}} K(L), \mathbb{C}) = \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}L, \mathbb{C}) \cong \mathbb{C}$. □

Theorem 5.25. *Let $k = \mathbb{C}$. Suppose we are given a nonnegative integer ℓ and let $V : i \in I \mapsto V_i$ assign to each $i \in I$ a finite dimensional irreducible representation V_i of $\mathfrak{g}$ of level $\leq \ell$. Then there exists a functor $H^\ell(-; V)$ from the groupoid of I-marked surfaces to the category of complex vector spaces with the property that for every such surface $(\Sigma, \tilde{S}, \tilde{\mathcal{L}})$ and every conformal structure $\mathfrak{c}$ on Σ we have a natural identification of $H^\ell(\Sigma, \tilde{S}, \tilde{\mathcal{L}}; V)$ with the associated vacuum space $\mathbb{H}^\ell(V)^{A_\mathfrak{c}\mathfrak{g}}$, where $A_\mathfrak{c}$ denotes the algebra of complex valued functions on $\Sigma - x(I)$ that are meromorphic relative to $\mathfrak{c}$. In particular, if Σ is diffeomorphic to S^2, then $H^\ell(\Sigma, \tilde{S}; V)$ is trivial for $|I| = 1$ and $|I| = 2$ unless V is the trivial representation (in which case we get a vector space canonically isomorphic to $\mathbb{C}$) resp. the two representations are dual to each other (in which case this vector space is of dimension one). The canonical generator of the covering transformation in $\tilde{S}_i \to S(T_{x(i)}\Sigma)$ resp. $\tilde{\mathcal{L}} \to \mathcal{L}(\Sigma)$ acts as scalar multiplication by $\exp\big(\frac{\pi\sqrt{-1}}{\ell+\check{h}} C_{V_i}\big)$ resp. $\exp\big(\frac{\pi\sqrt{-1}}{\ell+\check{h}} \ell \dim \mathfrak{g}\big)$.*

If $I' \subset I$ is such that V_i is the trivial representation for $i \in I - I'$, then we have a natural isomorphism $H^\ell(\Sigma, \tilde{S}, \tilde{\mathcal{L}}; V) \cong H^\ell(\Sigma, \tilde{S}|I', \tilde{\mathcal{L}}; V|I')$.

This construction is also functorial with respect to automorphisms of $\mathfrak{g}$: inner automorphisms acts as the identity and for every $\sigma \in \mathrm{Out}(\mathfrak{g})$ we have a natural isomorphism $H^\ell(\Sigma, \tilde{S}, \tilde{\mathcal{L}}; {}^\sigma V) \cong H^\ell(\Sigma, \tilde{S}, \tilde{\mathcal{L}}; V)$.

Proof. A conformal structure on Σ assigns to each tangent space of Σ an inner product up to scalar. The inner products up to scalar on a 2-dimensional real vector space are parametrized by a Lobatchewski disk and hence a conformal structure is just a section of a disk bundle over Σ. This shows that the space of conformal structures on Σ is contractible. An arc in the space of conformal structures yields a family of compact Riemann surfaces $\{C_s\}_{0 \leq s \leq 1}$. If we pull this family back over the projection

$$[0,1] \times \tilde{\mathcal{L}} \times \prod_{i \in I} \big(\tilde{S}_i \times_{S_i} (T_{x(i)}\Sigma - \{0\})\big) \to [0,1],$$

then Lemma 5.24 enables us to trivialize the associated determinant bundle, and the section defined by x_i comes with a tautological normal vector field. This is enough to have the WZW-connection in Corollary 5.14 trivialize the associated vacua bundle. Since the base space is contractible, it follows that we have a natural identification between the two vacua spaces. The proposition follows from this. □

Remark 5.26. We may drop the assumption that Σ be connected and let $\{\Sigma^j\}_{j\in J}$ enumerate its distinct connected components. If $\tilde{\mathcal{L}}^{(j)} \to \mathcal{L}(\Sigma^{(j)})$ is a universal cover, then the dependence of the tensor product $(\otimes_{\mathbb{C}})_{j\in J} H^\ell(\Sigma^{(j)}, \tilde{S}^{(j)}, \tilde{\mathcal{L}}^{(j)}; V^{(j)})$ on the covers $\tilde{\mathcal{L}}^{(j)} \to \mathcal{L}(\Sigma^{(j)})$ is only through their 'tensor product' $\tilde{\mathcal{L}} \to \mathcal{L}(\Sigma)$. So the preceding generalizes if we let $H^\ell(\Sigma, \tilde{x}, \tilde{\mathcal{L}}; V)$ be this tensor product.

Remark 5.27. The natural involution of $\mathfrak{g}$ with respect to a choice of root data takes every finite dimensional $\mathfrak{g}$-representation into one equivalent to its contra-gradient. Such an involution σ is unique up to inner automorphism and so we obtain a canonical isomorphism between $H_\ell(\Sigma, \tilde{S}, \tilde{\mathcal{L}}; V^*)$ and $H_\ell(\Sigma, \tilde{S}, \tilde{\mathcal{L}}; V)$. One expects that there exists a canonical perfect pairing $H_\ell(-\Sigma, \tilde{S}, \tilde{\mathcal{L}}; V^*) \otimes H_\ell(\Sigma, \tilde{S}, \tilde{\mathcal{L}}, V) \to \mathbb{C}$, where $-\Sigma$ stands for Σ with the opposite orientation.

For $g = 0$, we have a natural identification $\mathcal{L}(\Sigma) \cong \mathbb{R} - \{0\}$ so that we then have a well-defined vector space $H_\ell(\Sigma, V)$. Proposition 5.19 tells us what we get in some simple cases:

Proposition 5.28. *For Σ a disk (resp. a cylinder), $H_\ell(\Sigma, \tilde{S}; V) := H_\ell(\Sigma, \tilde{S}, \tilde{\mathcal{L}}; V)$ is zero unless V is the trivial representation (resp. the two representations attached to the boundary are each other's contra-gradient), in which case it is canonically equal to $\mathbb{C}$.*

6. Vacua modules in terms of polydifferentials

6.1. **Polydifferentials.** Let N be a nonempty finite set with cardinality n. Let U be a nonsingular curve over k. For every $r \in N$ we denote by $\pi_r : U^N \to U$ the corresponding projection. If $\mathcal{F}$ is a coherent sheaf on U, then we abbreviate the exterior tensor product on U^N (in the category of coherent sheaves) by $\otimes_{r\in N}\pi_r^*\mathcal{F}$ by $\mathcal{F}^{(N)}$. Notice that the permutation group $\mathcal{S}(N)$ acts on this sheaf. It is clear that $\mathcal{O}_U^{(N)} = \mathcal{O}_{U^N}$. On the other hand, we have a natural, $\mathcal{S}(N)$-equivariant, identification of $\Omega_U^{(N)}$ with $\Omega^n_{U^N} \otimes \mathrm{or}(N)$, where $\mathrm{or}(N)$ stands for the orientation module $\wedge^N\mathbb{Z}^N$. We define the *sheaf of polydifferentials* on U^N as the sheaf of graded $\mathcal{O}_{U^N}$-algebras $(\Omega_U^\bullet)^{(N)}$. Observe that here is a natural decomposition

$$\Omega_U^{\bullet}{}^{(N)} = \bigoplus_{J\subset N} \pi_J^*\Omega_U{}^{(J)},$$

where $\pi_J : U^N \to U^Y$ is the evident projection. For $\zeta \in \Omega_U^{\bullet}{}^{(N)}$, we write ζ^J for its component in $\pi_J^*\Omega_U^{(J)}$. Residue operators that involve a single factor have a meaning for polydifferentials. So if U is Zariski open in a nonsingular curve C, $p \in C - U$ and $i \in N$, then we have defined residue $\mathrm{Res}_{\pi_i=p} : \Omega_U^{(N)} \to \Omega_U^{(N-\{i\})}$.

There is an obvious extension of this notion to a relative situation $\mathcal{C}/S$ and its semi-local variants $\mathcal{O}/R$ and L/R.

6.2. **Residue maps for polydifferentials.** Suppose we are in the local case: $\mathcal{O} \cong R[[t]]$, where t is a generator of $\mathfrak{m}$ so that $L \cong R((t))$. Then $L^{(N)} \cong R[[\{t_r\}_{r\in N}]][\prod_r t_r^{-1}]$ and $\omega^{(N)} = L^{(N)}\prod_r dt_r$. We write $\mathbb{D} := \mathrm{Spec}(\mathcal{O})$ an $\mathbb{D}^* := \mathrm{Spec}(L)$.

For N as above, we have for every $r \in N$ an obviously defined residue in the ith factor:

$$\mathrm{Res}_i : \omega^{(N)} \to \omega^{(N-\{i\})}.$$

If $i, j \in N$ are distinct, then Res_i and Res_j commute, in contrast to the case of differential forms, where they anticommute. Now let $\Delta_{ij} \subset \mathbb{D}^N$ be the diagonal hyperplane defined by the factors i, j and consider the obvious extension

$$\mathrm{Res}_i : \omega^{(N)}(*\Delta_{ij}) \to \omega^{(N-\{i\})},$$

where $(*\Delta_{ij})$ means that we allow poles of arbitrary order along Δ_{ij}. Then Res_i and Res_j no longer commute:

Lemma 6.1. *On $\omega^{(N)}(*\Delta_{ij})$ we have $[\mathrm{Res}_i, \mathrm{Res}_j] = -\mathrm{Res}_i \mathrm{Res}_{j\to i}$, where*

$$\mathrm{Res}_{j\to i} : \omega^{(N)}(*\Delta_{ij}) \to \omega^{(N-\{i\})}(*\Delta_{ij})$$

is the residue at the diagonal divisor Δ_{ij} relative to the projection which forgets the jth component.

