ARTIN GROUPS AND THE FUNDAMENTAL GROUPS OF SOME MODULI SPACES

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ABSTRACT. We define for every affine Coxeter graph a certain factor group of the associated Artin group and prove that some of these groups appear as orbifold fundamental groups of moduli spaces. Examples are the moduli space of nonsingular cubic algebraic surfaces and the universal non-hyperelliptic smooth genus three curve. We use this to obtain a simple presentation of the mapping class group of a compact genus three topological surface with connected boundary. This leads to a modification of Wajnryb's presentation of the mapping class groups in the higher genus case that can be understood in algebro-geometric terms.

INTRODUCTION

Since our main results present *orbifold fundamental groups* of certain moduli spaces, we begin this introduction with recalling that notion; readers familiar with it may skip the first few paragraphs.

If G is a group (having the discrete topology) acting on a connected space X and $p \in X$ is a base point, then the set of pairs (α, g) with $g \in G$ and α a homotopy class of paths in X from p to gp forms a group under the composition law $(\alpha, g).(\beta, h) = (\alpha(g_*\beta), gh)$ (we adopt the geometric convention for the multiplication law for fundamental groupoids: if γ, δ are paths in X such that δ begins where γ ends, then $\gamma\delta$ stands for γ followed by δ). This is the *equivariant fundamental group* $\pi_1^G(X, p)$ of (X, p, G). The projection on the second component defines a surjective homomorphism $\pi_1^G(X, p) \to G$ with kernel $\pi_1(X, p)$.

Following an insight of Borel, one can also define $\pi_1^G(X, p)$ as the fundamental group of an actual space, the *homotopy orbit space* X_G of X by G. Such a space can be obtained by choosing a universal G-bundle $EG \to BG$ and take for X_G be the G-orbit space of $EG \times X$ (with G acting diagonally). The advantage of this definition is that it makes also sense if G is any topological group acting continuously on X. Since $X_G \to BG$ is a fiber bundle with fiber X, the Serre sequence gives rise to an exact sequence

$$\pi_1(G, e) \to \pi_1(X, p) \to \pi_1^G(X, p) \to \pi_0(G) \to 1$$

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with the first map induced by $g \in G \mapsto gp \in X$ (here we used that the Serre sequence of $EG \to BG$ yields an isomorphism $\pi_n(BG,*) \cong \pi_{n-1}(G,e)$).

Observe that there is natural projection $X_G \to G\backslash X$. Strictly speaking X_G is only defined up to a homotopy equivalence over $G\backslash X$ (we need a choice of a universal G-bundle, after all) and we therefore abuse notation a bit by letting X_G stand for *any* representative space within this homotopy type over $G\backslash X$ that suits the context. This allows us for instance to let X_G be $G\backslash X$ in case G acts freely. When X is a manifold and G is a Lie group acting smoothly and properly on X with finite stabilizers (which covers all cases considered here) it is convenient to think of X_G in more geometric terms as the orbit space $G\backslash X$ with a bit of extra structure given at the orbits with nontrivial stabilizer, as formalized by the notion of an *orbifold*. This is why $\pi_1^G(X,p)$ is also called the *orbifold fundamental group* of X_G. That group remembers the kernel K of the G-action on X, since it naturally appears as a normal subgroup of $\pi_1^G(X,p)$ (with quotient $\pi_1^{G/K}(X,p)$).

A fundamental example taken from algebraic geometry is the following: for integers $n \geq 0$ and $d \geq 3$, let $H^n(d)$ denote the discriminant complement in $\mathbb{P}(\mathrm{Sym}^d\,\mathbb{C}^{n+2})$, that is, the Zariski open subset of $\mathbb{P}(\mathrm{Sym}^d\,\mathbb{C}^{n+2})$ that parameterizes the *nonsingular* hypersurfaces in $\mathbb{P}^{n+1}$ of degree d. It supports the universal degree d hypersurface $X^n(d) \subset \mathbb{P}^{n+1}_{H^n(d)}$ (we adopt here the tradition of algebraic geometers of denoting the trivial bundle with base S and fiber F by F_S; confusion with our X_G above is not very likely). On both these algebraic manifolds the group $\mathrm{PGL}(n+2)$ acts properly (and hence with finite stabilizers, as the group is affine). So the orbifold fundamental group of $H^n(d)_{\mathrm{PGL}(n+2)}$ is a quotient of the fundamental group of $H^n(d)$ by the image of the fundamental group of $\mathrm{PGL}(n+2)$ (which may be identified with the center μ_{n+2} of its universal cover $\mathrm{SL}(n+2)$) and similarly for $X^n(d)_{\mathrm{PGL}(n+2)}$. For $n \geq 3$ a nonsingular hypersurface in $\mathbb{P}^n$ of degree $d \geq 3$ is simply connected and has in general no nontrivial automorphism. This implies that for such n and d, the projection $X^n(d)_{\mathrm{PGL}(n+2)} \to H^n(d)_{\mathrm{PGL}(n+2)}$ induces an isomorphism of orbifold fundamental groups. So the separate consideration of $X^n(d)_{\mathrm{PGL}(n+2)}$ is only worthwhile for $n = 0$ (d-element subsets of $\mathbb{P}^1$) or $n = 1$ (nonsingular plane curves of degree d).

Here is what we believe is known about these groups. There is the classical case of the coarse moduli space d-element subsets on a smooth rational curve, $H^0(d)_{\mathrm{PGL}(2)}$, whose orbifold fundamental group is isomorphic to the mapping class group of the 2-sphere less d points (where these points may get permuted) and for which a presentation is due to Birman [2] (although this presentation does not treat the d points on equal footing). The projection $X^0(d)_{\mathrm{PGL}(2)} \to H^0(d)_{\mathrm{PGL}(2)}$ is a ramified covering whose total space can be understood as the coarse moduli space of $(d-1)$-element subsets on an affine line. The orbifold fundamental group of this total space is in fact the usual Artin braid group with $d-1$ strands modulo its center. Another

case with a classical flavor is the moduli space of smooth genus one curves endowed with a divisor class of degree 3, $H^1(3)_{\mathrm{PGL}(3)}$. Dolgachev and Libgober showed [8] that the orbifold fundamental group of $H^1(3)_{\mathrm{PGL}(3)}$ is the semidirect product $\mathrm{SL}(2,\mathbb{Z}) \ltimes (\mathbb{Z}/(3))^2$. It is then easy to see that the orbifold fundamental group of the universal such curve, $X^1(3)_{\mathrm{PGL}(3)}$, is the semidirect product $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ with the projection $X^1(3)_{\mathrm{PGL}(3)} \to H^1(3)_{\mathrm{PGL}(3)}$ inducing the obvious map between these semidirect products. Finally, a concrete presentation for the fundamental group of any $H^n(d)$ has been recently obtained by Michael Lönne [13].

In this paper we shall, among other things, find a presentation for the orbifold fundamental groups of the moduli spaces of Del Pezzo surfaces of degree 3, 4 and 5 (in the last case there are no moduli and the orbifold fundamental group is simply the automorphism group of such a surface). We should perhaps recall that for $d = 3$ this is also the orbifold fundamental group of the moduli space of cubic surfaces $H^2(3)_{\mathrm{PGL}(4)}$. In all these cases the presentatation is obtained as a quotient of some Artin group of affine type (in this case $\hat{E}_6$, $\hat{D}_5$, $\hat{A}_4$). These quotients can in fact be defined in a uniform manner for all such Artin groups. For $\hat{E}_7$ (degree 2) it yields the orbifold fundamental group of the universal quartic curve $X^1(4)_{\mathrm{PGL}(3)}$ and we also have a result for $\hat{E}_8$ (degree 1).

By pushing our methods a bit further we get in fact an efficient presentation of the mapping class group of a compact genus three surface with a single boundary component relative to its boundary. This is obtained by purely algebro-geometric means. The presentation that we find is new and is similar (but not identical) to a M. Matsumoto's simplification of the presentation due to Hatcher-Thurston, Harer and Wajnryb. It leads also to a geometrically meaningful presentation of the mapping class group of any surface with a single boundary component and of genus at least three.

My interest in these questions were originally aroused by conversations with Domingo Toledo, while both of us were staying at the Institut des Hautes Études Scientifiques (IHÉS) in Bures. In fact, it was his asking for a presentation of the orbifold fundamental group of the moduli space of nonsingular cubic surfaces that is at the origin of this paper. He and his collaborators Allcock and Carlson have obtained a beautiful arithmetic representation of this fundamental group [1].

The essential part of this work was done in while I was enjoying the hospitality of the IHÉS in the Spring of 1997. The present paper is a substantial reworking of an earlier version, which was carried out during a visit of the Mittag-Leffler institute in 2007. I am very grateful to both institutions for their support and hospitality and each for offering its own characteristic ambiance that is so favourable for research. I also thank this journal's referee for his or her helpful comments.

Plan of the paper. Section 2, after a brief review of some basic facts about Artin groups, introduces certain factor groups of an Artin group of affine

type. This leads up to the formulation and discussion of our principal results. In Section 3 we investigate these factor groups in more detail. Here the discussion is still in the context of group theory, but in Section 4 we introduce natural orbifolds associated to Artin groups of affine type having such a group as orbifold fundamental group. In Section 5 we occupy ourselves with the following question: given a family of arithmetic genus one curves whose general fiber is an irreducible rational curve with a node and a finite set of sections, what happens to the relative positions of the images of these sections in the Picard group of the fibers, when we approach a special fiber? The answer is used to construct over the orbifolds introduced in Section 4 a family of what we have baptized Del Pezzo triples (of a fixed degree). When the degree is at least three, this moduli space can be understood as the normalization of the dual of the universal Del Pezzo surface of that degree. It contains an orbifold as constructed in Section 4 as the complement of a subvariety of codimension two and so has the same orbifold fundamental group. This leads to a proof of our main results with the exception of those which concern the mapping class group of genus three. The last section is devoted to that case.

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Some conventions. If a reflection group W acts properly on a complex manifold M, then M° will denote the locus of $x \in M$ not fixed by a reflection in W. The use if this notation is of course only permitted if there is no ambiguity on W.

If C is a complete curve, then $\mathrm{Pic}_k(C)$ stands for the set of divisor classes of degree k. We also follow the custom to denote the identity component of a topological group G by G^0. So $\mathrm{Pic}_0(C) \subset \mathrm{Pic}(C)^0$ and this is an equality when C is irreducible.

We write Δ for the spectrum of $\mathbb{C}[[t]]$; we denote its closed point by o and its generic point by Δ^*. (The symbol Δ with a subscript such as Δ_Γ, stands for something entirely different—namely an element of a certain group—but no confusion should arise.)

As a rule we work with the Hausdorff topology and so in general the adjective *open* must be understood in this sense. It should be clear from the context when the Zariski topology is involved.

1. Statement of the results

Artin groups and their reductions. Let Γ consist of a set I and a function m which assigns to each 2-element subset $\{i,j\} \subset I$ an element m_{ij} of $\{2,3,4,\dots\} \cup \infty$. We think of Γ as a weighted graph with vertex set I and an edge drawn between i and j of weight m_{ij} if $m_{ij} \geq 3$. We often simply refer to Γ as a graph when the weighting is understood. The graph Γ defines a presentation of group Art_Γ with generators $\{t_i : i \in I\}$ indexed by the vertex set and relations given as follows: if $i,j \in I$ are distinct, then impose the relation

$$t_i t_j t_i t_j \cdots = t_j t_i t_j t_i \cdots$$

with on both sides m_{ij} letters, whenever that number is finite (no relation is imposed when $m_{ij} = \infty$). It is called the *Artin group* attached to Γ. Such groups were first investigated by Brieskorn-Saito [5] and Deligne [6]. We shall shall refer to the pair $(Art_\Gamma, \{t_i\}_{i\in I})$ as an *Artin system*. The *length homomorphism* $\ell : Art_\Gamma \to \mathbb{Z}$ is the homomorphism which takes on each generator t_i the value 1. If $\Gamma' \subset \Gamma$ is a full subgraph of Γ, then according to Van der Lek [11], [10] the obvious homomorphism $Art_{\Gamma'} \to Art_\Gamma$ is injective and so we may regard $Art_{\Gamma'}$ as a subgroup of Art_Γ.

Imposing also the relations $t_i^2 = 1$ (all i) gives the *Coxeter group* W_Γ defined by Γ. If s_i denotes the image of t_i in W_Γ, then $(W_\Gamma, \{s_i\}_{i\in I})$ is a Coxeter system in the sense of Bourbaki [3]. It is shown there that W_Γ has a natural faithful representation in $\mathbb{R}^I$ with s_i acting as a reflection and and $s_i s_j$ ($i \neq j$) acting as a transformation of order m_{ij}. So Γ can be reconstructed from the Coxeter system and hence also from the Artin

system. The automorphism group of Γ acts in an obvious manner on Art_Γ and we can therefore form the semidirect product $Art_\Gamma \rtimes \mathrm{Aut}(\Gamma)$.

We say that the graph Γ is *of finite type* if W_Γ is finite. Brieskorn-Saito and Deligne have shown that for connected Γ of finite type the group of $u \in Art_\Gamma$ with the property that conjugation by u preserves the generating set $\{t_i\}_{i\in I}$ is infinite cyclic. The homomorphism ℓ is nontrivial on this subgroup, so there is a unique generator Δ_Γ on which ℓ takes a positive value. We call it the *Garside element*. Its square is always central. (The geometric meaning of these facts will be recalled in 2.1.) So the inner automorphism defined by the Garside element is in fact an automorphism of the system (of order at most two) and defines an automorphism of Γ of the same order, the *canonical involution* of Γ. The canonical involution is the identity if the associated Coxeter group contains minus the identity in its natural representation, and in the other cases ($A_{k\geq 2}$, D_{odd}, E_6, I_{odd}) it is the unique involution of Γ. If Γ is of finite type, but not necessarily connected, then Δ_Γ is by definition the product of the Garside elements of the connected components and the canonical involution of Δ_Γ is induced by conjugation with that element (and so the canonical involution on each component).

We say that a connected graph Γ is of *affine type* if the natural representation of W_Γ in $\mathbb{R}^I$ leaves invariant an affine hyperplane in the dual of $\mathbb{R}^I$ on which it acts faithfully and properly. (A characterization in terms of group theory is that W_Γ contains a nontrivial free abelian subgroup of finite index and is not a product of two nontrivial subgroups.) The classification of such graphs can be found in [3], Ch. VI, § 4, Thm. 4.

If Γ is connected, of finite type and given as the underlying graph of a connected Dynkin diagram (this excludes the types H_3, H_4 and $\mathrm{I}_2(m)$ for $m = 5$ or $m \geq 7$ and distinguishes the cases B_l and C_l), then there is a natural *affine completion* $\Gamma \subset \hat{\Gamma}$, where $\hat{\Gamma}$ is of affine type, has one vertex more than Γ, and contains Γ as a full subgraph. All connected graphs of affine type so arise (though not always in a unique manner, see below). The construction is canonical and so the natural involution of Γ extends to $\hat{\Gamma}$ (fixing the added vertex), but for what follows it is important to point out that this involution is *not* induced by conjugation with Δ_Γ.

In the remainder of this section we assume that Γ is connected and of affine type. The groups associated to Γ that we are about to define appear in the statements of our main results. This is the sole reason for introducing them now; a fuller discussion will be given in Section 3.

Given any $i \in I$, then the full subgraph Γ_i on $I - \{i\}$ is always of finite type and so we have defined a Garside element Δ_{Γ_i} which we here simply denote by Δ_i. We say that i is *special* if Γ is the natural completion of Γ_i. As we just noticed, a special vertex defines an involution g_i of Γ. We shall write I_{sp} for the set of special vertices. A glance at the tables tells us that I_{sp} is an orbit of the automorphism group of Γ (but as we will see in Section 3, it is in fact a consequence of a fuller treatment of the completion process) and

so the isomorphism type of Γ_i is determined by that of Γ. If Γ is of type $\hat{A}_l$ (a polygon with $l+1$ vertices in case $l > 1$), then all vertices are special. But for the other connected graphs of affine type there are only few of these: the graphs in question are trees and their special vertices are ends (i.e., lie on a single edge). To be precise, for those of type $\hat{D}_l$ we get all 4 ends, for $\hat{B}_l$ and $\hat{C}_l$ 2 ends, for $\hat{E}_6$, $\hat{E}_7$, $\hat{E}_8$ the ends on the longest branches (hence resp. 3, 2, 1 in number) and in the remaining cases $\hat{F}_4$ and $\hat{G}_2$ only one.

There is a similar property for pairs (i, j) of distinct special vertices in I: then there is an involution g_{ij} of Γ which interchanges i and j and is the canonical involution on the full subgraph Γ_{ij} of Γ spanned by $I - \{i, j\}$. If Γ has more than two vertices and is not a polygon, then Γ_{ij} is obtained from Γ by removing two ends and hence is still connected. The elements g_i and g_{ij} generate a subgroup $S(\Gamma) \subset \mathrm{Aut}(\Gamma)$, which is in fact all of $\mathrm{Aut}(\Gamma)$ unless Γ is of type $\hat{D}_{\mathrm{even}}$.

