RATIONAL SURFACES AND SINGULARITIES

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1. Del Pezzo surfaces

Fano manifolds. Let M be a connected complex manifold of complex dimension n. Its sheaf of holomorphic n-forms, Ω^n_M, often called the *canonical sheaf* of M, is the sheaf of holomorphic sections of the holomorphic line bundle defined by the nth exterior power of the holomorphic cotangent bundle of M. We however prefer the notation that refers to its a central role in Serre duality as a *dualizing sheaf* and write ω_M instead. We denote by K_M its class in $\mathrm{Pic}(M)$.

If M is compact, then the dimension of the *pluricanonical system* $|\omega_M^N|$ grows like a polynomial of degree $k \leq n$ and the smaller k, the more special M. From this point of view the most special compact complex manifolds are those for which the dual ω_M^{-1} of the canonical sheaf (called the *anticanonical sheaf*) is ample, for then $|\omega_M^N|$ is empty for all $N > 0$: after all, the dual of an ample bundle cannot have a nonzero section. This means that for some $k > 0$, the linear system $|\omega_M^{-k}|$ embeds M in a complex projective space, so that M is in fact a projective manifold. We call a compact connected complex manifold with this property a *Fano manifold*.

Example 1.1. $\mathbb{P}^n$ is a Fano manifold: the translation invariant n-form $dz_1 \wedge \cdots \wedge dz_n$ on $\mathbb{C}^n$ has, when viewed as a meromorphic n-form on $\mathbb{P}^n$, a pole of order $n+1$ along the hyperplane at infinity and so $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$. Hence $\omega_{\mathbb{P}^n}^{-1} \cong \mathcal{O}_{\mathbb{P}^n}(n+1)$ is ample. More generally, if $H \subset \mathbb{P}^{n+1}$ is a nonsingular hypersurface of degree d, then we observe that the residue on H of the meromorphic $(n+1)$-form $f^{-1}dz_1 \wedge \cdots \wedge dz_{n+1}$, where f is an equation of degree d for $H \cap \mathbb{C}^{n+1}$, vanishes of order $d-n-2$ at infinity, and so $\omega_H \cong \mathcal{O}_H(d-n-2)$. In particular, H is Fano if and only if $d \leq n+1$.

Observe that $\mathbb{P}^1$ is essentially the only Fano manifold of dimension 1, for if M is a compact connected Riemann surface of genus g, then ω_M has degree $2g-2$ and so ω_M^{-1} is ample if and only if $g = 0$. We will concentrate in this section on Fano manifolds of dimension 2. These are also called *Del Pezzo surfaces*. As we just observed, any surface in

$\mathbb{P}^3$ of degree ≤ 3 is a Fano surface. If S is a Del Pezzo surface, then the self-intersection number $K_S \cdot K_S = (-K_S) \cdot (-K_S)$ must be positive (for $-K_S$ is the class of an ample line bundle) and we call it the *degree* of S. This can be sometimes a bit confusing: for a quadric surface $S \subset \mathbb{P}^3$ we have $\omega_S \cong \mathcal{O}_S(2)$ and so S has degree 8 (rather than 2) as a Del Pezzo surface.

Exceptional curves of the first kind. If S is a smooth complex surface, then for $p \in S$ we have defined the blowup $\pi_p : S_p \to S$ of S at p. Here S_p is another smooth surface and π is a morphism that is an isomorphism over $S - \{p\}$, whereas the preimage of p is smooth curve E_p that can be identified with the projectivization $\mathbb{P}(T_pS)$ of the tangent space T_pS such that the normal bundle of E_p in M becomes the tautological line bundle over $\mathbb{P}(T_pS)$. So E_p is isomorphic to $\mathbb{P}^1$ and $E_p \cdot E_p = -1$.

Conversely, if $\tilde{S}$ is a smooth complex surface and $E \subset \tilde{S}$ is a smooth curve isomorphic to $\mathbb{P}^1$ with self-intersection number -1, then E can be holomorphically contracted to produce a smooth complex surface: $\pi : \tilde{S} \to S$, so that if $\pi(E) = \{p\}$, then there is an isomorphism $h : \tilde{S} \cong S_p$ such that $\pi = \pi_p h$. We then say that E is an *exceptional curve of the first kind*. It can be shown that if $\tilde{S}$ is projective, then so is S. The projection π identifies the (co)homology of the pair $(\tilde{S}, E)$ with that of the pair $(S, \{p\})$ and from this it readily follows that the map $\pi^* : H^2(S) \to H^2(\tilde{S})$ is injective with the supplement spanned by the fundamental class $[E]$ of E (or rather its image under the natural map $H_2(\tilde{S}) \to H^2(\tilde{S})$). So we have a natural splitting $H^2(\tilde{S}) \cong H^2(S) \oplus \mathbb{Z}[E]$. Likewise $H_2(\tilde{S}) \to H_2(S)$ is onto and intersection with E defines the second component of an isomorphism $H_2(\tilde{S}) \cong H_2(S) \oplus \mathbb{Z}[E]$. This splitting is orthogonal with respect to the intersection pairing.

A similar argument applies to the Picard groups. The projection π induces an injective map $\mathrm{Pic}(\pi) : \mathrm{Pic}(S) \to \mathrm{Pic}(\tilde{S})$ (pull-back of line bundles), which is of course compatible with the map $\pi^* : H^2(S) \to H^2(\tilde{S})$ via the first Chern class. The line bundle $\mathcal{O}_{\tilde{S}}(E)$ spans the complementary summand and splits $\mathrm{Pic}(\tilde{S})$ into $\mathrm{Pic}(S) \oplus \mathbb{Z}[\mathcal{O}_{\tilde{S}}(E)]$.

In terms of divisors this amounts to the following: if C is a reduced curve on S which has multiplicity m at p (so $m = 0$ if $p \notin C$), and $\tilde{C}$ is the *strict transform* of $\tilde{C}$ in $\tilde{S}$ (i.e., the closure of $C \setminus \{p\}$ in $\tilde{S}$), then the divisorial pull-back π^*C is $\tilde{C} + mE$. So in terms of the above splitting, $[\tilde{C}]$ corresponds to $([C], -m[E])$. Since the splitting is orthogonal, we find that $E \cdot \tilde{C} = -mE \cdot E = m$ and $\tilde{C} \cdot \tilde{C} = C \cdot C + m^2 E \cdot E = C \cdot C - m^2$.

If $\eta \in \omega_{S,p}$ is a generator, then $\pi^*\eta$ has a zero of order one along E. In other words, $\omega_{\tilde{S}} = \pi^*(\omega_S)(E)$. This implies that $K_{\tilde{S}} = \pi^*K_S + [E]$ in $\mathrm{Pic}(\tilde{S})$ and that $K_{\tilde{S}} \cdot [E] = [E] \cdot [E] = -1$. It also follows that $\omega_{\tilde{S}}^{-1} = \pi^*(\omega_S^{-1})(-E)$. This means that a global section of $\omega_{\tilde{S}}^{-1}$ can be thought of simply as a global section of ω_S^{-1} which vanishes in p so that the anticanonical system of $\tilde{S}$ is the subsystem of the anticanonical system of S defined by that property. In particular, $|\omega_{\tilde{S}}^{-1}|$ is of codimension one in $|\omega_S^{-1}|$ unless the latter has p as a fixed point (or is empty), in which case we have equality. A similar argument shows that for any $N > 0$, $|\omega_{\tilde{S}}^{-N}|$ is a subspace of $|\omega_S^{-N}|$ (of sections which vanish of order $\geq N$ at p) and a little more effort goes into proving that if $\tilde{S}$ is a Del Pezzo surface, then so is S. Hence, if our goal is to classify Del Pezzo surfaces, then we may start with restricting ourselves to the surfaces that are *relatively minimal*, meaning that they don't contain exceptional curves of the first kind. The proof of the following proposition (which we only outline) invokes a weak form of the classification of surfaces.

Proposition 1.2. *A Del Pezzo surface that is relatively minimal is isomorphic to $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.*

Sketch of proof. Let S be a Del Pezzo surface that is relatively minimal. Then $|\omega_S^N|$ will be empty for all positive N (for ω_S^{-1} is ample). According to the coarse classification of relatively minimal surfaces (see for instance [**?**]), S is then isomorphic to either $\mathbb{P}^2$ or to a $\mathbb{P}^1$-bundle over a curve B. One then verifies that in the last case, $|\omega_S^{-N}|$ is empty for all positive N unless B has genus zero: $B \cong \mathbb{P}^1$ and the $\mathbb{P}^1$-bundle over B is trivial. $\square$

We already observed that $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ (which embeds in $\mathbb{P}^3$ as a quadric surface) are Del Pezzo surfaces indeed.

If we blow up a point $p = (p_1, p_2)$ of $\mathbb{P}^1 \times \mathbb{P}^1$, then the strict transforms of the two factor lines passing through p become exceptional curves of the first kind and if blow each of them down we get a copy of the projective plane.

Corollary 1.3. *A Del Pezzo surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or to a surface obtained by successively blowing up points of $\mathbb{P}^2$.*

We now focus on the Del Pezzo surfaces that are blown-up projective planes.

Blown-up projective planes and the Manin lattice. Let us recall that $H_2(\mathbb{P}^2) \cong H^2(\mathbb{P}^2) \cong \mathrm{Pic}(\mathbb{P}^2)$ is infinite cyclic and generated by the class [L] of a line. As observed before, the anticanonical class of $\mathbb{P}^2$ is $3[L]$ and so its anticanonical system is the space of cubics in $\mathbb{P}^2$.

Let S be a surface and suppose that it can be obtained by successively blowing r points $p_1, \dots, p_r$ on (or rather, over) over a surface P isomorphic to $\mathbb{P}^2$:

$$P = P_\emptyset \leftarrow P_{p_1} \leftarrow P_{p_1p_2} \leftarrow \cdots \leftarrow P_{p_1p_2\cdots p_r} = S,$$

with $p_i \in P_{p_1p_2\cdots p_{i-1}}$. If $E_i \subset P_{p_1p_2\cdots p_i}$ denotes the exceptional divisor over p_i, then we think of its divisor class $[E_i]$ as lying in $\mathrm{Pic}(S)$ via pullback. Similarly the generator $[L]$ of $\mathrm{Pic}(P)$ is thought of as lying in $\mathrm{Pic}(S)$. We then find with induction on r that $\mathrm{Pic}(S)$ can be identified with $H^2(S)$ and has $[L], [E_1], \dots, [E_r]$ integral basis. It also follows that $-K_S = 3[L] - \sum_{i=1}^r [E_i]$.

This suggests to define the *Manin lattice*([1]) $\Lambda_{1,r}$ as $\mathbb{Z}^{1+r}$ with basis elements denoted $\ell, e_1, \dots, e_r$ and endowed with the symmetric bilinear form for which these basis elements are pairwise perpendicular, $\ell \cdot \ell = 1$ and $e_i \cdot e_i = 1$, for $i = 1, \dots, r$. It has evidently signature $(1, r)$. We put $d_r := 3\ell - \sum_{i=1}^r e_i$. Notice that $d_r \cdot d_r = 9 - r$. It is clear from the preceding:

Lemma 1.4. *The lattices* $H_2(S)$, $H^2(S)$ *and* $\mathrm{Pic}(S)$ *are all three as lattices canonically isomorphic to each other. If* S *is given as an iterated blow-up of a projective plane, then this identifies them with the Manin lattice* $\Lambda_{1,r}$ *such that* $[E_i]$ *corresponds to* e_i, $[L]$ *to* ℓ *and the anticanonical class to* $d_r = 3\ell - \sum_{i=1}^r e_i$.

Let us call an isomorphism $\phi : \Lambda_{1,r} \cong \mathrm{Pic}(S)$ as in this lemma a *geometric marking* of S. As the following example shows, for $r \geq 3$, S can usually be described as a blown up projective plane in more than one way and so may have different geometric markings.

Example 1.5. Suppose $r = 3$ and that p_1, p_2, p_3 have pairwise distinct image in P. If we identify p_i with its image in P, then the strict transform of the line p_ip_{i+1} is an exceptional divisor of the first kind, which we denote $E_{i,i+1}$. The three curves $E_{2,3}$, $E_{3,1}$, $E_{1,2}$ are pairwise disjoint and hence can be blown down simultaneously. The result will be a copy P' of $\mathbb{P}^2$ and so S is now obtained as a blown up projective plane in an essentially different way. The new marking now maps e_1 to $[E_{2,3}] = [L] - [E_2] - [E_3]$, e_2 to $[E_{3,1}] = [L] - [E_3] - [E_1]$ and e_3 to $[E_{1,2}] = [L] - [E_1] - [E_2]$.

[1]This is of course the simplest lattice of hyperbolic type, but we call it thus, because Manin was the first to exhibit in his book *Cubic Forms* the central role of this lattice in the theory of Del Pezzo surfaces.