Proof. It suffices to check this for $N = \{1, 2\}$ and $(i, j) = (2, 1)$. Choose a generator t of $\mathfrak{m}$. Then any $\zeta \in \omega^{(2)}(*\Delta_{ij})$ can be written $\zeta = (t_2 - t_1)^{-r} \sum_{k_1,k_2} a_{k_1,k_2} t_1^{k_1-1} t_2^{k_2-1} dt_1 dt_2$ for some $r \geq 0$. Let us concentrate on an individual term: $\zeta = (t_2 - t_1)^{-r} t_1^{k_1-1} t_2^{k_2-1} dt_1 dt_2$. We expand ζ near $t_1 = 0$:

$$\begin{aligned}\zeta &= (t_2 - t_1)^{-r} t_1^{k_1-1} t_2^{k_2-1} dt_1 dt_2 = t_2^{k_2-r-1}(1 - t_1/t_2)^{-r} t_1^{k_1-1} dt_1 dt_2 = \\ &= t_2^{k_2-r-1} t_1^{k_1-1} \sum_{n\geq 0} \binom{n+r-1}{n} (t_1/t_2)^n dt_1 dt_2 = \sum_{n\geq 0} \binom{n+r-1}{n} t_2^{k_2-r-n-1} t_1^{k_1+n-1} dt_1 dt_2.\end{aligned}$$

The residue in $t_1 = 0$ is zero unless $k_1 \leq 0$, in which case we get $\binom{r-k_1-1}{-k_1} t_2^{k_1+k_2-r-1} dt_2$. The residue of the latter is zero, unless $k_1 + k_2 = r$, in which case we get $\binom{r-k_1-1}{-k_1} = \binom{r-k_1-1}{r-1}$. So $\mathrm{Res}_1 \mathrm{Res}_2 \zeta$ is zero unless $r = k_1 + k_2$ and $k_1 \leq 0$, in which case we get $\binom{r-k_1-1}{r-1}$. Similarly, $\mathrm{Res}_1 \mathrm{Res}_2 \zeta$ is zero unless $r = k_1 + k_2$ and $k_2 \leq 0$: we then get $(-1)^r \binom{r-k_2-1}{r-1} = (-1)^r \binom{k_1-1}{k_1-r}$. These two cases exclude each other: $[\mathrm{Res}_2, \mathrm{Res}_1]\zeta$ is zero unless $r = k_1 + k_2$ and $k_1 \leq 0$ or $k_1 \geq r$, in which case we get $\binom{r-k_1-1}{r-1}$ resp. $(-1)^{r-1} \binom{k_1-1}{k_1-r}$.

In order to compute $\mathrm{Res}_{1\to 2} \zeta$, we put $s := t_2 - t_1$ so that we can write $\mathrm{Res}_{1\to 2} \zeta = \mathrm{Res}_{s=0} s^{-r} t_2^{k_2-1} (t_2 - s)^{k_1-1} d(-s) dt_2$. When $k_1 \geq r - 1$, this is $-(-1)^{r-1} \binom{k_1-1}{k_1-r} t_2^{k_2-1+k_1-r} dt_2$, whose residue is zero unless $k_1 + k_2 = r$, in which case we get $(-1)^r \binom{k_1-1}{k_1-r}$. When $1 \leq k_1 < r - 1$ we get zero. When $k_1 \leq 0$, we write

$$\begin{aligned}&- s^{-r} t_2^{k_2-1} (t_2 - s)^{k_1-1} ds dt_2 = -s^{-r} t_2^{k_1+k_2-2} (1 - s/t_2)^{k_1-1} ds dt_2 = \\ &\qquad = -s^{-r} t_2^{k_1+k_2-2} \sum_{n\geq 0} \binom{n-k_1}{n} (s/t_2)^n ds dt_2 = -\sum_{n\geq 0} \binom{n-k_1}{n} t_2^{k_1+k_2-2-n} s^{n-r} ds dt_2.\end{aligned}$$

Its residue in $s = 0$ is $-\binom{r-1-k_1}{r-1} t_2^{k_1+k_2-r-1} dt_2$ and the latter's residue is zero unless $k_1 + k_2 = r$ in which case we get $-\binom{r-k_1-1}{r-1} = -\binom{r-k_1-1}{-k_1}$. □

Two residues. We now focus on the case when the pole order along Δ_{ij} is at most 2. Consider the transposition involution σ_{ij} in $\mathcal{O}^{(N)}$ which interchanges the factors i and j. Then $\mathcal{O}^{(N)}(2\Delta_{ij})/\mathcal{O}^{(N)}$ is a sheaf of R-algebras of length 2 on Δ_{ij} and comes with an action of σ_{ij}. The line subbundle $\mathcal{O}^{(N)}(\Delta_{ij})/\mathcal{O}^{(N)}$ is precisely the (-1) eigensubmodule of σ_{ij}, so that $\mathcal{O}^{(N)}(2\Delta_{ij})/\mathcal{O}^{(N)}$ splits into modules of rank one:

$$\mathcal{O}^{(N)}(2\Delta_{ij})/\mathcal{O}^{(N)} = \mathcal{O}^{(N)}(\Delta_{ij})/\mathcal{O}^{(N)} \oplus \left(\mathcal{O}^{(N)}(2\Delta_{ij})/\mathcal{O}^{(N)}\right)^{\sigma_{ij}}.$$

We get a corresponding splitting of $\omega^{(N)}(2\Delta_{ij})/\omega^{(N)}$:

$$\omega^{(N)}(2\Delta_{ij})/\omega^{(N)} = \omega^{(N)}(\Delta_{ij})/\omega^{(N)} \oplus \big(\omega^{(N)}(2\Delta_{ij})/\omega^{(N)}\big)^{\sigma_{ij}}.$$

We use this splitting to define

$$\mathcal{R}_{ij} = (\mathcal{R}'_{ij}, \mathcal{R}''_{ij}) : \omega^{(N)}(2\Delta_{ij}) \to \omega^{(N-\{j\})} \oplus \omega^{(N-\{i,j\})}$$

as the reduction

$$\omega^{(N)}(2\Delta_{ij}) \to \omega^{(N)}(2\Delta_{ij})/\omega^{(N)} = \omega^{(N)}(\Delta_{ij})/\omega^{(N)} \oplus \big(\omega^{(N)}(2\Delta_{ij})/\omega^{(N)}\big)^{\sigma_{ij}}$$

postcomposed with the map which on the first summand is equal to $\mathrm{Res}_{j\to i}$ and which on the second summand is $\mathrm{Res}_i \mathrm{Res}_{j\to i}$. If we let Δ_N denote the union of the diagonal divisors Δ_{kl}, $k, l \in N$ distinct, then $\mathcal{R}_{ij}$ extends in an obvious manner to

$$\mathcal{R}_{ij} = (\mathcal{R}'_{ij}, \mathcal{R}''_{ij}) : \omega^{(N)}(2\Delta_N) \to \omega^{(N-\{i\})}(\Delta_{N-\{j\}}) \oplus \omega^{(N-\{i,j\})}(2\Delta_{N-\{i,j\}}).$$

Notice that $\mathcal{R}''_{ij} = -\mathcal{R}''_{ji}$. According to Lemma 6.1 we have $\omega^{(N)}(2\Delta_N)$

$$[\mathrm{Res}_i, \mathrm{Res}_j] = -\mathrm{Res}_i \mathcal{R}'_{ij} - \mathcal{R}''_{ij}\,.$$

The basic computation. Let us compute these residue operators in terms of coordinates (where we take $N = \{1, 2\}$ and $(i, j) = (2, 1)$ for simplicity). We have for $f_1, f_2 \in L$,

$$(\dagger)\qquad \mathcal{R}'_{21}\Big(\frac{f_1(z_1)f_2(z_2)dz_1dz_2}{z_2 - z_1}\Big) = \mathrm{Res}_{1\to 2}\frac{f_1(z_1)f_2(z_2)dz_1dz_2}{z_2 - z_1} = -f_1(z_2)f_2(z_2)dz_2,$$

which is indeed symmetric in f_1 and f_2 (even as a differential if we think of z_2 as a coordinate on Δ_{12}) and

$$(\ddagger)\quad \mathcal{R}''_{21}\Big(\frac{f_1(z_1)f_2(z_2)dz_1dz_2}{(z_2 - z_1)^2}\Big) = \mathrm{Res}_2\mathrm{Res}_{1\to 2}\frac{f_1(z_1)f_2(z_2)dz_1dz_2}{(z_2 - z_1)^2} =$$

$$= \mathrm{Res}_2 f_2(z_2)\big(\mathrm{Res}_{1\to 2}\frac{f_1(z_1)dz_1}{(z_2 - z_1)^2}\big)dz_2 = \mathrm{Res}_2 f_2(z_2)f_1'(z_2)dz_2 = \mathrm{Res}(f_2df_1) = -\mathrm{Res}(f_1df_2).$$