Definition 1.1. We define the *reduced Artin group* $\overline{Art}_\Gamma$ as the quotient of the semidirect product $Art_\Gamma \rtimes S(\Gamma)$ by the imposing the relations:

$$\Delta_i g_i \equiv 1 \quad \text{for } i \text{ special.} \tag{1}$$

$$\Delta_{ij} g_{ij} \equiv 1 \quad \text{for } i, j \text{ distinct and special.} \tag{2_m}$$

Here Δ_{ij} is the Garside element for the Artin subgroup defined by the full subgraph on $I - \{i, j\}$.

A few observations are in order. It is clear from the definition that the evident homomorphism $Art_\Gamma \to \overline{Art}_\Gamma$ is surjective. A description of $\overline{Art}_\Gamma$ as a quotient of Art_Γ amounts to saying that conjugation with any word w_Δ in the Garside elements Δ_i and Δ_{ij} permutes the generating set $\{t_i\}_{i \in I}$ as the corresponding word w_g in the g_i's and the g_{ij}'s and that w_Δ represents the identity element precisely when w_g does. In particular,

$$\Delta_i t_i \equiv t_i \Delta_i, \quad \Delta_i^{\mathrm{ord}\, g_i} \equiv 1 \quad \text{for } i \text{ special,}$$

$$\Delta_{ij} t_i \equiv t_j \Delta_{ij}, \quad \Delta_{ij}^{\mathrm{ord}\, g_{ij}} \equiv 1 \quad \text{for } i, j \text{ distinct and special.}$$

For a pair (i, j) as above, the identity $t_i \equiv \Delta_{\Gamma_{ij}} t_j \Delta_{\Gamma_{ij}}^{-1}$ shows that the image of t_i in $\overline{Art}_\Gamma$ can be expressed in terms of the images of the other generators t_j, $j \neq i$, so that $\overline{Art}_\Gamma$ is in fact a quotient of Art_{Γ_i}. But this breaks the symmetry and as a consequence the defining relations are no longer easy to describe.

As already mentioned, the affine Coxeter group W_Γ is naturally realized as an affine transformation group. The quotient of this affine transformation group by its translation subgroup is a finite (Coxeter) group W_Γ^{f} acting naturally in a vector space V. The center $Z(W_\Gamma^{\mathrm{f}})$ of W_Γ^{f} is the intersection of W_Γ^{f} with $\{\pm 1_V\}$. We shall show that the quotient of $\overline{Art}_\Gamma$ obtained by imposing the relations $t_i^2 \equiv 1$ for all $i \in I$ can be identified with $W_\Gamma^{\mathrm{f}}/Z(W_\Gamma^{\mathrm{f}})$.

We denote the latter group by $\overline{W}^{\mathrm{f}}_{\Gamma}$ (so it equals W^{f}_{Γ} in case Γ is of type $\hat{A}_{l\geq 2}$, $\hat{D}_{\mathrm{odd}}$ or $\hat{E}_6$). We will see that in general the homomorphism $\overline{Art}_{\Gamma} \to \overline{W}^{\mathrm{f}}_{\Gamma}$ has a nontrivial kernel. An exception is the case $\hat{A}_l$ (where $\overline{W}^{\mathrm{f}}_{\Gamma} = \mathcal{S}_{l+1}$):

Proposition 1.2. *For $l > 1$, the natural homomorphism $\overline{Art}_{\hat{A}_l} \to \mathcal{S}_{l+1}$ is an isomorphism.*

We will prove this (relatively easy) assertion in Section 3.

Some orbifold fundamental groups. We can now state some of our main results. These concern the moduli spaces of Del Pezzo surfaces of degree $d \leq 5$. Recall that such a surface can be obtained by blowing up $9-d$ points of the projective plane in general position (but is not naturally given this way). For $d = 2, 3, 4$ the moduli space in question is the space of projective equivalence classes of respectively smooth quartic curves in $\mathbb{P}^2$, of cubic surfaces in $\mathbb{P}^3$ and of complete intersections of two quadrics in $\mathbb{P}^4$. For $d = 5$ there are no moduli since the Del Pezzo surfaces of degree 5 are mutually isomorphic. But that does not imply that its orbifold fundamental group is trivial: this will be the (finite) automorphism group of that surface.

Theorem 1.3. *The orbifold fundamental group of the moduli space of Del Pezzo surfaces of degree resp. 5, 4 and 3 is isomorphic to a reduced Artin group of type resp. $\hat{A}_4$, $\hat{D}_5$ and $\hat{E}_6$. The natural surjection of this group on the (centerless) finite Weyl group of the corresponding type (resp. A_4, D_5 and E_6) describes the representation of orbifold fundamental group on the integral homology of the Del Pezzo surface.*

Notice that Proposition 1.2 implies that the automorphism group of a degree 5 Del Pezzo surface acts faithfully (via the finite Weyl group of type A_4) on its homology.

As remarked above, we can change the presentation of the reduced Artin group of type $\hat{E}_6$ into one which does not involve one of the special generators. We thus recover a theorem of Libgober [12], which says that the orbifold fundamental group of the moduli space of cubic surfaces is a quotient of an Artin group of type E_6. Our result says even which quotient, but as we noted, such a description does not do justice to the $\hat{E}_6$-symmetry.

For Del Pezzo moduli in degree 1 and 2 the situation is a little different. The graph of type $\hat{E}_7$ has two special vertices. Let us write simply E_7 and E'_7 for the corresponding subdiagrams and E_6 for their intersection (this labeling is of course of a selfreferencing nature).

Theorem 1.4. *The orbifold fundamental group of the universal smooth quartic plane curve, given up to projective equivalence is isomorphic to a reduced Artin group of type $\hat{E}_7$: it is the quotient of the Artin group of type $\hat{E}_7$ by the relations*

(i) $\Delta_{E_7} \equiv 1$,
(ii) $t_1\Delta_{E_6} \equiv \Delta_{E_6}t_7$ *and* $\Delta^2_{E_6} \equiv 1$.

The natural surjection of this group on the Weyl group of type E_7 modulo its center describes the representation of orbifold fundamental group on the homology of a smooth quartic curve with $\mathbb{Z}/2$-coefficients.

Perhaps we should point out that relation (ii) in this theorem implies that $\Delta_{E_7'} = \Delta_{E_6}\Delta_{E_7}\Delta_{E_6}^{-1}$ so that $\Delta_{E_7'} \equiv 1$ by relation (i) and we indeed obtain the reduced Artin group of type $\hat{E}_7$.

This theorem leads to a new simple presentation of the mapping class group of genus three. Recall that any smooth (complex) projective curve of genus three is either a quartic curve or is hyperelliptic. The hyperelliptic curves form in the moduli space of smooth projective genus three curves a divisor, so we expect the orbifold fundamental group of the universal smooth complex genus three curve to be a quotient of $\overline{Art}_{\hat{E}_7}$. This is the case: if $A_7 \subset \hat{E}_7$ denotes the unique subgraph of that type, then we have:

Theorem 1.5. *The orbifold fundamental group of the universal smooth complex genus three curve is naturally the quotient of the reduced affine Artin group of type $\hat{E}_7$ by the relation $\Delta_{A_7} \equiv \Delta_{E_6}$.*

Remark 1.6. We also find something of interest in the case $d = 1$: the reduced Artin group of type $\hat{E}_8$ is the orbifold fundamental group of pairs (S, K), where S is a Del Pezzo surface of degree 1 and K is a singular member of its anticanonical pencil (there are in general 12 such members).

Presentation of the mapping class group. Take a compact oriented surface Σ_g of genus g with connected boundary $\partial\Sigma_g$ (so $\partial\Sigma_g$ is diffeomorphic to a circle). We can consider the group of orientation preserving diffeomorphisms of Σ_g and its subgroup of diffeomorphisms that are the identity on the boundary. We follow Harer in denoting the groups of connected components by Γ_g^1 and $\Gamma_{g,1}$ respectively (although in view of a convention in algebraic geometry, a notation like $\Gamma_{g,1}$ resp. $\Gamma_{g,\vec{1}}$ would perhaps be more logical). The evident map $\Gamma_{g,1} \to \Gamma_g^1$ is surjective and its kernel is the (infinite cyclic) group generated by the Dehn twist along the boundary. Their Eilenberg-Maclane spaces have algebro-geometric incarnations that we now recall.

If $g \geq 1$, then the moduli space of pairs (X, p), where X is a complete nonsingular genus g curve and $p \in X$, is an orbifold (Deligne-Mumford stack) $\mathcal{M}_{g,1}$. Its orbifold fundamental group is Γ_g^1 and its orbifold universal covering (a Teichmüller space) is contractible. If in addition there is given a nonzero tangent vector to X at p, then the corresponding moduli space, which we propose to denote by $\mathcal{M}_{g,\vec{1}}$, is fine and smooth: this is because an automorphism of a projective nonsingular curve of positive genus which fixes a nonzero tangent vector must be the identity. Notice that it comes with an action of $\mathbb{C}^\times$ (acting as scalar multiplication on the tangent vector). That action is proper (but not free: finite nontrivial isotropy groups do occur) and its orbit space may be identified with $\mathcal{M}_{g,1}$. The smooth variety $\mathcal{M}_{g,\vec{1}}$ is an

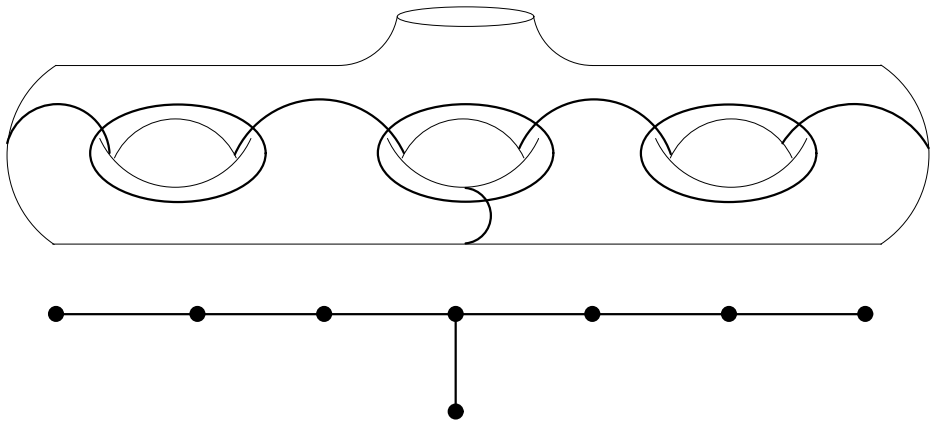

FIGURE 1. A holed genus three surface with a $\hat{E}_7$-configuration

Eilenberg-MacLane space for $\Gamma_{g,1}$: it has the latter as its fundamental group and its universal covering is contractible. The projection $\mathcal{M}_{g,\vec{1}} \to \mathcal{M}_{g,1}$ induces the natural map $\Gamma_{g,1} \to \Gamma_g^1$ and the positive generator of a fiber ($\cong \mathbb{C}^\times$) corresponds to a Dehn twist along the boundary. The fundamental group of $\mathcal{M}_{3,\vec{1}}$ can also be presented as a quotient of $Art_{\hat{E}_7}$:

Theorem 1.7. *The fundamental group of $\mathcal{M}_{3,\vec{1}}$ (which is also is the mapping class group $\Gamma_{3,\vec{1}}$ of a compact oriented genus three surface with a circle as boundary) is isomorphic to the quotient of $Art_{\hat{E}_7}$ by the relations*

(i) $\Delta_{E_7} \equiv \Delta_{E_6}^2$,
(ii) $\Delta_{A_7} \equiv \Delta_{E_6}$.

The proof will show that the Artin generators map to Dehn twists along embedded circles which meet like a $\hat{E}_7$-diagram as in Figure 1 (the Dehn twists along two curves α, β commute if the curves are disjoint, whereas in case they meet simply in a single point their isotopy classes D_α, D_β satisfy Artin's braid relation $D_\alpha D_\beta D_\alpha = D_\beta D_\alpha D_\beta$). It can be verified that the Garside elements Δ_{A_7} and Δ_{E_6} themselves can be represented by the same homeomorphism; this homeomorphism is the composite of an involution of $\Sigma_{3,1}$ with three (interior) fixed points (in Figure 1 this is the reflection with respect to the vertical axis of symmetry) and a half Dehn twist along $\partial\Sigma_{3,1}$ to ensure that it leaves $\partial\Sigma_{3,1}$ pointwise fixed. The Garside element Δ_{E_7} is the Dehn twist along $\partial\Sigma_{3,1}$.

A rather explicit, but still somewhat involved presentation of this group is due to Wajnryb [20]. Makoto Matsumoto [18] recently deduced with the help of a computer from that presentation a leaner one which differs from ours by just one relation (he also obtains the first relation) and involves Artin groups. Nevertheless, at present the equivalence of his and our presentation has not yet been established. In any case, the following geometric description of the generators and relations in the above theorem should

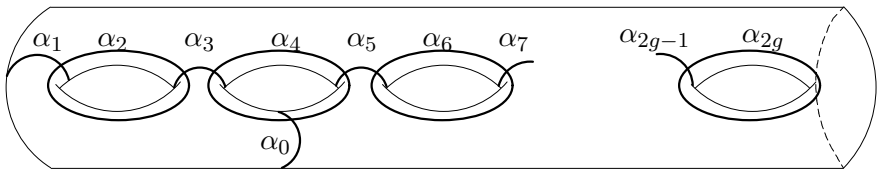

FIGURE 2. Humphries' Dehn twists generating $\Gamma_{g,1}$

make a direct comparison with the one of Wajnryb and Matsumoto possible. Another difference with his work is that we get the generators and relations handed to us by some classical algebraic geometry so that the presentation that we get is a very natural one.

Genus three is important because Wajnryb's presentation of $\Gamma_{g,1}$ shows that all the exotic (non Artin) relations manifest themselves here. To be precise, if $g \geq 3$ and $\alpha_0, \ldots, \ldots, \alpha_{2g}$ are the embedded circles in $\Sigma_{g,1}$ as indicated in Figure 2, then following Humphries [9] the associated Dehn twists $D_{\alpha_0}, \ldots, D_{\alpha_{2g}}$ generate $\Gamma_{g,1}$ and a theorem of Wajnryb [20] implies that the relations among them follow from the obvious Artin relations recalled above and the exotic relations that involve $D_{\alpha_0}, \ldots, D_{\alpha_6}$ only and define $\Gamma_{3,1}$. This fact is to an algebraic geometer rather suggestive: the locus in $\mathcal{M}_{g,\vec{1}}$ for which the underlying curve is hyperelliptic and the point is a Weierstraß point is a smooth closed subvariety $\mathcal{H}_{g,\vec{W}}$ of $\mathcal{M}_{g,\vec{1}}$. It is an Eilenberg-MacLane space for the *hyperelliptic mapping class group* of genus g: if ι is any involution of $\Sigma_{g,1}$ with $2g+1$ (interior) fixed points, for example in Figure 2 the symmetry in a horizontal axis, then this is the connected component group of the group of diffeomorphisms that leave its boundary pointwise fixed and commute with ι. This group has a presentation with generators the Dehn twists $D_{\alpha_1}, \ldots, D_{\alpha_{2g}}$ subject to the Artin relations (this is indeed Artin's braid group with $2g+1$ strands). Wajnryb's presentation can be interpreted as saying that the inclusion in $\mathcal{M}_{g,\vec{1}}$ of the union of $\mathcal{H}_{g,\vec{W}}$ and a regular neighborhood of a certain locus in the Deligne-Mumford boundary (that is easy to specify) intersected with $\mathcal{M}_{g,\vec{1}}$ induces an isomorphism on fundamental groups. It is an interesting challenge to algebraic geometers to prove this with methods indigenous to their field (such as those developed by Zariski and the later refinements by Fulton-Lazarsfeld). At the very least we would then have reproduced the above presentation of $\Gamma_{g,1}$ for $g \geq 3$ without the aid of a computer, but most likely this leads also to some conceptual gain.

2. ORBIT SPACES OF FINITE REFLECTION GROUPS–A BRIEF REVIEW

In this short section we recall some facts concerning the Artin groups attached to finite reflection groups. Let W be a finite group of transformations of a real, finite dimensional vector space V which is generated by reflections

and such that the fixed point set of W is reduced to $\{0\}$. The fixed point hyperplanes of the reflections in W decompose V into simplicial cones, called *faces*. An open face is called a *chamber*. Let us fix a chamber C and let $\{H_i\}_{i\in I}$ be the collection of its supporting hyperplanes. Each H_i is the fixed point set of a reflection $s_i \in W$. If m_{ij} denotes the order of s_is_j, then let Γ be the graph on the index set I that has $m_{ij}-2$ edges connecting i and j (with $i,j\in I$ distinct). The obvious homomorphism $W_\Gamma \to W$ is known to be an isomorphism [3] and we will therefore identify W with W_Γ. The element $w_0 \in W$ that sends C to $-C$ is called the *longest element of* W. Notice that $-w_0$ is a linear transformation of V which maps C to itself and hence induces a permutation in I. Since $w_0^2=1$, this permutation is called the *canonical involution* of Γ.

