

A PERFECT STRATIFICATION OF \mathcal{M}_g FOR $g \leq 5$

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ABSTRACT. We find for $g \leq 5$ a stratification of depth $g - 2$ of the moduli space of curves \mathcal{M}_g with the property that its strata are affine and the classes of their closures provide a \mathbb{Q} -basis for the Chow ring of \mathcal{M}_g . The first property confirms a conjecture of one of us. The way we establish the second property yields new (and simpler) proofs of theorems of Faber and Izadi which, taken together, amount to the statement that in this range the Chow ring is generated by the λ -class.

INTRODUCTION

The starting point of the present note is a beautiful topological result by Harer, who in [12] via a deep application of the theory of Strebel differentials showed that the (coarse) moduli space of smooth algebraic curves of genus $g \geq 2$, $\mathcal{M}_g(\mathbb{C})$, has the homotopy type of a complex of dimension $4g - 5$. A decade ago, the second author asked with Hain ([11], Problem 6.5) whether there is a Lefschetz type of proof of this fact. That would for instance be accomplished by finding a stratification of \mathcal{M}_g with all strata affine subvarieties of codimension $\leq g - 1$. One reason to make this desirable is that the relatively easy inductive algebraic approach developed by Arbarello-Cornalba in [4] to the rational cohomology of the Deligne–Mumford compactification $\overline{\mathcal{M}}_g(\mathbb{C})$ of $\mathcal{M}_g(\mathbb{C})$ has Harer’s theorem as the only nontrivial input from geometric topology. But as explained in [19], the existence of such a stratification has many other interesting cohomological consequences as well.

The classical candidate for an affine stratification of \mathcal{M}_g was the flag of subvarieties defined by Arbarello in terms of Weierstrass points in [2] and [3]. Its smallest stratum is the hyperelliptic locus and hence affine and Theorem 1 of [9] shows that the same is true for the open stratum. But the intermediate strata not always seem to be affine (although they are easily seen to be quasi-affine). Here we attempt to obtain an affine stratification by looking at properties of the linear system of quadrics passing through the canonical image of a smooth projective genus g curve. This also singles out the hyperelliptic locus as the smallest stratum, for if the curve is nonhyperelliptic, then this linear system is of dimension $\binom{g-2}{2}$, whereas in

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the hyperelliptic case its dimension is higher. But otherwise our approach yields strata that in general differ from those of Arbarello. For instance, the complement of our codimension 2 stratum is for $g = 4$ resp. $g = 5$ the locus of curves whose canonical model lies on a smooth quadric resp. is an intersection of quadrics (or equivalently, that are not trigonal). Although we do not offer a uniform recipe, we succeed in obtaining an affine stratification of \mathcal{M}_g of depth $g - 2$ for $g \leq 5$ (we give in fact two such for $g = 5$), thus answering Problem 6.5 in [11] affirmatively in this range.

The stratifications that we construct have one other nice property that, we feel, justifies calling them *perfect*, namely that the strata have trivial Chow ring and that their closures yield a \mathbb{Q} -basis for $A^\bullet(\mathcal{M}_g)$. In order to be precise, let us recall the structure of $A^\bullet(\mathcal{M}_g)$ in this range. If $f : \mathcal{C} \rightarrow S$ is a smooth projective family of curves of genus $g \geq 2$, then its i -th *tautological class* $\kappa_i = \kappa_i(\mathcal{C}/S)$ is $f_*(c_1(\omega_{\mathcal{C}/S})^{i+1}) \in A^i(S)$ (here $A^\bullet(S)$ stands for the Chow ring of S with rational coefficients). According to Mumford [15], the subalgebra of $A^\bullet(S)$ generated by these classes (denoted by $\mathcal{R}^\bullet(\mathcal{C}/S)$ or more often simply by $\mathcal{R}^\bullet(S)$) contains the i th Chern class $\lambda_i = \lambda_i(\mathcal{C}/S)$ of the Hodge bundle $f_*\omega_{\mathcal{C}/S}$ with λ_1 being a nonzero multiple of κ_1 . A theorem of one of us [14] implies that $\lambda_1^{g-1} = 0$ always. On the other hand, Faber [8] proved that for the universal family over the moduli stack \mathcal{M}_g , $\lambda_1^{g-2} \neq 0$, so that we have an injection

$$\mathbb{Q}[\lambda_1]/(\lambda_1^{g-1}) \rightarrow \mathcal{R}^\bullet(\mathcal{M}_g) \subset A^\bullet(\mathcal{M}_g).$$

The composite map $\mathbb{Q}[\lambda_1]/(\lambda_1^{g-1}) \rightarrow A^\bullet(\mathcal{M}_g)$ is for $g \leq 5$ also onto; this is due to Faber [7] for $g = 3, 4$ and to Izadi [13] for $g = 5$. We find in every codimension $k \leq g - 2$ a single stratum and so the closure of that stratum must represent a nonzero multiple of λ_1^k . We prove this property independent of the two works just cited and thus obtain a new (and considerably simpler) proof that $A^\bullet(\mathcal{M}_g) = \mathbb{Q}[\lambda_1]/(\lambda_1^{g-1})$ in this range.