The passage from one model to the other, $P \leftarrow S \rightarrow P'$, defines a birational map $P \dashrightarrow P'$, which in terms of suitable coordinates in P and P' is given by $[z_0 : z_1 : z_2] \mapsto [z_0^{-1} : z_1^{-1} : z_2^{-1}] = [z_1z_2 : z_2z_0 : z_0z_1]$. Such a map is called an *elementary Cremona transformation*. More intrinsically, $P \dashrightarrow P'$ is defined by the linear system of conics in P passing through p_1, p_2, p_3 (so that P' is the projective space dual to it). This is why this is also sometimes called an elementary quadratic transformation.

The Fano condition. The following proposition gives necessary conditions for the Fano property (which turn out to be also sufficient).

Proposition 1.6. *If $S = P_{p_1 \cdots p_r}$ is a Del Pezzo surface, then $r < 9$ and the points $p_1, \dots, p_r$ are in general position in the sense that they have distinct image in P (so that a simultaneous blowup is possible) and if we identify them with their images in P, then no 3 lie on a line, no 6 on a conic and no 8 on a cubic that has a singular point at some p_i.*

Proof. Since $-K_S$ is ample, we have $0 < (-K_S) \cdot (-K_S) = d_r \cdot d_r = 9 - r$, and so $r < 9$.

We show that there exists an effective divisor D on S with $D \cdot K_S = 0$ (so that $-K_C$ cannot be ample) if one of the mentioned nondegeneracy conditions is not met. If $1 \leq i < j \leq r$ are such that p_j lies on E_i, but $p_{i+1}, \dots, p_{j-1}$ do not, then the strict transform of E_i in $S_{p_1 \cdots p_j}$ is an effective divisor D which represents $[E_i] - [E_j]$ and we have $K_S \cdot D = K_S \cdot ([E_i] - [E_j]) = 0$. This proves that $p_1, \dots, p_r$ must have distinct image in P. For the other cases, one may take for D the strict transform of a line through 2 p_i's, a conic through 6 p_i's, or a cubic through 8 p_i's having a singular point at one of them; after renumbering its class is represented by $\ell - \sum_{i=1}^{3} e_i$, $2\ell - \sum_{i=1}^{6} e_i$, $3\ell - 2e_1 - \sum_{i=2}^{8} e_i$ respectively and these have zero inner product with d_r. ☐

Suppose S is as in the preceding proposition. Then by our previous discussion the anticanonical system $|\omega_S^{-1}|$ can be identified with the linear system of cubics in P passing through $p_1, \dots, p_r$. The complete linear system of cubics on $\mathbb{P}^2$ is of dimension 9 and so $|\omega_S^{-1}|$ has dimension $\geq 9 - r$. One can show that the condition for a cubic C to pass through each p_i imposes r independent conditions on C as long as $r < 9$. So for $r = 8$ we get a one dimensional linear system. When C and C' are distinct smooth members of this system, then they must meet in a ninth point p_9 and hence each member of the linear system has that property. In particular, the conditions are no longer independent for $r = 9$, as there is a pencil of cubics passing through $p_1, \dots, p_9$.

Remark 1.7. So if S is a Del Pezzo surface of degree $d > 1$, then its anticanonical system is without fixed points and hence provides a morphism f_S from S to a projective space of dimension d. This is an embedding for $d \geq 3$. In fact, for $d = 4$ the image is a complete intersection of two quadrics in projective 4-space and for $d = 3$ a smooth cubic surface in projective 3-space. We already noticed that a cubic surface is a Del Pezzo surface. The same is true for any nonsingular complete intersection S of two quadrics in $\mathbb{P}^4$ (for a smooth complete intersection $M \subset \mathbb{P}^{n+k}$ of k hypersurfaces of degree $1+a_1, \dots, 1+a_k$ we have $\omega_M \cong \mathcal{O}_M(-n-1+\sum_{i=1}^k a_i)$).

For $d = 2$, the morphism f_S maps to a projective plane and must be of degree 2. It will ramify along a smooth quartic curve. Again there is a converse: for a double cover $f : S \to \mathbb{P}^2$ which ramifies along a quartic curve C, we have $\omega_S = f^*(\omega_{\mathbb{P}^2})(C)$. Since C is a divisor for $f^*(\mathcal{O}_{\mathbb{P}^2}(2))$, it follows that $\omega_S^{-1} \cong f^*(\omega_{\mathbb{P}^2}^{-1})(-C) \cong f^*(\mathcal{O}_{\mathbb{P}^2}(3)) \otimes_{\mathcal{O}_S} f^*(\mathcal{O}_{\mathbb{P}^2}(-2)) = f^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and so ω_S^{-1} is ample and has degree 2.

When $d = 1$, the linear system $|\omega_S^{-1}|$ may be regarded as a pencil of cubics through 8 points in a projective plane P in general position. As noted above, the members of this pencil pass through a ninth point, $p = p_9$ say. If we blow up that point also, then the resulting surface fibers over a projective line with the members of the pencil as fibers: we get a rational elliptic surface on which the exceptional curve E_9 now appears as a section. The general position condition amounts to requiring that there are no reducible cubics in this pencil. Conversely, if $\tilde{S} \to \mathbb{P}^1$ is a rational elliptic surface all of whose fibers are irreducible and is endowed with a section E, then E is necessarily an exceptional curve of the first kind and contraction of that curve yields a Del Pezzo surface of degree 1.

Root data in the Manin lattice. We assume $r \geq 3$ and introduce a number of notions attached to $\Lambda_{1,r}$. Let us write Q_r for the set of vectors $v \in \Lambda_{1,r}$ with $v \cdot d_r = 0$. Since $d_r \cdot d_r = 9 - r$, we see that Q_r is negative definite for $r < 9$, negative semidefinite with radical spanned by d_r for $r = 9$ and of signature $(1, r-1)$ for $r > 9$. If $\alpha \in Q_r$ is such that $\alpha \cdot \alpha = -2$, then we have defined a reflection s_α in $\Lambda_{1,r}$ which fixes d_r by the formula:

$$s_\alpha(v) := v + (v \cdot \alpha)\alpha.$$

Notice that this reflection is orthogonal (it is in $O(\Lambda_{1,r})$) and fixes d_r. Now Q_r admits a basis of such vectors, namely $\{\alpha_0, \alpha_1, \dots, \alpha_{r-1}\}$ with $\alpha_0 = \ell - e_1 - e_2 - e_3$ and $\alpha_i := e_i - e_{i+1}$ for $i = 1, \dots, r-1$. Notice that two distinct elements have inner product 0 or 1. This means that $\big(-(\alpha_i\cdot\alpha_j)\big)_{i,j}$ is a (symmetric) generalized Cartan matrix (the type of which is denoted by E_r). This is the point of departure of the construction of a Kac-Moody algebra. We follow a number of steps toward this construction that is relevant in this context (but the Kac-Moody algebra will not appear here).

We first observe that for $i = 1,\dots,r$, the reflection s_{α_i} simply interchanges the basis elements e_i and e_{i+1}, leaving each of the remaining basis elements fixed. If for a geometrically marked surface S as above, $p_{i+1} \notin E_i$, then we could interchange the order of blowing up: first blow up p_{i+1} and then p_i and the effect on the geometric marking is just composition with s_{α_i}.

The reflection s_{α_0} does not affect $e_4,\dots,e_r$ and we have $s_{\alpha_0}(e_1) = \ell - e_2 - e_3$, $s_{\alpha_0}(e_2) = \ell - e_3 - e_1$, $s_{\alpha_0}(e_3) = \ell - e_1 - e_2$ and $s_{\alpha_0}(\ell) = 2\ell - e_1 - e_2 - e_3$. This is precisely the transformation formula for the elementary Cremona transformation that we found above.

Now let W_r denote the subgroup of $O(\Lambda_{1,r})_{d_r}$ generated by the simple reflections $s_{\alpha_0},\dots,s_{\alpha_{r-1}}$. Notice that the subgroup of W_r generated by the last $r-1$ reflections $s_{\alpha_1},\dots,s_{\alpha_{r-1}}$ fixes ℓ and can be identified with the permutation group on $e_1,\dots,e_r$. The preceding discussion shows (inductively) that for a *general* choice of $(p_1,\dots,p_r) \in (\mathbb{P}^2)^r$, that is, avoiding a countable union of subvarieties of $(\mathbb{P}^2)^r$, we have that for every $w \in W_r$,

(i) ϕw is a geometric marking,
(ii) $\phi w(e_r)$ represents the class of an exceptional curve of the first kind,
(iii) $\phi w(\alpha_i)$ $(i = 0,\dots,r-1)$ can be represented by the difference $[E]-[E']$, where E and E' are disjoint exceptional curves of the first kind.
(iv) $\phi w(\ell)$ represents a class that defines a blowdown of S to a projective plane.

We denote

$$\mathcal{R}_r := W_r\{\alpha_0,\dots,\alpha_{r-1}\},\quad \mathcal{E}_r := W_r e_r,\quad \mathcal{L}_r := W_r\ell.$$

Observe that if we put $e_{12} := \ell - e_1 - e_2$, then $(e_{12}, e_1,\dots,e_r)$ is basis of $\Lambda_{1,r}$ contained in $\mathcal{E}_r$ and $\alpha_0 = e_{12} - e_3$, so that every α_i is now written as a difference of two perpendicular elements of $\mathcal{E}_r$. We may think of $\mathcal{R}_r$ as a (generalized) root system and W_r as its Weyl group. Clearly, every $\alpha \in \mathcal{R}_r$ has the property that $\alpha\cdot\alpha = -2$ and $s_\alpha \in W_r$, but the converse need not hold: it may happen that $\alpha \in Q_r$ is such that $\alpha\cdot\alpha = -2$, but $\alpha \notin \mathcal{R}_r$ and $s_\alpha \notin W_r$. But for $r = 3,4,5,6,7,8$, Q_r is negative definite and $\mathcal{R}_r$ is a finite root system of type $A_2 + A_1$,

A_4, D_5, E_6, E_7, E_8 respectively. In these cases, it is not hard to show that we have characterizations of the subsets defined above solely in terms of the abstract pair $(\Lambda_{1,r}, d_r)$ (and not involving the given basis of $\Lambda_{1,r}$): every $\alpha \in Q_r$ with $\alpha \cdot \alpha = -2$ lies in $\mathcal{R}_r$, every $e \in \Lambda_{1,r}$ with $e \cdot e = -e \cdot d_r = -1$ lies in $\mathcal{E}_r$, and every $\ell' \in \Lambda_{1,r}$ with $\ell'. \cdot \ell' = 1$ and $\ell' \cdot d_r = 3$ lies in $\mathcal{L}_r$. Moreover, the inclusion $W_r \subset O(\Lambda_{1,r})_{d_r}$ is an equality.

Let us list for $r \leq 8$, the elements of $\mathcal{E}_r$ up to a permutation of indices. In other words, we determine the orbits of the permutation group generated by $s_{\alpha_1}, \dots, s_{\alpha_r}$ in $\mathcal{E}_r$. For $r \leq 4$, these are represented by e_1 and e_{12}. For $r \geq 5$ we also get $s_{\alpha_0}(\ell - e_4 - e_5) = 2\ell - e_1 - \cdots - e_5$ and for $r \geq 7$, $s_{\alpha_0}(2\ell - e_1 - e_4 - e_5 - e_6 - e_7) = 3\ell - 2e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7$. This completes their description.

We do the same for the roots, or rather (since the opposite of a root is also a root) for those roots for which ℓ has a nonnegative coefficient. For $r \leq 4$ we have $e_1 - e_2$ and $\ell - e_1 - e_2 - e_3$. For $r \geq 6$ we also get $s_{\alpha_0}(\ell - e_4 - e_5 - e_6) = 2\ell - e_1 - \cdots - e_6$ and for $r = 8$, $s_{\alpha_0}(2\ell - e_1 - e_4 - e_5 - e_6 - e_7 - e_8) = 3\ell - 2e_1 - e_2 - \cdots - e_8$. Since these classes are perpendicular to d_r, they should under a geometric marking of a Del Pezzo surface *not* go to classes which represent positive divisors. For instance, there should be no conic in P passing through $p_1, \dots, p_6$ as that class is represented by $2\ell - e_1 - \cdots - e_6$. We thus not only recover the assertion of Proposition 1.6, but also obtain a more appealing formulation: it simply says that the Fano property implies that no root must map to the class of a positive divisor. This last property (appearing as (v) below) is equivalent to a number of others:

Theorem 1.8. *Suppose* $3 \leq r \leq 8$, *S a surface obtainable from a projective plane by blowing up successively* r *points and let* $\phi : \Lambda_{1,r} \cong \mathrm{Pic}(S)$ *be an isometry which takes* $-K_S$ *to* ω_r. *Then the following are equivalent:*

(i) *S is a Del Pezzo surface,*
(ii) *for every* $w \in W_r$, ϕw *is a geometric marking,*
(iii) *every element of* $\phi(\mathcal{E}_r)$ *represents the class of an exceptional curve of the first kind,*
(iv) *every element of* $\phi(\mathcal{R}_r)$ *can be represented as* $[E] - [E']$, *where* E *and* E' *are disjoint exceptional curves of the first kind,*
(v) *no element of* $\phi(\mathcal{R}_r)$ *is represented by a positive divisor,*
(vi) *every element of* $\phi(\mathcal{L}_r)$ *represents the line class of a blowdown* $S \to \mathbb{P}^2$,
(vii) *any irreducible curve on* S *with negative self-intersection is an exceptional curve.*

The proof, which we omit, can be reconstructed from our earlier observations.