The Artin group Art_Γ has now an interpretation as a fundamental group: The action of W on

$$\mathbb{V}^\circ := \mathbb{V} - \cup\{\mathbb{H} \,:\, H \text{ refl. hyperpl. of } W\}$$

is free (the use blackboard font indicates that we have complexified the subspace of V of that same name). Brieskorn's theorem says that the fundamental group the W-orbit space $\mathbb{V}^\circ_W$ is the Artin group attached to W. For a precise statement (and for some related properties discussed below) it is convenient to use a contractible open set $\mathbb{U}$ if $\mathbb{V}^\circ$ that we now describe. Given $x\in V$, let C_x denote the intersection of halfspaces containing C bounded by a reflection hyperplane passing through x. So C_x is an open cone containing C; it is all of V if x is not in any reflection hyperplane. Now let $\mathbb{U}\subset\mathbb{V}$ be the subset consisting of the $x+\sqrt{-1}y$ with the property that $y\in C_x$. This is an subset of $\mathbb{V}^\circ$ that is open for the Hausdorff topology. Moreover, $\mathbb{U}$ is starlike with respect to any point in $\sqrt{-1}C$ and hence contractible. Now fix a base point $*\in C$ and regard it as an element of $\mathbb{U}$. Then for every $w\in W$, $w(*)\in\mathbb{U}$ as well and there is a well-defined relative homotopy class of curves in $\mathbb{U}$ of curves connecting $*$ with $w(*)$. Denote the image of that class in the orbit space $\mathbb{V}^\circ_W$ by $t(w)$ so that $t(w)\in\pi_1(\mathbb{V}^\circ_W,*)$.

Proposition 2.1 (Brieskorn [4]). *The group $\pi_1(\mathbb{V}^\circ_W,*)$ is generated by the elements $t(s_i)$. These generators satisfy the Artin relations*

$$t(s_i)t(s_j)t(s_i)\cdots = t(s_j)t(s_i)t(s_j)\cdots \quad (m_{ij} \text{ letters on both sides})$$

and the resulting homomorphism $Art_\Gamma\to\pi_1(\mathbb{V}^\circ_W,*)$ is an isomorphism.

Via this isomorphism, the map $w\in W_\Gamma\mapsto t(w)\in\pi_1(\mathbb{V}^\circ_W,*)$ becomes a section of the natural epimorphism $Art_\Gamma\to W_\Gamma$; it sends the longest element $w_0\in W$ (that maps C to $-C$) to the Garside element Δ_Γ and has the property that $t(ww')=t(w)t(w')$, provided that the length of the product ww' equals the sum of the lengths w and w'.

In particular, for distinct i and j, we have $t(s_i)t(s_j)t(s_i)\cdots = t(s_is_js_i\cdots)$ (m_{ij} letters) and this is also the Garside element of the Artin subgroup generated by $t(s_i)$ and $t(s_j)$. Such a section exists for any Artin system (wether it is of finite type or not).

It is well-known that if W is irreducible in the sense that it cannot be written as a product of two nontrivial reflection groups, then the center of W is $W \cap \{\pm 1_V\}$. Brieskorn-Saito and Deligne show that the center of Art_Γ is then generated by Δ_Γ or Δ_Γ^2 according to whether or not $-1_V \in W$.

3. Affine reflection groups

In this section A is a real, finite dimensional affine space with translation space V. The latter is a vector space, but since one usually writes a transformation group multiplicatively, we write $\tau_v : A \to A$ for translation over $v \in V$. We are also given a group W of affine-linear transformations of A generated by reflections which acts properly discretely with compact orbit space. (For a group generated by reflections this amounts to its translation subgroup being of finite index and defining a discrete cocompact lattice in V.) In order to avoid irrelevant complications, we assume that W acts irreducibly in V.

Realization as a Coxeter group. We may identify W as a Coxeter group in much the same way as for a finite reflection group. The general reference for this section is [3]. The fixed point hyperplanes of the reflections in W decompose A into bounded (relatively open) polyhedra, which turn out to be simplices. The group W acts simply transitively on the collection of open simplices (the chambers). We fix one such chamber C. Then $\bar{C}$ is a fundamental domain for W in A. Let $\{H_i\}_{i\in I}$ be the collection of the supporting hyperplanes of C, and let $s_i \in W$ be the reflection that has H_i as its fixed point hyperplane. If m_{ij} denotes the order of $s_i s_j$, then let Γ be the graph on the index set I that has $m_{ij} - 2$ edges connecting i and j (with $i, j \in I$ distinct). Then Γ is connected and of affine type and the obvious homomorphism $W_\Gamma \to W$ is an isomorphism [3]. We shall therefore identify W_Γ with W. The Tits construction [3] shows that any connected graph of affine type so arises.

We denote the translation lattice of W by $Q \subset V$ and the image of W in $GL(V)$ by W^{f}. Then W is isomorphic to the semidirect product $Q \rtimes W^{\mathrm{f}}$, for there exist points $a \in A$ whose stabilizer W_a maps isomorphically onto W^{f}. Such a point is called a *special point* for this action.

Lemma-Definition 3.1. *The group of translations in A that normalize W acts transitively on set of special points in A. It is called the* weight lattice *and denoted P.*

Proof. If $a, b \in A$ are special, then clearly, the translation over $b - a$ normalizes W. Conversely, a translation that normalizes W permutes the special points. □

It is clear that P contains Q. (In fact, W^{f} is the Weyl group of a finite, irreducible, reduced root system whose span is Q and P is the lattice dual to the span of the dual root system.) The notions of a special point and of

a special vertex are related as follows: if for $i \in I$, a_i denotes the unique vertex of $\bar{C}$ that is not contained H_i, then i is special precisely when a_i is.

Consider the normalizer of W in the group of affine transformations of A. Then the stabilizer of C in this normalizer acts faithfully on C, and hence likewise on the Coxeter system $(W, \{s_i\}_{i\in I})$ and the graph Γ. The Tits construction makes evident that the image is all of $\mathrm{Aut}(\Gamma)$ and so the normalizer is in fact $W \rtimes \mathrm{Aut}(\Gamma)$. The translation subgroup of the normalizer is the lattice P and hence $\mathrm{Aut}(\Gamma)$ contains P/Q as a distinguished subgroup. Now the Corollary to Prop. 6 of Ch. VI, §2 of [3] can be stated as follows:

Lemma 3.2. *The group P/Q acts simply transitively on I_{sp}.*

The characterization of $\mathrm{Aut}(\Gamma)$ as the stabilizer of C in the normalizer of W leads to a geometric interpretation of the elements g_i and g_{ij} of Γ (with i and j special) defined earlier. For every (nonempty) face F of C we denote by W_F the subgroup of W that leave F pointwise fixed. This group is generated by the s_i, $i \in I$, with $H_i \supset F$ and $(W_F, \{s_i\}_{H_i\supset F})$ is a Coxeter system of finite type. Let w_F denote the element of W_F that maps C to the chamber that is opposite F. This is an element of order two and is the 'longest' element of W_F. In case F consists of a single vertex a_i or is the span of two such vertices a_i, a_j, then we also write w_i resp. w_{ij} instead of W_F. Denote by ι_j resp. ι_{ij} the central symmetry of A in the vertex a_j resp. the midpoint of a_i and a_j: $\iota_j(a) = a - 2(a - a_j)$ and $\iota_{ij}(a) = a - (2a - a_i - a_j)$.

Lemma 3.3. *Assume that $j \in I$ has the property that ι_j normalizes W. Then $w_j\iota_j$ is an involution g_j of C whose image in $\mathrm{Aut}(\Gamma)$ fixes j. It preserves each connected component of Γ_j and acts there as the canonical involution. The assumption is satisfied when j is special (and in that case Γ_j is connected, so that g_j acts on Γ_j as the canonical involution).*

Proof. The first two statements are clear from the definitions. If $j \in I$ is special, then $W = Q \rtimes W_j$ and so ι_j normalizes W. □

It may happen that ι_j normalizes W for some nonspecial $j \in I$. An example that will appear here is when Γ is of type $\hat{E}_7$ and j is the end of the short branch.

If two special points of A belong to the same chamber (i.e., if the open interval they span is a one dimensional face), then in accordance with received terminology, their difference in V is called a *minuscule weight*. That is why we shall refer to the corresponding pair of special points as a *minuscule pair*. Since $\iota_{ij}\iota_i$ is translation over $a_j - a_i$, the involution ι_{ij} normalizes W. Define g_{ij} by the property that $\iota_{ij} = w_{ij}g_{ij}$. This is an affine linear transformation of A which preserves C.

Recall that $S(\Gamma)$ stands for the subgroup of $\mathrm{Aut}(\Gamma)$ generated by the elements g_i and g_{ij} with i special resp. $\{i, j\}$ a minuscule pair. We shall write

$$SW := W \rtimes S(\Gamma) \text{ and } SArt_\Gamma := Art_\Gamma \rtimes S(\Gamma).$$

We also recall that we introduced the reduced Artin group $\overline{Art}_\Gamma$ (Definition 1.1) as a quotient of $SArt_\Gamma$. It is clear that SW can be obtained from $SArt_\Gamma$ by imposing the relations $t_i^2 \equiv 1$ (all i). We put

$$SW^{\mathrm{f}} := \{\pm 1_V\}.W^{\mathrm{f}} \subset \mathrm{GL}(V), \quad \overline{W^{\mathrm{f}}} := \{\pm 1_V\}.W^{\mathrm{f}}/\{\pm 1_V\}.$$

So the first group or the second group equals W^{f}, depending on whether or not $-1_V \in W^{\mathrm{f}}$.

Lemma 3.4. *The group $S(\Gamma)$ contains P/Q as a normal subgroup. The quotient is cyclic of order two if W^{f} does not contain -1_V and is trivial otherwise, in other words, the natural homomorphism $SW \to \mathrm{GL}(V)$ has kernel P and image SW^{f}. Moreover, $\overline{W^{\mathrm{f}}}$ is obtained as a quotient of the reduced Artin group $\overline{Art}_\Gamma$ by imposing the relations $t_i^2 \equiv 1$ (all i).*

Proof. We use the fact SW is generated by W and the central symmetries ι_i and ι_{ij}. We have seen that SW contains the translation over $a_j - a_i$ if (i,j) is a minuscule pair. So SW contains P as a translation subgroup. Since P is the full group of translations normalizing W, it follows that P is the kernel of the natural homomorphism $SW \to \mathrm{GL}(V)$. Since ι_i and ι_{ij} act in V as minus the identity it follows that the image is $\{\pm 1_V\}.W^{\mathrm{f}}$, as asserted.

Finally, the set of relations imposed on Art_Γ to get $\overline{Art}_\Gamma$ yield in SW the relations $w_i g_i \equiv 1$ (i special) and $w_i g_i \equiv w_{ij} g_{ij}$ ($\{i,j\}$ a minuscule pair). Since $w_i g_i = \iota_i$ and $w_{ij} g_{ij} = \iota_{ij}$, we see that imposing these relations amount to killing P (so that we get a quotient of SW^{f}) and subsequently the common image -1_V of the central symmetries. The result is indeed $\overline{W^{\mathrm{f}}}$. $\square$

Since conjugation by g_{ij} interchanges g_i and g_j, the g_i's make up a single conjugacy class in $S(\Gamma)$. So if P/Q is cyclic and W^{f} does not contain -1_V, then $S(\Gamma)$ has the structure of a dihedral group whose group of rotations is P/Q and whose reflections are the g_i's and the g_{ij}'s.

Let us go through the list of connected graphs of affine type and see what we get (see for instance [3] for a classification). When there is just one special point ($\hat{E}_8, \hat{F}_4, \hat{G}_2$) or two ($\hat{A}_1$, $\hat{B}_{l\geq 2}$, $\hat{C}_{l\geq 3}$, $\hat{E}_7$), there is little to say: in these cases, the involutions g_i must be trivial and $S(\Gamma) = \mathrm{Aut}(\Gamma)$ (and of order at most two).

In case $\hat{D}_{l\geq 4}$ the special vertices are the four end vertices. When $l \geq 5$ the graph $\hat{D}_l$ has two branch points, each connected with two end vertices. When l is odd, then we arrange the four end vertices as the vertices of a square such that vertices separated by a single branch vertex are opposite and $S(\Gamma)$ can be identified with the corresponding dihedral group of order 8. When l is even, the involutions g_i are all trivial and so $S(\Gamma) = P/Q$. The latter is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

The remaining cases are $\hat{A}_{l\geq 2}$ and $\hat{E}_6$. In the first case, Γ is an $(l+1)$-gon all of whose points are special and $S(\Gamma) = \mathrm{Aut}(\Gamma)$ is a dihedral group of order $2(l+1)$. In case $\hat{E}_6$ the situation is basically the same: we have three

special points, the g_i's are the transpositions and $g_i = g_{jk}$ if i, j, k are special and mutually distinct. From this description we deduce:

Conclusion 3.5. *The group $S(\Gamma)$ equals $\mathrm{Aut}(\Gamma)$ (the latter being understood as the group of permutations of the vertex set of Γ that preserve the weights of the edges) and is a dihedral group, unless Γ is of type $\hat{D}_{\mathrm{even}}$, in which case $S(\Gamma) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.*

Affine Artin groups as fundamental groups. We continue with the situation of the previous section. The complexification $\mathbb{A}$ of the affine space A is naturally isomorphic to $A \times V$ (the affine transformation group of A acts after complexification on the imaginary part through $GL(V)$).

Let $\mathbb{A}^\circ \subset \mathbb{A}$ be the complement of the union of all the complexified reflection hyperplanes of W. Then W acts freely on $\mathbb{A}^\circ$ so that its orbit space $\mathbb{A}^\circ_W$ is a complex manifold. Its fundamental group can be determined in much the same way as in the finite reflection group case. We begin with defining a contractible W-open neighborhood $\mathbb{U}$ of the W-orbit of $C \times \{0\}$ in $\mathbb{A}^\circ$ that is the analogue of its namesake introduced in Section 2. Given $x \in A$, then the intersection of halfspaces containing C and bounded by a reflection hyperplane passing through x is of the form $x + C_x$ for some open linear convex cone C_x in V and we let $\mathbb{U}$ to be the set of $(x, y) \in \mathbb{A}$ with $y \in C_x$. This is indeed an open subset of $\mathbb{A}^\circ$. The projection $\mathbb{U} \to A$ is surjective and the preimage of the star of every face of A is starlike (see the argument used for a finite reflection group) and hence contractible. This implies that $\mathbb{U}$ is contractible.

Let $x_o \in C$ be the barycenter (and hence fixed under any automorphism of C) and let $* := (x_o, 0) \in \mathbb{U}$. For every $w \in W$, there is a unique homotopy class of curves in $\mathbb{U}$ from $*$ to $w(*)$; we denote the image of that homotopy class in $\pi_1(\mathbb{A}^\circ_W, *)$ by $t(w)$. In analogy to 2.1 we have:

Proposition 3.6 (Nguyêñ Viêt Dũng [19])**.** *The group $\pi_1(\mathbb{A}^\circ_W, *)$ is generated by the elements $t(s_i)$. These generators satisfy the Artin relations*

$$t(s_i)t(s_j)t(s_i)\cdots = t(s_j)t(s_i)t(s_j)\cdots \quad (m_{ij} \text{ letters on both sides})$$

and the resulting homomorphism $Art_\Gamma \to \pi_1(\mathbb{A}^\circ_W, *)$ is an isomorphism.

Via this isomorphism, the map $w \in W_\Gamma \mapsto t(w) \in Art_\Gamma$ becomes a section of the projection $Art_\Gamma \to W_\Gamma$ with the property that $t(ww') = t(w)t(w')$ if the length of ww' is the sum of the length of w and of w'.

The group SW acts also on $\mathbb{A}^\circ$ and we thus find

Corollary 3.7. *The orbifold fundamental group of $\mathbb{A}^\circ_{SW}$ is $SArt_\Gamma$.*

Under this identification (as defined in the introduction of this paper), an element $g \in S(\Gamma)$ is represented by the pair consisting of the constant path $[*]$ at $*$ and the element g. So $t(w).g$ is represented by the pair $(t(w), wg)$. On the other hand, $gt(w)$ is represented by $(t({}^g w), gw)$, We extend the section $W \to Art$ to a section t of $SArt_\Gamma \to SW$ by putting $t(wg) := t(w)g$.