In the first section we collect or prove some general results pertaining to \mathcal{M}_g . As is well-known, a canonical curve possessing an effective even theta characteristic is contained in a rank 3 quadric. This is a codimension one phenomenon and so this suggests to take as our open stratum the locus of curves having no such theta characteristic. We indeed establish that for $g \geq 3$ this open subset is affine and can be used as the open stratum for $g \leq 5$. The cases $g = 3, 4, 5$ each have their own section. (The case $g = 2$ is trivial: \mathcal{M}_2 is affine and has trivial Chow ring.)

We work here over a fixed algebraically closed field k . We take it to be of characteristic zero since some of the results we invoke are (still) only available in that characteristic (the computations themselves only seem to exclude characteristic 2 and 3).

Notation. If C is a connected nonsingular projective curve, we write V_C for $H^0(\omega_C)^*$ and denote by $f : C \rightarrow \mathbb{P}V_C$ the canonical map. We recall that when C is hyperelliptic of genus ≥ 2 , $f(C)$ is a smooth rational curve of degree $g - 1$ in the $(g - 1)$ -dimensional $\mathbb{P}V_C$ and identifies the orbits of the hyperelliptic involution. Otherwise f is an embedding with image a curve of degree $2g - 2$ and we then identify C with $f(C)$.

We also follow the custom to write λ for λ_1 .

We shall need the following basic result of geometric invariant theory (which is essentially due to Hilbert).

Lemma 0.1. *Let U be an affine variety on which operates a reductive group G . If every G -orbit is closed (a condition that is obviously fulfilled if these orbits are fibers of a morphism of varieties with domain U), then the orbit space $G \backslash U$ is in a natural manner an affine variety. If in addition G acts with finite stabilizers, then $\dim(G \backslash U) = \dim U - \dim G$.*

Proof. It is a classical theorem that, G being reductive, $k[U]^G$ is noetherian and separates the closed orbits. So the assumption that every orbit is closed implies that a closed point of $\text{Spec}(k[U]^G)$ corresponds to a G -orbit in U and vice versa. If G also acts with finite stabilizers, then the fibers of $U \rightarrow G \backslash U$ will have the same dimension as G and the last assertion easily follows. \square

As a modest application of this lemma we recover the well-known fact that the hyperelliptic locus $\mathcal{H}_g \subset \mathcal{M}_g$ is affine: we may identify \mathcal{H}_g with $\mathcal{S}_{2g+2} \backslash \mathcal{M}_{0,2g+2}$ and $\mathcal{M}_{0,2g+2}$ can be identified with an open affine subset of affine $(2g + 1)$ -space. It is also well-known that \mathcal{H}_g is an irreducible and closed suborbifold of \mathcal{M}_g of codimension $g - 2$ and has trivial Chow ring. In particular, $\mathcal{M}_2 = \mathcal{H}_2$ has these properties.

1. SOME GENUS INDEPENDENT RESULTS

The Thetanull divisor. For $g \geq 3$, the locus in \mathcal{M}_g parametrizing curves with an effective even theta characteristic is an irreducible hypersurface in \mathcal{M}_g that we shall refer to as the *Thetanull* divisor and denote by \mathcal{M}'_g .

Proposition 1.1. *The Thetanull divisor \mathcal{M}'_g is irreducible and its complement in \mathcal{M}_g is affine for every $g \geq 4$.*

Proof. The irreducibility is due to Teixidor i Bigas [18], Proposition (2.4). For the other property we argue as for the proof of [9], Theorem 1. Indeed, if we recall that the rational Picard group of $\overline{\mathcal{M}}_g$ is generated by the Hodge class λ and the classes δ_i of the irreducible components Δ_i ($i = 0, \dots, [\frac{1}{2}g]$) of the boundary divisor $\partial \overline{\mathcal{M}}_g$, then according to Teixidor i Bigas (*op. cit.*, Proposition (3.1)), the class of the closure $\overline{\mathcal{M}'_g}$ of the Thetanull divisor in \mathcal{M}_g is $a\lambda - \sum_i b_i \delta_i$ with $a > 11$ and $b_i > 1$ for every i . As a consequence, the divisor

$$D := \overline{\mathcal{M}'_g} + \sum_i (b_i - 1) \Delta_i$$

is effective of class $a\lambda - \delta$ and has support $\overline{\mathcal{M}}'_g \cup \partial\overline{\mathcal{M}}_g$. Since $a > 11$, we deduce from the Cornalba–Harris criterion ([6], Theorem 1.3) that D is ample on $\overline{\mathcal{M}}_g$. Hence the complement of its support (which is also the complement of \mathcal{M}'_g in \mathcal{M}_g) is affine. \square

We shall find that this proposition also holds for $g = 3$ (as is well-known, $\mathcal{M}'_3 = \mathcal{H}_3$).

Halfcanonical pencils and quadrics. We need the following lemma.

Lemma 1.2. *Let C be a nonsingular projective curve of genus $g \geq 3$. Any effective even theta characteristic determines a rank 3 quadric in $\mathbb{P}V$ that contains $f(C)$. Conversely, a rank 3 quadric $Q \subset \mathbb{P}V$ whose smooth part contains $f(C)$ defines a halfcanonical pencil on C .*

Proof. Let $L, \phi : L^2 \cong \omega_C$ be an effective even theta characteristic. If $\alpha, \beta \in H^0(C, L)$ are linearly independent, then $u := \phi(\alpha^2)$, $v := \phi(\beta^2)$ and $w := \phi(\alpha\beta)$ elements of V_C^* that satisfy $uv = w^2$. The quadric defined by this equation evidently contains $f(C)$.