2. Moduli of Del Pezzo pairs

Four points in $\mathbb{P}^2$ in general position 'have no moduli': we can always find a projective coordinate system such they are given by $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, $[1:1:1]$. So Del Pezzo surfaces of degree $\geq 9-4=5$ have no moduli either. Each additional point to blow up increases the number of moduli by 2 and so we find that the Del Pezzo surfaces of degree d have $2(r-4)_+ = 2(5-d)_+$ moduli. In view of the remarks above, we get for $d=4,3,2,1$ the (coarse) moduli space of smooth complete intersections of 2 quadrics in $\mathbb{P}^4$, of smooth cubic surfaces in $\mathbb{P}^3$, of smooth plane quartics and of smooth rational elliptic surfaces without reducible fibers and endowed with a section. It is however not so easy to get a good direct description of these moduli spaces. The situation is much better if instead we consider the moduli space of pairs (S,D), where S is a Del Pezzo surface (of a fixed degree d) and D is an anticanonical positive divisor (of fixed type). This means that there exists a meromorphic 2-form ω on S with divisor $-D$ (this makes $S-D$ in a sense an affine Calabi-Yau surface). The meromorphic 2-form is of course unique up to scalar. It is often convenient to look at pairs (S,ω) instead and this is what we will do first. We need the following general fact:

Lemma 2.1. *Let S be a smooth compact complex surface and let D be a curve on S. Then we have a natural exact sequence*

$$H_1(D_{reg}) \to H_2(S-D) \to H_2(S) \to H_0(D_{reg}),$$

where the first map is given by the Lefschetz tube construction (recalled in the proof), the middle map by the inclusion and the last map by intersecting 2-cycles transversally with D. If moreover $H_1(S)=0$ and D has at most one singular point, then the first map is injective and the last one surjective.

Proof. Let U be an open regular neighborhood of D in S with smooth boundary. Then Alexander duality plus excision yield the identification

$$H_k(S,S-D) \cong H_k(S,S-U) \cong H_k(\bar{U},\partial U) \cong H^{4-k}(\bar{U}) \cong H^{4-k}(D).$$

If we combine this with the duality isomorphism $H_3(S) \cong H^1(S)$, the exact sequence for the pair $(S,S-D)$ yields the exact sequence

$$H^1(S) \to H^1(D) \to H_2(S-D) \to H_2(S) \to H^2(D) \to H_1(S).$$

Since the singular set D_{sg} of D is finite, $H_0(D_{reg}) \cong H^2(D, D_{sg}) \to H^2(D)$ is an isomorphism and $H_1(D_{reg}) \cong H^1(D, D_{sg}) \to H^1(D)$ is onto. This establishes the exact sequence as stated.

If $H_1(S) = 0$, then $H^1(S) = 0$ and so $H^1(D) \to H_2(S-D)$ is injective. And if D_{sg} is empty or a singleton, the restriction $H^0(D) \to H^0(D_{sg})$ is onto and so $H_1(D_{reg}) \cong H^1(D, D_{sg}) \to H^1(D)$ is injective. The lemma now follows, except perhaps the identification of the maps. We only do this for the first map. Retracing our steps, we find that the map $H_1(D_{reg}) \to H_2(S - D)$ is given as follows: if $\bar{U}$ admits a retraction $\bar{\pi} : \bar{U} \to D$ such that $\bar{\pi}$ is locally trivial over D_{reg}, then assign to a 1-cycle c on D_{reg} the boundary of the 3-chain $\bar{\pi}^{-1}c$ (this boundary has its support in $\partial U \subset S - D$). This is known as the Lefschetz tube construction. $\square$

The cuspidal case. Let S be a Del Pezzo surface of degree $d = 9-r \leq 6$ (i.e., $r \geq 3$) and ω a meromorphic 2-form on S whose divisor is of the form $-D$ with D a cuspidal curve. We invoke Lemma 2.1 to determine $H_2(S - D)$. Since $D_{reg} \cong \mathbb{C}$, this lemma implies that $H_2(S - D)$ can be identified with the sublattice of $H_2(S) \cong H^2(S)$ that is orthogonal to $[D]$. The cohomology class of ω can be understood as a linear map $[\omega] : H_2(S - D) \to \mathbb{C}$.

The form ω has a residue on the smooth part D_{reg} of D: this will be a nowhere zero differential on D_{reg}. Since D_{reg} is isomorphic to the affine line $\mathbb{C}$, and since a nowhere zero differential on $\mathbb{C}$ is of the form λdz, we can find an isomorphism $z : D_{reg} \cong \mathbb{C}$ such that $\mathrm{Res}_D\, \omega = dz$. This function z is of course unique up to a constant.

Now let be given a geometric marking $\phi : \Lambda_{1,r} \cong H_2(S)$. Then the preceding shows that ϕ maps Q_r onto $H_2(S-D)$. So we have defined

$$\chi = \chi(S, \omega, \phi) := \phi^*\omega \in \mathrm{Hom}(Q_r, \mathbb{C}).$$

We will show that $\chi(S, \omega, \phi)$ is a complete invariant of the isomorphism type of the triple (S, ω, ϕ). We do this by computing its values on the basis $(\alpha_0, \dots, \alpha_{r-1})$. In fact, let $\alpha \in \mathcal{R}_r$ and represent $\phi(\alpha)$ as $[E] - [E']$ with E and E' mutually disjoint exceptional curves of the first kind. Since $E \cdot (-K_C) = 1$, E meets D transversally in a point $p \in D_{reg}$. Likewise E' meets D transversally in some $p' \in D_{reg}$ and we have $p' \neq p$. We must find a 2-cycle A on $S - D$ that is homologous in S to $E - E'$. This is done as follows: let $\bar{U}$ be a closed regular neighborhood of D in S and $\bar{U} \to D$ a deformation retraction that is a C^∞-locally trivial disk bundle over D_{reg} and for which the fiber over p resp. p' lies in E resp. E'. Let γ be a smooth path in D_{reg} from p' to p and put $A := (E - U_p) + (\partial U)_\gamma - (E' - U_{p'})$. Then A is a 2-cycle and since $A - (E - E')$ bounds $\bar{U}_\gamma$, A represents $\phi(\alpha)$ in $H_2(S)$. So

$$[\omega](\phi(\alpha)) = \int_A \omega = \int_{E-U_p} \omega + \int_{(\partial U)_\gamma} - \int_{E'-U_{p'}} \omega = \int_{(\partial U)_\gamma} \omega$$

(for a holomorphic 2-form on a curve is zero) and the last integral equals by the residue theorem $2\pi\sqrt{-1}\int_\gamma \mathrm{Res}_D\, \omega = 2\pi\sqrt{-1}\big(z(p) - z(p')\big)$. In particular, $[\omega](\phi(\alpha)) \neq 0$ (for $p \neq p'$). Now let $\pi : S \to P$ be the blow down defined by the marking (so P is isomorphic to $\mathbb{P}^2$). Then $\pi(D)$ is a cuspidal curve in P. Since D is the strict transform of $\pi(D)$ under π we identify D with $\pi(D)$. Then E_i meets D in p_i and we have for $i = 1, \dots, r-1$, $[\omega](\phi(\alpha_i)) = 2\pi\sqrt{-1}(z(p_{i+1}) - z(p_i))$. If L_{12} is line in P through p_1 and p_2, then L_{12} meets D in a third point p_{12}. The strict transform E_{12} of L_{12} in S is an exceptional curve of the first kind and then $E_{12} - E_3$ represents $\phi(\alpha_0) = \phi(\ell - e_1 - e_2 - e_3)$. We thus find that $[\omega](\phi(\alpha_0)) = 2\pi\sqrt{-1}(z(p_{12}) - z(p_3))$.

In Hodge theory one accounts for the factor $2\pi\sqrt{-1}$ by regarding $[\omega]$ as a linear map from Q_r to the Tate Hodge structure $\mathbb{C}(1)$ (which has weight -2), or what amounts to the same, as an element of $\mathrm{Hom}(Q_r(-1), \mathbb{C})$ (and where Q_r has weight -2).

In order to see that these integrals enable us to completely reconstruct the triple (S, ϕ, ω) up to unique isomorphism, we first go in the opposite direction: we write V_r for $\mathrm{Hom}(Q_r, \mathbb{C})$ and construct for any $\chi \in V_r$ an associated triple $(S_\chi, \omega_\chi, \phi_\chi)$ as follows.

There is a unique $(z_{12}, z_1, z_2, \dots, z_r) \in \mathbb{C}^{1+r}$ with $z_{12} + \sum_{i=1}^r z_i = 0$ such that $\chi(\alpha_0) = (2\pi\sqrt{-1})(z_{12} - z_3)$ and $\chi(\alpha_i) = (2\pi\sqrt{-1})(z_i - z_{i+1})$ for $i = 1, \dots, r-1$. Consider the embedding

$$f_\chi : \mathbb{C} \to \mathbb{C}^2, \quad f_\chi(z) := (z, (z - z_1)(z - z_2)(z - z_{12})).$$

Notice that the closure $D_\chi := \overline{f_\chi(\mathbb{C})}$ of its image in $\mathbb{P}^2$ is the cubic curve given by $X_2X_0^2 = (X_1 - z_1X_0)(X_1 - z_2X_0)(X_1 - z_3X_0)$. It has a cusp at $[0 : 0 : 1]$ whose tangent is given by $X_0 = 0$. Moreover the images of z_1, z_2, z_{12} lie on the coordinate line $X_2 = 0$. There is a unique meromorphic 2-form ω_χ on $\mathbb{P}^2$ that has as divisor $-D_\chi$ and whose residue on $f_\chi(\mathbb{C})$ pulled back to $\mathbb{C}$ is dz. Now put $S_\chi := \mathbb{P}^2_{f_\chi(z_1)\cdots f_\chi(z_r)}$ and identify D_χ resp. ω_χ with its strict transform in resp. pull-back to S_χ. We have arranged that

$$[\omega_\chi](\phi_\chi(\alpha_i)) = \chi(\alpha_i), \quad i = 0, \dots, r-1.$$

so that $\phi^*[\omega_\chi] = \chi$. We can do this 'with parameters' and thus obtain a (trivial) projective plane bundle $\mathcal{P}_{V_r} = \mathbb{P}^2 \times V_r$ over V_r, a bundle of cuspidal curves $\mathcal{D}_{V_r} \subset \mathcal{P}_{V_r}$ over V_r and sections $p_1, \dots, p_r$ of $\mathcal{D}_{V_r} \to V_r$ such that we get a surface bundle $\mathcal{S}_{V_r}$ over V_r by blowing up these sections successively. The construction also comes with a meromorphic relative 2-form on $\mathcal{S}_{V_r}/V_r$ whose divisor is $-\mathcal{D}_{V_r}$ and whose cohomology class over $\chi \in V_r$ is represented by χ.

If S_χ is a Del Pezzo surface, then for every $\alpha \in \mathcal{R}_r$, $\phi_\chi(\alpha)$ is representable by a difference of disjoint exceptional curves of the first kind so that $\chi(\alpha) \neq 0$. Conversely, if S_χ is not a Del Pezzo surface, then there exists a $\alpha \in \mathcal{R}_r$ representable by an effective divisor A. This divisor must be disjoint with D_χ (for $A \cdot D_\chi = 0$ and $D_\chi \cdot D_\chi > 0$) and so $\chi(\alpha) = 0$. Hence S_χ is a Del Pezzo surface if and only if $\chi|\mathcal{R}_r$ is never zero. We denote the set of such χ by V_r°; it is the complement of the union of the reflection hyperplanes in V_r.