6.3. **Moderate poles.** An element of $(F^0\omega)^{(N)}(2\Delta_N)$ has by definition a denominator of the form $\prod_i z_i \prod_{i\neq j}(z_i - z_j)$. We will here be interested in a submodule for which the polar part is milder. Let us first note that a subproduct of $\prod_i z_i \prod_{i\neq j}(z_i - z_j)$ can be completely described in terms of a graph with vertex set N that has some double bonds and has some of its vertices marked: draw an edge connecting i and j with the same multiplicity as the factor $z_i - z_j$ appears in the subproduct and mark the i-th vertex if z_i is a factor. We now let $(F^0\omega)^{(N)}(\log \Delta_N)$ resp. $(F^0\omega)^{(N)}(2\log \Delta_N)$ be the submodule spanned by the elements in $(F^0\omega)^{(N)}(2\Delta_N)$ whose denominator is a subproduct of $\prod_i z_i \prod_{i\neq j}(z_i - z_j)$ for which the associated marked graph has the property that each connected component is a tree resp. a tree or a polygon (so a double bond can here only appear as a polygon with two vertices) and that any marked vertex has degree one (and hence can only appear on a tree component). Notice that our residue maps preserve these submodules. In fact, our residue operators Res_i, $\mathcal{R}'_{ji}$, $\mathcal{R}''_{ji}$ affect the graph as follows: Res_i contracts the unique edge containing the marked vertex i, $\mathcal{R}'_{ji}$ contracts the simple edge connecting i and j, and $\mathcal{R}''_{ji}$ removes the connected component with two vertices labeled i, j (in case this operation does not apply, all edges and markings are removed). The module $(F^0\omega)^{(N)}(\log \Delta_N)$ can also be understood as the elements for which the polar part stays reduced after arbitrary blowing up. It is clear that both modules are $\mathcal{S}(N)$-invariant.

6.4. **The dual of a representation.** We will apply the preceding to the cases when $N = \{1, \dots, n\}$ and write then n for N. We give here an alternative definition of vacua modules and then show that this definition coincides with the one we gave earlier.

Let ℓ be a nonnegative integer and V be a finite dimensional irreducible representation of $\mathfrak{g}$ of level $\ell_V \le \ell$. Denote by $\widetilde{\mathbb{H}}^\ell(\widehat{L\mathfrak{g}}, V^*)$ the space of sequences of equivariant linear maps

$$\zeta_\bullet = \big(\zeta_n \in \mathrm{Hom}_{k[\mathcal{S}_n]}(\mathfrak{g}^{\otimes n} \otimes V, (F^0\omega)^{(n)}(2\log\Delta_n))\big)_{n=0}^\infty$$

(so $\zeta_0 \in V^*$) satisfying the following two properties:

(i) for $n \ge 1$,

$$\mathrm{Res}_1\, \zeta_n(Y_n \otimes \cdots \otimes Y_1 \otimes v) = \zeta_{n-1}(Y_n \otimes \cdots \otimes Y_2 \otimes Y_1 v),$$

(ii) For $n \ge 2$,

(ii$_a$) $$\mathcal{R}'_{n,n-1}\, \zeta_n(Y_n \otimes \cdots \otimes Y_1 \otimes v) = \zeta_{n-1}([Y_n, Y_{n-1}] \otimes Y_{n-2} \otimes \cdots \otimes Y_1 \otimes v),$$

(ii$_b$) $$\mathcal{R}''_{n,n-1}\, \zeta_n(Y_n \otimes \cdots \otimes Y_1 \otimes v) = \ell\check{c}(Y_n \otimes Y_{n-1})\zeta_{n-2}(Y_{n-2} \otimes \cdots \otimes Y_1 \otimes v),$$

Note that these conditions completely describe the polar part of ζ_n in terms of its predecessors. On the other hand, given these predecessors, then two choices of ζ_n can differ by an arbitrary $k[\mathcal{S}_n]$-homomorphism from $\mathfrak{g}^{\otimes n} \otimes V$ to $(F^1\omega)^{(n)}$.

Example 6.2. Let us work this out for $n = 1$ and $n = 2$. Clearly, $\zeta_0 \in V^*$. We have that

$$\zeta_1(Y \otimes v) = \zeta_0(Yv)\frac{dz}{z} + \eta_1(Y \otimes v)$$

with $\eta_1(Y \otimes v) \in F^1\omega$ so that the polar part of $\zeta_1(Y \otimes v)$ is simply $\zeta_0(Yv)\frac{dz}{z}$. Likewise the polar part of $\zeta_2(Y_2 \otimes Y_1 \otimes v)$ is computed as follows:

$$\zeta_2(Y_2 \otimes Y_1 \otimes v) \equiv$$
$$\frac{\ell\check{c}(Y_2 \otimes Y_1)\zeta_0(v)dz_1dz_2}{(z_1-z_2)^2} + \frac{\zeta_1([Y_2,Y_1] \otimes v)(z_2)dz_1}{z_1-z_2} + \zeta_1(Y_2 \otimes Y_1 v)(z_2)\frac{dz_1}{z_1} + \zeta_1(Y_1 \otimes Y_2 v)(z_1)\frac{dz_2}{z_2},$$

where we used the symmetry property of ζ_2. We determine the residues:

$$\mathrm{Res}_{1\to 2}\, \zeta_2(Y_2 \otimes Y_1 \otimes v) = \zeta_1([Y_2, Y_1] \otimes v)(z_2),$$

so that $\mathrm{Res}_2 \mathrm{Res}_{1\to 2}\, \zeta_2(Y_2 \otimes Y_1 \otimes v) = \zeta_0([Y_2, Y_1]v)$. Next we observe that

$$\mathrm{Res}_1\, \zeta_2(Y_2 \otimes Y_1 \otimes v) = \zeta_1(Y_2 \otimes Y_1 v)(z_2) + \zeta_0(Y_1Y_2v)\frac{dz_2}{z_2}$$

so that $\mathrm{Res}_2 \mathrm{Res}_1(Y_2 \otimes Y_1 \otimes v) = \zeta_0(Y_2Y_1v + Y_1Y_2v)$. Finally,

$$\begin{aligned}
&\mathrm{Res}_2\, \zeta_2(Y_2 \otimes Y_1 \otimes v) = \\
&\quad \mathrm{Res}_2 \Big(\frac{\zeta_1([Y_2,Y_1] \otimes v)(z_2)dz_1}{z_1-z_2} + \zeta_1(Y_2 \otimes Y_1 v)(z_2)\frac{dz_1}{z_1} + \zeta_1(Y_1 \otimes Y_2 v)(z_1)\frac{dz_2}{z_2}\Big) \\
&\quad = \zeta_0([Y_2,Y_1]v)\frac{dz_1}{z_1} + \zeta_0(Y_2Y_1v)\frac{dz_1}{z_1} + \zeta_1(Y_1 \otimes Y_2 v)(z_1) \\
&\quad = \zeta_0(2Y_2Y_1v)\frac{dz_1}{z_1} + \eta_1(Y_1 \otimes Y_2 v)(z_1),
\end{aligned}$$

so that $\mathrm{Res}_1 \mathrm{Res}_2(Y_2 \otimes Y_1 \otimes v) = \zeta_0(2Y_2Y_1v)$.

Note that this shows that $(z_1-z_2)^{-1}\zeta_1([Y_2,Y_1]\otimes v)(z_2)dz_1$ accounts for the polar part of $\zeta_2(Y_2\otimes Y_1\otimes v)$ that only depends on $Y_1\wedge Y_2$; the remaining sum only depends on the symmetric part of $Y_1\otimes Y_2$.

Let $\zeta_\bullet\in\tilde{\mathbb{H}}^\ell(\widehat{L\mathfrak{g}},V^*)$. We extend ζ_n to an R-linear map:

$$\zeta_n : L\mathfrak{g}^{\otimes n}\otimes V\to\omega^{(n)}(2\log\Delta_n)$$

by $\zeta_n(X_n(f_n)\otimes\cdots\otimes X_1(f_1)\otimes v):=\pi_n^*f_n\cdots\pi_1^*f_1\zeta_n(X_n\otimes\cdots\otimes X_1\otimes v)$. The following proposition has its origin in a theorem of Beilinson-Drinfeld [2]:

Proposition 6.3. *The pairing* $\tilde{\mathbb{H}}^\ell(\widehat{L\mathfrak{g}},V^*)\times(L\mathfrak{g})^{\otimes_R\bullet}\otimes_k V\to R$*,*

$$\big\langle\zeta_\bullet|X_n(f_n)\otimes\cdots\otimes X_1(f_1)\otimes v\big\rangle:=\operatorname{Res}_{z_n=0}\cdots\operatorname{Res}_{z_1=0}\pi_n^*f_n\cdots\pi_1^*f_1\zeta_n(X_n\otimes\cdots\otimes X_1\otimes v),$$

drops to a pairing

$$\langle\ |\ \rangle:\tilde{\mathbb{H}}^\ell(\widehat{L\mathfrak{g}},V^*)\times\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}},V)\to R$$

which is topologically perfect: it identifies $\tilde{\mathbb{H}}^\ell(\widehat{L\mathfrak{g}},V)$ *with the topological dual of* $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}},V)$*.*

Proof. We first verify that the pairing is well-defined. The PBW theorem and the symmetry properties of $\zeta_\bullet$ tell us that it suffices to check that for $v\in V$, $X,Y\in\mathfrak{g}$, $\xi\in(L\mathfrak{g})^{\otimes m}$ and $\eta\in(L\mathfrak{g})^{\otimes n}$ we have

$$\langle\zeta_\bullet|\eta\otimes(Y\otimes v-Yv)\rangle=0$$

and

$$\big\langle\zeta_\bullet|\eta\otimes\big(Y_2f_2\otimes Y_1f_1-Y_1f_1\otimes Y_2f_2-[X_2,Y_1]f_2f_1-\check{c}(Y_2,Y_1)\operatorname{Res}_0 f_1df_2.c\big)\otimes\xi\big\rangle=0.$$