Now let $Q \subset \mathbb{P}V_C$ be a rank 3 quadric that contains $f(C)$. So its singular set Q_{sg} is a codimension 3 linear subspace of $\mathbb{P}V_C$. The dual variety \check{Q} of Q is a conic in the projective plane in $\mathbb{P}V_C^*$ dual to Q_{sg} . Any $p \in \check{Q}$ defines a hyperplane in $\mathbb{P}V_C$ that meets Q in a codimension 2 linear subspace in $\mathbb{P}V_C$ which contains Q_{sg} . Such a hyperplane cuts out on C a 2-divisible divisor, i.e., one of the form $2D_p$. It is clear that D_p halfcanonical and moves in a pencil. \square

The trigonal locus. We denote by $\mathcal{T}_g \subset \mathcal{M}_g$ the locus which parameterizes the genus g curves admitting a g_3^1 . It is well-known (and easy to prove) that $\mathcal{T}_g = \mathcal{M}_g$ for $g = 2, 3, 4$. The following assertion is also well-known, but we give a proof for completeness.

Proposition 1.3. *For $g \geq 5$, the trigonal locus $\mathcal{T}_g \subset \mathcal{M}_g$ is a closed irreducible subvariety of \mathcal{M}_g of codimension $g - 4$. It contains the hyperelliptic locus \mathcal{H}_g (which is a subvariety of codimension $g - 2$).*

Proof. Denote by $\tilde{\mathcal{T}}_g$ the moduli space of pairs (C, P) , where C is a nonsingular projective genus g curve of genus g and P is a g_1^3 on C . We first show that the forgetful map $\tilde{\mathcal{T}}_g \rightarrow \mathcal{M}_g$ is proper. If $\mathcal{C}_g \rightarrow \mathcal{M}_g$ is the universal genus g curve and $\text{Pic}^3(\mathcal{C}_g/\mathcal{M}_g)$ the degree 3 component of its Picard variety, then we have an obvious morphism $\tilde{\mathcal{T}}_g \rightarrow \text{Pic}^3(\mathcal{C}_g/\mathcal{M}_g)$. The fiber over a point $[(C, \ell)]$ of $\text{Pic}^3(\mathcal{C}_g/\mathcal{M}_g)$ is the Grassmannian of 2-planes in $H^0(C, \ell)$. From this it is easily seen that $\tilde{\mathcal{T}}_g \rightarrow \text{Pic}^3(\mathcal{C}_g/\mathcal{M}_g)$ is proper. Hence so is $\tilde{\mathcal{T}}_g \rightarrow \mathcal{M}_g$.

The preimage of the hyperelliptic locus \mathcal{H}_g in $\tilde{\mathcal{T}}_g$ can be identified with the universal hyperelliptic curve $\mathcal{C}_{\mathcal{H}_g}$: if (C, P) represents a point of $\tilde{\mathcal{T}}_g$ and P has a fixed point, then C is hyperelliptic and the moving part of P is

the hyperelliptic pencil. Conversely, a hyperelliptic curve and a point on it determine a point of $\tilde{\mathcal{T}}_g$ and so $\tilde{\mathcal{T}}_g$ contains the universal hyperelliptic curve.

We next show that a smooth projective curve C of genus $g \geq 5$ cannot have two distinct g_1^3 's. Suppose it did so that we have a morphism $\phi : C \rightarrow \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of bidegree $(3, 3)$. Since a reduced curve in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (a, b) has for positive a, b (by the adjunction formula) arithmetic genus resp. $(a-1)(b-1)$, ϕ cannot be of degree one onto its image. It is then easy to see that both pencils have a common fixed point. But then they are hyperelliptic and coincide. This contradicts our assumption.

So $\tilde{\mathcal{T}}_g \setminus \mathcal{C}_{\mathcal{H}_g}$ maps bijectively onto $\mathcal{T}_g \setminus \mathcal{H}_g$. A standard deformation argument shows that this is étale and is hence is an isomorphism.

Any element of $\mathcal{T}_g \setminus \mathcal{H}_g$ is represented by a triple cover $C \rightarrow \mathbb{P}^1$ that is unique up to an automorphism of \mathbb{P}^1 . The discriminant of this cover has degree $2g+4$. From this we see that \mathcal{T}_g has dimension $2g+4-3=2g+1$.

According to Fulton [10], the moduli space of smooth connected triple coverings \mathbb{P}^1 of genus g is irreducible and so \mathcal{T}_g is irreducible as well. \square

2. THE CASE $g = 3$

In the theorem below the second part is due to Faber [7]. The proof given here is not only simpler, but also has the virtue of yielding the first part (which is probably known).

Theorem 2.1. *An affine stratification of \mathcal{M}_3 of depth 1 is given by its hyperelliptic locus (which is also its Thetanull divisor): $\mathcal{M}_3 \setminus \mathcal{H}_3$ and \mathcal{H}_3 are affine and irreducible. Moreover the strata $\mathcal{M}_3 \setminus \mathcal{H}_3$ and \mathcal{H}_3 have the Chow ring of a point and so $A^\bullet(\mathcal{M}_3)$ is \mathbb{Q} -spanned by the classes of \mathcal{M}_3 and \mathcal{H}_3 .*

Proof. We know that \mathcal{H}_3 is affine and that $A^\bullet(\mathcal{H}_3)$ is spanned by $[\mathcal{H}_3]$. So we must prove that $\mathcal{M}_3 \setminus \mathcal{H}$ is affine and $A^\bullet(\mathcal{M}_3 \setminus \mathcal{H})$ is spanned by $[\mathcal{M}_3 - \mathcal{H}]$.