We return to our triple (S, ω, ϕ). If we take $\chi = \chi(S, \omega, \phi) = \phi^*[\omega]$, then we claim that here is a unique isomorphism $h : \mathbb{P}^2 \cong P$ that takes D_χ to D, $f_\chi(z_1)$ to p_1 and $f_\chi(z_2)$ to p_2 and for which $h^*\omega = \omega_\chi$. To see this, first observe that there is a unique isomorphism of the completed x-axis onto L_{12} which takes $f_\chi(z_1)$ to p_1, $f_\chi(z_2)$ to p_2 and $f_\chi(z_{12})$ to a point of D. We may extend this to an isomorphism $h : \mathbb{P}^2 \cong P$ that takes the cusp of D_χ ($[0 : 0 : 1]$) and its tangent line ($X_0 = 0$) to the cusp of D and its the tangent line. Then h is determined up to scalar multiplication in the X_1-direction, but becomes unique if we also demand that $h(D_\chi) = D$. Then $h^*\omega = \omega_\chi$ and h will also take $f_\chi(z_i)$ to p_i for $i > 2$. So h lifts to an isomorphism of geometrically marked Del Pezzo surfaces $S_\chi \cong S$ with $h^*\omega = \omega_\chi$.

We now quote a famous theorem, the essential part of which is due to Chevalley:

Theorem 2.2. *Let V be a complex vector space of dimension r and W a finite subgroup of* GL(V) *generated by complex reflections (i.e. transformations of finite order whose fixed point set is a hyperplane). Then:*

(i) *the W-stabilizer of any point of V is generated by complex reflections, so that W acts freely on $V^\circ := V - \Delta$, where $\Delta \subset V$ denotes the union of the fixed point hyperplanes of complex reflections in V and*

(ii) *the graded $\mathbb{C}$-algebra of W-invariant regular functions on V, $\mathbb{C}[V]^W$, has r algebraically independent homogeneous generators and Δ is defined by a homogeneous element of $\mathbb{C}[V]^W$*

Property (ii) says that if we pick homogeneous generators $f_1, \dots, f_r$ for $\mathbb{C}[V]^W$, say of degrees $d_1, \dots, d_r$, then $f := (f_1, \dots, f_r) : V \to \mathbb{C}^r$ factors through an isomorphism $V/W \cong \mathbb{C}^r$ of affine varieties. The $\mathbb{C}^\times$-action on the orbit space V/W (coming from scalar multiplication on V) then corresponds in $\mathbb{C}^r$ to the $\mathbb{C}^\times$-action $t.(u_1, \dots, u_r) = (t^{d_1}u_1, \dots, t^{d_r}u_r)$ and if $F \in \mathbb{C}[u_1 \dots, u_r]$ is homogeneous and such that $F(f_1, \dots, f_r)$ defines Δ, then the image of Δ in V/W corresponds to the hypersurface $(F = 0) \subset \mathbb{C}^r$. This means that $(F = 0)$ defines the discriminant of the finite morphism $V \to V/W \cong \mathbb{C}^r$; in particular, the morphism $V^\circ \to V^\circ/W \cong \mathbb{C}^r - (F = 0)$ is unramified. We further note that the orbit space $\mathbb{P}(V)/W$ gets identified with weighted projective space $\mathbb{P}(d_1, \dots, d_r)$ and that F defines a hypersurface in it whose complement corresponds to $\mathbb{P}(V^\circ)/W$.

Returning to our situation, we see that W_r acts freely on V_r°, so that passing to the W_r-orbit spaces yields an object over V_r°/W_r: a bundle $\mathcal{S}_{V_r^\circ/W_r} \to V_r^\circ/W_r$, a subbundle of cuspidal curves $\mathcal{D}_{V_r^\circ/W_r}$ and a relative 2-form $\omega_{V_r^\circ/W_r}$ on $\mathcal{S}_{V_r^\circ/W_r} \to V_r^\circ/W_r$ with $-\mathcal{D}_{V_r^\circ/W_r}$ as divisor. We sum up:

Theorem 2.3. *We have obtained a universal family* $(\mathcal{S}_{V_r^\circ/W_r}, \omega_{V_r^\circ/W_r})$ *over* V_r°/W_r *for the pairs* (S, ω)*, where* S *is a Del Pezzo surface of degree* $9 - r \leq 6$ *and* ω *a meromorphic differential on* S *whose divisor is minus a cuspidal curve. The* $\mathbb{C}^\times$*-orbit space of the base,* $\mathbb{P}(V_r^\circ)/W_r$*, is a variety whose underlying set can be identified with the set of isomorphism classes of pairs* (S, D)*, where* D *is an anticanonical cuspidal curve on* S*.*

Since the action of W_r on V_r/W_r has fixed points, we should not expect it to extend to $\mathcal{S}_{V_r/W_r}$. Indeed, for a fixed point $\chi \in V_r$ of $s_{\alpha_{r-1}}$, we have $p_{r-1} = p_r$. This means that the fiber S_χ is of the form $P_{p_1 \dots p_r}$ with $p_{r-1} = p_r$. Then the strict transform C of E_{r-1} represents the class $[E_{r-1}] - [E_r] = \phi(e_r - e_{r-1})$. But if $s_{\alpha_{r-1}}$ were to extend to an involution of S_χ, then it would send C to a class representing $-C$. This is not possible, for on a projective surface a positive divisor cannot be represented by a negative divisor as well (any ample class has positive intersection product with such a class).

It is interesting to see what happens for the most extreme case $\chi = 0 \in V_r$. Then $z_{12} = z_1 = \cdots = z_{r-1} = 0$ and hence $f_0(z) = (z, z^3)$. In particular, $f_0(0) = (0, 0)$ is a flex point of $f_0(\mathbb{C}) = D_{\chi,reg}$. We then must blow up r times over the strict transform of D_0 over this flex point. The result is perhaps somewhat of a surprise: we see that every basis element α_i of Q_r is represented by (a smooth) (-2)-curve

C_i (the curve C_2 is the strict transform of the tangent line of D_0 in $\mathbb{P}^2$ at $(0,0)$. These curves make up a normal crossing curve C and intersect according tot the Dynkin diagram of the root system R_r. The connected components of this curve can be contracted (since R_3 is of type $A_2 + A_1$, we get two connected components in that case) to produce a normal singular surface $\bar{S}_0$. The curve C is disjoint with D_0 and so D_0 then still lies on the smooth part of $\bar{S}_0$.

Running ahead a bit, we mention here that we can do something similar in every fiber of $\mathcal{S}_{V_r} \to V_r$. For instance, if $\alpha \in \mathcal{R}_r$ is a positive root (in the sense that it is a linear combination of $\alpha_0, \dots, \alpha_{r-1}$ with positive coefficients), then for a generic χ in the hyperplane defined by $\chi(\alpha) = 0$, we have that the class on S_χ defined by α is a (-2)-curve (a smooth rational curve with self-intersection -2). This is fairly clear if $\alpha = \alpha_i$ for some i and the general case is deduced from that by applying an appropriate element of W_r to this situation. This can be generalized to yield a more precise statement (which is typical for other situations that we consider below):

Theorem 2.4. *Given $\chi \in V_r$, then every connected component of the union of the (-2)-curves contained in $S_\chi - D_\chi$ is a configuration of type A, D or E. The set of classes of these (-2)-curves consists of positive roots in $\mathcal{R}$ and makes up a root basis for the root subsystem $\mathcal{R}_{r,\chi} \subset \mathcal{R}_r$ consisting of the $\alpha \in \mathcal{R}_r$ with $\chi(\alpha) = 0$.*

We can contract each of the connected components of the union of the (-2)-curves in $S_\chi - D_\chi$ and thus obtain a surface with Kleinian singularities $S_\chi \to \bar{S}_\chi$ (we return to this later). This can be done 'in a family' so as obtain a bundle $\bar{\mathcal{S}}_{V_r} \to V_r$ on which W_r acts (with the action of s_α on $\bar{\mathcal{S}}_{V_r}$ over the hyperplane in V_r defined by $\chi(\alpha) = 0$ is the identity). We now have defined a family $\bar{\mathcal{S}}_{V_r/W_r}$ over V_r/W_r which extends $\mathcal{S}_{V_r^\circ/W_r}$ over V_r°/W_r. The family of 2-forms extends as well, where it is understood that these forms are only defined on the smooth part of the fibers (we get a generating section of the relatively dualizing sheaf of $\bar{\mathcal{S}}_{V_r/W_r} \to V_r/W_r$ twisted by the cuspidal divisor at infinity).

The nodal case. Let us now assume that the anticanonical divisor D is a nodal curve: it has a normal crossing singularity. We will find this to be multiplicative version of the cuspidal case.

The nodal curve D has at its singularity two branches and we fix an ordering of them. This amounts to a choice of a generator of the infinite cycle group $H_1(D)$. We then choose an isomorphism $z: D_{reg} \cong \mathbb{C}^\times$ so that $z = 0$ is the first branch and $z = \infty$ the second. This isomorphism is unique up to multiplication by a scalar in $\mathbb{C}^\times$. It also defines a generator of $\delta \in H_1(D_{reg})$, as being the class of the positively oriented circle $|z| = 1$. Let ω be a meromorphic 2-form on S with divisor $-D$. Then we can take its residue twice: first on D_{reg} and then at the singular point on the first branch. If we require that this double residue is 1, then ω will be unique. So then $\mathrm{Res}_D\,\omega = \frac{dz}{z}$. Suppose S geometrically marked by $\phi : \Lambda_{1,r} \cong H_2(S)$. (We shall refer to the combination of ϕ and the order of the branches of the node of D as a geometric marking of the *pair* (S, D).) It then follows from Lemma 2.1 that we have a natural exact sequence

$$0 \to \mathbb{Z}\delta \to H_2(S - D) \to Q_r \to 0.$$

Our choice of ω ensures that the linear map $[\omega] : H_2(S{-}D) \to \mathbb{C}$ takes on δ the value $(2\pi\sqrt{-1})^2$. So we have a well-defined homomorphism

$$\chi = \chi(S, \omega, \phi) : Q_r \to \mathbb{C}^\times, \quad \chi(a) := \exp\Big(\frac{1}{2\pi\sqrt{-1}}\int_{\widetilde{\phi(a)}} \omega\Big),$$

where $\widetilde{\phi(a)}$ is a lift of $\phi(a)$ in $H_2(S - D)$. If $a = \alpha \in \mathcal{R}_r$, then we proceed as in the cuspidal case: we represent $\phi(\alpha)$ as the difference $[E] - [E']$ of two disjoint exceptional curves of the first kind. They will meet D_{reg} in p resp. p' say, and if we choose a path γ in $\mathbb{C}^\times$ from $z(p')$ to $z(p)$, then we find a 2-cycle A in $S - D$ which lifts $\phi(\alpha)$ and for which

$$\int_A \omega = 2\pi\sqrt{-1}\int_\gamma \frac{dz}{z} \equiv 2\pi\sqrt{-1}\log(z(p)/z(p')) \pmod{(2\pi\sqrt{-1})^2\mathbb{Z}}.$$

Hence we have $\chi(\alpha) = z(p)/z(p')$. We thus find $\chi(\alpha_0) = z(p_{12})/z(p_3)$ and $\chi(\alpha_i) = z(p_i)/z(p_{i+1})$ for $i = 1, \dots, r - 1$.

In a way quite similar to the cuspidal case, we deduce that $\chi(S, \omega, \phi)$ is a complete invariant for the triple (S, ω, ϕ) together with the order of branches at of D at its singular point. Let us do the opposite construction. We put $T_r := \mathrm{Hom}(Q_r, \mathbb{C}^\times)$ and let $\chi \in T_r$. Choose $(z_{12}, z_1, z_2, \dots, z_{r-1}) \in (\mathbb{C}^\times)^{r+1}$ such that

$$z_{12}/z_3 = \chi(\alpha_0) \text{ and } z_i/z_{i+1} = \chi(\alpha_i) \text{ for } i = 1, \dots, i - 1.$$

Note that $(z_{12}, z_1, z_2, \dots, z_{r-1})$ is uniquely defined up to a common factor (the above formulae define a surjective homomorphism $(\mathbb{C}^\times)^{r+1} \to T_r$ whose kernel is the main diagonal of $(\mathbb{C}^\times)^{r+1}$). Now let $f_\chi : \mathbb{C}^\times \to \mathbb{P}^2$ be the embedding defined by

$$f_\chi(z) := [z : z^2 : (z - z_1)(z - z_2)(z - z_{12})].$$

The closure of its image is a nodal curve D_χ which has its double point in $[0 : 0 : 1]$ (a value obtained for $z = 0$ and $z = \infty$) and has $f_\chi(\mathbb{C})$ as its smooth part. Notice that $f_\chi(z_1)$, $f_\chi(z_2)$ and $f_\chi(z_{12})$ lie on the line at infinity. We let ω_χ be the meromorphic 2-form on $\mathbb{P}^2$ characterized by the property that it has divisor $-D_\chi$ and that $\mathrm{Res}_{z=0} f_\chi^* \mathrm{Res}_{D_\chi} \omega_\chi = 1$. Then $S_\chi := \mathbb{P}^2_{f(z_1)f(z_2)\cdots f(z_r)}$ is the blow-up $\mathbb{P}^2$ that we associate to χ. It comes with a nodal curve (which we identify with D_χ as its strict transform) and a meromorphic 2-form (the pull-back of ω_χ). The same argument as in the cuspidal case shows that S_χ is a Del Pezzo surface if and only if $\chi \in T_r^\circ$, where T_r° is the set of $\chi \in T_r$ which have no root in their kernel. The geometric markings of the pair (S_χ, D_χ) are now permuted by $\{\pm 1\}.W_r \subset \mathrm{GL}(Q_r)$. We can carry this out over T_r to obtain a family of geometrically marked pairs $(\mathcal{S}_{T_r}, \mathcal{D}_{T_r}, \tilde\phi_{T_r})$.