The first assertion easily reduces the case $\eta=1$ and is then immediate from the ordinary residue theorem. Similarly for the proof of the second identity we may without loss of generality assume that $\eta=1=\xi$. Then the desired property follows from

$$\begin{aligned}
&\operatorname{Res}_{z_2=0}\operatorname{Res}_{z_1=0}\big(\zeta_2(Y_2f_2\otimes Y_1f_1\otimes v)-\zeta_2(Y_1f_1\otimes Y_2f_2\otimes v)\big)\\
&=[\operatorname{Res}_2,\operatorname{Res}_1]\pi_2^*(f_2)\pi_1^*f_1.\zeta_2(Y_2\otimes Y_1\otimes v)=-\operatorname{Res}_2\operatorname{Res}_{1\to2}\pi_2^*(f_2)\pi_1^*f_1.\zeta_2(Y_2\otimes Y_1\otimes v)\\
&=(-\operatorname{Res}_2\mathcal{R}'_{2,1}-\mathcal{R}''_{2,1})\pi_2^*(f_2)\pi_1^*f_1.\zeta_2(Y_2\otimes Y_1\otimes v)\\
&=\operatorname{Res}_2\pi_2^*(f_2f_1)\,\mathcal{R}'_{2,1}\,\zeta_2(Y_2\otimes Y_1\otimes v)+\operatorname{Res}(f_1df_2)\,\mathcal{R}''_{2,1}\,\zeta_2(Y_2\otimes Y_1\otimes v)\\
&=\zeta_1([Y_2,Y_1]f_2f_1\otimes v)+\zeta_0(v)\ell\check{c}(Y_2,Y_1)\operatorname{Res}f_1df_2,
\end{aligned}$$

where we invoked the formulae (†) and (‡).

Next we show that the pairing is topologically perfect. We have an increasing filtration $F_\bullet$ on $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}},V)$ by letting F_n be the image of $\widehat{L\mathfrak{g}}^{\otimes n}\otimes V$ and a decreasing filtration $G^\bullet$ on $\tilde{\mathbb{H}}^\ell(\widehat{L\mathfrak{g}},V^*)$ by letting G^n be the $\zeta_\bullet$ with $\zeta_{n-1}=0$.

By the PBW theorem, $F_n/F_{n-1}\cong\operatorname{Sym}^n_R(\widehat{L\mathfrak{g}}/\widehat{\mathcal{O}\mathfrak{g}})\otimes V\cong\operatorname{Sym}^n_R(\mathfrak{g}\otimes_k(L/\mathcal{O}))\otimes V$. The residue pairing identifies the latter's dual with $\operatorname{Sym}^n_R(\operatorname{Hom}(\mathfrak{g},F^1\omega))\otimes V^*$, or what amounts to the same, the space of $\mathcal{S}_n$-equivariant linear maps from $\mathfrak{g}^{\otimes n}\otimes_k V$ to $\operatorname{ann}(\mathcal{O})=F^1\omega^{(n)}$. But this is just G^n/G^{n+1}. □

Recall that $\mathbb{H}_\ell(\widehat{L\mathfrak{g}},V)$ is the quotient of $\tilde{\mathbb{H}}_\ell(\widehat{L\mathfrak{g}},V)$ by the $\widehat{L\mathfrak{g}}$-submodule generated by $X_{\tilde\alpha}(t^{-1})^{1+\ell-\ell_V}V(\lambda)$. Consider for $Y_1,\dots,Y_n\in\mathfrak{g}$ and a generator $v_\lambda\in V(\lambda)$, the poly-differential of degree $n+\ell+1-\ell_V$, $\zeta_{n+\ell+1-\ell_V}(Y_n\otimes\cdots\otimes Y_1\otimes X_{\tilde\alpha}^{\otimes(n+\ell+1-\ell_V)}\otimes v_\lambda$. Since $X_{\tilde\alpha}v_\lambda=0$, this has no poles along the divisors $z_i=0$, $i=1,\dots,\ell+1-\ell_V$. Furthermore,

$\check{c}(X_{\tilde\alpha}, X_{\tilde\alpha}) = 0$ and evidently $[X_{\tilde\alpha}, X_{\tilde\alpha}] = 0$. So this form is regular in has neither a pole along $z_i = z_j$ when $1 \le i < j \le \ell+1-\ell_V$. In other words, $\zeta_{\bullet+\ell+1-\ell_V}|\mathfrak{g}^{\otimes\bullet} \otimes X_{\tilde\alpha}^{\otimes \ell+1-\ell_V} \otimes V(\lambda)$ takes values in polydifferentials that are regular in the generic point of $z_1 = \cdots = z_{\ell+1-\ell_V} = 0$. But then more is true: the Lie algebra $\widehat{L\mathfrak{g}}$ acts on $\mathbb{H}^\ell(\widehat{L\mathfrak{g}}, V^*)$ and hence so does $\mathfrak{g}$. The subrepresentaion of $\mathfrak{g}^{\otimes \ell+1-\ell_V} \otimes V$ generated by $X_{\tilde\alpha}^{\otimes \ell+1-\ell_V} \otimes V(\lambda)$ is irreducible with highest weight $(\ell+1-\ell_V)\tilde\alpha + \lambda_V$. Its level is $c(\ell+1-\ell_V)\tilde\alpha + \lambda_V, \tilde\alpha) = (2(\ell+1-\ell_V) + \ell_V = 2\ell+2-\ell_V$. Indeed, this is the unique subrepresentation of $\mathfrak{g}^{\otimes \ell+1-\ell_V} \otimes V$ of that level (and of this weight). Let us denote this subrepresentation by $(\mathfrak{g}^{\otimes(\ell+1-\ell_V)} \otimes V)_{\max}$.

Proposition 6.4. *The topological dual $\mathbb{H}^\ell(\widehat{L\mathfrak{g}}, V^*)$ of $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, V)$ is the subspace $\mathbb{H}^\ell(\widehat{L\mathfrak{g}}, V) \subset \tilde{\mathbb{H}}^\ell(\widehat{L\mathfrak{g}}, V)$ that consists of the $\zeta_\bullet$ with the property that $\zeta_{\bullet+\ell+1-\ell_V}|\mathfrak{g}^{\otimes\bullet} \otimes (\mathfrak{g}^{\otimes(\ell+1-\ell_V)} \otimes V)_{\max}$ maps to polydifferentials that vanish on $z_1 = \cdots = z_{\ell+1-\ell_V} = 0$.*

Proof. From the above discussion it is clear that $\zeta_\bullet \in \tilde{\mathbb{H}}^\ell(\widehat{L\mathfrak{g}}, V)$, if and only if it has zero pairing with $Y_n(f_n)\cdots Y_1(f_1)X_{\tilde\alpha}(t^{-1})^{\ell+1-\ell_V}v_\lambda$ for all $Y_i \in \mathfrak{g}$ and $f_i \in L$. This pairing has the value

$$\mathrm{Res}_{n+\ell+1-\ell_V}\cdots\mathrm{Res}_1 \frac{\zeta_{n+\ell+1-\ell_V}(Y_n \otimes \cdots \otimes Y_1 \otimes X_{\tilde\alpha}^{\otimes\ell+1-\ell_V} \otimes v_\lambda)}{z_1 z_2 \cdots z_{\ell+1-\ell_V}} \prod_{i=1}^n f_i(z_{i+\ell+1-\ell_V}).$$

This is zero for all $Y_n(f_n) \otimes \cdots \otimes Y_1(f_1)$ if and only if the polydifferential of degree n

$$\mathrm{Res}_{\ell+1-\ell_V}\cdots\mathrm{Res}_1 \frac{\zeta_{n+\ell+1-\ell_V}(Y_n \otimes \cdots \otimes Y_1 \otimes X_{\tilde\alpha}^{\otimes\ell+1-\ell_V} \otimes v_\lambda)}{z_1 z_2 \cdots z_{\ell+1-\ell_V}} \prod_{i=1}^n f_i(z_{i+\ell+1-\ell_V}).$$

is zero for all $Y_n \otimes \cdots \otimes Y_1 \in \mathfrak{g}^{\otimes n}$. This is equivalent to $\zeta_{n+\ell+1-\ell_V}(Y_n \otimes \cdots \otimes Y_1 \otimes X_{\tilde\alpha}^{\otimes\ell+1-\ell_V} \otimes v_\lambda)$ vanishing in $z_1 = \cdots = z_{\ell+1-\ell_V} = 0$. □

Remark 6.5. The symmetry property of $\zeta_\bullet$ implies that we have similar vanishing properties if we permute the arguments. But we can say more: the Lie algebra $\widehat{L\mathfrak{g}}$ acts on $\mathbb{H}^\ell(\widehat{L\mathfrak{g}}, V^*)$ and hence so does $\mathfrak{g}$. This implies in particular, that the vanishing property in Proposition 6.4 above will in fact hold on .

6.5. **The global counterpart.** We first place ourselves in the setting of Definition 3.9. We then have for every $i \in I$ an irreducible representation V_i of $\mathfrak{g}$. We denote by $\lambda_i \in \mathfrak{h}^*$ the highest weight of V_i so that $\lambda := \sum_i \lambda_i$ is the highest weight of $\otimes V_i$. Let $V^{(i)} \subset \otimes_j V_j$ be the tensor subproduct of the V_j, $j \neq i$ and the highest weight space $V_i(\lambda_i)$.