Let C be a nonhyperelliptic genus 3 curve and let $p_1, p_2 \in C$ be in the support of two different odd theta characteristics of C . We shall use these theta characteristics to find normal coordinates in $\mathbb{P}V_C$. If we think of C as canonically embedded as a quartic curve in the projective plane $\mathbb{P}V_C$, then the projective tangent line $l_i \subset \mathbb{P}V_C$ to C at p_i is a bitangent of C and $l_1 \neq l_2$. Denote the point of intersection of these bitangents by o and by l_∞ the line through p_1 and p_2 . It is clear that $o \notin C$. The line l_∞ meets C in p_1, p_2 and two other points (which may coincide but differ from p_1 and p_2). Choose coordinates $[Z_0 : Z_1 : Z_2]$ such that $o = [0 : 0 : 1]$, $p_1 = [0 : 1 : 0]$, $p_2 = [0 : 0 : 1]$ and the two other points of $C \cap l_\infty$ are $\{[0 : 1 : k], [0 : k : 1]\}$ for some $k \neq 0$. In terms of the affine coordinates $(z_1, z_2) = (Z_1/Z_0, Z_2/Z_0)$, an equation f for C has the form

$$f(z_1, z_2) = 1 + a_1 z_1^2 + a_2 z_2^2 + b z_1 z_2 + (c_1 z_1 + c_2 z_2) z_1 z_2 + d(z_1^2 + z_2^2) z_1 z_2 + e z_1^2 z_2^2,$$

and hence is given by $(a_1, a_2, b, c_1, c_2, d, e) \in \mathbb{A}^7$. The coordinate system is unique up to the action of $\{\pm 1\} \times \mathbb{G}_m$ -action defined by $(\varepsilon, t) \cdot (Z_0, Z_1, Z_2) = (tZ_0, Z_1, \varepsilon Z_2)$. The corresponding action on \mathbb{A}^7 is given by

$$(\varepsilon, t) \cdot (a_1, a_2, b, c_1, c_2, d, e) = (t^2 a_1, t^2 a_2, \varepsilon t^2 b, \varepsilon t^3 c_1, t^3 c_2, \varepsilon t^4 d, t^4 e).$$

We use the discriminant to eliminate the \mathbb{G}_m -action: For a polynomial F homogeneous of degree 4 in Z_0, Z_1, Z_2 we have defined its discriminant $\Delta(F) \in k$. It is nonzero if and only if F defines a nonsingular quartic and it is a semi-invariant for the action of $\mathrm{GL}(3)$. So its restriction to \mathbb{A}^7 , $\delta \in k[a_1, \dots, e]$, will be a semi-invariant for the group $\{\pm 1\} \times \mathbb{G}_m$. Denote by \mathbb{P} the weighted projective space $\mathbb{G}_m \backslash (\mathbb{A}^7 \setminus \{0\})$ and by $H \subset \mathbb{P}$ the hypersurface defined by $\delta = 0$. Since the morphism $\mathbb{P} \setminus H \rightarrow \mathcal{M}_3 \setminus \mathcal{H}_3$ is finite (of degree $2^3 \cdot 28$), Lemma 2.2 below implies that $\mathcal{M}_3 \setminus \mathcal{H}_3$ is affine and $A^\bullet(\mathcal{M}_3 \setminus \mathcal{H}_3) = \mathbb{Q}$. \square

Lemma 2.2. *Let \mathbb{P} be a weighted projective space (i.e., the proj of a graded polynomial algebra whose generators have positive weights) and let $H \subset \mathbb{P}$ be a hypersurface. Then $\mathbb{P} \setminus H$ (and hence any variety admitting a finite cover isomorphic to $\mathbb{P} \setminus H$) is affine and has the Chow ring of a point.*

Proof. If $\mathbb{P} = \mathrm{Proj} k[z_0, \dots, z_n]$ with z_i having weight $d_i > 0$, then the substitution $z_i = w_i^{d_i}$ defines a finite morphism $\mathbb{P}^n \rightarrow \mathbb{P}$. So without loss of generality we may assume that $\mathbb{P} = \mathbb{P}^n$. Clearly, $\mathbb{P}^n \setminus H$ is affine. Since $A^\bullet(\mathbb{P}^n)$ is generated by the hyperplane class, it is also generated by $[H]$ (which is a multiple of that class). It follows that $A^\bullet(\mathbb{P}^n \setminus H) = \mathbb{Q}$. \square

Since we have an injective homomorphism $\mathbb{Q}[\lambda]/(\lambda)^2 \rightarrow A^\bullet(\mathcal{M}_3)$, we find:

Corollary 2.3. *The map $\mathbb{Q}[\lambda]/(\lambda)^2 \rightarrow A^\bullet(\mathcal{M}_3)$ is an isomorphism and the class $[\mathcal{H}_3] \in A^1(\mathcal{M}_3)$ is nonzero and a multiple of λ .*

3. THE CASE $g = 4$

According Lemma 1.2 a smooth projective curve of genus 4 possesses a halfcanonical pencil if and only if its canonical model of C lies on a singular conic. The corresponding locus \mathcal{M}'_4 in \mathcal{M}_4 is the Thetanull divisor which by Proposition 1.3 contains the hyperelliptic locus \mathcal{H}_4 .