Theorem 2.5. *The restriction of this family to* T_r°, $(\mathcal{S}_{T_r^\circ}, \mathcal{D}_{T_r^\circ}, \tilde\phi_{T_r^\circ})$ *is a universal family of triples* $(S, D, \tilde\phi)$, *where* S *is a Del Pezzo surface of degree* $9 - r \leq 6$, D *a nodal anticanonical divisor of* S *and* $\tilde\phi$ *is a geometric marking of the pair* (S, D). *The closed points of the orbit space* T_r°/W_r *are in bijective correspondence with the isomorphisms classes of pairs* (S, D) *endowed with a generator* $H_1(D)$.

Note that taking the opposite generator induces an involution in T_r°/W_r which is the identity only when $-1 \in W_r$, i.e., when $r = 7$ or $r = 8$.)

Remark 2.6. If $S \subset \mathbb{P}^3$ is a cubic surface, then as we have seen, any plane section of S is an anticanonical divisor of S and vice versa. For $p \in S$, the projective plane $P_pS \subset \mathbb{P}^3$ tangent to S at p meets S in a curve D_p which will have at p a singular point. Generically this will be nodal curve. It follows that we may interpret $T_6^\circ/(\{\pm 1\}.W_6)$ resp. $\mathbb{P}(V_6^\circ)/W_6$ as the space of PGL(4)-orbits of pairs (S, p), where $S \subset \mathbb{P}^3$ is a smooth cubic surface and D_p a nodal resp. cuspidal curve.

In particular, we have a natural map from $T_6^\circ/(W_6.\{\pm 1\})$ to the moduli space of smooth cubic surfaces whose fibers are Zariski open subsets of such surfaces. It would be interesting to characterize the fibers of this map solely in terms of the root data. This is even of interest in the birational category, for apparently $\mathbb{C}(T_6)^{\{\pm 1\}.W_6}$ contains naturally the field of rational functions of the moduli space of cubic surfaces as a subfield (which is known to be rational) and the question is whether we can describe it explicitly.

Similarly, if $C \subset \mathbb{P}^2$ is a smooth quartic curve, and $f : S_C \to \mathbb{P}^2$ is the double cover of $\mathbb{P}^2$ which ramifies along C, then for every line $L \subset \mathbb{P}^2$, f^*L is an anticanonical curve. For $p \in C$, the line $L_pC \subset \mathbb{P}^2$, f^*L_pC which will have at p a singular point and generically this will be nodal curve or a cuspidal curve according to whether L_pC is generic (i.e., meets C in 2 other points) or is a simple flex point. It follows that we may interpret T_7°/W_7 resp. $\mathbb{P}(V_7^\circ)/W_7$ accordingly (the involution of S_C defined by its cover of $\mathbb{P}^2$ defines $-1 \in W_7$ and so $\{\pm 1\}.W_7 = W_7$). We have the same question as for cubic surfaces: can we explicitly describe the subfield of $\mathbb{C}(T_7)^{W_7}$ that we thus obtain as the field of rational functions of the moduli space of quartic curves in $\mathbb{C}(T_7)^{W_7}$? This subfield is known to be rational, too, but the known proof (due to Katzylo) is hard and very computational—perhaps that this can lead to a less computational proof.

Finally, T_8°/W_8 resp. $\mathbb{P}(V_8^\circ)/W_8$ can be interpreted as the moduli space of rational elliptic fibrations with only irreducible fibers and endowed with a section and a nodal resp. cuspidal fiber singled out.

Theorem 2.5 does not assert that T_r°/W_r carries a universal family. The reason is that the action of W_r on T_r° may fixed points. We show that this is remedied by passing to a finite cover. For this we recall that the nodal curve $D \subset S$ satisfies $D \cdot D = 9 - r = d$. So the normal bundle $\nu_{D/S}$ of D in S has degree d. We further observe that the identity component T_D of $\mathrm{Aut}(D)$ is isomorphic to $\mathbb{C}^\times$ and acts simply transitively on the smooth part D_{reg} of D.

We now change the moduli problem a bit as follows. We fix a nodal curve D endowed a line bundle ν on D of degree d. We consider pairs $(S, i : D \hookrightarrow S, u : \nu_i \cong \nu)$ with S a Del Pezzo surface of degree d, i an isomorphism onto an anticanonical divisor and u an isomorphism of the normal bundle of i onto ν given up to scalar. We can proceed to construct these as before as follows:

Recall that $(\varepsilon_{12}, \varepsilon_1, \dots, \varepsilon_r)$ is a basis of $\Lambda_{1,r}$. In terms of that basis

$$d_r = 3(\varepsilon_{12} + \varepsilon_1 + \varepsilon_2) - \varepsilon_1 - \cdots - \varepsilon_r = 3\varepsilon_{12} + 2\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3 - \cdots - \varepsilon_r.$$

Consider in D_{reg}^{r+1} the subset $\mathcal{T}_r$ of $p_\bullet = (p_{12}, p_1, \dots, p_r)$ with the property that $3p_{12} + 2p_1 + 2p_2 - p_3 - \cdots - p_r$ is a divisor for ν. For every $p_\bullet \in \mathcal{T}$ we use the divisor $p_{12} + p_1 + p_2$ to embed D in a projective plane P: P is then the plane of lines in $|p_{12} + p_1 + p_2|$ and the images of p_{12}, p_1, p_2 lie on a line by construction. Observe that the normal bundle of this embedding has $3(p_{12} + p_1 + p_2)$ as divisor. Now blow up P in the images of $p_1, \dots, p_r$ to get a surface $S_{p_\bullet}$ endowed with an embedding $i_{p_\bullet} : D \hookrightarrow S_{p_\bullet}$. Then the normal bundle of $i_{p_\bullet}$ has $3(p_{12} + p_1 + p_2) - \sum_{i=1}^r p_i = 3p_{12} + 2p_1 + 2p_2 - p_3 - \cdots - p_r$ as divisor. The product D_{reg}^{r+1} is principal homogeneous space (torsor) of the torus T_D^{r+1} and so $\mathcal{T}_t$ is a torsor for the kernel of the homomorphism

$$(t_{12}, t_1, \dots, t_r) \in T_D^{r+1} \mapsto 3t_{12} + 2t_1 + 2t_2 - t_3 - \cdots - t_r \in T_D.$$

But we use the basis $(\varepsilon_{12}, \varepsilon_1, \dots, \varepsilon_r)$ of $\Lambda_{1,r}$ to identify T_D^{r+1} with $T_D \otimes \Lambda_{1,r}$, then we see that this map is just obtained by tensoring the homomorphism $\nu \in \Lambda - 1, r \mapsto \nu \cdot d_r$ with T_D, so that this kernel may be identified with $Q_r \otimes T_D$. We conclude that $\mathcal{T}$ is a $Q_r \otimes T_D$-torsor. The Weyl group W_r acts via Q_r on $Q_r \otimes T_D$, but this action lifts naturally to $\mathcal{T}$ if we require that the intersection of $\mathcal{T}$ with the main diagonal to T_D^{r+1} (these are the $(p,\dots,p)$ with $d(p)$ a divisor for ν; there are d solutions of these). are fixed points of W_r. (Indeed, we have a natural W_r-equivariant map $\mathcal{T}_r \to \mathrm{Hom}(Q_r, T_D)$ of which this is the fiber over the identity element.)

Let us now recall that the exponential invariant theory for root systems gives the following toric analogue of Chevalley's theorem:

Theorem 2.7. *Let R be a root system, W its Weyl group and Q the lattice spanned by its roots. If we put $T := Q \otimes \mathbb{C}^\times$, then:*

(i) *the W-stabilizer of any point of T is generated by reflections, so that T acts freely on $T^\circ := T - \Delta$, where $\Delta \subset T$ denotes the union of the fixed point hypertori of reflections in W and*
(ii) *the $\mathbb{C}$-algebra of W-invariant regular functions on T, $\mathbb{C}[V]^W$, has r algebraically independent homogeneous generators and Δ is defined by an element of $\mathbb{C}[V]^W$.*

We conclude:

Theorem 2.8. *We have a universal family*

$$(\mathcal{S}_{\mathcal{T}_r^\circ/W_r}, i_{\mathcal{T}_r^\circ/W_r} : D_{\mathcal{T}_r^\circ/W_r} \hookrightarrow \mathcal{S}_{\mathcal{T}_r^\circ/W_r})$$

over $\mathcal{T}_r^\circ/W_r$ for the pairs $(S, i : D \hookrightarrow S)$, where S is a Del Pezzo surface of degree $9 - r \leq 6$ and $i(D)$ an anticanonical curve on S such that the normal bundle ν_i is D-isomorphic to ν.

Again, we can extend this family across $\mathcal{T}_r/W_r$ if we allow the fibers to have Kleinian singularities.

3. Affine surfaces as Milnor fibers

Normal surface singularities. A (complex-analytic) *singularity* is a punctual germ of an analytic space. This is defined by a neighborhood X of a closed point o of an analytic space with the understanding that any other neighborhood of x in that space defines the same singularity. It is algebraically given by its local $\mathbb{C}$-algebra (of local-analytic functions) $\mathcal{O}_{X,o}$. We will focus on the case of a normal surface singularity: $\dim_o X = 2$ and X is normal at o. That last property implies that a punctured neighborhood of o in X is nonsingular: X has o as isolated singular point (we then say that the singularity is *genuine*) or is smooth in o. We say that a normal surface singularity is *Gorenstein* if there exists a nowhere zero 2-form on punctured neighborhood of o. (We then always assume that X is chosen so small that $X - \{o\}$ is nonsingular and carries a nowhere zero 2-form.) The easiest way to generate normal surface singularities is by taking an $f \in \mathbb{C}\{x, y, z\}$ with $f(o) = 0$ and with the property that the common zero set of the partial derivatives of f has o as an isolated point: then the zero set of f at o is a normal surface singularity with local $\mathbb{C}$-algebra $\mathbb{C}\{x,y,z\}/(f)$. These are in fact Gorenstein: the residue of $f^{-1}dx \wedge dy \wedge dz$ on the smooth part of $f^{-1}(0)$ is a a nowhere zero 2-form on that smooth part.

Any genuine normal surface singularity (X, o) admits a *normal crossing resolution*, i.e, there exists a nonsingular surface $\tilde{X}$, a normal crossing curve $C \subset \tilde{X}$ and a proper holomomorpic map $\pi : \tilde{X} \to X$ that is an isomorphism over $X - \{o\}$ and for which $C = \pi^{-1}(o)$ (so C is compact). Any irreducible component of C that is an exceptional curve of the first kind can be contracted and still gives normal crossing resolution. If there are no such irreducible components, then we call the normal crossing resolution *minimal* and it is a remarkable fact that a minimal normal crossing resolution is unique (this is however particular to dimension two). If $(C_i)_{i\in I}$ is the collection of irreducible components of C, then the intersection matrix $(C_i \cdot C_j)_{i,j}$ is negative definite. (And conversely, a compact curve on a complex manifold of dimension 2 can be analytically contracted to produce a normal surface singularity, if the intersection matrix of its irreducible components (whether they are smooth or not) is negative definite.) If (X, o) is Gorenstein, then the pull-back of the 2-form on $X - \{o\}$ yields on $\tilde{X}$ a meromorphic 2-form whose divisor $K_{\tilde{X}}$ is supported by C: $\omega_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}(K_{\tilde{X}})$.

Let us now describe the simplest normal (Gorenstein) surface singularities.

Kleinian singularities. If $\Gamma \subset SL(2,\mathbb{C})$ is a nontrivial finite subgroup, then Γ, then the image $\bar{\Gamma}$ of Γ in $PSL(2,\mathbb{C})$ acts on the Riemann sphere and acts there orthogonally with respect to a spherical (Fubini-Study) metric. Thus $\bar{\Gamma}$ becomes the group of special symmetries of a regular polyhedron and this is the way we usually classify the finite subgroups of $PSL(2,\mathbb{C})$ up to conjugacy. This also leads to a classification of finite subgroups of $SL(2,\mathbb{C})$ up to conjugacy, for either Γ is the preimage of $PSL(2,\mathbb{C})$ in $SL(2,\mathbb{C})$ or is a subgroup of index two thereof.