Theorem 6.6. *Let $\pi : \mathcal{C} \to S$ and $(x = (x_i : S \to \mathcal{C})_{i\in I})$ be as in Subsection 3.5. The iterated residue pairing identifies the prevacua bundle $\tilde{\mathcal{H}}^\ell(\widehat{\mathcal{L}\mathfrak{g}}, V^*)^{\mathcal{A}\mathfrak{g}}$ with the bundle of graded $\mathcal{O}_S$-modules $\zeta_\bullet = \Big(\zeta_n \in \mathrm{Hom}_{k[\mathcal{S}_n]}\big(\mathfrak{g}^{\otimes n} \otimes \bigotimes_{i\in I} V_i, \pi_*^{(n)}\omega_{\mathcal{C}/S}(x)^{(n)}(2\log\Delta_n)\big)\Big)_{n=0}^\infty$ such that*

(i) *if $v \in \otimes_{i\in I} V_i$, then we have for every $n \ge 1$ and every $i \in I$,*

$$\mathrm{Res}_{z_1=x_i}\zeta_n(Y_n \otimes \cdots \otimes Y_1 \otimes v) = \zeta_{n-1}(Y_n \otimes \cdots \otimes Y_2 \otimes Y_1^{(i)} v),$$

(ii$_a$) *for $n \ge 2$,*

$$\mathcal{R}'_{n,n-1}\zeta_n(Y_n \otimes \cdots \otimes Y_1 \otimes v) = \zeta_{n-1}([Y_n, Y_{n-1}] \otimes Y_{n-2} \otimes \cdots \otimes Y_1 \otimes v),$$

(ii$_b$) *for $n \ge 2$,*

$$\mathcal{R}''_{n,n-1}\zeta_n(Y_n \otimes \cdots \otimes Y_1 \otimes v) = \ell\check{c}(Y_n \otimes Y_{n-1})\zeta_{n-2}(Y_{n-2} \otimes \cdots \otimes Y_1 \otimes v).$$

The subbundle of vacua $\mathcal{H}^\ell(\widehat{\mathcal{L}\mathfrak{g}}, V^*)^{\mathcal{A}\mathfrak{g}}$ gets identified with the graded submodule of $\zeta_\bullet$ for which

$$\zeta_{\ell+1-\ell_{V_i}+\bullet}|\mathfrak{g}^{\otimes\bullet} \otimes (\mathfrak{g}^{\otimes(\ell+1-\ell_{V_i})} \otimes V_i)_{\max} \otimes_{j\neq i} V_j \quad \textit{resp.} \quad \zeta_{\ell+1+\bullet}|\mathfrak{g}^{\otimes\bullet} \otimes (\mathfrak{g}^{\otimes(\ell+1)})_{\max} \otimes V$$

takes values in polydifferentials that vanish on $z_1 = \cdots = z_{\ell+1-\ell_{V_i}} = x_i$ ($i \in I$) resp. on the diagonal $z_1 = \cdots = z_{\ell+1}$.

Proof. This is immediate from the definitions, except perhaps the last assertion where we claim the vanishing on the diagonal. This follows from the propagation principle, for we may add a new section x_o to which we associate the trivial representation and nothing changes. □

Proposition 6.7. *Let $\pi : \mathcal{C} \to S$ and $x : S \to \mathcal{C}$ be as in Subsection 3.5. The iterated residue pairing identifies the prevacua bundle $\tilde{\mathcal{H}}^\ell(\widehat{\mathcal{L}\mathfrak{g}}, V)^{\mathcal{A}\mathfrak{g}}$ with the bundle of graded $\mathcal{O}_S$-modules*

$$\zeta_\bullet = \Big(\zeta_n \in \operatorname{Hom}_{k[\mathcal{S}_n]}\big(\mathfrak{g}^{\otimes n}, \pi_*^{(n)}\omega_{\mathcal{C}/S}(x)^{(n)}(2\log\Delta_n)\big)\Big)_{n=0}^{\infty},$$

where ζ_n is $\mathcal{S}_n$-equivariant and satisfies Properties (ii$_a$) and (ii$_b$) of 6.6. The subbundle of vacua $\mathcal{H}^\ell(\widehat{\mathcal{L}\mathfrak{g}}, V)^{\mathcal{A}\mathfrak{g}}$ gets identified with the graded submodule with the property that for every $i \in I$, $\zeta_{\ell+1-\ell_{V_i}+\bullet}|\mathfrak{g}^{\otimes\bullet} \otimes (\mathfrak{g}^{\otimes(\ell+1)} \otimes V_i)_{\max} \otimes \otimes_{j\neq i} V_j$ resp. $\zeta_{\ell+1+\bullet}|\mathfrak{g}^{\otimes\bullet} \otimes (\mathfrak{g}^{\otimes(\ell+1)} \otimes V)_{\max}$ takes values in polydifferentials that vanish on $z_1 = \cdots = z_{\ell+1-\ell_{V_i}} = x_i$ resp. $z_1 = \cdots = z_{\ell+1}$.

Remark 6.8. There will be other vanishing properties than we have stated here (although they must be necessarily consequences of these). For example, if x is a local section of $\mathcal{C}/S$ defined by $t \in \mathcal{O}_\mathcal{C}$, then $\mathfrak{g}(t^{-1})$ acts in a locally nilpotent manner on $\widehat{L_x\mathfrak{g}}$, where L_x is associated to x as usual. The same argument as used above implies that there exists an an integer $d > 0$ such that for any $\zeta_\bullet$ in the vacua bundle has the property that $\zeta_{d+\bullet}|\mathfrak{g}^{\otimes\bullet} \otimes \operatorname{Sym}^d(\mathfrak{g}) \otimes V$ vanishes on $z_1 = \cdots = z_d$. It would be nice to connect this with the discussion of the genus zero case in [] and [**?**].

6.6. **A canonical 2-form on the self-product of a compact Riemann surface.** In order to discuss the vacua modules on families on curves, we prove a lemma on curves that has some interest in its own right. Let C be a projective nonsingular curve. We take $k = \mathbb{C}$ and treat C as a compact Riemann surface. Singular (Betti) cohomology is taken with $\mathbb{Q}$-coefficients. We will prove that C^2 carries a canonical symmetric polydifferential with a pole of order 2 along the diagonal.

We first observe that the Künneth formula yields the isomorphism

$$H^2(C^2) \cong \big(H^2(C)\otimes 1 \oplus 1\otimes H^2(C)\big) \oplus \big(H^1(C) \otimes H^1(C)\big).$$

Let us write Δ_C for the diagonal in C^2. Then under the above decomposition, its class $[\Delta_C]$ is equal to $\mu\otimes 1 + 1\otimes\mu + \delta$, where $\mu \in H^2_{DR}(C)$ is the natural generator, and $\delta \in (H^1_{DR}(C) \otimes H^1_{DR}(C))^{-\sigma}$ is defined by the intersection pairing. Notice that both μ and δ are of type $(1,1)$.

Lemma 6.9. *The map $H^2(C^2) \to H^2(C^2 - \Delta_C)$ is surjective with kernel $\mathbb{Z}[\Delta_C]$.*

Proof. Consider the Gysin sequence

$$H^0(\Delta_C) \to H^2(C^2) \to H^2(C^2 - \Delta_C) \to H^1(\Delta_C) \to H^3(C^2).$$

The first arrow has image $\mathbb{Z}[\Delta_C]$. The last arrow is via duality identified with $H_1(\Delta_C) \to H_1(C^2)$. Under the identification $H_1(C^2) = H_1(C) \oplus H_1(C)$ this is given by $a \mapsto a\otimes 1 + 1\otimes a$ and hence is injective. □

We already observed that we have a natural σ-anti-invariant trivialization $\Omega^2_{C^2}(2\Delta_C)\otimes\mathcal{O}_{\Delta_C}\cong\mathcal{O}_{\Delta_C}$: if z is a local coordinate at $x\in C$, so that (z_1,z_2) is a chart at $(x,x)\in C^2$, then the restriction of $(z_2-z_1)^{-2}dz_2\wedge dz_1$ to Δ_C is independent of that choice. If we replace the $\wedge$-symbol by a dot so that it becomes a polydifferential, then the identification becomes σ-invariant. According to Biswas-Raina [4] there exists a $\eta\in H^0(C^2,\Omega^2(2\Delta_C))$ whose restriction to Δ_C yields this trivialization. We may, of course, take this generator to be anti-invariant under σ: $\sigma^*\eta=-\eta$. So if the think of η as a polydifferential, then it is σ-invariant. It is unique up an element of $\operatorname{Sym}^2 H^0(C,\Omega_C)=H^0(C^2,\Omega^2)^{-\sigma}\subset H^2_{DR}(C^2)^{-\sigma}$. Notice that defines a cohomology class $[\eta]\in H^2_{DR}(C^2-\Delta_C)^{-\sigma}$.

Lemma 6.10. *The image $[\eta]$ of η in $H^2_{DR}(C^2-\Delta_C)^{-\sigma}$ satisfies*

$$[\eta]\equiv 2\pi\sqrt{-1}\delta \pmod{\operatorname{Sym}^2 H^0(C,\Omega_C)}.$$

In particular, there is in $H^0(C^2,\Omega^2_{C^2}(2\Delta_C))^{-\sigma}$ a unique (De Rham) representative of $2\pi\sqrt{-1}\delta$.