Theorem 3.1. *A stratification of \mathcal{M}_4 of depth 2 is given by the Thetanull divisor and the hyperelliptic locus: $\mathcal{M}_4 \supset \mathcal{M}'_4 \supset \mathcal{H}_4$. This is a filtration by subvarieties whose successive differences are affine and have the Chow ring of a point (so that $A^\bullet(\mathcal{M}_4)$ is spanned by the classes of $\mathcal{M}_4, \mathcal{M}'_4, \mathcal{H}_4$).*

Since we already know that \mathcal{H}_4 is irreducible, affine and has the Chow ring of a point, the theorem follows from the two statements below.

Proof that $\mathcal{M}_4 \setminus \mathcal{M}'_4$ is affine and has the Chow ring of a point. Let be given a projective curve C of genus 4 that represents a point of $\mathcal{M}_4 \setminus \mathcal{M}'_4$, i.e., has no effective even theta characteristics. Then C is in $\mathbb{P}V_C$ the complete

intersection of a nonsingular quadric surface Q and a cubic surface. We recall that such a quadric has two rulings (it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$). We shift our attention to Q and regard C as a divisor on Q .

We choose three items attached to C : an ordering of the two rulings of Q , an effective degree 3 divisor D on C that is an odd theta characteristic and a point p in the support of D . Then D determines a plane section $k \subset Q$ with the property that within Q , k meets C in $2D$. Let m be the line in the second ruling of Q through p . This line and C meet inside Q at p simply and hence they meet in two other (possibly coinciding) points $\neq p$ as well. Denote by l the line in the first ruling through the barycenter in $m \setminus \{0\}$ of these two points. Now choose an affine coordinate system (x, y) for Q such that the lines k, l, m are defined by resp. $x = y, y = 0, x = \infty$. This coordinate system is unique up to the action of \mathbb{G}_m -action defined by $t \cdot (x, y) = (t^{-1}x, t^{-1}y)$. If

$$f(x, y) = \sum_{i,j=0}^3 a_{ij} x^i y^j$$

is an equation for C , then $t \in \mathbb{G}_m$ acts on a_{ij} as multiplication by t^{i+j} . The conditions imposed imply that $f(x, 0)$ is of the form $x(u + vx^2)$, in other words, $a_{0,0} = a_{2,0} = 0$, and that $f(x, x)$ is of the form $x^2(a + bx^2)^2$, in other words, $\sum_{i+j=k} a_{i+j}$ equals 0 for $k = 1, 3, 5$, and $a^2, 2ab, b^2$ for $k = 2, 4, 6$. We may therefore parameterize such equations by affine 10-space:

$$\left((a_{ij})_{i=1,2;j=1,2,3}, a_{1,0}, a_{3,0}, a, b \right) \in \mathbb{A}^{10}$$

But beware: $(a^2, 2ab, b^2)$ is read off from the equation, not (a, b) , and so (a, b) is defined up to a common sign. We give a resp. b weight 1 resp. 3 so that \mathbb{G}_m now acts on \mathbb{A}^{10} with positive weights. From here on we proceed as in the case $g = 3$: the \mathbb{G}_m -action on \mathbb{A}^{10} defines a weighted projective space \mathbb{P} of dimension 9, the discriminant $\delta \in k[\mathbb{A}^{10}]$ determines a hypersurface $H \subset \mathbb{P}$. The morphism $\mathbb{P} \setminus H \rightarrow \mathcal{M}_4 \setminus \mathcal{M}'_4$ is finite. It then follows from Lemma 2.2 that $\mathcal{M}_4 \setminus \mathcal{M}'_4$ is affine and has trivial Chow ring. \square

Proof that $\mathcal{M}'_4 \setminus \mathcal{H}_4$ is affine and has the Chow ring of a point. Let be given a projective curve C that represents an element of $\mathcal{M}'_4 \setminus \mathcal{H}_4$. Then the canonical model of C is complete intersection of a quadric cone Q and a cubic surface. Denote the vertex of Q by o . Since C is smooth, $o \notin C$. We choose two items attached to C : an effective degree 3 divisor D on C that is an odd theta characteristic and a point p in the support of D . Since $2D$ is canonical, it is the section of a plane $H \subset \mathbb{P}V_C$. This plane will not pass through o : otherwise H would be tangent to Q , hence D would lie in the halfcanonical pencil defining Q , and this would contradict our assumption that D is odd. Since the line l through o and p is not contained in H , l and C , viewed as curves on Q , meet simply in p . In particular, $l - \{o, p\}$ meets C in two points p_1, p_2 (which may coincide). We put $Q_\infty := Q \cap H$.

Now C is defined by a regular function f on $Q \setminus Q_\infty$ which has a pole of order three along Q_∞ . We normalize f by requiring that $f(o) = 1$. Regard $\mathbb{P}V_C \setminus H$ as a vector space with origin o so that we can write $f = f_3 + f_2 + f_1 + 1$ with f_i homogeneous of degree i . The quadratic part f_2 defines an effective degree 4 divisor E on Q_∞ . It follows from the preceding that the restriction of f to $l \setminus \{p\}$ must be of exact degree two (defining the divisor $(p_1) + (p_2)$) and so p is not in the support of E . Regard E as a divisor on $Q_\infty \setminus \{p\}$ (which is isomorphic to an affine line) and denote by m the line through the barycenter of E in $Q_\infty \setminus \{p\}$ and o . We now have coordinate hyperplanes in $\mathbb{P}V_C \setminus H$ spanned by l and m and the two tangent planes of Q containing l and m .