The Γ-orbit space $X := \Gamma\backslash\mathbb{C}^2$ has at (the image of) the origin a singular point. Such a singularity (and any analytic germ isomorphic to it) is called, depending on the background (or nationality) of the speaker, a *rational double point*, a *Kleinian singularity*, a *Du Val singularity* or an *A-D-E singularity*. We shall opt for 'Kleinian singularity'. Since $\Gamma \subset SL(2,\mathbb{C})$, Γ acts freely on $\mathbb{C}^2 - \{(0,0)\}$. It also follows that X is Gorenstein singularity: Γ leaves invariant the 2-form $dz_1 \wedge dz_2$, and so this form descends to a 2-form (which we shall denote ω) on $X - \{o\}$. In all these cases $\mathbb{C}[w_1, w_2]^\Gamma$ has 3 homogeneous generators

TABLE 1. The Kleinian singularities

subgroup of $SL(2,\mathbb{C})$	Dynkin	equation	weights
cyclic group of order $n+1$	$A_{n\geq 1}$	$x^{n+1}+y^2+z^2$	$(2,n+1,n+1)$
dihedral group	$D_{n\geq 4}$	$xy^2+x^{n-1}+z^2$	$(2,n-2,n-1)$
binary tetrahedral group	E_6	$x^4+y^3+z^2$	$(3,4,6)$
binary octadral group	E_7	$x^3y+y^3+z^2$	$(4,6,9)$
binary icosahedral group	E_8	$x^5+y^3+z^2$	$(6,10,15)$

obeying one equation. In table 1 we denoted these generators x,y,z, mention their degrees (weights) and give the equation they satisfy. So these are surface singularities that can be realized in $\mathbb{C}^3$. For example, if Γ is the cyclic group of matrices $\left(\begin{smallmatrix}\zeta & 0\\ 0 & \zeta\end{smallmatrix}\right)$, where $\zeta^{n+1}=1$, then $\mathbb{C}[w_1,w_2]^\Gamma$ is generated by $x := w_1w_2$, $y' := w_1^{n+1}$ and $z' := w_2^{n+1}$. These obey the relation $x^{n+1}-y'z'=0$. A linear transformation $y' := y+\sqrt{-1}z$, $z' := -y+\sqrt{-1}z$, then turns this equation into $x^{n+1}+y^2+z^2=0$.

The A-D-E designation has to do with the fact that the minimal normal crossing resolution of such a singularity has as exceptional divisor a curve whose irreducible components are smooth of genus zero (so isomorphic to $\mathbb{P}^1$) of self-intersection -2 and cross each other normally so that the intersection matrix is minus that of a simply laced Dynkin diagram, (which is of type $A_{r\geq 1}$, $D_{r\geq 4}$ or E_r for $r=6,7,8$).

Lemma 3.1. *The lift of ω on $\tilde{X}$ is holomorphic on $\tilde{X}$ and has no zeroes and so $\omega_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}$.*

Proof. Let $K_{\tilde{X}}$ denote the divisor of the lift of ω. Since every irreducible component C_i of C is smooth of genus zero with $C_i\cdot C_i=-2$, the adjunction formula says that $K_{\tilde{X}}\cdot C_i=(2g(C_i)-2)-C_i\cdot C_i=-2+2=0$. Since the intersection matrix $(C_i\cdot C_j)$ is nondenegerate (it is in fact negative definite), it follows that $K_{\tilde{X}}=0$. □

Simple elliptic singularities. A normal surface is called simply elliptic of degree $d>0$ if it admits a resolution whose exceptional curve is a smooth genus one curve C with $C\cdot C=-d$. One can show that the isomorphism type of C completely determines the isomorphism type of the singularity. So such a singularity is always obtained by taking the total space $\tilde{X}$ of a line bundle over ℓ over C of degree $-d$ (all such are isomorphic if we allow the isomorphism to induce a translation in C), and contracting its zero section: the underlying variety is an

TABLE 2. Simply-elliptic singularities embeddable in $\mathbb{C}^3$

d	Dynkin	equation	weights
1	$\tilde{E}_8 = \tilde{T}_{2,3,6}$	$x^6 + y^3 + z^2 + \lambda xyz$	(2,3,6)
2	$\tilde{E}_7 = \tilde{T}_{2,4,4}$	$x^4 + y^4 + z^2 + \lambda xyz$	(2,4,4)
3	$\tilde{E}_6 = \tilde{T}_{3,3,3}$	$x^3 + y^3 + z^3 + \lambda xyz$	(3,3,3)

affine cone, namely

$$X := \operatorname{Spec}\big(\oplus_{k=0}^{\infty} H^0(C, \ell^{-k})\big).$$

Since the isomorphism type of C comes in a one parameter family, so does the isomorphism type of these singularities. From the adjunction formula one sees that $\omega_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}(C)$ and so a simply-elliptic singularity is Gorenstein. They embed in $\mathbb{C}^3$ for $d \leq 3$ only (see Table 2); the j-invariant of the associated genus one curve is a rational function of the parameter λ (which therefore must omit certain values).

Cusps. This class of singularities is easiest characterized by means of the type of their minimal normal crossing resolution, but that is not the way they arose (and which explains their name): a normal surface singularity is called a *cusp singularity* if its minimal normal crossing resolution has as exceptional divisor a nodal curve or a cycle $C = \cup_{i\in\mathbb{Z}/n} C_i$ (with $n \geq 2$) of smooth rational curves such that $C_i \cdot C_{i+1} = 1$ and $C_i \cdot C_j = 0$ if $j \notin \{i-1, i, i+1\}$. The negative definiteness of $(C_i \cdot C_j)_{i,j}$ then implies that for wll i, $C_i \cdot C_i \leq -2$ and that the is inequality is strict for at least some i. One can show that the cyclically ordered set of self-intersection numbers $(C_i \cdot C_i)_{i\in\mathbb{Z}/n}$ completely determines the isomorphism type of the singularity.

The alternative definition goes like this. Let L be a free abelian group of rank 2 (so $L \cong \mathbb{Z}^2$) and put $L_\mathbb{R} := \mathbb{R}|otimes L$ and $T := \mathbb{C}^\times \otimes L$ (so $T \cong (\mathbb{C}^\times)^2$). The map $z \in \mathbb{C}^\times \mapsto w := -\log|z| \in \mathbb{R}$ is a group homomorphism with kernel the circle group $U(1)$. This map induces a $SL(L)$-equivariant homomorphism $I : T \to L_\mathbb{R}$ with kernel the compact 2-torus $U(1) \otimes L$. For $\phi \in \operatorname{Hom}(L, \mathbb{Z})$ we denote by $\phi_\mathbb{R} : L_\mathbb{R} \to \mathbb{R}$ the 'realification' and by $e^\phi : T = \mathbb{C}^\times \otimes L \to \mathbb{C}^\times \otimes \mathbb{Z} = \mathbb{C}^\times$ the corresponding character. Note that $-\log|e^\phi| = \phi_\mathbb{R} I$ and so $|e^\phi| = \exp(-\phi_\mathbb{R} I)$.

Let $S \in SL(L)$ be such that $\operatorname{Tr}(S) > 2$. Since the characteristic polynomial of S is $\lambda^2 - \operatorname{Tr}(S)\lambda + 1$, S has two distinct eigenvalues $\lambda_\pm = \frac{1}{2}(\operatorname{Tr}(S) \pm \sqrt{\operatorname{Tr}(S)^2 - 4})$ that are both positive quadratic irrationalities. So S acts freely and properly discontinuously on $L_\mathbb{R} - \{0\}$ (for no eigenvalue is ± 1) and the two eigenlines corresponding to the two eigenvalues decompose $\mathbb{R} \otimes L$ into four sectors, each of which is invariant under S (for both eigenvalues are positive). We fix such a sector $U \subset \mathbb{R} \otimes L$ and put $\mathcal{U} := I^{-1}U$. Then S leaves $\mathcal{U}$ invariant. We put $X^\circ := S^{\mathbb{Z}} \backslash \mathcal{U}$. Analytic functions on X° are obtained as follows. We claim that

$$f_\phi := \sum_{n \in \mathbb{Z}} e^{(S^n)^* \phi}$$

converges uniformly and absolutely on compact subsets of $\mathcal{U}$. To see this, let $\check{U}$ be the set of $\phi \in L^*_{\mathbb{R}}$ with $\phi | U > 0$. This is clearly a sector in $L^*_{\mathbb{R}}$. If $u \in U$ is arbtritrary, then the set of $\psi \in \check{U} \cap \mathrm{Hom}(L, \mathbb{Z})$ with $\psi \le N$ is the set of lattice points in a solid triangle of proportionality size N and hence has cardinality $\le c_u N^2$ for some $c_u > 0$. So if $m_u := \min_{\psi \in \check{U} \cap \mathrm{Hom}(L,\mathbb{Z})} \psi(u)$, then

$$\sum_{n \in \mathbb{Z}} \sup_{I^{-1}u} |e^{(S^n)^* \phi}| \le \sum_{n \in \mathbb{Z}} \exp(-(S^n)^* \phi_{\mathbb{R}}(u)) \le$$
$$\le \sum_{\psi \in \check{U} \cap \mathrm{Hom}(L,\mathbb{Z})} \exp(-\psi_{\mathbb{R}}(u)) \le \sum_{N \ge m_u} c_u N^2 \exp(-N) < \infty.$$

But then this same bound holds if we replace $I^{-1}(u)$ by $I^{-1}(u + U)$. Since any compact subset K of U is contained in $u + U$ for some u, we have such an estimate on $I^{-1}K$. It then holds also on the $S^{\mathbb{Z}}$-orbit of $I^{-1}K$. You can verify that the convex hull $[U \cap L]$ of $U \cap L$ can be written as the $S^{\mathbb{Z}}$-orbit of $K + U$ for some compact K and so f_ϕ will converge uniformly and absolutely on $I^{-1}[U \cap L]$. The invariance properties of f_ϕ ensure that it factors through a holomorphic function on X°. One can show that there are enough of such functions to separate the points of X° (even infinitesimally). We also note that $t > 0$, then $m_{tu} = t m_u$ and c_{tu} can be taken to be $t^2 c_u$. Hence the series $\sum_{N \ge t m_u} c_u t^2 N^2 \exp(-N)$ tends to 0 as $t \to \infty$ and so if $f_\phi(z)$ tends to 0, when $I(z) \in U$ tends to infinity in U. In fact, as we have observed, f_ϕ is bounded on $I^{-1}[U \cap L]$ and $\lim_{t \to \infty} \sup_{I^{-1}(t[U \cap L])} f_\phi = 0$. This suggests that we can extend X° to a space X by adding a point o "at ∞": we give the disjoint union $X := X^\circ \sqcup \{o\}$ a topology at o by taking for a neighborhood basis of o the subsets of the form $I^{-1}(t[U \cap L]) \sqcup \{o\}$, $t > 0$. We then endow X with an analytic structure by stipulating that the functions f_ϕ are analytic on X when we let them take the value zero in o. This makes that X has at o a normal surface singularity. This is the original definition of a cusp singularity (they appear naturally on the Baily-Borel compactification of so-called Hilbert modular surfaces). Such a singularity is in fact Gorenstein: the choice of a generator of $\wedge^2 \mathrm{Hom}(L, \mathbb{Z})$ defines a translation invariant 2-form on T which is also invariant under $\mathrm{SL}(L)$ and so its restriction to $\mathcal{U}$ descends to a 2-form on X° which is nowhere zero and meromorphic on X.

The theory of torus embeddings furnishes a minimal normal crossing resolution of the above type: the vertices on the boundary of $[U \cap L]$ is a set Σ of integral vectors; this set is totally ordered in the sense that each $v \in \Sigma$ has a successor v_1 in the cone spanned by v and the half line bounding U with eigenvalue λ_+. Then there exists an $n \ge 0$ such that $S(v) = v_n$ for all

TABLE 3. Cusp singularities embeddable in $\mathbb{C}^3$

Dynkin	equation
$\tilde{T}_{p,q,r}$	$x^p + y^q + z^r + xyz$

$\nu \in \Sigma$. This give rise to a torus nonsingular embedding $T \subset T_\Sigma$: every basis (ν, ν_1) of L as above gives rise to an isomorphism $T \cong (\mathbb{C}^\times)^2$ and we then add to T the normal crossing curve $C_\nu \cup C_{\nu_1}$ corresponding to the embedding $(\mathbb{C}^\times)^2 \subset \mathbb{C}^2$ (which adds the coordinate axes). A more precise way to index the boundary strata is as follows: the group of $\phi \in \mathrm{Hom}(L, \mathbb{Z})$ which vanish on ν is infinite cyclic; if ϕ_ν is the generator which takes the value 1 on ν_1, then $\chi_\nu := e^{\phi_\nu} \in \mathrm{Hom}(T, \mathbb{C}^\times)$ extends to a coordinate on C_ν such that χ_ν takes the value zero on $C_\nu \cap C_{\nu_1}$ and ∞ on $C_\nu \cap C_{\nu_{-1}}$. So the map above $T \cong (\mathbb{C}^\times)^2$ has coordinates $(\chi_\nu, \chi_{\nu_1}^{-1})$.