Lemma 3.8. *If (i,j) is a minuscule pair, then $t(\tau_{a_j-a_i}) = (\Delta_{ij}g_{ij})^{-1}\Delta_i g_i$.*

Proof. From $\tau_{a_i-a_j} = \iota_{ij}\iota_i = w_{ij}g_{ij}w_i g_i = w_{ij}w_j g_{ij} g_i = w_{ij}^{-1}w_j g_{ij} g_i$ we see that $t(\tau_{a_i-a_j}) = t(w_{ij}^{-1}w_j)g_{ij}g_i$. Since the length of w_j is the sum of the lengths of w_{ij} and $w_{ij}^{-1}w_j$, we have $\Delta_j = \Delta_{ij}t(w_{ij}^{-1}w_j)$. Substituting this in the preceding gives $t(\tau_{a_i-a_j}) = \Delta_{ij}^{-1}\Delta_j g_{ij}g_i = \Delta_{ij}^{-1}g_{ij}\Delta_i g_i = (\Delta_{ij}g_{ij})^{-1}\Delta_i g_i$. □

Toric structure of the orbifold. We can also arrive at $\mathbb{A}^\circ_{SW}$ by starting from the intermediate orbit space $\mathbb{A}_P$. Notice that $\mathbb{A}_P$ is naturally an algebraic torus whose identity element is the image of the special orbit. As such it is canonically isomorphic the 'adjoint' torus $\mathbb{C}^\times \otimes_{\mathbb{Z}} P$, where the adjective *canonically* guarantees (via Lemma 3.4) equivariance relative to $SW^{\mathrm{f}} = SW/P$. We shall write $\mathbb{T}$ for $\mathbb{C}^\times \otimes_{\mathbb{Z}} P$. The open subset $\mathbb{T}^\circ$ is the complement of the union of the reflection hypertori (relative to the SW^{f}-action). It is clear that

$$\mathbb{A}^\circ_{SW} \cong \mathbb{T}^\circ_{SW^{\mathrm{f}}}.$$

We shall use this identification to describe certain extensions of $\mathbb{A}^\circ_{SW}$ as an orbifold by means of SW^{f}-equivariant extensions of $\mathbb{T}^\circ$.

We take the occasion to prove Proposition 1.2.

Proof of Proposition 1.2. In view of Lemma 3.4 it suffices to prove that the square of every generator of $\hat{A}_l$ becomes 1 in $\overline{W}_{\hat{A}_l}$.

To this end we label the vertices of the $(l+1)$-gon $\hat{A}_l$ in cyclic order by $\mathbb{Z}/(l+1)$. If ζ denotes the rotation of this polygon that sends the vertex with index i to the one with index $i+1$, then the relations imposed on $\overline{Art}_{\hat{A}_l}$ imply that $\Delta_{\mathbb{Z}/(l+1)-\{i\}} \equiv \Delta_{\mathbb{Z}/(l+1)-\{i,j\}}\zeta^{i-j}$ for all distinct pairs (i,j) in $\mathbb{Z}/(l+1)$. If we use that

$$\Delta_{A_k} = t_1(t_2t_1)(t_3t_2t_1)\cdots(t_kt_{k-1}\cdots t_1),$$

(which follows for instance from the fact that the corresponding product of transpositions is a reduced expression for the longest element of $\mathcal{S}_{k+1}$), we see that for $(i,j) = (0,l)$ resp. $(i,j) = (0,l-1)$ we get

$$t_lt_{l-1}\cdots t_1 \equiv \zeta \quad \text{resp.} \quad (t_{l-1}t_{l-2}\cdots t_1)(t_lt_{l-1}\cdots t_1) \equiv t_l\zeta^2.$$

If we left-multiply the last congruence by t_l and substitute therein the first congruence, we find that $t_l^2 \equiv 1$. Any other generator t_i is conjugate to t_l and so we also have that $t_i^2 \equiv 1$. □

Effect of a divisorial extension on the fundamental group. Let X be a connected complex manifold, and $Y \subset X$ a smooth hypersurface and write U for its complement $X - Y$. We fix a base point $* \in U$. Then the natural homomorphism $\pi_1(U,*) \to \pi_1(X,*)$ is surjective with with its kernel normally generated by the simple (positively oriented) loops around Y. Two such loops are conjugate in $\pi_1(U,*)$ if they encircle the same connected component. So we may say that $\pi_1(X,*)$ is obtained from $\pi_1(U,*)$ by imposing a

relation $\rho_i \equiv 1$ for every connected component Y_i of Y (with ρ_i represented by a simple loop around Y_i).

This is still true in an orbifold setting: if a group G acts properly discontinuously on X and preserves Y, then the natural homomorphism

$$\pi_1^G(U,*) \to \pi_1^G(X,*)$$

is surjective with its kernel normally generated by the ρ_i's, where i now runs over a system of representatives of the G-action on $\pi_0(Y)$.

Toric extensions. Any indivisible $p \in P$ defines an injective homomorphism $\rho_p : \mathbb{C}^\times \to \mathbb{T}$. This defines a smooth partial torus embedding $\mathbb{T} \subset \mathbb{T} \times^{\rho_p} \mathbb{C}$ which adds to $\mathbb{T}$ a divisor $\mathbb{T}(p)$ canonically isomorphic to the cokernel of ρ_p. This can be done independently (and simultaneously) for any finite set $\Sigma \subset P$ of indivisible vectors. The corresponding partial torus embedding, denoted $\mathbb{T}_\Sigma$, is smooth. If Σ is SW^{f}-invariant, then we can form the orbifold $(\mathbb{T}_\Sigma)_{SW^{\mathrm{f}}}$.

Lemma 3.9. *Let $p \in P$ be indivisible and nonzero and denote by Σ_p its SW^{f}-orbit. Then the inclusion of orbifolds $\mathbb{T}^\circ_{SW^{\mathrm{f}}} \subset (\mathbb{T}^\circ_{\Sigma_p})_{SW^{\mathrm{f}}}$ introduces on orbifold fundamental groups the relation $t(\tau_{-p}) \equiv 1$ in $SArt_\Gamma$.*

Proof. Let $x_o \in C$ be as chosen earlier and let $y \in V$ be not in any reflection hyperplane. Consider the path ω_y in $\mathbb{A}$ from $* = (x_o, 0)$ to $\tau_p(*) = (x_o - p, 0)$ that traverses the segments successively connecting $(x_o, 0)$, (x_o, y), $(x_o - p, y)$ and $(x_o - p, 0)$. It need not lie in $\mathbb{A}^\circ$, but if we replace y by $y + \lambda p$ with λ large, it will even lie in $\mathbb{U}$ (as defined in 3.6) and will hence represent $t(\tau_{-p})$. The image of this path in $\mathbb{T}$ defines a simple negatively oriented loop around $\mathbb{T}(p)$. $\square$

An extension of $\mathbb{T}^\circ_{SW^{\mathrm{f}}}$ as above will be refered to as a *toric extension*. Lemmas 3.9 and 3.8 imply:

Corollary 3.10. *The toric extension of $\mathbb{T}^\circ_{SW^{\mathrm{f}}}$ defined by the set of minuscule weights has orbifold fundamental group canonically isomorphic to the quotient of the extended Artin group $Art \rtimes S(\Gamma)$ by the relations $\Delta_{ij} g_{ij} = \Delta_i g_i$ with i, j special and distinct.*

Blowup extensions. We next consider certain equivariant blowups. The image of A in $\mathbb{T}$ is a compact torus and the decomposition of A in faces gives a similar decomposition of that torus. This decomposition is finite; in particular, we have finitely many vertices (of finite order). A SW^{f}-invariant set of vertices in $\mathbb{T}$ determines an $S(\Gamma)$-invariant subset of I and vice versa.

Fix a $S(\Gamma)$-invariant subset $J \subset I$, let $v(J)$ denote the corresponding SW^{f}-invariant set of vertices in $\mathbb{T}$ and consider the blowup of $\mathbb{T}$ at $v(J)$:

$$\mathrm{Bl}_{v(J)}(\mathbb{T}) \to \mathbb{T}.$$

The exceptional set E_j over the image of a_j, $j \in J$, in $\mathbb{T}$ can be identified with the projective space $\mathbb{P}(\mathbb{V})$. This exceptional set is of course invariant

under SW_{a_j} and the identification with $\mathbb{P}(\mathbb{V})$ is equivariant. So the intersection of $\mathrm{Bl}_{v(J)}(\mathbb{T})^\circ$ with E_j can be identified with the complement of the union of reflection hyperplanes of W_{Γ_j} in $\mathbb{P}(V)$. Since the finite group SW^{f} acts on $\mathrm{Bl}_{v(J)}(\mathbb{T})^\circ$ we can form $\mathrm{Bl}_{v(J)}(\mathbb{T})^\circ_{SW^{\mathrm{f}}}$; it contains $\mathbb{T}^\circ_{SW^{\mathrm{f}}}$ as the complement of a closed hypersurface.

Lemma 3.11. *Suppose that $j \in I$ is special, or more generally, suppose that $j \in I$ is such that $\iota_j \in SW$, so that we can define $g_j \in S(\Gamma)$ by the proprerty $w_j g_j = \iota_j$. Then the open immersion of orbifolds $\mathbb{T}^\circ_{SW^{\mathrm{f}}} \subset \mathrm{Bl}_{v(J)}(\mathbb{T})^\circ_{SW^{\mathrm{f}}}$ introduces on orbifold fundamental groups the relation $\Delta_{\Gamma_j} g_j \equiv 1$ in $SArt_\Gamma$.*

Proof. An element of SW^{f} which leaves E_j pointwise fixed must lie in SW_{a_j} and induce a homothety in V. Such element must be the central symmetry relative to $v(j)$ (or the identity). So the relation in the orbifold fundamental group defined by E_j comes from a half loop in $\mathbb{T}^\circ$ around $v(j)$ in $\mathbb{T}$. This loop is represented in $SArt_\Gamma$ by $\Delta_{\Gamma_j} g_j$. □

It is clear that toric and blowup extensions can be performed independently. Our main interest concerns the case when $J = I_{\mathrm{sp}}$ (so that $v(J) = \{1\}$)) and Σ is the set of all minuscule weights. In that case we abbreviate

$$\hat{\mathbb{T}} := \mathrm{Bl}_{\{1\}}(\mathbb{T}_\Sigma),$$

so that $\hat{\mathbb{T}}^\circ \supset \mathbb{T}^\circ$. From 3.10 and 3.11 we immediately get:

Corollary 3.12. *The open embedding of orbifolds $\mathbb{T}^\circ_{SW^{\mathrm{f}}} \subset \hat{\mathbb{T}}^\circ_{SW^{\mathrm{f}}}$ yields on orbifold fundamental groups the epimorphism $SArt_\Gamma \to \overline{Art}_\Gamma$.*

When Γ is of type $\hat{E}_7$, we will also consider the case when J consists of the three end vertices, denoted here s, s', n, where s, s' are the special vertices. Then n satisfies the hypotheses of Lemma 3.11 with $g_n = g_{s,s'}$ and we thus obtain the group appearing in the statement of Theorem 1.5 as orbifold fundamental group:

Corollary 3.13. *Let $\tilde{\mathbb{T}}$ be obtained from $\mathbb{T}$ by taking for the toric data the set of all minuscule weights and for the the blownup data the set of the three end vertices. Then the open embedding of orbifolds $\mathbb{T}^\circ_{SW^{\mathrm{f}}} \subset \tilde{\mathbb{T}}^\circ_{SW^{\mathrm{f}}}$ gives on orbifold fundamental groups the relation $\Delta_{A_7} \equiv \Delta_{E_6}$ in $SArt_\Gamma$.*

4. Moduli spaces of marked Del Pezzo surfaces

Let S be a Del Pezzo surface. By definition this means that S is a complete smooth surface whose anticanonical bundle ω_S^{-1} is ample. Its *degree* $d = \omega_S \cdot \omega_S$ is then a positive integer ≤ 8. The linear system $|\omega_S^{-1}|$ is d-dimensional and its general member is a genus one curve. It has no base points when $d \geq 2$ and when $d \geq 3$ it defines an embedding. For $d = 3$ the image is of this map is a cubic surface and for $d = 4$ a complete intersection of two quadrics. When $d = 2$, the map $|\omega_S^{-1}|$ is a twofold covering of a projective plane which ramifies along a smooth curve of degree four and this defines a natural

involution of S. Conversely, any smooth cubic surface, complete intersection of two quadrics or double cover of a projective plane ramifying along smooth quartic curve is a Del Pezzo surface (of degree $4, 3, 2$ respectively) obtained in this manner. A Del Pezzo surface of degree d is isomorphic to the blowup of $r := 9-d$ points in a projective plane in general position or (when $d = 8$) to a product two projective lines. So there is just one isomorphism type for $d = 5, 6, 7, 9$ and two for $d = 8$. The classes e of the exceptional curves of the first kind are characterized by the properties $e \cdot e = e \cdot \omega_S = -1$. We recall the structure of the Picard group of a Del Pezzo surface.

Del Pezzo lattices. Let r be a nonnegative integer and denote the standard basis vectors of $\mathbb{Z}^{r+1}$ by $l, e_1, \dots, e_r$. We put an inner product on $\mathbb{Z}^{r+1}$ by requiring that the basis is orthogonal, $l \cdot l = 1$ and $e_i \cdot e_i = -1$. This is the standard Lobatchevski lattice of rank $r+1$, which we shall denote by $\Lambda_{1,r}$.

Following Manin [17], we put $k := 3l - e_1 - \cdots - e_r \in \Lambda_{1,r}$. So $k \cdot k = 9 - r$. The orthogonal complement Q_r of k in $\Lambda_{1,r}$ is negative definite precisely when $r \leq 8$. Then the elements $\alpha \in Q_r$ with $\alpha \cdot \alpha = -2$ define a (finite) root system R_r with the reflection s_α associated to $\alpha \in R_r$ being the one which orthogonal relative to the given inner product. We denote its Weyl group by W_r.

The set of *exceptional vectors*, $\mathcal{E}_r$, is the set of $e \in \Lambda_{1,r}$ with $-e \cdot e = e \cdot k = 1$. This set is W_r-invariant and consists of the elements e_i, $l - e_i - e_j$, $2l - \sum_{i \in I} e_i$ with I a 5-element subset of $\{1, \dots, r\}$ (so this occurs only when $r \geq 5$) and $3l - 2e_j - \sum_{i \in I} e_i$ with I a 6-element subset of $\{1, \dots, r\} - \{j\}$ (occuring only when $r \geq 7$).

For $r \geq 3$, a root basis of R_r is $\alpha_1 := l - e_1 - e_2 - e_3, \alpha_2 := e_1 - e_2, \dots \alpha_r := e_{r-1} - e_r$. This is also a basis for Q_r and so Q_r may then be regarded as the root lattice of R_r. The root system R_r is for $r = 3, \dots, 8$ of type $A_1 \sqcup A_2$, A_4, D_5, E_6, E_7, E_8 respectively. (For $r = 2$, $R_r = \{\pm\alpha_2\}$ and for $r \leq 1$, R_r is empty.) Notice that the subgroup of W_r generated by the roots $\alpha_2 = e_1 - e_2, \dots \alpha_r = e_{r-1} - e_r$ is of type A_{r-1} and can be identified with the symmetric group of the e_i's. It is also the W_r-stabilizer of l. We shall concentrate on the range $4 \leq r \leq 8$.

Markings. Let S be a Del Pezzo surface of degree $d \leq 5$ and put $r := 9 - d$. It is clear that there exists an isometry $f : \Lambda_{1,r} \to \mathrm{Pic}(S)$ which maps k onto ω_S^{-1}. Such an isometry will be called a *marking* of S. Since two such isometries differ by an element of W_r, the latter group permutes simply transitively the markings of S. Notice that a marking f maps an exceptional vector to the class of exceptional curve of the first kind and vice versa. It is also known (and again, easy to verify) that $\alpha \in Q_r$ is a root (i.e., satisfies $\alpha \cdot \alpha = -2$) if and only if it is the difference of two mutually perpendicular exceptional vectors, or equivalently, if and only if $f(\alpha)$ can be represented by $E - E'$, where E and E' are disjoint exceptional curves of the first kind.

The complete linear system $|f(l)|$ is two dimensional without base points and defines a birational morphism to a projective plane. Its contracts r exceptional (disjoint) curves whose classes are $f(e_1), \ldots, f(e_r)$. Their images will be r points in general position in the sense that no three are on a line, no six are on a conic and no eight lie on a cubic which is singular at one of these points. The linear system $|\omega_S^{-1}|$ is mapped onto the linear system of cubics passing through these points.

Conversely, if we blow up r points in in a projective plane in general position we get a (canonically) marked Del Pezzo surface of degree d. If $d \leq 5$, then we can take the first 4 of these $r = 9 - d$ points to be $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, $[1:1:1]$, so that an open subset of $(\mathbb{P}^2)^{5-d}$ is the base of a fine moduli space of marked Del Pezzo surfaces of degree d. The group W_{9-d} acts on this open set and the associated orbifold is a coarse moduli space for Del Pezzo surfaces of that degree.

Definition 4.1. A *Del Pezzo pair of degree* d consists of a Del Pezzo surface S of degree d and a *singular* anticanonical curve K of S. If there is also given a singular point p of K, then we call (S, K, p) a *Del Pezzo triple of degree* d.

It is clear that a Del Pezzo pair can be extended to a Del Pezzo triple and that this can be done in only a finite number of ways.