It is then easy to see that there exist coordinates (x, y, z) in $\mathbb{P}V_C \setminus H$ such that l is the y -axis, m is the x -axis, Q is given by $xy = z^2$ and $y(p_1)y(p_2) = 1$. This system of coordinates is unique up to the action of the group $\{\pm 1\} \times \mathbb{G}_m$ given by $(\epsilon, t) \cdot (x, y, z) = (t^{-2}x, \epsilon y, t^{-1}z)$. Since $Q_\infty \cdot C = 2D$, f_3 can be given the form $x(ax + by + cz)^2$, where $ax + by + cz = 0$ defines in H the line that is spanned by the degree 2 divisor $D - (p)$ (so the factor $ax + by + cz$ is unique up to sign). We represent f_2 as a polynomial without the monomial z^2 . The condition imposed on f_2 means that $f_2(1, t^2, t)$ is a quartic polynomial with vanishing t^3 -coefficient. This means that f_2 has vanishing yz -coefficient. The fact that $f(0, y, 0)$ is quadratic with constant term 1 and the product of its roots equal to 1 implies that the coefficient of y^2 is 1. So we have now written

$$f(x, y, z) = 1 + (a_2x + b_2y + c_2z) + (a_1x^2 + b_1xy + y^2 + c_1xz) + x(ax + by + cz)^2.$$

These equations are parameterized by $(a, b, c, a_1, b_1, c_1, a_2, b_2, c_2) \in \mathbb{A}^9$, where we should bear in mind that $(-a, -b, -c; a_1, b_1, c_1, a_2, b_2, c_2)$ yields the same equation. The $\{\pm 1\} \times \mathbb{G}_m$ -action lifts to

$$\begin{aligned} (\epsilon, t) \cdot (a, b, c; a_1, b_1, c_1, a_2, b_2, c_2) &= \\ &= (t^3a, \epsilon tb, t^2c; t^4a_1, \epsilon t^2b_1, t^3c_1, t^2a_2, \epsilon b_2, tc_2). \end{aligned}$$

We proceed as before: the \mathbb{G}_m -action on \mathbb{A}^9 defines a weighted projective space \mathbb{P} of dimension 8 and the discriminant $\delta \in k[\mathbb{A}^9]$ determines a hypersurface $H \subset \mathbb{P}$ so that we have a finite morphism $\mathbb{P} \setminus H \rightarrow \mathcal{M}'_4 \setminus \mathcal{H}_4$. It remains to apply Lemma 2.2. \square

Combining this with the injectivity of $\mathbb{Q}[\lambda]/(\lambda)^3 \rightarrow A^\bullet(\mathcal{M}_4)$ this yields:

Corollary 3.2. *The map $\mathbb{Q}[\lambda]/(\lambda)^3 \rightarrow A^\bullet(\mathcal{M}_4)$ is an isomorphism and the classes $[\mathcal{M}'_4]$ resp. $[\mathcal{H}_4] \in A^1(\mathcal{M}_4)$ are nonzero multiples of λ and λ^2 respectively.*

In fact, according to [18], Proposition (3.1), $[\mathcal{M}'_4] \in A^1(\mathcal{M}_4) = 34\lambda$. On the other hand, we have by [15], §7, $[\mathcal{H}_4] = 3\kappa_2 - 15\lambda^2$, and from the two formulas for the \mathbb{Q} -class of the hyperelliptic locus given in [16], Example 5.7 (a), we deduce the relation $2\kappa_2 = 27\lambda^2$ in the case $g = 4$.

4. THE CASE $g = 5$

Let C be a smooth projective curve C of genus 5. If C is nonhyperelliptic, then the quadrics passing through its canonical model form a net and either that net defines C (so that C is a complete intersection of three quadrics) or a rational scroll (for this and what follows, see [17]).

In the last case, the scroll is a Hirzebruch surface of type F_1 (= a projective plane blown up in a point) embedded by the linear system $|e + 2f|$, where $e \in \text{Pic}(F_1)$ is the class of the exceptional curve $E \subset F_1$ and $f \in \text{Pic}(F_1)$ the class of the ruling, and we have $C \in |3e + 5f|$. So C is trigonal: $C \cdot f = 3$ and the ruling of the scroll yields a g_3^1 on C . It also follows that $C \cdot E = 2$. So upon contracting E in F_1 we get a projective plane curve in which the image of C becomes a quintic with either an ordinary double point or a cusp (according to whether C meets E in two points or one). The converse is also true: a quintic plane curve C' with a singular point of arithmetic genus one defines a genus 5 curve with a g_3^1 defined by the pencil of lines through that point. In the cusp case, our g_3^1 on C is the moving part of a halfcanonical pencil that has the unique point of intersection of C and E as its fixed point. (According to Accola [1], any halfcanonical pencil on a trigonal curve is necessarily of this form and so C has no such pencils if C' has an ordinary double point.)