Thus the boundary $C_\Sigma := T_\Sigma - T$ is an infinite chain of normal crossing curves: each $\nu \in \Sigma$ determines such a curve C_ν and C_{ν_1} and C_{ν_1} meet in a normal crossing point, whereas $C_\nu \cap C_{\nu_k} = \emptyset$ for $k \geq 2$. The self-intersection $C_\nu \cdot C_\nu$ is computed as follows: if we write ν_1 in terms of basis (ν, ν_{-1}), then this takes the form $\nu_1 = -\nu_{-1} + k\nu$. We claim that $C_\nu \cdot C_\nu = -k$. For then $\phi_{\nu_{-1}}(\nu_1) = k$ and so $\chi_{\nu_{-1}}$ has order of vanishing 0 on $C_{\nu_{-1}}$, 1 on C_ν and k on C_{ν_1}. For the principal divisor D of $\chi_{\nu_{-1}}$ we must have $0 = D \cdot C_\nu = (C_\nu + kC_{\nu_1}) \cdot C_\nu$ and hence $C_\nu \cdot C_\nu = -kC_{\nu_1} \cdot C_\nu = -k$.

Now T_Σ comes with an action of $S^{\mathbb{Z}}$. We let $\mathcal{U}_\Sigma$ be the interior of the closure of $\mathcal{U}$ in T_Σ. Then $\mathcal{U}_\Sigma$ is a S-invariant neighborhood of C_Σ in T_Σ and the action of $S^{\mathbb{Z}}$ on $\mathcal{U}_\Sigma$ is proper and free. So the orbit space $\tilde{X} := S^{\mathbb{Z}} \backslash \mathcal{U}_\Sigma$ is a complex manifold of dimension 2. The toric boundary $C := \tilde{X} - X^\circ = S^{\mathbb{Z}} \backslash C_\Sigma$ is now a cycle of n smooth rational curves (when $n > 1$) and a nodal curve when $n = 1$. There is a natural map $\tilde{X} \to X$ which is the identity on X° and maps C onto o. This is our minimal normal crossing resolution.

The cusp singularities that are embeddable in $\mathbb{C}^3$ have an equation of the form as in table 3. The corresponding S (or its inverse) is then conjugate to

$$\begin{pmatrix} 0 & 1 \\ -1 & p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & r \end{pmatrix}.$$

Deformations of an isolated singularity. We briefly review the theory of deformations. Let us fix an isolated singularity (X, o) of complex dimension n (so a punctured neighborhood of o in X is a complex manifold of dimension n). A *deformation* of (X, o) over a base germ (S, s) is an embedding of (X, o) as a fiber in a 'family' of n-dimensional varieties parametrized by (S, s), to be precise: it is given by a flat map-germ $f : (\mathcal{X}, x) \to (S, s)$ between germs of analytic spaces and an identification ι of the fiber over s with (X, o). In algebraic terms:

we are given a flat $\mathbb{C}$-algebra homomorphism $\mathcal{O}_{S,s} \to \mathcal{O}_{\mathcal{X},x}$ plus an isomorphism of $\mathbb{C}$-algebras $\mathcal{O}_{\mathcal{X},x}/\mathfrak{m}_{S,s}\mathcal{O}_{\mathcal{X},x} \cong \mathcal{O}_{X,o}$. We can represent the map-germ f by an actual map whose fibers are complex analytic spaces of pure dimension n having isolated singularities only. We shall see that is not always possible to find a deformation of (X,o) general fiber is without singularities, but if that is the case, we say that (X,o) is *smoothable* (and the deformation in question is then called a *smoothing*). If (X',o') is another singularity, then we say that (X,x) *deforms into* (X',o') if there exists a one-parameter deformation $f:(\mathcal{X},x)\to(\mathbb{C},0)$ of (X,o) plus a section $\sigma:(\mathbb{C},0)\to(\mathcal{X},x)$ of f such that the generic fiber has along this section a singularity isomorphic to (X',o'). In this sense our list of Gorenstein surface singularities is ordered according to this relation: a Kleinian singularity only deforms into Kleinian singularity, a simply-elliptic singularity can only deform into a simply-elliptic singularity (by simply changing the isomorphism type of the genus one curve) or into a Kleinian singularity, and a cusp singularity can only deform into another cusp singularity, a simply-elliptic singularity or into Kleinian singularity. (For those that embed in $(\mathbb{C}^3,o)$ this was established by Arnol'd.) It can be shown that if (X,o) is a normal Gorenstein surface singularity, then so are the singularities in the fibers of f: the fibers of f are normal and there exists a relative meromorphic 2-form on $\mathcal{X}/S$ which is regular and nonzero away from the critical set of f.

The deformations of (X,o) are the objects of a category: given deformations $(f:(\mathcal{X},x)\to(S,s),\iota:X\hookrightarrow\mathcal{X})$ and $(f':(\mathcal{X}',x')\to(S',s'),\iota:X\hookrightarrow\mathcal{X}')$, then a morphism $\Phi:(f',\iota')\to(f',\iota')$ is given by a Cartesian diagram

$$\begin{array}{ccc}
(\mathcal{X}',x') & \xrightarrow{\phi} & (\mathcal{X},x)\\
{\scriptstyle f'}\big\downarrow & & {\scriptstyle f}\big\downarrow\\
(S',s') & \xrightarrow{\bar\phi} & (S,s)
\end{array}$$

with the property that $\phi\iota'=\iota$. This means that every fiber of f' already appears as a fiber f (for ϕ maps it isomorphically onto one of f). A deformation (f,ι) of (X,o) is called *versal* if it has the property that for any deformation (f',ι') of (X,o) there exists a morphism $\Phi=(\phi,\bar\phi):(f',\iota')\to(f,\iota)$ in the sense above. We call it *semi-universal* if in addition $\bar\phi$ is unique up to first order (i.e., the induced map $\mathfrak{m}_{S,s}/\mathfrak{m}_{S,s}^2\to\mathfrak{m}_{S',s'}/\mathfrak{m}_{S',s'}^2$ is unique). A basic result of the theory asserts that such semi-universal deformations exist. It can be deduced from the definitions that a semi-universal deformation is unique up to isomorphism (the isomorphism need not be unique, but its first order part will). In particular, the Zariski tangent space T_sS of the base of a semi-universal deformation (f,ι) at its closed point $s \in S$ (which is just the dual of $\mathfrak{m}_{S,o}/\mathfrak{m}_{S,o}^2$) is invariantly defined. Indeed, to every deformation (f,ι) of (X,o) there is functorially associated a linear map $T_sS \to \mathrm{Ext}^1_{\mathcal{O}_{X,o}}(\Omega^1_{X,o},\mathcal{O}_{X,o})$ (so compatible with morphisms of deformations) which is surjective resp. an isomorphism if and only if (f,ι) is versal resp. semi-universal.

It is also a fact that versality is open: if (f,ι) is versal, then f admits a representative which defines a versal deformation at each of its singularities. So the relation "Singularity A deforms into singularity B" is transitive.

Remark 3.2. Suppose $(X,o) \subset (\mathbb{C}^{n+1},o)$ is an isolated hypersurface singularity, that is, given by a single nonzero $f \in \mathfrak{m}_{\mathbb{C}^{n+1},0}$. Since the critical values of a function are nowhere dense, f is a smoothing. The singularity isolated and so the ideal $I_f \subset \mathcal{O}_{\mathbb{C}^{n+1},o}$ generated by f and its partial derivatives of f defines 0. We may then view f as a one-parameter deformation of (X,o). But is it also easy two write down a semi-universal deformation (F,ι) of X_o. First observe that by the local Nulstellensatz, $I_f \supset \mathfrak{m}^N_{\mathbb{C}^{n+1},0}$ for some N. This implies that the $\mathbb{C}$-algebra $\mathcal{O}_{\mathbb{C}^{n+1},o}/I_f$ is finite dimensional. Then if $\phi_1,\dots\phi_k \in \mathcal{O}_{\mathbb{C}^{n+1},o}$ are such that $(1=\phi_1,\dots\phi_k)$ maps to basis of $\mathcal{O}_{\mathbb{C}^{n+1},o}/I_f$, then a semi-universal deformation of (X,o) is given by taking for $\mathcal{X},o)$ the zero set in $(\mathbb{C}^{n+1+k}$ (with coordinates $(z_0,\dots,z_n,u_1\dots,u_k)$) of $f(z)+\sum_{i=1}^k u_k\phi_k(z)$ (the partial derivative of this expression with respect to ϕ is 1 and so $(\mathcal{X},o)$ will be smooth by the the implicit function theorem) and letting $F:(\mathcal{X},o)\to(\mathbb{C}^k,o)$ be the projection on the u-coordinates, together with the obvious identification of (X,o) with $F^{-1}(o)=X\times\{o\}$. We also see that we have a morphism of deformations $\Phi=(\phi,\bar\phi)$ from f to F with $\phi(z)=(z,-f(z),0,\dots,0)$ and $\bar\phi(t)=(-t,o)$.

Let us also illustrate for this class of examples how to the tangent space $T_o\mathbb{C}^{1+k}$ is identified with $\mathrm{Ext}^1_{\mathcal{O}_{X,o}}(\Omega^1_{X,o},\mathcal{O}_{X,o})$. First note that differentiating the expression $f(z)+\sum_{i=1}^k u_k\phi_k(z)$ with respect to the u-variables establishes the isomorphism $T_o\mathbb{C}^k \cong \mathcal{O}_{\mathbb{C}^{n+1},o}/I_f$ via $\frac{\partial}{\partial u_j}\big|_o \leftrightarrow \phi_j$ $(j=1,\dots k)$. Now $\Omega^1_{X,o}$ is obtained from $\Omega^1_{\mathbb{C}^{n+1},o}=\mathcal{O}_{\mathbb{C}^{n+1},o}\{dz_0,\dots,dz_n\}$ by dividing it out by the $\mathcal{O}_{\mathbb{C}^{n+1},o}$-submodule generated by $df=\sum_{i=0}^n\frac{\partial f}{\partial z_i}dz_i$ and reducing modulo the ideal $f\mathcal{O}_{\mathbb{C}^{n+1},o}$. This amounts to a presentation of $\Omega^1_{X,o}$ as a

$\mathcal{O}_{X,o}$-module:

$$\mathcal{O}_{X,x} \xrightarrow{\cdot df} \mathcal{O}_{X,x}\{dz_0, \dots, dz_n\} \to \Omega^1_{X,o} \to 0.$$

But the first map is easily seen to be injective, so that we have in fact a free resolution $0 \to \mathcal{O}_{X,x} \xrightarrow{\cdot df} \mathcal{O}_{X,x}\{dz_0, \dots, dz_n\} \to 0$ (of length 1) of $\Omega^1_{X,o}$ as a $\mathcal{O}_{X,o}$-module. If we apply $\mathrm{Hom}_{\mathcal{O}_{X,o}}(-, \mathcal{O}_{X,o})$ to this resolution, we get

$$0 \leftarrow \mathcal{O}_{X,x} \xleftarrow{df} \mathcal{O}_{X,x}\Big\{\frac{\partial}{\partial z_0}, \dots, \frac{\partial}{\partial z_n}\Big\} \leftarrow 0,$$

with df sending $\sum_i g_i \frac{\partial}{\partial z_i}$ to $\sum_i g_i \frac{\partial f}{\partial z_i}$. The cokernel of this map is by definition $\mathrm{Ext}^1_{\mathcal{O}_{X,o}}(\Omega^1_{X,o}, \mathcal{O}_{X,o})$ and we see that it can also be identified with $\mathcal{O}_{\mathbb{C}^{n+1},o}/I_f \cong T_o\mathbb{C}^k$.