Proof. A class $K\in H^2_{DR}(C^2)(1)$ defines a map $H^\bullet_{DR}(C)\to H^\bullet_{DR}(C)$ by $a\mapsto\pi_{2*}(K\cup\pi_1^*(a))$. We regard δ as an element of $H^2_{DR}(C^2)(1)$ and then we get zero in degrees 0 and 2 and the identity in degree 1. We next do this computation for $[\eta]$. Given $x\in C$, then identify $H^1_{DR}(C)$ with the space of differentials of the second kind on C modulo the exact ones. The cup product pairing $H^1_{DR}(C)\otimes H^1_{DR}(C)\to H^2_{DR}(C)\cong\mathbb{C}(-1)$ is then given as follows: if $\alpha,\beta\in H^0(C-\{x\},\Omega^1_C)$, then $\alpha=df$, $\beta=dg$ for some meromorphic functions f,g near x and then $\langle\alpha,\beta\rangle=-\sum_{x\in C}\operatorname{Res}_x(f\beta)=\sum_{x\in C}\operatorname{Res}_x g\alpha$ (multiply by $2\pi\sqrt{-1}$ to get the Betti cup product).

We first observe that any element of $\operatorname{Sym}^2 H^0(C,\Omega_C)$ induces an endomorphism of $H^\bullet_{DR}(C)$ that is zero in degree $\neq 1$ and factors in degree 1 as $H^1_{DR}(C)\twoheadrightarrow H^1(C,\mathcal{O}_C)\to H^0(C,\Omega_C)\hookrightarrow H^1_{DR}(C)$. We show that the cohomology class of η defines a map $H^0(C,\Omega_C)$ to itself that is the identity. This suffices, for $[\eta]$ induces the zero map in degree $\neq 1$. Moreover, the adjoint of this map (relative to Serre duality) is obtained by interchanging factors and hence given by $-[\eta]$. The same is true for δ (its adjoint is obtained by interchanging factors and hence equal to $-\delta$), and so $[\eta]$ and δ induce the same map on $H^1(C,\mathcal{O}_C)$.

Let β be an abelian differential on C. For $x\in C$, choose a local coordinate z at x so that

$$\eta=(z_1-z_2)^{-2}dz_2\wedge dz_1+\text{a form regular at }(x,x).$$

We can integrate at p with respect to z_1 and find that

$$\eta=dz_2\wedge d\Big(\frac{-1}{z_1-z_2}+\text{a regular function at }(x,x)\Big).$$

Now if β is an abelian differential on C, then $\beta=gdz$ for some meromorphic function g on C defined at x. The residue pairing relative to the z_1-coordinate (with z_2 as parameter) yields

$$dz_2\wedge-\operatorname{Res}_{z_1=z_2}\frac{-\beta(z_1)}{z_1-z_2}=dz_2\wedge\operatorname{Res}_{z_1=z_2}\frac{g(z_1)dz_1}{z_1-z_2}=dz_2\,g(z_2)=\beta(z_2).$$

So η induces the identity in $H^0(C,\Omega_C)$. □

Remark 6.11. This argument works *mutatis mutandis* for a smooth family $\mathcal{C}\to S$ of projective curves. However, some care is needed with the statement: if $\pi^{(2)}:\mathcal{C}^2_{/S}=\mathcal{C}\times_S\mathcal{C}\to S$ is the projection, then we have a natural symmetric C^∞-section ζ of $\pi^{(2)}_*\Omega^{(2)}_{\mathcal{C}}(2\log\Delta_C)^\sigma$ which spans supplement of $(\pi^{(2)}_*\Omega^{(2)}_{\mathcal{C}})^\sigma$ in $\pi^{(2)}_*\Omega^{(2)}_{\mathcal{C}}(2\log\Delta_C)^\sigma$. We cannot expect ζ to be complex-analytic, because we used the Hodge decomposition to define it.

6.7. Cohomology of the configuration space of a curve. In order to interpret the logarithmic polydifferentials in terms of Hodge theory, we digress to discuss the configuration space of a curve.

The partitions $\mathscr{P}(N)$ of a finite set N are partially ordered: $\mathcal{P} \le \mathcal{P}'$ means that $\mathcal{P}$ refines $\mathcal{P}'$. For $\mathcal{P} \in \mathscr{P}(N)$, we define the codimension $c(\mathcal{P})$ as the difference between N and the number of members of $\mathcal{P}$. So $c(\mathcal{P}) = 0$ corresponds to the finest partition N. We define the *Orlik-Solomon functor* as the contravariant functor A from $(\mathscr{P}(N), \le)$ to the category of free abelian groups of finite rank, characterized by the following properties:

(i) when $c(\mathcal{P}) \le 1$, then $A(\mathcal{P}) = \mathbb{Z}$,
(ii) for $c(\mathcal{P}) > 1$, the following sequence is exact

$$0 \to A(\mathcal{P}) \to \oplus_{\mathcal{P}' \in \mathscr{P}(1)} A(\mathcal{P}') \to \oplus_{\mathcal{P}'' \in \mathscr{P}(2)} A(\mathcal{P}'') \to \cdots, \tag{1}$$

where $\mathscr{P}(k)$ denotes the set of $\mathcal{Q} \in \mathscr{P}$ with $\mathcal{Q} > \mathcal{P}$ and $c(\mathcal{Q}) = c(\mathcal{P}) + k$. We also put $A^k(N) := \oplus_{c(\mathcal{P})=k} A(\mathcal{P})$. The direct sum $A^\bullet(N) = \oplus_k A^k(N)$ has the structure of a graded-commutative ring, known as the *Orlik-Solomon ring*. It is generated in degree 1: if for every 2-element subset $\{a, b\} \subset N$, $\mathcal{P}_{ab}$ denotes the partition which has $\{i, j\}$ as the unique member that is not a singleton (so that $c(\mathcal{P}_{ab}) = 1$), then $A^\bullet(N)$ is generated by the $A_{ij} := A(\mathcal{P}_{ij})$ (see also [22]) and is subject to the Arnol'd relations $A_{ij}A_{jk} + A_{jk}A_{ki} + A_{ki}A_{ij} = 0$. This is a complete set of relations for $A^\bullet(N)$ as a graded-commutative ring. The characterizing property (1) shows that we have a natural map $A^r(N) \to A^1 \otimes A^{r-1}(N)$. It assigns to a monomial $A_{i_1j_1} \cdots A_{i_rj_r}$ the tensor $\sum_{s=1}^r (-1)^{s-1} A_{i_sj_s} \otimes A_{i_1j_1} \cdots \widehat{A}_{i_sj_s} \cdots A_{i_rj_r}$. It is injective when $r > 0$ for composition with the multiplication map followed by division by r is the identity. In view of the following topological interpretation (due to Arnol'd) it is natural to endow $A^p(N)$ with the Tate Hodge structure of type (p, p): denoting for any set X by $\mathring{X}^N$ the subset of X^N given by injective maps $N \hookrightarrow X$, then the cohomology ring of $\mathring{\mathbb{C}}^N$ (with its mixed Hodge structure) can be identified with $A^\bullet_N$. This interpretation also implies that the Poincaré series $\sum_{k \ge 0} \mathrm{rk}(A^k(N)) t^k$ is equal to the polynomial $\prod_{k=1}^{|N|-1}(1 + kt)$ [?].

We now take $N = \{1, \dots, n\}$ and write $A^\bullet_n$ for $A^\bullet(N)$.

Lemma 6.12. *Let C be a connected projective curve. Let $x : I \hookrightarrow C$ embed a nonempty finite subset and put $U := C - x(I)$. Then the natural maps $H^0(C^n, \Omega_C(x)^{(n)}(\log 2\Delta_n)) \to H^0(\mathring{U}^n)$ and $H^0(C^n, \Omega_C(x)^{(n)}(\log \Delta_n)) \to H^0(\mathring{U}^n)$ are injective with the latter taking values in $F^n H^0(\mathring{U}^n)$. Moreover, the natural maps $H^0(C, \Omega_C)^{\otimes n} \to H^0(C^n, \Omega^n_C(\log \Delta_n)) \to F^n H^0(\mathring{C}^n)$ are isomorphisms.*

Proof. The diagonals of U^n define a decomposition of U^n into strata with each stratum being given by a partition of n. For every partition $\mathcal{P}$ of n we denote by $U_\mathcal{P}$ the corresponding stratum closure in U^n. Note that its codimension is $c(\mathcal{P})$ and hence is isomorphic to $U^{n-c(\mathcal{P})}$. In [14] we set up a spectral sequence of mixed Hodge structures

$$E_1^{-p,q} = \oplus_{c(\mathcal{P})=p} H^{q-2p}(U_\mathcal{P}; \mathbb{Z}) \otimes A(\mathcal{P}) \Rightarrow H^{q-p}(\mathring{U}^n; \mathbb{Z}).$$

This remains a spectral sequence if we apply F^n:

$$F^n E_1^{-p,q} = \oplus_{c(\mathcal{P})=p} F^{n-p} H^{q-2p}(U_\mathcal{P}) \otimes A(\mathcal{P}) \Rightarrow F^n H^{q-p}(\mathring{U}^n).$$

Since $U_\mathcal{P} \cong U^{n-p}$ and $F^r H^k(U) =$ for $k < r$, the term $F^n E_1^{-p,q}$ is zero unless $q - 2p \ge n - p$, or equivalently, $q \ge n + p$. In case x is nonempty, U is affine, and then U^{n-p} has no cohomology in degree $> n-p$ so that $F^{n-p}E_1^{-p,q}$ is zero unless $q-2p \le n-p$, or equivalently, $q \le n+p$. Hence the sequence degenerates: we have

$$F^n E_\infty^{-p,q} = \oplus_{c(\mathcal{P})=p} F^{n-p}H^{q-2p}(U_{\mathcal{P}}) \otimes A(\mathcal{P}) \cong F^{n-p}H^{q-2p}(U^{n-p}) \otimes A_n^p.$$

If we take $q = n+p$, then this shows that $F^nH^n(\mathring{U}^n)$ has a filtration whose associated graded vector space is $\oplus_p F^{n-p}H^{n-p}(U^{n-p}) \otimes A_n^p \cong \oplus_p H^1(C, \Omega(x))^{\otimes(n-p)} \otimes A_n^p$.