If S is a Del Pezzo surface of degree d and $p \in S$, then denote by L_p the linear subsystem of $|\omega_S^{-1}|$ of anticanonical curves which have a singular point at p. For $d \geq 3$, this subsystem has dimension $d - 3$ always (it is the projective space of hyperplanes in the receiving projective space of its canonical embedding that contain the tangent plane of S at p). For $d = 2$, L_p is nonempty if and only if p lies on the fixed curve (and is then a singleton and defined by the tangent line of the fixed curve at p). For $d = 1$, generically 12 points p of S have nonempty L_p.

A *marking* of a Del Pezzo pair resp. triple will be simply a marking of the first item. There is an evident fine moduli space $\tilde{M}(d)$ of marked Del Pezzo triples of degree d. It is easy to see that $\tilde{M}(d)$ is smooth when $d \leq 2$. We shall also find that this is so when $d = 1$. The evident action of the group W_r on $\tilde{M}(d)$ therefore defines an orbifold $M(d) := \tilde{M}(d)_{W_r}$, that we interpret as the coarse moduli space of such triples. We can similarly define a moduli space for Del Pezzo pairs (though we will not introduce a notation for it) and it is easy to see that the forgetful map from $M(d)$ to this moduli space is a normalization. Let us record these remarks as a

Lemma 4.2. *For $d = 3, 4, 5$, $M(d)$ is a $\mathbb{P}^{d-3}$ bundle over the universal Del Pezzo surface of degree d, $M(2)$ is the universal smooth nonhyperelliptic curve of genus 3 and $M(1)$ is a 12-fold (ramified) covering of the coarse moduli space of Del Pezzo surfaces of degree 1.*

We make the connection with the setting of Section 3 although the affine Coxeter group will not appear here in a natural manner. We introduce for $4 \leq r \leq 8$ the affine transformation group $\hat{W}_r$ generated by W_r and the

translations over Q_r. This group satisfies the assumptions at the beginning of Section 3. The quadratic form on Q_r identifies Q_r with a sublattice of $P_r := \mathrm{Hom}(Q_r, \mathbb{Z})$. The latter is then the group of special points of $\mathrm{Hom}(Q_r, \mathbb{R})$ and

$$\mathbb{T}_r := \mathrm{Hom}(Q_r, \mathbb{C}^\times) = \mathbb{C}^\times \otimes P_r$$

is the 'adjoint torus'. We are in the setting of Section 3 with $W^{\mathrm{f}} = W_r$; in particular

$$SW_r := \{\pm 1\}.W_r \text{ and } \overline{W}_r := \{\pm 1\}.W_r/\{\pm 1\}.$$

We have defined $\hat{\mathbb{T}}_r^\circ$ and in case $r = 7$ (W_7 is of type E_7), also $\tilde{\mathbb{T}}_7^\circ$. The following theorem makes the connection between the results of Section 3 and some of the theorems stated in Section 1.

Theorem 4.3. *For $d \leq 5$ there is a $\overline{W}_r$-equivariant open embedding of $(\hat{\mathbb{T}}_r^\circ)_{\{\pm 1\}}$ (recall that $r = 9 - d$) in the moduli space $\tilde{M}(d)$ of marked Del Pezzo triples of degree d whose complement is of codimension ≥ 2 so that the corresponding open embedding of $(\hat{\mathbb{T}}_r^\circ)_{SW_r}$ in $M(9-r)$ induces an isomorphism between the orbifold fundamental group of $(\hat{\mathbb{T}}_r^\circ)_{SW_r}$ and the orbifold fundamental group of $M(d)$. The strata of $(\hat{\mathbb{T}}_r^\circ)_{SW_r}$ record the Kodaira type of the anticanonical curve: the open stratum $(\mathbb{T}_r)^\circ_{SW_r}$ parameterizes the Del Pezzo triples (S, K, p) for which K is an irreducible rational curve with a simple node at p (Kodaira type I_1), the blowup stratum those for which K is a cuspidal curve (Kodaira type II) and the toric stratum (which is empty for $r = 8$) those for which K is of Kodaira type I_2.*

We prove this theorem in Section 5. It immediately gives two of our main results:

Proof that Theorem 4.3 implies Theorems 1.3 and 1.4. This follows from Theorem 4.3 and Corollary 3.12. □

In case $r = 7$, $M(2)$ can be identified with the coarse moduli space of smooth pointed nonhyperelliptic genus three curves.

Theorem 4.4. *The open embedding of orbifolds $(\hat{\mathbb{T}}_7^\circ)_{W_7}$ in $M(2)$ extends to an open embedding of the orbifolds $(\tilde{\mathbb{T}}_7^\circ)_{W_7}$ in the coarse moduli space $\mathcal{M}_{3,1}$ of pointed smooth genus three curves with the new stratum (of projective type A_7) parameterizing the hyperelliptic curves pointed with a non-Weierstraß point. The complement of this immersion is everywhere of codimension ≥ 2 and so the immersion induces an isomorphism of the orbifold fundamental group of $(\tilde{\mathbb{T}}_7^\circ)_{SW_r}$ onto the pointed mapping class group $\Gamma_{3,1}$.*

This theorem, which will be proved in Section 6, yields another main result:

Proof that Theorem 4.4 implies Theorem 1.5. Just combine Theorem 4.4 with Corollary 3.13. □

Reduced Kodaira curves. We use the following defining property.

Definition 4.5. A *reduced Kodaira curve* is a complete connected curve of arithmetic genus one whose automorphism group acts transitively on its regular part.

Here are some basic properties of a singular Kodaira curve K. We have two classes:

(add) K is simply connected and has exactly one singular point; it is then of Kodaira type II (a rational curve with a cusp), III (two smooth rational curves meeting with multiplicity two) or IV (isomorphic to the union of three concurrent lines in the plane), or

(mult) K has first Betti number one; it is then of Kodaira type $\mathrm{I_{n\geq 1}}$ (for $n \geq 2$ this is an n-gon consisting of smooth rational curves; for $n = 1$ this means a rational curve with a node).

We then say that K is of *additive* resp. *multiplicative* type. This is explained by the fact that in the short exact sequence

$$1 \to \mathrm{Pic}(K)^0 \to \mathrm{Pic}(K) \to H^2(K;\mathbb{Z}) \to 0$$

we have that $\mathrm{Pic}(K)^0$ is isomorphic to the additive group $\mathbb{G}_a$ or the multiplicative group $\mathbb{G}_m$ according to its type. There is more to this sequence: the abelian group $H^2(K;\mathbb{Z})$ is freely generated by the irreducible components of K and if C is one such component, then assigning to $p \in K_{\mathrm{reg}} \cap C$ the class of (p) defines an isomorphism of $K_{\mathrm{reg}} \cap C$ on the preimage of $[C] \in H^2(K;\mathbb{Z})$ in $\mathrm{Pic}(K)$. In other words, each connected component of K_{reg} is a principal homogeneous space (torsor) of $\mathrm{Pic}(K)^0$ and hence $\mathrm{Pic}(K)^0$ appears as a normal subgroup of $\mathrm{Aut}(K)$. The group $G_K := \mathrm{Aut}(K)/\mathrm{Pic}(K)^0$ acts on the 'dual graph' Γ_K of K and the resulting homomorphism $G_K \to \mathrm{Aut}(\Gamma_K)$ is onto. This is an isomorphism in case K is of multiplicative type (so that G_K is a dihedral group of order $2n$ when K is of type $\mathrm{I_n}$), but in case K is of additive type, the kernel of this homomorphism is isomorphic to the multiplicative group $\mathbb{G}_m$. For instance if K is of type I_1, then G_K is of order two (acting faithfully on the infinite cyclic $H_1(K;\mathbb{Z})$). If K is of type II, then K_{reg} is an affine line and $\mathrm{Aut}(K)$ is its affine-linear group having $\mathrm{Pic}(K)^0$ as its group of translations.

5. Del Pezzo structures on reduced Kodaira curves

Homological del Pezzo structures. Recall that we defined the Lobatchevski lattice $\Lambda_{1,r}$ with basis $(l, e_1, \ldots, e_r)$ and $k = 3l - e_1 - \cdots - e_r$ as a distinguished element. We write d for $9-r$. If (S, K) is a Del Pezzo pair of degree d with K a reduced Kodaira curve, then the restriction map $\mathrm{Pic}(S) \to \mathrm{Pic}(K)$ assigns to an exceptional curve E on S the class of $E \cdot K$. It has degree one and hence represented by the singleton $E \cap K$, unless E is an irreducible component of K. This motivates the following definition.

Definition 5.1. Let K be a reduced Kodaira curve. We say that a homomorphism $\psi : H_2(K) \to \Lambda_{1,9-d}$ defines a *homological Del Pezzo structure* (of degree d) on K if

(i) the fundamental class $[K] = \sum_{C \text{irr comp}} [C]$ maps to k,
(ii) if C, C' are distinct irreducible components of K, then their intersection number in K is equal to $\psi[C] \cdot \psi[C']$ and
(iii) for an exceptional vector e and an irreducible component C of K we have $e \cdot \psi[C] \geq 0$, unless $\psi[C] = e$,

Notice that if K is irreducible, condition (ii) is empty and (iii) follows from (i). The following proposition shows that this notion helps us to parameterize Del Pezzo triples.

Proposition 5.2. *Let K be a reduced Kodaira curve and let $\chi : \Lambda_{1,9-d} \to \mathrm{Pic}(K)$ be a homomorphism such that*

(i) *the dual $\psi : H_2(K) \to \Lambda_{1,9-d}$ of the composite $\deg \circ \chi : \Lambda_{1,9-d} \to H^2(K)$ defines a homological Del Pezzo structure of degree d on K,*
(ii) *χ is nonzero on the roots in that lie in the kernel of $\deg \circ \chi$.*

Then there is a marked Del Pezzo surface (S, f) of degree d and an embedding $i : K \to S$ inducing χ. The triple (S, f, i) is unique up to unique isomorphism.

Proof. We first notice that $\chi(l)$ has degree $l \cdot k = 3$ on K. Since $\chi(l)$ has nonnegative degree on every irreducible component, the associated linear system is nonempty and maps K onto a reduced cubic curve in a projective plane P. The image $\bar{K} \subset P$ is obtained from K by collapsing each irreducible component on which l has degree zero. According to property (iii) of Definition 5.1 this happens when C is mapped to some e_i.

We claim that the converse also holds: if an irreducible component C of K is contracted under $S \to P$, then C is mapped to some e_i. For in that case $\psi[C]$ is of the form $\sum_i n_i e_i$. If $\psi[C] \neq e_i$ for some i, then we must have $n_i \leq 0$ for all i. But the inner product of $\psi[C]$ with an exceptional vector of the form $l - e_i - e_j - e_k$ must also be ≥ 0. From this we readily see that all n_i must be zero. This contradicts the fact that C has nonzero intersection with another irreducible component (in case K is reducible) or represents K (in case K is irreducible). In this situation we denote by $p_i \in \bar{K}$ the point that is the image of C (so with $\psi[C] = e_i$). The points $p_i \in \bar{K}$ that we thus obtain are pairwise distinct: if $p_i = p_j$ with $i \neq j$, then $p_i - p_j$ is a root in the kernel of both $\deg \circ \chi$ and χ, which is excluded by (ii).

Any e_j that is not represented by a degree zero component of K defines an element $\chi(e_j) \in \mathrm{Pic}(K)$ which has degree ≥ 0 on each component of K. Since the total degree is 1, $\chi(e_j)$ is represented by a unique smooth point of K. Denote the image of that point in $\bar{K}$ by p_j. The points $p_1, \ldots, p_r$ that we now have defined are still pairwise distinct: First of all no two of the newly added points concide since otherwise χ would vanish on a root in the kernel of $\deg \circ \chi$. And no newly added p_j equals an old p_i that is associated to an

irreducible component C of K: since $e_j.\chi^*(C) = e_j.e_i = 0$, $\chi(e_j)$ has zero degree on C, hence is represented by a point of $K - C$.

Blowing up $p_1, \ldots, p_r$ in P yields a surface S, a marking $f : \Lambda_{1,r} \cong \mathrm{Pic}(S)$ and an embedding $i : K \to S$ such that $\chi = i^* f$. One verifies that S is a Del Pezzo surface of degree d. The uniquess is left as an exercise. □

Let K be a reduced Kodaira curve, p a singular point of K and $\psi : H_2(K) \to \Lambda_{1,9-d}$ a homological Del Pezzo structure. This defines in an evident manner a locus $\widetilde{M}(K, p; \psi)$ in the moduli space of marked Del Pezzo triples.

Corollary 5.3. *Let K be a reduced Kodaira curve, p a singular point of K and $\psi : H_2(K) \to \Lambda_{1,9-d}$ a homological Del Pezzo structure. Denote by $Q_\psi \subset \Lambda_{1,9-d}$ the kernel of ψ^*. If K is of multiplicative type, then $\widetilde{M}(K, p; \psi)$ is isomorphic to the toric arrangement complement $\mathrm{Hom}(Q_\psi, \mathbb{C}^\times)^\circ$ (with the isomorphism given up to ± 1), unless K is of type I_1, in which case it is naturally isomorphic to $\mathrm{Hom}(Q_\psi, \mathbb{C}^\times)^\circ_{\{\pm 1\}}$. If K is of additive type, then $\widetilde{M}(K, p; \psi)$ is naturally isomorphic to the projective arrangement complement $\mathbb{P}(\mathrm{Hom}(Q_\psi, \mathbb{C})^\circ)$. Both $\mathrm{Hom}(Q_\psi, \mathbb{C}^\times)^\circ$ and $\mathbb{P}(\mathrm{Hom}(Q_\psi, \mathbb{C})^\circ)$ support families of Del Pezzo triples.*

Proof. Denote the space of $\chi \in \mathrm{Hom}(\Lambda_{1,9-d}, \mathrm{Pic}(K))$ satisfying the two conditions of Proposition 5.2 by $\mathrm{Hom}(\Lambda_{1,9-d}, \mathrm{Pic}(K))^\circ_\psi$ so that we have an obvious map $\mathrm{Hom}(\Lambda_{1,9-d}, \mathrm{Pic}(K))^\circ_\psi \to \widetilde{M}(d)$. This map is essentially the passage to the orbit space of the action of the group of automorphisms of K that act trivially on $H_2(K)$ and fix p. This group is easily identified as $\mathrm{Aut}(K)^0$, except when K is of type I_1: it is then all of $\mathrm{Aut}(K)$ (which has $\mathrm{Aut}(K)^0$ as a subgroup of index 2).

Denote by $Q_\psi \subset \Lambda_{1,9-d}$ the kernel of ψ^*. We can find for every irreducible component C of K a class $e_C \in \Lambda_{1,9-d}$ such that $\psi^*(e_C)$ takes the value 1 on $[C]$ and zero on the other irreducible components. These vectors map to a basis of $\Lambda_{1,9-d}/Q_\psi$. Since the identity component $\mathrm{Aut}(K)^0$ acts simply transitively on K_{reg}, two elements of $\mathrm{Hom}(\Lambda_{1,9-d}, \mathrm{Pic}(K))^\circ_\psi$ lie in the same $\mathrm{Aut}(K)^0$-orbit if and only if they have the same restriction to Q_ψ. Therefore the above map factors through $\mathrm{Hom}(Q_\psi, \mathrm{Pic}(K)^0)^\circ$. The action of $\mathrm{Aut}(K)^0$ on $\mathrm{Hom}(Q_\psi, \mathrm{Pic}(K)^0)^\circ$ is via $\mathrm{Pic}(K)^0$. This is trivial in the multiplicative case and is via scalar multiplication in the additive case. All the assertions of the corollary but the last now follow easily.

Proposition 5.2 (in its relative form) shows that $\mathrm{Hom}(\Lambda_{1,9-d}, \mathrm{Pic}(K))^\circ_\psi$ carries a family of of Del Pezzo triples. Since the group $\mathrm{Aut}(K)^0$ acts freely on the base of this family, the passage to $\mathrm{Aut}(K)^0$-orbit spaces still yields a family. This yields the last assertion. □

Glueing constructions. When K is irreducible, both ψ and p become irrelevant and we therefore simply write $\widetilde{M}(d,\mathrm{I}_1)$ resp. $\widetilde{M}(d,\mathrm{II})$ for the corresponding strata. We have $\widetilde{M}(d,\mathrm{I}_1) \cong \mathrm{Hom}(Q_r,\mathbb{C}^\times)^\circ_{\{\pm 1\}} = (\mathbb{T}_r^\circ)_{\{\pm 1\}}$ and $\widetilde{M}(d,\mathrm{II}) \cong \mathbb{P}(\mathrm{Hom}(\mathrm{Q_r},\mathbb{C})^\circ)$. Notice that in either case the center of W_r acts trivially so that the action is via $\overline{W}_r$. Both $\mathbb{T}_r^\circ$ and $\mathbb{P}(\mathrm{Hom}(Q_r,\mathbb{C})^\circ)$ carry families of Del Pezzo triples. Recall that the blowup with reflection loci removed, $(\mathrm{Bl}_1\,\mathbb{T}_r)^\circ$, comes with an SW_r-action and is as such the union of the invariant strata $\mathbb{T}_r^\circ$ and $\mathbb{P}(\mathrm{Hom}(Q_r,\mathbb{C})^\circ)$.