Recall from Proposition 1.3 that the trigonal locus $\mathcal{T}_5 \subset \mathcal{M}_5$ contains the hyperelliptic locus. Since a hyperelliptic curve possesses halfcanonical pencils, we have in fact $\mathcal{H}_5 \subset \mathcal{T}_5 \cap \mathcal{M}'_5$. The main result of this section is:

Theorem 4.1. *An affine stratification of \mathcal{M}_5 of depth 3 consists of the Thetanull divisor, the intersection of the trigonal locus with the Thetanull divisor and the hyperelliptic locus; another one is obtained by replacing the Thetanull divisor by the trigonal locus. So if we put $\mathcal{T}'_5 := \mathcal{T}_5 \cap \mathcal{M}'_5$, then these are given by the following filtrations by closed irreducible subvarieties:*

$$\mathcal{M}_5 \supset \mathcal{M}'_5 \supset \mathcal{T}'_5 \supset \mathcal{H}_5 \text{ resp. } \mathcal{M}_5 \supset \mathcal{T}_5 \supset \mathcal{T}'_5 \supset \mathcal{H}_5.$$

Moreover, the successive differences $\mathcal{M}_5 \setminus \mathcal{T}_5$, $\mathcal{T}_5 \setminus \mathcal{T}'_5$ and $\mathcal{T}'_5 \setminus \mathcal{H}_5$ have trivial Chow ring.

We know that \mathcal{H}_5 is a closed irreducible subvariety of \mathcal{M}_5 that is also affine. Following Proposition 1.1, \mathcal{M}'_5 is an irreducible hypersurface in \mathcal{M}_5 whose complement is affine. We discuss the remaining strata separately.

The proof that the Chow ring of $\mathcal{M}_5 \setminus \mathcal{T}_5$ is tautological, requires a few general remarks about Chern classes attached to families of curves. Recall that if C is a nonhyperelliptic projective smooth curve, then the natural map $\text{Sym}^2 H^0(C, \omega_C) \rightarrow H^0(C, \omega_C^2)$ is surjective and its kernel defines the linear system of quadrics through C in $\mathbb{P}V_C$.

Lemma 4.2. *Let $f : \mathcal{C} \rightarrow S$ be a smooth projective family of nonhyperelliptic curves. Then we have a corresponding surjection of vector bundles*

$$\text{Sym}^2 f_* \omega_{\mathcal{C}/S} \rightarrow f_* \omega_{\mathcal{C}/S}^2$$

whose kernel has the property that its Chern classes (taken in the Chow ring of S) are in the subalgebra generated by the tautological classes $\kappa_i(C/S)$.

Proof. The i -th Chern class λ_i of the Hodge bundle $f_*\omega_{C/S}$ is according to Mumford a universal expression in the κ -classes. Since the Chern classes of a symmetric power of a vector bundle are universally expressed in terms of those of the vector bundle, the same is true for $\text{Sym}^2 f_*\omega_{C/S}$. It therefore remains to show the corresponding property for $f_*\omega_{C/S}^2$.

The proof of this is a minor variation of Mumford's. Since the higher direct images of $\omega_{C/S}^2$ vanish, the total Chern class of the bundle $f_*\omega_{C/S}^2$ can be computed by means of Grothendieck-Riemann-Roch: it tells us that this Chern class is the direct image on S of the product of the relative Todd class of f and the Chern character of $\omega_{C/S}^2$. The Todd class is a polynomial in $c_1(\omega_{C/S})$ and so is the Chern character of $\omega_{C/S}^2$ (for it equals $\exp(2c_1\omega_{C/S})$). The desired property then follows. \square

Proof that $\mathcal{M}_5 \setminus \mathcal{T}_5$ is affine and has trivial Chow ring. Let V be a fixed vector space of dimension 5 and denote by $\text{Sym}^2 V^*$ the 15-dimensional space of quadratic forms on V . A net of quadrics in $\mathbb{P}V$ is given by a 3-plane in $\text{Sym}^2 V^*$. By the purity of the discriminant, the nets that fail to define a smooth complete intersection make up a hypersurface H in the Grassmannian $G_3(\text{Sym}^2 V^*)$ of 3-planes in $\text{Sym}^2 V^*$. The Picard group of $G_3(\text{Sym}^2 V^*)$ is generated by an ample class and so the complement $U := G_3(\text{Sym}^2 V^*) \setminus H$ is affine and $A^1(U) = 0$. Since U parametrizes canonically embedded genus 5 curves, we have a morphism $U \rightarrow \mathcal{M}_5$ whose fibers are $\text{SL}(V)$ -orbits. It then follows from Lemma 0.1 that $\text{SL}(V) \setminus U$ is affine. The image of U in \mathcal{M}_5 is clearly $\mathcal{M}_5 \setminus \mathcal{T}_5$. So $\mathcal{M}_5 \setminus \mathcal{T}_5$ is affine.

Next we prove the tautological nature of the Chow ring. Notice that U carries a family of genus 5 curves to which Lemma 4.2 applies. So $A^\bullet(U)$ is generated by the tautological classes of this family. In other words, the map $U \rightarrow \text{SL}(V) \setminus U \cong \mathcal{M}_5 \setminus \mathcal{T}_5$ induces a surjection of $\mathcal{R}^\bullet(\mathcal{M}_5 \setminus \mathcal{T}_5) \rightarrow A^\bullet(U)$.