Assume now C projective. The above spectral sequence also exists for U. Then $E_1^{-p,q}$ is then pure of weight q and so the sequence degenerates at E_2: $E_\infty^{-p,n+p}$ is the homology of the sequence of Gysin maps

$$H^{n-p-2}(C^{n-p-1}) \otimes A^{p+1}(n) \to H^{n-p}(C^{n-p}) \otimes A^p(n) \to H^{n+2-p}(C^{n+p+1}) \otimes A^{p-1}(n)$$

Then $F^nH^n(\mathring{C}^n)$ is the homology of the sequence

$$\begin{aligned} F^{n-p-1}H^{n-p-2}(C^{n-p-1}) \otimes A^{p+1}(n) \to F^{n-p}H^{n-p}(C^{n-p}) \otimes A^p(n) \to \\ \to F^{n-p+1}H^{n-p+2}(C^{n-p+1}) \otimes A^{p-1}(n). \end{aligned}$$

Note that $F^{n-p-1}H^{n-p-2}(C^{n-p-1}) = 0$, and by the Künneth theorem, $F^{n-p}H^{n-p}(C^{n-p}) \cong H^0(C, \Omega_C)^{\otimes(n-p)}$ and $F^{n-p+1}H^{n-p+2}(C^{n-p+1}) \cong \sum_{i=1}^{n-p+2} \pi_i^*\mu \cup H^0(C, \Omega_C)^{\otimes(n-p)}$. In fact, the second map is obtained by tensoring $H^0(C, \Omega_C)^{\otimes(n-p)}$ with the map $A^p(n) \to A^1(n) \otimes A^p(n-1)$ defined above. As this map is injective for $p > 0$, it follows that $F^nH^n(\mathring{C}^n) \cong H^0(C, \Omega_C)^{\otimes n}$. □

6.8. **Application to vacua bundles.** We apply this to our description of vacua bundles. We use Lemma 6.10 to define for $n \ge 2$,

$$\begin{aligned} \zeta_n^+(Y_n \otimes \cdots \otimes Y_1 \otimes v) &:= \zeta_n(Y_n \otimes \cdots \otimes Y_1 \otimes v) \\ &- \sum_{i>j} \ell\check{c}(Y_i \otimes Y_j)\pi^{ij*}\zeta_{n-2}(Y_n \otimes \cdots \otimes \widehat{Y_i} \otimes \cdots \otimes \widehat{Y_j} \otimes \cdots \otimes Y_1 \otimes v)\pi_{ij}^*\eta, \end{aligned}$$

(where $\pi_{ij} : \mathcal{C}^n_{/S} \to \mathcal{C}^2_{/S}$ and $\pi^{ij} : \mathcal{C}^n_{/S} \to \mathcal{C}^{n-2}_{/S}$ retain (resp. ignore) the factors with indices i, j) has at most a pole of order one along a relative diagonal. If we let $\zeta_i^+ = \zeta_i$ for $i = 0, 1$, then we find that the system $\zeta_\bullet$ satisfies the properties (i), (ii$_a$) and (ii$_b$) above if and only if $\zeta_\bullet^+$ satisfies the properties (i) and (ii$_a$). The advantage of $\zeta_\bullet^+$ over $\zeta_\bullet$ is that it takes values in the space of logarithmic polydifferentials. It follows from Lemma's 6.10 and 6.12 that both $\zeta_\bullet$ and $\zeta_\bullet^+$ can be understood as taking values in the cohomology of configuration spaces.

Corollary 6.13. *In the situation of Proposition 6.7, the iterated residue pairing identifies the prevacua bundle as a C^∞-bundle with the bundle of graded $\mathcal{O}_S$-modules*

$$\zeta_\bullet^+ = \left(\zeta_n^+ \in \mathrm{Hom}_{k[\mathcal{S}_n]}\left(\mathfrak{g}^{\otimes n}, \pi_*^{(n)}\omega_{\mathcal{C}/S}(x)^{(n)}(\log \Delta_n)\right)\right)_{n=0}^\infty,$$

where ζ_n^+ satisfies the residue properties (i) and (ii$_a$) of 6.6. This identifies subbundle of vacua $\mathbb{H}^\ell(\widehat{L\mathfrak{g}}, V)^{A\mathfrak{g}}$ with the submodule of $\zeta_\bullet$ with the property that for every $i \in I$, $\zeta_{\ell+1-\ell_{V_i}+\bullet}|\mathfrak{g}^{\otimes\bullet} \otimes (\mathfrak{g}^{\otimes(\ell+1)} \otimes V_i)_{\max} \otimes \otimes_{j\neq i} V_j$ resp. $\zeta_{\ell+1+\bullet}|\mathfrak{g}^{\otimes\bullet} \otimes (\mathfrak{g}^{\otimes(\ell+1)} \otimes V)_{\max}$ takes values in polydifferentials that vanish on $z_1 = \cdots = z_{\ell+1-\ell_{V_i}} = x_i$ resp. $z_1 = \cdots = z_{\ell+1}$. We may replace here $H^0(C^n, \Omega_C^{(n)}(\log \Delta_n))$ by the Hodge space $F^nH^n(\mathring{C}^n) \otimes \det(\mathbb{Z}^n)$, and in case, $I = \emptyset$, even by $H^0(C^n, \Omega_C^n) \otimes \det(\mathbb{Z}^n) = F^nH^n(C^n) \otimes \det(\mathbb{Z}^n)$.

6.9. **The WZW-action on vacua bundles.** Recall that

$$T_{\mathfrak{g}}(\hat{D}_k) = \frac{-1}{\ell + \check{h}} \sum_{\kappa} \sum_{k_1+k_2=k} \tfrac{1}{2} : X_\kappa t^{k_2} \circ X_\kappa t^{k_1} : .$$

and that its action on on $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, k)$ is given by

$$T_{\mathfrak{g}}(\hat{D}_k)\big(Y_r f_r \circ \cdots \circ Y_1 f_1 \circ r\big) = \\ = \sum_{i=1}^{r} Y_r f_r \circ \cdots \circ Y_i D_k(f_i) \circ \cdots \circ Y_1 f_1 \circ r + Y_r f_r \circ \cdots \circ Y_1 f_1 \circ r T_{\mathfrak{g}}(\hat{D}_k)$$

.

In order to determine the induced action on the space of polydifferentials $\mathbb{H}_\ell(\widehat{L\mathfrak{g}}, k)$ we first note that the action of θ on ω by Lie derivation, $\mathcal{L}_D(\alpha) := d(\langle D, \alpha\rangle)$, is minus the adjoint of D acting on L (with respect to the residue pairing): $r(\alpha, Df) + r(\mathcal{L}_D(\alpha), f) = 0$. We extend the Lie action on ω in the obvious way to $\omega^{(n)}$ by letting $\mathcal{L}_D$ act as $\sum_i \mathcal{L}_D^{(i)}$, where $\mathcal{L}_D^{(i)}$ acts in the ith factor. It then follows that

$$\langle \zeta| - T_{\mathfrak{g}}(\hat{D}_k)\big(Y_r f_r \circ \cdots \circ Y_1 f_1 \circ r\big)\rangle = \\ = \langle \mathcal{L}_D \zeta_n | Y_r f_r \circ \cdots \circ Y_1 f_1 \circ r\rangle - \langle \zeta_{n+2} | Y_r f_r \circ \cdots \circ Y_1 f_1 \circ r T_{\mathfrak{g}}(\hat{D}_k)\rangle.$$

We analyze the last term by computing

$$\text{(2)} \quad \operatorname{Res}_2 \operatorname{Res}_1 \zeta_2(T_{\mathfrak{g}}(D_k) \otimes v) = \frac{-1}{\ell + \check{h}} \operatorname{Res}_2 \operatorname{Res}_1 \sum_{\kappa} \zeta_2(X_\kappa \otimes X_\kappa \otimes v) \sum_{k_1+k_2=k} : \tfrac{1}{2} z_2^{k_2} z_1^{k_1} :$$