Lemma 5.4. *The families of Del Pezzo triples over these strata can be glued to form one over* $(\mathrm{Bl}_1\,\mathbb{T}_r)^\circ$. *This family comes with an action of* SW_r.

Proof. It is best to start out with a parameterized nodal curve: the curve K obtained from $\mathbb{P}^1$ by identifying 0 and ∞, so that $K_{\mathrm{reg}} = \mathbb{C}^\times$.

Every $e \in \mathcal{E}_r$ defines a homomorphism $\mathbb{T}_r \to \mathbb{C}^\times$. If think of s_e as a section of $K \times \mathbb{T}_r \to \mathbb{T}_r$, then these sections are the restriction of homomorphism from $\Lambda_{1,r}$ to the relative Picard group of $K \times \mathbb{T}_r \to \mathbb{T}_r$; over $\mathbb{T}_r^\circ$ this yields a family of degree d Del Pezzo structures on K. But for the moment we rather regard s_e as a section of $\mathbb{P}^1 \times \mathbb{T}_r \to \mathbb{T}_r$. We pull back the latter family with sections to the blowup $\mathrm{Bl}_1\,\mathbb{T}_r$. Over the exceptional locus they all take the value $1 \in \mathbb{C}^\times \subset \mathbb{P}^1$. Blow up $\{1\} \times \mathbb{P}(\mathrm{Hom}(Q_r,\mathbb{C}))$ in $\mathbb{P}^1 \times \mathrm{Bl}_1\,\mathbb{T}_r$. The sections s_e will extend across this blow up. Subsequently we contract the strict transform of $\mathbb{P}^1 \times \mathbb{P}(\mathrm{Hom}(Q_r,\mathbb{C}))$ in the ambient space to its intersection with the new exceptional divisor (this is possible, since locally this this the standard situation over a higher dimensional base). Then identify the sections 0 and ∞ (or rather their images) and the result is a family of Kodaira curves over $\mathrm{Bl}_1\,\mathbb{T}_r$. Its restriction to $\mathbb{T}_r^\circ$ resp. $\mathbb{P}(\mathrm{Hom}(Q_r,\mathbb{C}))$ is the universal family. In the first case this is clear, in the second case one can verify this easily by restricting to a 'linear' curve germ in $(\mathrm{Bl}_1\,\mathbb{T}_r)^\circ$ transversal to the exceptional locus. A relative version of Proposition 5.2 produces a universal family of marked Del Pezzo triples over $(\mathrm{Bl}_1\,\mathbb{T}_r)^\circ$, together with an action of SW_r. $\square$

The type I_2 case. It is clear that $(\mathrm{Bl}_1\,\mathbb{T}_r)^\circ_{\{\pm 1\}}$ may be identified with an open subset of $\tilde{M}(d)$. We wish to extend this to a description of $\tilde{M}(d)$ up to codimension two. This means we will allow also the degeneration of a nodal curve into other Kodaira curves. The codimension condition restricts the possibilities to I_2 and II and as we have already dealt with II we now focus on I_2.

We begin with determining the homological Del Pezzo structures on such a curve. It is clear that they are permuted by W_r and the connected component group of $\mathrm{Aut}(K)$ and that these actions commute. We want to classify the homological Del Pezzo structures $\psi : H_2(K) \to \Lambda_{1,9-d}$ with $d \le 4$ up to this action.

Lemma 5.5. *Suppose that K is of type I_2 and let be given a homological Del Pezzo structure $\psi : H_2(K) \to \Lambda_{1,9-d}$ of degree d. Then we have $d \geq 2$ and there is an element of W_r that takes the classes of the irreducible components of K to the following systems of vectors in $\Lambda_{1,r}$ (we also denoted the image of the system in P_r):*

(d=2) $(e_7, k - e_7) \mapsto (\varpi_7, -\varpi_7)$ *with Q_ψ of type E_6,*
(d=3) $(e_6, k - e_6) \mapsto (\varpi_6, -\varpi_6)$ *with Q_ψ of type D_5,*
(d=4) $(e_5, k - e_5) \mapsto (\varpi_5, -\varpi_5)$ *with Q_ψ of type D_4 or*
$(l - e_1, k - (l - e_1)) \mapsto (\varpi_2, -\varpi_2)$ *with Q_ψ of type A_4,*
(d=5) $(e_4, k - e_4) \mapsto (\varpi_4, -\varpi_4)$ *with Q_ψ of type $A_2 + A_1$ or*
$(l - e_1, k - (l - e_1)) \mapsto (\varpi_2, -\varpi_2)$ *with Q_ψ of type A_3,*

In brief, the irreducible components of K map to pairs of opposite minuscule weights and each such pair occurs.

Proof. An irreducible component C of K that is not mapped to some e_i is mapped to an element of the form $c = nl - \sum_i n_i e_i$ with $n_0 \geq 0$. We also know that $c \cdot e_i \geq 0$ for all i and so $n_i \geq 0$. The same must hold for the other curve $K - C$: $k - c = (3 - n)l - \sum_i (1 - n_i))e_i$. So $n \in \{0, 1, 2, 3\}$ and $n_i \in \{0, 1\}$. So there are only finitely many possibilities for c. A straightforward analysis yields the lemma. □

We recall that the set of minuscule weights in $\mathrm{Hom}(Q_r, \mathbb{Z})$ of Q_r defines a toric extension $\hat{\mathbb{T}}_r$ of $\mathrm{Bl}_{\{1\}}(\mathbb{T}_r)^\circ$. So $\hat{\mathbb{T}}_r^\circ$ contains $\mathrm{Bl}_{\{1\}}(\mathbb{T}_r)^\circ$ as an open subset and the complement is the union of the $\widetilde{M}(\mathrm{I}_2, \psi)$, where I_2 represents a fixed Kodaira curve of that type endowed with a fixed singular point singled out. We have the expected analogue of Lemma 5.4:

Lemma 5.6. *The families of marked Del Pezzo triples over these strata can be glued to form one over $\hat{\mathbb{T}}_r^\circ$. This family comes with an action of SW_r.*

Proof. As in Lemma 5.4 we work with the product $\mathbb{T}_r \times \mathbb{P}^1 \to \mathbb{T}^r$ and the sections s_e (e an exceptional vector) that take values in $\mathbb{T}_r \times \mathbb{C}^\times$. For every minuscule weight c we have a one parameter subgroup $\rho_c : \mathbb{C}^\times \to \mathbb{T}_r$ and a torus embedding $\mathbb{T}_r \times^{\rho_p} \mathbb{C}$ which adds to $\mathbb{T}_r$ a copy $\mathbb{T}_r(c)$ of the cokernel of ρ_c. Given an exceptional vector e, then the corresponding homomorphism $s_e : \mathbb{T}_r \to \mathbb{C}^\times$ extends to a morphism $\mathbb{T}_r \times^{\rho_c} \mathbb{C} \to \mathbb{P}^1$ such that its value over the toric divisor $\mathbb{T}_r(c)$ is 0 if $c(e) > 0$, lies in $\mathbb{C}^\times$ if $c(e) = 0$, and is ∞ if $c(e) < 0$.

We now regard each s_e as a section of the trivial $\mathbb{P}^1$-bundle over $\mathbb{T}_r \times^{\rho_p} \mathbb{C}$. Blow up the smooth codimension two locus $\{0\} \times \mathbb{T}_r(p)$ in this bundle and identify the strict transforms of the constant sections 'zero' and '∞'. The result is a family of Kodaira curves over $\mathbb{T}_r \times^{\rho_p} \mathbb{C}$, of type I_2 over the toric divisor $\mathbb{T}_r(c)$ and of type I_1 elsewehere. The section s_e is now extends to this family when $c(e) \geq 0$: for $c(e) = 0$ it will avoid the exceptional divisor and for $c(e) > 0$ it will avoid the strict transform of $\mathbb{P}^1 \times \mathbb{T}_r(p)$. In either case, we denote the resulting divisor D_e. □

Proof of Theorem 4.3. Lemma 5.6 shows that $\hat{\mathbb{T}}_r^\circ$ is the parameter space of a family of marked Del Pezzo triples of degree $d = 9 - r$. The ones that are missing are those for which the Milnor numbers of the singular points of K add up to at least 3. This locus is indeed of codimension ≥ 2. $\square$

6. Moduli of degree two Del Pezzo surfaces and genus three curves

In this section we exclusively deal with the degree two case and so the relevant root system will be R_7 (of type E_7). *We therefore often suppress the subscript 7, it being understood that R stands for R_7, Q for Q_7, $\mathbb{T}$ for $\mathbb{T}_7$ etc.*

Tacnodal degenerations. We first take up an aspect on degenerating Kodaira curves needed for what follows.

Let $b, c \in \mathbb{C}[[t]]$ have positive (finite) order and consider the double cover of $\mathcal{K} \to \mathbb{P}^1_\Delta$ defined by $y^2 = x^2(x^2 + 2bx + c)$ (here x is the affine coordinate of $\mathbb{P}^1$). We shall assume that both c and $d := b^2 - c$ are nonzero. This ensures that the general fiber of $\mathcal{K} \to \Delta$ is a nodal curve. The fibre K over the closed point is a union of two copies $K^\pm$ of $\mathbb{P}^1$ having in common a single point, where it displays a *tacnodal singularity* (given by $y^2 - x^4 = 0$).

Definition 6.1. We call a family of reduced Kodaira curves $\mathcal{K}/\Delta$ with general fiber of type I_1 a *good tacnodal degeneration* if the special fiber is of type II and the family can be given by the equation as the double cover of $\mathbb{P}^1_\Delta$ defined by $y^2 = x^2(x^2 + 2bx + c)$ with $b, c \in \mathbb{C}[[t]] - \{0\}$ such that $\mathrm{ord}(b^2) > \mathrm{ord}(c) > 0$.

Assume first that $d := b^2 - c$ has even order, so that $\sqrt{d}$ is defined. We may then write $x^2 + bx + c = (x - u)(x - v)$ with $u, v \in \mathbb{C}[[t]]$. Having a good tacnodal degeneration the amounts to $(u + v)^2$ having order than uv. This comes down to saying that the two roots have the same positive order and have opposite initial coefficients: vu^{-1} is regular and takes the value -1 in 0. The existence of $\sqrt{d}$ also ensures that the family has two sections $\sigma_\pm : t \mapsto (-b \pm \sqrt{d}, 0)$ whose generic point lies in the smooth part. So we have an isomorphism $\mathrm{Pic}^0(\mathcal{K}_{\Delta^*}) \cong \mathbb{C}((t))^\times$ (given up to inversion).

If $d := b^2 - c$ has odd order, then adjoin $\sqrt{t}$ to $\mathbb{C}[[t]]$ and argue as before.

Lemma 6.2. *Suppose we have a good tacnodal degeneration as above that is* split *is the sense that $\sqrt{d} \in \mathbb{C}[[t]]$.*

Then the map from the set of sections of $\mathcal{K}/\Delta$ that meet K_o in $K_{o,\mathrm{reg}}$ to the set $\mathrm{Pic}_1(\mathcal{K}/\Delta)$ of relative divisor classes of degree one is a bijection and the specialization map $\mathrm{Pic}_1(\mathcal{K}/\Delta) \to \mathrm{Pic}_1(K_o) = K_{o,\mathrm{reg}}$ is a surjective map of torsors (the underlying homomorphism $\mathrm{Pic}_0(\mathcal{K}/\Delta) \to \mathrm{Pic}_0(K_o)$ is onto).

If σ_1 and σ_2 are sections of $\mathcal{K}/\Delta$ meeting K_o in $K_{o,\mathrm{reg}}$, then the image of $(\sigma_1) - (\sigma_2)$ in $\mathrm{Pic}(\mathcal{K}_{\Delta^*})^0 \cong \mathbb{C}((t))^\times$ lies in $\pm 1 + t\mathbb{C}[[t]]$ with the sign depending on whether $(+)$ or not $(-)$ σ_1 and σ_2 hit the same component of $K_{o,\mathrm{reg}}$.

Proof. The first assertion is well-known and is only included for the record. To prove the second assertion, we introduce the coordinates $\tilde{y} := y/x$, $\tilde{x} := x+b$, so that the generic fiber has the equation $\tilde{y}^2 = \tilde{x}^2 - d$, where $d = b^2 - c$. Notice that the two branches of $\mathcal{K}_{\Delta^*}$ at its singular point have tangents $y = x \pm \sqrt{c}$, so that the points $(\tilde{x}, \tilde{y}) = (b, \pm\sqrt{c})$ must be omitted. Since $(\tilde{x} - \tilde{y})(\tilde{x} + \tilde{y}) = d$, $\tilde{x} + \tilde{y}$ is a relative paramaterization $\mathcal{K}_{\Delta^*}$. To be precise, $\tilde{x} + \tilde{y}$ maps the smooth part of $\mathcal{K}_{\Delta^*}$ to $\mathbb{P}^1_{\Delta^*}$ with image the complement of $b \pm \sqrt{c}$. This suggests to use the coordinate

$$u := \frac{(\tilde{x} + \tilde{y}) - (b - \sqrt{c})}{(\tilde{x} + \tilde{y}) - (b + \sqrt{c})} = \frac{x^2 + y + x\sqrt{c}}{x^2 + y - x\sqrt{c}}.$$

instead, for which the two points in question are now 0 and ∞. This defines an isomorphism $\mathrm{Pic}(\mathcal{K}_{\Delta^*})^0 \cong \mathbb{C}((t))^\times$ that maps $(\sigma_1) - (\sigma_2)$ to $\sigma_1^* u / \sigma_2^* u$.

The computation is now straightforward. Let σ stand for some σ_i. We have that $\sigma(o)$ (assuming it is not a point at infinity) is of the form $(a, \varepsilon a^2)$ with $a \neq 0$ and $\varepsilon \in \{\pm 1\}$ (depending on the branch we are on). So if $\varepsilon = 1$, then $\sigma^* u(0) = (2a^2 + 0)(2a^2 - 0)^{-1} = 1$. If $\varepsilon = -1$, we need to be a bit more careful. Then

$$\sigma^*(x^2 + y)s = \sigma^*(x^2 - x\sqrt{x^2 + 2bx + c}) = \sigma^*(-bx + c) + \text{higher order terms},$$

and so

$$\sigma^* u = \sigma \frac{(\sqrt{c} - b)\sigma^* x + c}{(-\sqrt{c} - b)\sigma^* + c} + \text{higher order terms}.$$

Since b has higher order than $\sqrt{c}$, it follows that $\sigma^* u(0) = -1$.

These computations remain valid if a tends to infinity and so the assertion follows. □

The universal curve of genus three. Let us begin with a simple observation. Let C be a nonsingular complete genus 3 curve endowed with a basis of its canonical system. This defines a morphism $C \to \mathbb{P}^2$ whose image we denote by $\bar{C}$. For C nonhyperelliptic, $C \to \bar{C}$ is an isomorphism and $\bar{C}$ is a quartic curve. For C hyperelliptic, $\bar{C}$ is a conic and $C \to \bar{C}$ is dividing out by the hyperelliptic involution. In that case the ramification points of $C \to \bar{C}$ form an 8-element subset D of C. If we place ourselves in $\mathbb{P}^2$, then clearly we can reconstruct C from the pair $(\bar{C}, D)$ (strictly speaking, up to its hyperelliptic involution). It is best regard as $\bar{C}$ as a conic with multiplicity 2, so that it is a degenerate quartic. The following lemma is well-known.

Lemma 6.3. *Let $\mathcal{C}/\Delta$ be a smooth family genus three curves whose special fiber is hyperelliptic, but whose generic fiber is not. Let $\mathcal{C} \to \mathbb{P}^2_\Delta$ be a relative canonical map and denote by $\bar{\mathcal{C}}$ its image and by $D \subset \bar{C}_o$ the ramification set of $C_o \to \bar{C}_o$. Then every bitangent of the generic fiber of $\bar{\mathcal{C}}/\Delta$ specializes over o to a line through to two distinct points of D and every such line is thus obtained.*

Proof. An odd theta characteristic Θ of $\mathcal{C}/\Delta$ is smooth of degree 2 over Δ. The generic fiber Θ_{Δ^*} is given by a bitangent of $\bar{\mathcal{C}}_{\Delta^*}$ in the sense that $2\Theta_{\Delta^*}$ is the pull-back of a line in $\mathbb{P}^2_{\Delta^*}$ that is a bitangent of $\bar{\mathcal{C}}_{\Delta^*}$ and conversely,

any odd theta characteristic of $\mathcal{C}_{\Delta^*}$ so arises. The special fiber Θ_o is a sum of two Weierstraß points of C_o and all pairs of Weierstraß points of C_o arise in this manner. The lemma easily follows this. $\square$

Let $X \subset \mathbb{P}^2$ be a nonsingular quartic and let $S \to \mathbb{P}^2$ be the double cover that ramifies along X. Then S is a Del Pezzo surface of degree 2. The preimage of a bitangent of X in S consists of two exceptional curves (together forming a Kodaira curve of type I_2 or III) and we thus get all its $2.28 = 56$ exceptional curves. Now let $p \in X$ be such that X has a simple tangent at p in the sense that the projective tangent line L meets X simply in two other points. Then the preimage $\tilde{L}$ of L in S is an anticanonical curve of type I_1 (its double point is p). Conversely, any anticanonical curve on X of that type is so obtained. The 28 bitangents of S meet L in as many points. These are the images of points of intersection with $\tilde{L}$ of exceptional curves of S. Since these exceptional curves generate $\mathrm{Pic}(S)$, we expect that the relative position of the 28 points on L contains the data that describe the map $\mathrm{Pic}(S) \to \mathrm{Pic}(\tilde{L})$.