Since the group $\text{SL}(V)$ acts on U with finite stabilizers and has $\mathcal{M}_5 \setminus \mathcal{T}_5$ as orbit space, it follows from a theorem of Vistoli [20] that the kernel of the algebra homomorphism $A^\bullet(\mathcal{M}_5 \setminus \mathcal{T}_5) \rightarrow A^\bullet(U)$ is the $\mathcal{R}^\bullet(\mathcal{M}_5 \setminus \mathcal{T}_5)$ -submodule generated by the 'Chern classes' of the almost $\text{SL}(V)$ -principal bundle $U \rightarrow \mathcal{M}_5 \setminus \mathcal{T}_5$. These Chern classes are readily identified as the Chern classes of the Hodge bundle over $\mathcal{M}_5 \setminus \mathcal{T}_5$ and so the latter's Chow ring is tautological as asserted. Since the first Chern class of a $\text{SL}(V)$ -bundle is trivial, it also follows that $A^1(\mathcal{M}_5 \setminus \mathcal{T}_5) = 0$, in particular $\lambda = 0$ in $A^\bullet(\mathcal{M}_5 \setminus \mathcal{T}_5)$.

Now in order to conclude we may argue as in [13], proof of Corollary 3.3 on p. 293: by [15] p. 309 and [7] p. 447, $\mathcal{R}^\bullet(\mathcal{M}_5 \setminus \mathcal{T}_5)$ is generated by κ_2 . On the other hand, from formula (7.7) in [15] by keeping into account the vanishing of λ it follows that the class of the locus of curves with a point p with $h^0(3p) \geq 2$ is $81\kappa_2$. Since this locus is contained in \mathcal{T}_5 , its

class vanishes in $A^\bullet(\mathcal{M}_5 \setminus \mathcal{T}_5)$, hence our claim follows. (Alternatively, we can invoke Faber's verification in [8] that $\mathcal{R}(\mathcal{M}_5)$ is generated by λ . Since $\lambda|_{\mathcal{M}_5 \setminus \mathcal{T}_5} = 0$, it follows that $\mathcal{R}^\bullet(\mathcal{M}_5 \setminus \mathcal{T}_5) = \mathbb{Q}$.) \square

Proof that $\mathcal{T}_5 \setminus \mathcal{T}'_5$ and $\mathcal{T}'_5 \setminus \mathcal{H}_5$ are affine and have trivial Chow ring. Let C' be a quintic plane curve in a projective plane P whose singular set consists of a node or a cusp, say at $p \in P$. A generic line ℓ in P through p meets C' in three other points. Then it is easily verified that the barycenter in ℓ of these three points in the affine line $\ell \setminus \{p\}$ traverses, as ℓ varies, a cubic curve $K \subset P$ which has the same 3-jet in p as C' . If K is a nodal cubic curve, then it has 3 flex points, otherwise there is just one. Choose a flex point and denote by l the tangent to $K \setminus \{p\}$ at its unique flex point. There exist affine coordinates (x, y) for $P \setminus l$ such that K is given by $y^2 + x^2 + x^3$ resp. $y^2 + x^3 = 0$. In the nodal case this system is unique up to a sign: $(x, -y)$ works also, in the cuspidal case it is unique up to the $\{\pm 1\} \times \mathbb{G}_m$ -action defined by $(\epsilon, t) \cdot (x, y) = (t^{-2}x, \epsilon t^{-3}y)$. Let us continue with the cuspidal case (the nodal case being easier). We then find that C' has an equation of the form

$$x^3 + y^2 + \sum_{i+j=4,5} a_{ij}x^i y^j.$$

The coefficient space is an 11-dimensional affine space \mathbb{A}^{11} . The group $\{\pm 1\} \times \mathbb{G}_m$ acts on this space componentwise by letting (ϵ, t) send a_{ij} to $\epsilon^j t^{2i+3j} a_{ij}$. The \mathbb{G}_m -action defines a weighted projective space \mathbb{P} and the discriminant yields a hypersurface $H \subset \mathbb{P}$ such that we get a finite morphism $\mathbb{P} \setminus H \rightarrow \mathcal{T}'_5 \setminus \mathcal{H}_5$. The assertion then follows from Lemma 2.2. \square

Corollary 4.3. *The map $\mathbb{Q}[\lambda]/(\lambda^4) \rightarrow A^\bullet(\mathcal{M}_5)$ is an isomorphism and for each of the two filtrations of Theorem 4.1, the classes of its members define a \mathbb{Q} -basis for $A^\bullet(\mathcal{M}_5)$.*

Proof. We know that the map $\mathbb{Q}[\lambda]/(\lambda^4) \rightarrow A^\bullet(\mathcal{M}_5)$ is injective. Since the filtration $\mathcal{M}_5 \supset \mathcal{T}_5 \supset \mathcal{T}'_5 \supset \mathcal{H}_5$ consists of varieties with the property that the restriction of λ to each successive difference is zero and are irreducible (except the last), we have that λ^k is proportional to the class of the k th member of this filtration. The theorem of Teixidor i Bigas [18] cited earlier implies that this remains true if we replace \mathcal{T}_5 by \mathcal{M}'_5 . \square

Question 4.4. Since for $g \geq 5$, \mathcal{T}_g contains \mathcal{H}_g as a closed affine subvariety of codimension 2, we wonder whether for such g there exists an affine stratification of \mathcal{T}_g of depth 2. The answer may very well be *no*, but the preceding theorem shows it is *yes* when $g = 5$.

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