We may write $\zeta_2(C \otimes v) = 2\ell\zeta_0(v)(z_1 - z_2)^{-2} dz_2 dz_1 + \zeta_2^+(C \otimes v)$ with

$$\zeta_2^+(C \otimes v) = \sum_{\kappa} \sum_{r_1 \geq 0, r_2 \geq 0} \zeta^\kappa_{r_1, r_2}(v) z_1^{r_1 - 1} z_2^{r_2 - 1} dz_1 dz_2$$

for certain $\zeta^\kappa_{r_1,r_2} \in \operatorname{Hom}_k(V, L^{(n)}(2\log \Delta_n))$, where we note that $\zeta^\kappa_{r_1,r_2} = \zeta^\kappa_{r_2,r_1}$ by $\mathcal{S}_2$-invariance. Let k_1, k_2 be such that $k_1 + k_2 = k$. Then

$$\operatorname{Res}_{z_2=0} \operatorname{Res}_{z_1=0} \frac{z_2^{k_2} z_1^{k_1}}{(z_1 - z_2)^2} dz_2 dz_1 = \begin{cases} k_2 = -k_1 & \text{if } k = 0 \text{ and } k_2 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here we only have such terms for which $k_2 \leq k_1$, and so it follows that

$$\langle (T_{\mathfrak{g}}(D_k) v | (z_1 - z_2)^{-2} dz_2 dz_1\rangle = 0.$$

If we replace in Equation (2), $\zeta_2(C \otimes v)$ by $\zeta_2^+(C \otimes v)$, then, since $\zeta_2^+(C \otimes v)$ is $\mathcal{S}_2$-invariant and there is no pole along the diagonal, the order of the residues is immaterial and so we can here omit the normal ordering symbol. Hence

$$\operatorname{Res}_2 \operatorname{Res}_1 \zeta_2^+(T_{\mathfrak{g}}(D_k) \otimes v) = \\ = \frac{-1}{\ell + \check{h}} \operatorname{Res}_2 \operatorname{Res}_1 \Big(\sum_{\substack{r_1 \geq 0 \\ r_2 \geq 0}} \zeta^\kappa_{r_1, r_2} z_1^{r_1 - 1} z_2^{r_2 - 1} dz_2 dz_1 . \big(\tfrac{1}{2} \sum_{k_1+k_2=k} z_2^{k_2} z_1^{k_1}\big)\Big) = \tfrac{1}{2} \sum_{k_1+k_2=k} \zeta^\kappa_{-k_1, -k_2}.$$

On the other hand,

$$\langle D_k^{(1)}, \zeta_2^+(C\otimes v)\rangle = \sum_{r_1,r_2\geq N} \zeta^\kappa_{r_1,r_2} z_1^{k+r_1} z_2^{r_2-1} dz_2,$$

where the upper index in $D_k^{(1)}$ refers to the position of the argument with which D_k is contracted. The restriction of this expression to the diagonal $z_1=z_2$ is $\sum_{r_1,r_2\geq N}\zeta^\kappa_{r_1,r_2} z^{r_1+r_2+k-1}dz$ of which the residue at $z=0$ is $\sum_{r_1+r_2=-k}\zeta^\kappa_{r_1,r_2}=\sum_{k_1+k_2=k}\zeta^\kappa_{-k_1,-k_2}$. Notice that this is also what we get if $D_k^{(1)}$ is replaced by $D_k^{(2)}$. Thus we find that

$$\zeta_2^+(C\otimes v) := \zeta_2(\eta\otimes c\otimes v) - \ell\dim\mathfrak{g}.\zeta_0(v)\frac{dz_1dz_2}{(z_1-z_2)^2},$$

has no pole along Δ_{12} and that for every k-derivation D of L we have

$$\mathrm{Res}_2\,\mathrm{Res}_1\,\zeta_2(T_\mathfrak{g}(D)v) = \frac{-1}{2(\ell+\check{h})}\,\mathrm{Res}\left(\langle D^{(1)}, \zeta_2^+(C\otimes v)\rangle|_{\Delta_{12}}\right).$$

Corollary 6.14. *The Sugawara action on $\tilde{\mathbb{H}}(\widehat{L\mathfrak{g}},k)$ is given by:*

$$T_\mathfrak{g}(\hat{D})(\zeta_n) = -\mathcal{L}_D(\zeta_n) + S_\mathfrak{g}(D)(\zeta_{n+2}),$$

where

$$S_\mathfrak{g}(D)(\zeta_{n+2})(Y_nf_n\otimes\cdots Y_1f_1\otimes v) := \frac{-1}{2(\ell+\check{h})}\,\mathrm{Res}\left(\langle D^{(1)}, \zeta_{n+2}^+(Y_n\otimes\cdots Y_1\otimes C\otimes v)\rangle|_{\Delta_{12}}\right.$$

with

$$\begin{aligned}\zeta_{n+2}^+(Y_n\otimes\cdots Y_1\otimes C\otimes v) &:= \\ &= \zeta_{n+2}(Y_n\otimes\cdots Y_1\otimes C\otimes v) - \ell\dim\mathfrak{g}.\zeta_n(Y_n\otimes\cdots Y_1\otimes C\otimes v)(z_3,\dots,z_{n+2})\frac{dz_1dz_2}{(z_1-z_2)^2}.\end{aligned}$$

REFERENCES

[1] A. Beauville: *Conformal blocks, fusion rules and the Verlinde formula,* Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry, 75–96, Israel Math. Conf. Proc. **9**, Bar-Ilan Univ., Ramat Gan (1996). 31, 34

[2] A.A. Beilinson, V. Drinfeld: *Affine Kac-Moody Algebras and Polydifferentials,* International Mathematics Research Notices (1994), **1**, 1–11 40

[3] A.A. Beilinson, Yu.I. Manin, V.V. Schechtman: *Sheaves of the Virasoro and Neveu-Schwarz algebras*, in: *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, 52–66, Lecture Notes in Math. **1289**, Springer, Berlin (1987). 6

[4] I. Biswas, A.K. Raina: *Projective structures on a Riemann surface I, II, III,* Internat. Math. Res. Notices (1996) **15**, 753–768; Internat. Math. Res. Notices (1999) **13**, 685–716; Differential Geom. Appl. **15** (2001), 203–219. 8, 43

[5] A. Boer, E. Looijenga: *On the unitary nature of abelian conformal blocks,* J. Geom. Phys. 60 (2010), no. 2, 205–218.

[6] G. Faltings: *A proof for the Verlinde formula,* J. Algebraic Geom. 3 (1994), no. 2, 347–374.

[7]

[8] R.C. Gunning: *Lectures on Riemann surfaces,* Princeton Mathematical Notes, Princeton University Press, Princeton, N.J. 1966 iv+254 pp. 8

[9] V.G. Kac: *Infinite-dimensional Lie algebras,* 3rd ed. Cambridge University Press, Cambridge (1990). 20, 21, 25, 27

[10] V.G. Kac, A.K. Raina: *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras,* Advanced Series in Mathematical Physics, 2. World Scientific Publishing Co., Inc., Teaneck, NJ (1987). 3, 20, 25
[11] T. Kohno: *Integrable connections related to Manin and Schechtman's higher braid groups,* Ill. J. Math. **34** (1990), 476–484. 33
[12] Y. Laszlo: *Hitchin's and WZW connections are the same,* J. Differential Geom. 49 (1998), 547–576.
[13] Y. Laszlo, C. Sorger: *The line bundles on the moduli of parabolic G-bundles over curves and their sections,* Ann. Sci. École Norm. Sup. (4) 30 (1997), 499–525.
[14] E. Looijenga: *Cohomology of $\mathcal{M}_3$ and $\mathcal{M}_3^1$,* in: Mapping class groups and moduli spaces of Riemann surfaces (Gttingen, 1991/Seattle, WA, 1991), 205–228, Contemp. Math. **150**, Amer. Math. Soc., Providence, RI, 1993. 44
[15] D. Mumford: *On the equations defining abelian varieties I,* Invent. Math. 1 1966 287–354. 11
[16] P. Scheinost, M. Schottenloher: *Metaplectic quantization of the moduli spaces of flat and parabolic bundles,* J. Reine Angew. Math. 466 (1995), 145– 219.
[17] G. Segal: *The definition of conformal field theory,* in: Topology, geometry and quantum field theory, 421–577, London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge, 2004.
[18] R. Steinberg: *Générateurs, relations et revêtements de groupes algébriques,* in Colloque sur la Théorie des Groupes Algébriques, held in Brussels, 1962, Paris, Gauthier-Villars, 1962, pp. 113–127. 22
[19] Ch. Sorger: *La formule de Verlinde,* Sém. Bourbaki Exp. 794,Astérisque 237 (1996), 87–114.
[20] X. Sun, I-Hsun Tsai: *Hitchin's connection and differential operators with values in the determinant bundle,* J. Differential Geom. 67 (2004), 335–376.
[21] J. Tate: *Residues of differentials on curves,* Ann. Sci. École Norm. Sup. **1** (1968) 149–159. 3
[22] B. Totaro: *Configuration spaces of algebraic varieties,* Topology **35** (1996), 1057–067. 44
[23] Y. Tsuchimoto: *On the coordinate-free description of the conformal blocks,* J. Math. Kyoto Univ. 33 (1993), no. 1, 29–49.
[24] A. Tsuchiya, K. Ueno, Y. Yamada: *Conformal field theory on universal family of stable curves with gauge symmetries,* in: Integrable systems in quantum field theory and statistical mechanics, 459–566, Adv. Stud. Pure Math. **19**, Academic Press, Boston, MA (1989). 28
[25] V.G. Turaev, *Quantum invariants of knots and 3-manifolds,* de Gruyter Studies in Mathematics, 18. Walter de Gruyter & Co., Berlin, 1994.
[26] K. Ueno: **Q**-*structure of conformal field theory with gauge symmetries,* in: The moduli space of curves (Texel Island, 1994), 511–531, Progr. Math., 129, Birkhäuser Boston, Boston, MA, 1995.
[27] K. Ueno: *Conformal field theory and modular functor,* in Advances in algebra and combinatorics, 335–352, World Sci. Publ., Hackensack, NJ, 2008.
[28] K. Ueno: *Conformal field theory with gauge symmetry.* Fields Institute Monographs, 24. American Mathematical Society, Providence, RI; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2008.
[29] G.E. Welters: *Polarized abelian varieties and the heat equations,* Compositio Math. 49 (1983), 173–194. 14