In order to understand what happens, let $\mathcal{X} \subset \mathbb{P}^2_\Delta$ be a family of quartic curves that is smooth over Δ^* and whose closed fiber is a double conic $2C$. We suppose that this family defines a reduced degree 8 divisor D on C. We form the double cover $\mathcal{S} \to \mathbb{P}^2_\Delta$ with ramification locus $\mathcal{X}$. Evidently, a fibers S_t is a degree 2 Del Pezzo surface when $t \neq 0$. The fiber S_0 is the double cover of $\mathbb{P}^2$ with ramification $2C$ and hence is the union of two copies (denoted $S_0^\pm$) of $\mathbb{P}^2$ which meet in C and cross normally along C. For every 2-element subset A of D denote by $E_A \subset \mathbb{P}^2$ the line passing through A. This gives us copies $E_A^\pm$ in $S_o^\pm$. It follows from Lemma 6.3 that each of the 56 elements of the collection $E_A^\pm$ is the specialization of a family $\mathcal{E}_A^\pm \subset \mathcal{S}$ over Δ whose generic fiber is an exceptional curve on the generic fiber of $\mathcal{S}/\Delta$.

Lemma 6.4. *The intersection numbers in the generic fiber of two distinct exceptional curves given as above are as follows. Let A, A' be 2-element subsets of D.*

(i) *If $A \cap A'$ is a singleton, then $E_{A,t}^\pm \cdot E_{A',t}^\pm = 0$ and $E_{A,t}^\pm \cdot E_{A',t}^\mp = 1$,*
(ii) *if $A \cap A' = \emptyset$, then $E_{A,t}^\pm \cdot E_{A',t}^\pm = 1$ and $E_{A,t}^\pm \cap E_{A',t}^\mp = 0$ and*
(iii) $E_{A,t}^+ \cdot E_{A,t}^- = 2$.

Proof. It is a priori clear from Lemma 6.3 that such intersection numbers acquire their contributions from points near D. In order to compute the way a given $p \in D$ contributes, we resolve $\mathcal{S}$. It is not hard to verify that $\mathcal{S}$ is smooth outside D. To see what happens at $p \in D$, we notice that a local equation for $\mathcal{X} \subset \mathbb{P}^2_\Delta$ at such a point is $y^2 + tx = 0$, where (x, y, t) is an analytic coordinate system on a neighborhood of p in $\mathbb{P}^2_\Delta$. So a local equation for $\mathcal{S}$ at that point is $y^2 + tx = z^2$. This is a cone over a smooth quadric Σ in $\mathbb{P}^3$ and hence gets resolved by a single blowup $\hat{\mathcal{S}} \to \mathcal{S}$ with a

copy of Σ as exceptional divisor. The closed fiber S_0 is given by $z = \pm y$. The strict transform $\hat{S}_0^{\pm}$ of $S_0^{\pm}$ is a copy of $\mathrm{Bl}_p(\mathbb{P}^2)$ and the three smooth varieties $\hat{S}_0^+$, $\hat{S}_0^-$ and Σ cross normally. The intersection $S_0^{\pm} \cap \Sigma$ corresponds to the exceptional divisor of $\mathrm{Bl}_p(\mathbb{P}^2) \to S_0^{\pm}$. If we denote the latter by F and by $o \in F$ the point defined by the tangent line of C at p, then we may canonically identify Σ with $F \times F$ in such a manner that $S_0^+ \cap \Sigma = F \times \{o\}$ and $S_0^- \cap \Sigma = \{o\} \times F$. The involution on $\mathcal{S}$ (defined by $(x, y, z; t) \mapsto (x, y, -z; t)$) lifts to $\hat{\mathcal{S}}$ and acts on $\Sigma = F \times F$ by interchanging the factors.

A pair of exceptional curves $E_t^{\pm}$ on S_t is the preimage of a bitangent of X_t. If that bitangent specializes to a line L through p, then L defines a point $[L] \in F - \{o\}$ and the exceptional curve E_t^+ specializes in $\hat{S}_0$ to the sum of the strict transform of E_0^+ in $\hat{S}_0^+$ and the rule $[L] \times F$ on Σ. Similarly, E_t^- specializes in $\hat{S}_0$ to the sum of the strict transform of E_0^- in $\hat{S}_0^+$ and $F \times [L]$. In particular, the two specializations meet in our coordinate patch only in $([F], [F])$. This a point of Σ where $\hat{\mathcal{S}} \to \Delta$ is smooth. Since the intersection is simple, the contribution of p towards $E_t^+ \cdot E_t^-$ is 1.

For another pair $E_t'^{\pm}$ of exceptional curves (distinct from $E_t^{\pm}$) we find an $[L'] \neq [L]$ and we see that the specializations of E_t^+ and $E_t'^+$ in $\hat{S}_0^+$ do not meet, whereas those of E_t^- and $E_t'^+$ simply meet in $[L'] \times [L]$. The same argument as above now shows that the contribution of p towards their intersection number is 0 resp. 1. This completes the proof. □

Corollary 6.5. *A marking of $\mathcal{S}_{\Delta^*}$ is obtained by numbering the points of D: $D = \{p_0, \ldots, p_7\}$ and the rule*

$$\begin{aligned} e_i &\mapsto E^-_{p_0,p_i}, && 1 \le i \le 7 \\ (2l - e_1 - \cdots - e_7) + e_i + e_j &\mapsto E^-_{p_i,p_j}, && 1 \le i < j \le 7, \\ l - e_i - e_j &\mapsto E^+_{p_i,p_j}, && 1 \le i < j \le 7, \\ (3l - e_1 - \cdots - e_7) - e_i &\mapsto E^+_{p_0,p_i}, && 1 \le i \le 7. \end{aligned}$$

Proof. Following Lemma 6.4 this assignment respect the inner products. Since the collection of exceptional vectors generates $\Lambda_{1,7}$, the first part of the corollary follows. □

Consider the map that sends $E_A^{\pm}$ to ± 1. Via the marking defined by corollary 6.5 we see that it is simply given by the parity of the coefficient of l. It therefore defines a homomorphism $\Lambda_{1,7} \to \{\pm 1\}$. Now observe that the restriction of that homomorphism to Q (which is an element of $\mathrm{Hom}(Q, \pm 1) = \mathrm{Hom}(Q, \mathbb{C}^\times) \subset \mathbb{T}$) is the weight e^{ϖ_1} defined by the nonspecial end vertex of the $\hat{E}_7$-diagram. Its stabilizer in W is $\{\pm 1\}.W_{\varpi_1}$, where W_{ϖ_1} is of type A and hence isomorphic tot the symmetric group of 8 elements. This group has two orbits in $\mathcal{E}_7$ that are distinguished by the parity of the coefficient of l and in terms of the indexing by the $E_A^{\pm}$'s by the sign.

Let us now fix $p \in C$ distinct from the support of the associated degree 8 divisor D on C and let γ be a section of $\mathcal{X}/\Delta$ with $\gamma(o) = p$.

Lemma 6.6. *The generic fiber X_{Δ^*} has a simple tangent at $\gamma(\Delta^*)$ and if we denote that line by $\mathcal{L}_{\Delta^*} \subset \mathbb{P}^2_{\Delta^*}$, then this family extends to $\mathcal{L}/\Delta$ with the fiber over o being the tangent line of C at p. Moreover, the preimage $\mathcal{K}/\Delta \subset \mathcal{S}/\Delta$ of $\mathcal{L}/\Delta$ defines a good tacnodal degeneration in the sense of Definition 6.1.*

Proof. We choose affine coordinates (x, y) for $\mathbb{P}^2_\Delta$ such that $p = (0, 0; 0)$ and C is given by $(y - x^2)^2 = t = 0$. So $\mathcal{X}$ has an equation of the form $F(x, y) = 0$, with

$$F(x, y) = (y - x^2)^2 + t^r G(x, y), \quad r \geq 1,$$

where $G \in \mathbb{C}[[t]][x, y]$ is a quartic polynomial in (x, y) that is not divisible by t and $G_o(x, y)$ is not divisible by $y - x^2$. The degree 8 divisor on C is defined by $G_0 = 0$ and so we must have $G_o(0, 0) \neq 0$. So if $\gamma = (\gamma_x, \gamma_y)$, with $\gamma_x, \gamma_y \in \mathbb{C}[[t]]$, then the identity

$$(\gamma_y - \gamma_x^2)^2 + t^r G(\gamma_x, \gamma_y) = 0$$

shows that that r must be even and that $\gamma_y - \gamma_x^2$ has order $r/2$. The line through $(0, 0)$ that is parallel to the tangent line of X_t at γ is given by

$$2(\gamma_y - \gamma_x^2)(dy - 2\gamma_x)dx) + t^r dG(\gamma_x, \gamma_y) = 0$$

If we divide this expression by $t^{r/2}$ and the put $t = 0$, we find the line with equation $dy = 0$. This is the slope of the limiting line. That line also passes through $(0, 0)$ and so the first part of the lemma follows.

For the second part, we parameterize $\mathcal{L}$ in the obvious way: $u \mapsto (\gamma_x + u, \gamma_y + \alpha(t)u)$ where we know that α has order ≥ 1. Substitution of this parameterization in $F(x, y; t)$ gives

$$F(\gamma_x + u, \gamma_y + \alpha u) = ((\gamma_y + \alpha u) - (\gamma_x + u)^2)^2 + t^r G(\gamma_x + u, \gamma_y + \alpha u).$$

The lefthand side must be divisible by u^2 and will therefore have the form $u^2(au^2 + bu + c)$ with $a, b, c \in \mathbb{C}((t))$. Modulo t^r this must be equal to

$$\Big(\gamma_y + \alpha u) - (\gamma_x + u)^2\Big)^2 = \Big((\gamma_y - \gamma_x^2) + (\alpha - 2\gamma_x)u - u^2\Big)^2$$

and thus we find that

$$\begin{aligned}
a &\equiv -1 \pmod{t^r},\\
b &\equiv -2(\alpha(t) - 2\gamma_x) \pmod{t^r},\\
c &\equiv -2(\gamma_y - \gamma_x^2) + (\alpha - 2\gamma_x)^2 \pmod{t^r},\\
0 &\equiv 2(\alpha - 2\gamma_x)(\gamma_y - \gamma_x^2) \pmod{t^r}.
\end{aligned}$$

So a has order zero. Since $\gamma_y - \gamma_x^2$ has exact order $r/2$, it follows from the last equation that $\alpha - 2\gamma_x$ has order $> r/2$. Hence $b/2a$ has order $> r/2$ and c/a has exact order $r/2$ and so the last assertion of the lemma follows. $\square$

We can now relate this to our torus $\mathbb{T}$.

Lemma 6.7. *If A, A' are 2-element subsets of D whose intersection is a singleton, then Lemma 6.4 associates to $E_A^+ - E_{A'}^+$ a root (denoted $\beta(A, A')$) whose reflection fixes e^{ϖ_1}. All roots with that property are thus obtained and make up a subsystem of type A_7 that has*

$$(\beta_1, \dots, \beta_7) := (e_2 + \cdots + e_7 - 2l, e_1 - e_2, e_2 - e_3, \dots, e_6 - e_7)$$

as a root basis.

Proof. With the help of Corollary 6.5 we easily verify that $\beta(A, A')$ runs over all the roots for which the coefficient of l is even. These are precisely the roots whose reflection fixes e^{ϖ_1}. The rest is straightforward. □

Recall that our marking determines a group homomorphism $\chi : Q \to \mathrm{Pic}(\mathcal{K}_{\Delta^*})^0 \cong \mathbb{C}((t))^\times$.

Proposition 6.8. *For a section γ of $\mathcal{X}/\Delta$ as above, the resulting family of Del Pezzo triples over Δ^* defines an Δ^*-valued point of $\mathbb{T}^\circ$ that extends to $\Delta \to \mathbb{T}$ with o mapping to e^{ϖ_1}. Its lift to $\mathrm{Bl}_{e^{\varpi_1}}(\mathbb{T})$ sends o to a point that does not lie on the strict transform of a reflection hypertorus (hence is in $\tilde{\mathbb{T}}^\circ$) and which describes the relative position of $p_0, \dots, p_7$ on the punctured conic $C - \{p\}$ (which we think of as an affine line) as follows: the mutual ratio of $[p_0 - p_1 : \cdots : p_6 - p_7] \in \mathbb{P}^7$ is the value of $[\chi(\beta_1) - 1 : \cdots : \chi(\beta_7) - 1]$ in o.*

Proof. We have seen that each $E_A^\pm$ is the specialization of a unique exceptional curve $\mathcal{E}_A^\pm$ over Δ. The latter meets $\mathcal{K}$ in a section of $\mathcal{K}/\Delta$ that we shall denote by $\sigma_A^\pm$. The value of this section in o lies over the point p_A where the line through A and the tangent to C at p meet and since $p_A \neq p$, this value is a smooth point of K_o. Two such sections hit the same same component of $K_{o,\mathrm{reg}}$ if and only they have the same sign in their labeling and in that case, according to Lemma 6.2, their difference in $\mathrm{Pic}(\mathcal{K}/\Delta)$ then specializes over o to 1 (it will be -1 otherwise). This implies the first part of the proposition: the closed point of Δ will go to a point of $\mathbb{T}$ where all the characters defined by exceptional vectors take only two (opposite) values, the value only depending on the parity of the coefficient of l in the exceptional vector. Since roots are differences of exceptional vectors, this means that the character defined by any root takes there the value ± 1, depending on the parity of l in the root. So this must be e^{ϖ_1}.

If $A \cap A'$ is a singleton, then the lines through A and A' clearly do not meet outside $A \cap A'$ and hence $p_A \neq p_{A'}$. Our marking associates to such a pair the root $\beta_{A,A'} \in R$. If we fix A' for the moment, then the entries of $(p_A - p_{A'})_{|A \cap A'|=1}$ are all nonzero (these differences makes sense if we regard $L_o - p$ as an affine line) and it follows from the first part of 6.2 that their mutual ratio is equal to that the collection $\big((\sigma_A^+) - (\sigma_{A'}^+) - 1\big)_{|A \cap A_0|=1}$ evaluated in o. But then this remains true if we let both A and A' vary subject to the condition $|A \cap A'| = 1$. For such a pair (A, A') we have $(\sigma_A^+) - (\sigma_{A'}^+) = \chi(\beta_{A,A'})$ and since any root whose reflection fixes ϖ_0 is of the form $\beta_{A,A'}$ all assertions of the proposition but the last follow.

To prove the last assertion, we note that assigning to a point $a \in C - \{p\}$ the point a' of intersection of its tangent with L_o defines an isomorphism $C - \{p\} \cong L_o - \{p\}$. For $a, b \in C - \{p\}$, the line spanned by a, b (the tangent to C if $a = b$) meets $L - \{p\}$ in a point which is easily verified to be the barycenter of a' and b'. Thus $E_{p_i,p_j} \cap L_o = \frac{1}{2}(p'_i + p'_j)$. Now $E_{p_i,p_j} \cap L_o$ is the image of $E^+_{p_i,p_j} \cap K_o = \sigma^+_{p_i,p_j}(o)$ and this is associated to the exceptional vector e_j if $0 = i < j$ and to $(2l - e_1 - \cdots - e_7) + e_i + e_j$ if $1 \leq i < j$. This makes $\beta_1 = e_2 + \cdots + e_7 - 2l = e_2 - ((2l - e_1 - \cdots - e_7) + e_1 + e_2)$ correspond to $\frac{1}{2}(p'_0 + p'_2) - \frac{1}{2}(p'_1 + p'_2) = \frac{1}{2}(p'_0 - p'_1)$ and $\beta_i = e_i - e_{i+1}$ for $i = 1, \ldots, 6$ to $\frac{1}{2}(p'_0 + p'_i) - \frac{1}{2}(p'_0 + p'_{i+1}) = \frac{1}{2}(p'_i - p'_{i+1})$. The last clause follows. □

Recall that we have a family of marked Del Pezzo triples of $\hat{\mathbb{T}}^\circ$ with the underlying family of Del Pezzo surfaces being a double cover over $\mathbb{P}^2_{\hat{\mathbb{T}}^\circ}$. The ramification locus $\mathcal{X}_{\hat{\mathbb{T}}^\circ} \subset \mathbb{P}^2_{\hat{\mathbb{T}}^\circ}$ of that double cover is then a family of nonsingular quartic curves over $\hat{\mathbb{T}}^\circ$. It comes with a section, making it a family of pointed quartic curves. We can now fill in a family of pointed hyperelliptic genus 3 curves over $\tilde{\mathbb{T}}^\circ - \hat{\mathbb{T}}^\circ$ as to obtain a family of pointed smooth genus 3 curves over $\tilde{\mathbb{T}}^\circ$.

Corollary 6.9. *The closure of $\mathcal{X}_{\hat{\mathbb{T}}^\circ}$ in $\mathbb{P}^2_{\hat{\mathbb{T}}^\circ}$ yields after normalization a family of pointed genus 3 curves $\mathcal{X}_{\tilde{\mathbb{T}}^\circ}$ over $\tilde{\mathbb{T}}^\circ$, which is over $\tilde{\mathbb{T}}^\circ - \hat{\mathbb{T}}^\circ$ hyperelliptic. The section extends across $\tilde{\mathbb{T}}^\circ$ and avoids the Weierstraß points.*

Proof. It suffices to deal with the stratum over e^{ϖ_1}. This stratum can be identified with $\mathbb{P}(V^\circ)$, where V is the span of an A_7 root system with root basis $\beta_1, \ldots, \beta_7$. An element of $\mathbb{P}(V^\circ)$ defines an affine line with a configuration of 8 distinct numbered points $p_0, \ldots, p_7$, given up to translation. We take as affine line a punctured conic $C - \{p\} \subset \mathbb{P}^2$. If $F \in \mathbb{C}[Z_0, Z_1, Z_2]$ is an equation for C and choose $G \in \mathbb{C}[Z_0, Z_1, Z_2]$ homogeneous of degree 4 such that $C \cap (G = 0) = \{p_0, \ldots, p_7\}$. Then $F^2 + tG$ defines a degenerating family of quartic curves $\mathcal{X}/\Delta$ whose normalization $\tilde{\mathcal{X}}/\Delta$ is a family of genus three curves. Choose also a section γ of $\tilde{\mathcal{X}}/\Delta$ such that $\gamma(o) = p$. According to Proposition 6.8 these data define a morphism $\Delta \to \tilde{\mathbb{T}}^\circ$ whose generic point lies in $\hat{\mathbb{T}}^\circ$ and whose special point goes to the given element of $\mathbb{P}(V^\circ)$. The universal nature of this construction yields the corollary. □

Proof of Theorem 4.4. Most of what the theorem states is immediate from Corollary 6.9. The assertion that remains to prove is that the complement of $(\tilde{\mathbb{T}}^\circ_7)_{W_7}$ in $\mathcal{M}_{3,1}$ is of codimension ≥ 2. That complement parameterizes pointed genus three curves (X, p) for which, in case is X a quartic curve, p is a hyperflex point and in case X is hyperelliptic, p is a Weierstraß point (in either case this amounts to: $4(p)$ is a canonical divisor). It it clear that this defines a locus in $\mathcal{M}_{3,1}$ of codimension ≥ 2. □

Remark 6.10. If S is a Del Pezzo surface of degree 2 and $X \subset S$ is its fixed point set of the involution, then a marking $\Lambda_{1,7} \to \mathrm{Pic}(S)$ of S composed

with the restriction map yields a homomorphism $\Lambda_{1,7} \to \mathrm{Pic}(X)$. Now an odd theta characteristic of X is of the form $(p) + (q)$, with $2(p) + 2(q)$ a canonical divisor and so this is where X meets a bitangent (relative to its canonical embedding). Such a bitangent corresponds to a pair of exceptional curves on S whose sum is anticanonical and hence this homomorphism sends an exceptional vector to an odd theta characteristic of X and k to the canonical class $[K_X] \in \mathrm{Pic}_4(X)$. We thus obtain a bijection between the set $\Theta_{\mathrm{odd}}(X) \subset \mathrm{Pic}^2(X)$ of odd theta characteristics of X and pairs of exceptional vectors with sum k.

An exceptional vector defines an element of the weight lattice $Q^* := \mathrm{Hom}(Q, \mathbb{Z})$ and the bijection is in fact the restriction of a map between two torsors: namely from $Q[1] := \{v \in \Lambda_{1,7} \,|\, v \cdot k = 1\}$ (a Q-torsor) to the set $\Theta(C)$ of all theta characteristics of X (a torsor under the group of order two elements in $\mathrm{Pic}(X)$, $\mathrm{Pic}(X)_2$). This map is surjective and factors through an isomorphism of torsors

$$Q[1]/2Q^* \cong \Theta(C).$$

The underlying group isomorphism $Q/2Q^* \cong \mathrm{Pic}(X)_2$ had been observed by Van Geemen as well as the fact that this is an isomorphism of symplectic modules: the evident $\mathbb{Z}/2$-valued bilinear form on $Q/2Q^*$ corresponds to the natural pairing on $\mathrm{Pic}(X)_2$. This also identifies $\overline{W} = W/\{\pm 1\}$ with the symplectic group of $\mathrm{Pic}(X)_2$. So the orbifold $\tilde{\mathbb{T}}^\circ_{\{\pm 1\}}$ can be regarded is an open subset in the moduli space $\mathcal{M}_{3,1}[2]$ of pointed genus three curves with level two structure.

The marked Del Pezzo surfaces of degree 2 define a connected degree two covering of a connected component of $\mathcal{M}_{1,3}[2]$ and a W-covering of $\mathcal{M}_{3,1}$ (which has $\tilde{\mathbb{T}}^\circ$ as an open set). This is not an unramified covering, even if $\mathcal{M}_{3,1}$ is regarded as a stack, but one may wonder whether this still has some meaning in terms of a structure on X.

The genus three surface with a boundary circle. Recall that $\mathcal{M}_{3,\vec{1}}$ denotes the moduli space of pairs (X, v), where X is projective nonsingular genus three curve and v a nonzero tangent vector of X and that it comes with a proper $\mathbb{C}^\times$-action whose orbit space is $\mathcal{M}_{3,1}$. We have indentified $\tilde{\mathbb{T}}^\circ_W$ with an open subset of $\mathcal{M}_{3,1}$ whose complement has codimension ≥ 2. So it enough to determine the fundamental group of the restriction of $\mathcal{M}_{3,\vec{1}}$ to $\tilde{\mathbb{T}}^\circ_W$. We find it convenient to work W-equivariantly with the pull-back of $\mathcal{M}_{3,\vec{1}}$ to $\tilde{\mathbb{T}}^\circ$. Over that variety it is a genuine $\mathbb{C}^\times$-bundle with W-action that we shall denote by $\mathcal{V}^\times \to \tilde{\mathbb{T}}^\circ$; we reserve the symbol $\mathcal{V}$ for the associated line bundle over $\tilde{\mathbb{T}}^\circ$.

Let us begin with the following observation. If (S, K, p) is a Del Pezzo triple of degree two, then the natural involution ι of S has the property that its fixed point set $X \subset S$ is a quartic curve passing through p so that T_pX is the fixed point line of ι in T_pS. Since T_pS is also the Zariski tangent space T_pK of K at p, we can reconstruct T_pX from the triple (X, p, ι). Our

construction of $\mathbb{T}^\circ$ yields a trivial bundle of type I_1 curves with involution over $\mathbb{T}^\circ$: if K denotes such a curve with involution ι so that the family is $K \times \mathbb{T}^\circ \to \mathbb{T}^\circ$, then the action of W respects this decomposition with $w \in W$ acting on K by ι if $\det(w) = -1$. So from the above discussion we see that $\mathcal{V}\big|_{\mathbb{T}^\circ}$ is simply $(T_pK)^\iota \times \mathbb{T}^\circ \to \mathbb{T}^\circ$ with W acting trivially on the first factor. In particular, $(\mathcal{V}^\times\big|\mathbb{T}^\circ)_W$ can be identified with $\mathbb{C}^\times \times \mathbb{T}^\circ_W$. Since $(\mathcal{V}^\times\big|\mathbb{T}^\circ)_W$ is open-dense in $\mathcal{V}^\times$, the fundamental group of $\mathcal{V}^\times_W$ (which is also the one of $\mathcal{M}_{3,\vec{1}}$) is a quotient of the direct product of an infinite cyclic group $r^{\mathbb{Z}}$ (here we think of r as the positive generator of $\pi_1(\mathbb{C}^\times, 1)$) and the orbifold fundamental group $\{1, \iota_1\} \ltimes Art_{\hat{E}_7}$ of $\mathbb{T}^\circ_W$ (here ι_1 denotes the nontrivial involution of the $\hat{E}_7$-diagram). In fact, the complement of $(\mathcal{V}^\times\big|\mathbb{T}^\circ)_W$ in $\mathcal{V}^\times_W$ consist of three connected divisors, and each of these defines a relation in $r^{\mathbb{Z}} \times (\{1, \iota_1\} \ltimes Art_{\hat{E}_7})$ so that $\pi_1(\mathcal{V}^\times)$ is obtained by imposing these relations. So we need to identify these relations.

To this end, we write the complement $\tilde{\mathbb{T}}^\circ - \mathbb{T}^\circ$ as a sum of three of divisors: $D(E_7)$, the divisor over the unit element of $\mathbb{T}$ (and over which K degenerates into a type II curve), $D(A_7)$ the sum of the divisors over the A_7-points (over which K degenerates into a type III curve) and $D(E_6^{\mathrm{tor}})$, the sum of the toric divisors (over which K degenerates into a type I_2 curve).

Lemma 6.11. *The W-line bundle $\mathcal{V}$ over $\tilde{\mathbb{T}}^\circ$ is W-equivariantly isomorphic to*

$$\mathcal{O}_{\tilde{\mathbb{T}}^\circ}(-2D(E_7) - D(A_7) + D(E_6^{\mathrm{tor}})).$$

Proof. Since W acts transitively on the connected components of a divisor of a given type, the bundle $\mathcal{V}$ will a priori be given by a linear combination of the three divisors. Our task is therefore to find the coefficients. We do this with a valuative test: we pull back along a morphism $\gamma : \Delta \to \tilde{\mathbb{T}}^\circ_W$ whose special point goes to one of the added divisors and which is transversal to that divisor. Since $\tilde{\mathbb{T}}^\circ \to \tilde{\mathbb{T}}^\circ_W$ has order two ramification along these divisors (they are pointwise fixed under the involution ι), a base change of order two ($\tilde{t}^2 = t$) yields a lift $\tilde{\gamma} : \tilde{\Delta} \to \tilde{\mathbb{T}}^\circ$. The resulting degeneration $\mathcal{K}/\tilde{\Delta}$ is split ($\mathcal{K}_{\tilde{\Delta}^*/\tilde{\Delta}^*}$ is trivial in the sense that it is $\tilde{\Delta}^*$-isomorphic to $K \times_{\mathrm{Spec}(\mathbb{C})} \tilde{\Delta}^*$) and an affine part of $\mathcal{K}$ admits the following weighted homogeneous equation:

(II) $y^2 = x^3 + \tilde{t}^2 x^2$ with $(x, y, \tilde{t})$ having weights $(2, 3, 1)$, ($\omega = \tilde{t}^{-2}dx$);
(III) $y^2 = x^4 + \tilde{t}^2 x^2$ with (x, y, s) having weights $(1, 2, 1)$, ($\omega = \tilde{t}^{-1}dx$);
(I_2) $y^2 = x^2 + \tilde{t}^2 x^4$ with (x, y, t) having weights $(1, 1, -1)$, ($\omega = \tilde{t}dx$).

Here the form ω has (and is up to a scalar in $\mathbb{C}^\times$ characterized by) the following three properties: it is ι-invariant, it is nonzero on the Zariski tangent space of the singular point of $\mathcal{K}_{\tilde{\Delta}^*}$ and it has weight zero. The last property ensures that if we trivialize $K_{\tilde{\Delta}^*/\tilde{\Delta}^*}$, then ω gets trivialized, too. So ω might be regarded as the pull-back of a constant section of $\mathcal{V}^*\big|_{\mathbb{T}^\circ}$. On the other hand in all three cases the form dx is ι-invariant and nonzero on the Zariski tangent space of the singular point of the central fiber as well. So in the

first case, $\tilde{t}^2\omega$ defines an isomorphism of the pull-back of $\mathcal{V}$ with $\tilde{t}^2\mathbb{C}[[\tilde{t}]]$ and likewise for the other cases. The lemma follows. $\square$

We will find it convenient to work on the quotient of $\mathcal{V}^\times$ by the involution given by scalar mulitplication in the fibers by -1. This quotient is of course simply $(\mathcal{V}\otimes\mathcal{V})^\times$. In this way $r^{\frac{1}{2}}$ can be understood as an element of the orbifold fundamental group of $(\mathcal{V}\otimes\mathcal{V})^\times_W$ and we identify the fundamental group of $\mathcal{V}^\times_W$ with a subgroup of index two of the orbifold fundamental group of $(\mathcal{V}\otimes\mathcal{V})^\times_W$.

Lemma 6.12. *The orbifold fundamental group of $(\mathcal{V}\otimes\mathcal{V})^\times_W$ is obtained from $r^{\frac{1}{2}\mathbb{Z}}\times SArt_{\hat{E}_7}$ (the fundamental group of $(\mathcal{V}\otimes\mathcal{V}|_{\mathbb{T}^\circ})^\times_W$) by imposing the following three relations (corresponding to the three added divisors):*

$$D(E_7):(r,\Delta_{E_7})\equiv 1;$$
$$D(A_7):(r^{\frac{1}{2}},\iota_1\Delta_{A_7})\equiv 1;$$
$$D(E_6^{\mathrm{tor}}):(r^{-\frac{1}{2}},(\iota_1\Delta_{E_6})^{-1})\equiv 1.$$

Proof. We use our fixed trivialization over of $\tilde{\mathbb{T}}^\circ$.

Let us begin with the first case, where the divisor is $D(E_7)$. In terms of our fixed trivialization, the section $\tilde{t}=1$ of $\tilde{\gamma}^*\mathcal{V}^\times$ is over $\tilde{\Delta}^*$ given by $(\tilde{t}^2,\tilde{t})\in\mathbb{C}^\times\times\tilde{\Delta}^*$. The homotopy class of this restriction is $(r^2,\Delta^2_{E_7})$ and so this represents a trivial element of the fundamental group of $\tilde{\mathbb{T}}^\circ$. But $\tilde{\gamma}^*\mathcal{V}^\times$ is obtained from $\gamma^*\mathcal{V}^\times_W$ by the substitution $\tilde{t}^2=t$ and so (r,Δ_{E_7}) represents a trivial element of the fundamental group of $\mathcal{V}^\times_W$.

The second case is dealt with in the same way, except that we now work in $r^{\frac{1}{2}\mathbb{Z}}\times SArt_{\hat{E}_7}$ (the orbifold fundamental group of $(\mathcal{V}\otimes\mathcal{V})^\times_W$).

The difference between the third case and the previous two cases is that we need to explain why we get in the second factor not $\iota_1\Delta_{E_6}$, but its inverse. The answer to that is that in $\mathbb{C}^\times$ simple positive loops around 0 and ∞ represent in the fundamental group of $\mathbb{C}^\times$ elements that are each others inverse. To make this precise, consider the weight $\varpi_7\in\mathrm{Hom}(Q,\mathbb{Z})$ (given by taking the inner product with e_7). Then we have a cocharacter $e^{\varpi_7}:\mathbb{C}^\times\to\mathrm{Hom}(Q,\mathbb{C}^\times)=\mathbb{T}$. The preimage of $\mathbb{T}^\circ$ is $\mathbb{C}^\times-\{1\}$ and the composite map $\mathbb{C}^\times\to\mathbb{T}\to\mathbb{T}_W$ factors through a closed embedding of the quotient of $\mathbb{C}^\times$ by the involution $z\mapsto z^{-1}$. Let us denote the orbit space of $\mathbb{P}^1$ by this involution C and denote the image of 0 resp. 1 by p_0 resp. p_1. Then a simple loop around 0 maps to simple loop around p_0, whereas a simple loop around 1 maps to the square a simple loop around p_1. Since a simple positive loop around p_1 represents $\iota_1\Delta_{E_7}$, a simple positive loop around 0 represents its inverse $(\iota_1\Delta_{E_7})^{-1}$. This completes the proof. $\square$

Proof of Theorem 1.5. The fundamental group of $\mathcal{M}_{3,\vec{1}}$ is that of $\mathcal{V}^\times_W$ and Lemma 6.12 shows that this equals the image of $r^{\mathbb{Z}}\times SArt_{\hat{E}_7}$ in the quotient

of $r^{\frac{1}{2}\mathbb{Z}} \times SArt_{\hat{E}_7}$ by the three relations of that lemma. It is clear that this is simply the quotient $SArt_{\hat{E}_7}$ by the relations $\Delta_{A_7} \equiv \Delta_{E_6}$ and $\Delta_{E_6}^2 \equiv \Delta_{E_7}$. $\square$

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