Remark 3.3. One might wonder why we did not require that the morphism Φ be unique, thus defining the notion of a *universal* deformation. The answer is that this notion would be empty: such deformations simply do not exist. But if they were to exist, then any automorphism g of (X, o) would extend uniquely to an automorphism of a universal deformation (f, ι): $(f, \iota g)$ is another deformation of (X, o) and must also be universal; so there exists a unique isomorphism $\Phi = (\phi, \bar{\phi})$ of (f, ι) onto $(f, \iota g)$ and this is precisely the extension we asked for. Thus we would have $\mathrm{Aut}(X, o)$ act on (f, ι) with image the full automorphism group of (f, ι). Although this argument is invalid (as it depends on a false assumption) one can nevertheless show that this extension is possible for any a reductive subgroup $G \subset \mathrm{Aut}(X, o)$: the G-action on (X, o) may be extended to (f, ι), not uniquely in general, but any two such extensions will be conjugate in $\mathrm{Aut}(f, \iota)$.

A case of particular interest is when $G = \mathbb{C}^\times$ and G acts on (X, o) with positive weights. This can only happen if (X, o) is represented by an affine variety whose ring of regular functions is graded such that every homogeneous element that is not constant has positive degree. This is the case for the Kleinian singularities and the simply elliptic singularities (the $\mathbb{C}^\times$-action is then evident from the construction), but not for the cusp singularities. The $\mathbb{C}^\times$-action on $(\mathcal{X}, x)$ or (S, s) need not be with positive weights (for the action of $\mathbb{C}^\times$ on $\mathrm{Ext}^1_{\mathcal{O}_{X,o}}(\Omega^1_{X,o}, \mathcal{O}_{X,o})$ may need not have that property). But if that is the case, then the whole semi-universal deformation is naturally represented by a morphism of affine varieties. This is the case for Kleinian singularities, as one can easily verify. If the weights in (S, s) are only non-negative (which turns out be the case for simply-elliptic singularities), then the whole semi-universal deformation is naturally represented by an affine morphism of complex-analytic spaces: there exists an N such that we can embed our total space $\mathcal{X}$ as a closed analytic subset in $S \times \mathbb{C}^N$ such that its closure in $S \times \mathbb{P}^N$ is also analytic (the fibers are then affine varieties).

4. GLOBAL MODELS FOR SEMI-UNIVERSAL DEFORMATIONS

Semi-universal deformation of a Kleinian singularity. A Kleinian singularity admits, as any isolated singularity, a semi-universal deformation. Our goal is to construct a nice model for it. Part of the construction that follows will also be useful for the study of semi-universal deformations of the more complex simply-elliptic and cusp singularities and so we start out with any integer triple of the form $r_1 \geq r_2 \geq r_3 \geq 1$. Let P' a projective plane on which we are given a three concurrent lines $\overline{D}_1, \overline{D}_2, \overline{D}_3$ and a line L not passing through the common point of the $\overline{D}_i$'s. Let q_i denote the point of intersection $\overline{D}_i \cap L$ and low up each q_i. Then the strict transform of L has self-intersection -2. Denote by $q_i^{(1)}$ the intersection point of the strict transform of $\overline{D}_i$ with the exceptional curve over q_i and blow up this point. Continue this process until we reach $q_i^{(r_i)}$. The result is then a blowup $\pi : \hat{S} \to P'$ such that the preimage of L is the union a T_{r_1,r_2,r_3}-curve C and three exceptional curves of the first kind (each connected to an 'end component' of C). The strict transform $\hat{D}_i$ of $\overline{D}_i$ has self-intersection $1 - r_i$. Note that $\hat{D} := \hat{D}_1 + \hat{D}_2 + \hat{D}_3$ is an anticanonical divisor whose support does not meet C. We then let $\hat{S} \to S$ be obtained by contracting C and blowing down any exceptional curves of the first kind below $\hat{D}$.

To be precise, if $r_3 = 2$, then $\hat{D}_1$ is an exceptional curve of the first kind and so we blow it down. The images $\hat{D}_1$ resp. $\hat{D}_2$ are then smooth and have one point in common (with intersection multiplicity 2) and self-intersection $2 - r_1$ resp. $2 - r_2$. Let us also assume that $r_2 > 3$ so that neither is exceptional. Then denote the blowdown by $\hat{S} \to S$ and the image of $\hat{D}_i$ under this map by D_i $(i = 1, 2)$. Their sum $D = D_1 + D_2$ is an anticanonical divisor for S. It is a fun exercise to see that we could have obtained the pair (S, D) in an alternative manner by just starting out with a projective plane P, a conic Q in P and a tangent line L of Q. If $p \in Q \setminus L$ and $q \in L \setminus Q$ are such that pq is another tangent of Q, then (S, D) is obtained by blowing up over p (and the strict transforms of Q) $r_1 + 2$ times and blowing up over q (and on the strict transforms of L) $r_2 - 1$ times, so that the strict transform of Q resp. L has self-intersection $4 - (r_1 + 2) = 2 - r_1$ resp. $1 - (r_2 - 1) = 2 - r_2$.

If in addition $r_2 = 3$, then the image of D_2 is also an exceptional curve of the first kind and we blow it down as well. We then reserve the notation S for this twofold blowdown $\hat{S} \to S$ and note that the image D of D_1 is a cuspidal curve with self-intersection $6 - r_3$. We may write this as $9-r$, where r is the number of nodes $(r_1+r_2+r_3-2)$ in a T_{r_1,r_2,r_3}-graph. A similar argument shows that the pair (S, D) is obtained by starting out with a projective plane P, a cuspidal curve $K \subset P$ and blowing up r times over the flex point of K (and the strict transforms of K). This is indeed the construction we gave in the discussion following Theorem 2.8.

Now assume that that the T_{r_1,r_2,r_3}-curve C can be contracted to a Kleinian singularity: we have for A_r we take $(r_1, r_2, r_3) = (r, 1, 1)$ $(r \geq 1)$, for D_r $(r-1, 2, 2)$ $(r \geq 4)$ and for E_r $(2, 3, r-3)$ $(r = 6, 7, 8)$. In these cases the corresponding anticanonical divisor D has three components with an ordinary planar triple point with self-intersection numbers $(1-r, 0, 0)$, two components with a common tangent with self-intersection numbers $(4-r, 0)$ and a cuspidal curve with self-intersection number $9-r$.

We denote the resulting (contracted) surface S_o, but we keep denoting the image of D in S_o by the same symbol. Notice, that the $\mathbb{C}^\times$-action on P' which has $\{p\} \cup L$ as its fixed point set lifts to S and hence also to S_o. If we fix a 2-form ω on P' with divisor $-\sum_i \overline{D}_i$, then ω lifts to a generating of $\omega_X(D)$ and then descends to a generating section $\omega_{S_o}(D)$, where ω_{S_o} denotes the dualizing sheaf of S_o. This shows again that a Kleinian singularity is Gorenstein and that the 2-form lift to the resolution without poles or zeroes across the exceptional divisor.

We shall consider pairs (S, ω) for which S is a normal surface whose dualizing sheaf ω_S is invertible and ω is a meromorphic section of ω_S whose divisor is of the form $-D$ with D near S like it is in S_o, so a cuspidal curve of self-intersection $9-r$ if we are in the E_r-case $(r = 6, 7, 8)$, two smooth genus zero curves meeting in a point with multiplicity 2 and with self-intersections 0 and $4-r$ in the D_r-case $(r \geq 4)$ and three concurrent smooth genus zero curves with self-intersections are 0, 0 and $1-r$ in the A_r-case. We assume that D supports an ample divisor, which amounts to requiring that $S - D$ is affine. We just constructed such a pair (although we did not really check there that $S - D$ is affine). For the E_r cases, the construction described in Section 2 produced a whole family. We shall see that such a family can also be obtained for the other cases.

The A_r-case. Start out with a projective plane P with three distinct lines $\overline{D}_1$, $\overline{D}_2$, $\overline{D}_3$ in P through a point $p \in P$ and a meromorphic 2-form ω on P with divisor $-\overline{D}_1 - \overline{D}_2 - \overline{D}_3$. Now let $S \to P$ be obtained by first blowing up points $p_1 \in \overline{D}_2 - \{p\}$ and $p_2 \in \overline{D}_3 - \{p\}$ and then successively blowing up points $p_3, \dots, p_{r+2}$ over $\overline{D}_1 - \{p\}$. If we denote by D_i the strict transform of $\overline{D}_i$ in S and identify ω with its pull-back to S, then we have $D_2 \cdot D_2 = D_3 \cdot D_3 = 0$, $D_1 \cdot D_1 = 1 - r$, and $-D = -(D_1 + D_2 + D_3)$ is also the divisor of ω on S. If E_i denotes the divisor on S defined by the blowing up of p_i and E_{12} the class of the strict transform of the line p_1p_2 in $P_{p_1p_2}$, then $([E_{12}] - [E_3], [E_3] - [E_4], \dots, [E_{r+1}] - [E_{r+2}])$ is a basis for the orthogonal complement of the image of $H_2(D) \to H_2(S)$. Its intersection graph is of type A_r. If the $p_3, \dots p_{r+2}$ are pairwise distinct and none of them is on the line p_1p_2, then D is ample and the pair (S, ω) is as desired. This is also true in the degenerate cases, provided we blow down the (-2) curves that then arise.

There is obvious notion of a marking ϕ of (S, D) relative the Manin lattice $\Lambda_{1,r+2}$ with a triple $\mathbf{d} = (d_1, d_2, d_3)$ of distinguished elements $d_1 := \ell - e_3 - \cdots - e_{r+2}$, $d_2 := \ell - e_1$, $d_3 := \ell - e_2$; we then require that $\phi(d_i) = [D_i]$. If we denote by $Q_{\mathbf{d}}$ the orthogonal complement of $\mathbb{Z}d_1 + \mathbb{Z}d_2 + \mathbb{Z}d_3$ in $\Lambda_{1,r+2}$, then

$$(\alpha_0 = \ell - e_1 - e_2 - e_3, \alpha_3 = e_3 - e_4, \dots, \alpha_{r+1} := e_{r+1} - e_{r+2})$$

is a basis of $Q_{\mathbf{d}}$ consisting of roots. It generates a root system of type A_r whose Weyl group $W_{\mathbf{d}}$ is the subgroup of $O(\Lambda_{1,r+2})$ which fixes each d_i. If we put $V_{\mathbf{d}} := \mathrm{Hom}(Q_{\mathbf{d}}, \mathbb{C})$, then in a fashion similar to the cuspidal case, we find:

Theorem 4.1. *We have a universal family for the pairs (S, ω) as above over $V_{\mathbf{d}}/W_{\mathbf{d}}$:*

$$(f : (\mathcal{S}, \mathcal{D}) \to V_{\mathbf{d}}/W_{\mathbf{d}}, \omega).$$

It comes with a $\mathbb{C}^\times$-action such that over the fixed point $o \in V_{\mathbf{d}}/W_{\mathbf{d}}$ we have the pair (S_o, ω_o) above. Moreover, the restriction $f : \mathcal{S} - \mathcal{D} \to V_{\mathbf{d}}/W_{\mathbf{d}}$ is affine and may serve as a model for the semi-universal deformation of the Kleinian singularity of type A_r in the central fiber.

As we have seen the pull-back of f over $V^\circ_{\mathbf{d}}$ extends to $\hat{\mathcal{S}} \to V_{\mathbf{d}}$ with fibers blown up projective spaces. This family is evidently topologically trivial and this is still the case is we restrict if we restrict to of $\hat{\mathcal{S}} - \hat{\mathcal{D}}$. Thus we find that the pull-back of $f : \mathcal{S} - \mathcal{D} \to V_{\mathbf{d}}/W_{\mathbf{d}}$ over $V^\circ_{\mathbf{d}}$ is topologically trivial. This implies that the monodromy representation (the action of the fundamental group of $V^\circ_{\mathbf{d}}/W_{\mathbf{d}}$ on H_2 of a fiber) factors through $W_{\mathbf{d}}$ (which is here simply the permutation group of $r + 1$ items) and is in fact given by the $W_{\mathbf{d}}$ action on $Q_{\mathbf{d}}$.

The D_r**-case.** Here we take a projective plane P in which we fix a conic Q and a line L tangent to Q at some point $p \in Q$. We choose a meromorphic 2-form ω on P with divisor $-Q-L$. Choose $p_1 \in L-\{p\}$. Then blow up p_1 and successively points $p_2, \dots, p_{r+1}$ over $Q-\{p\}$ and obtain $S = P_{p_1 \dots p_r p_{r+1}}$. If we denote by D_0 resp. D_1 the strict transform in S of Q resp. L and identify ω with its pull-back to S, then we have $D_0 \cdot D_0 = 4-r$ and $D_1 \cdot D_1 = 0$ and $-D$ is also the divisor of ω on S.

If E_i denotes the divisor on S defined by the blowing up of p_i and E_{12} the class of the strict transform of the line p_1p_2 in S, then $([E_{12} - [E_3], [E_2] - [E_3], \dots, [E_{r-1}] - [E_r])$ is a basis for the orthogonal complement of the image of $H_2(D) \to H_2(S)$ whose intersection graph is of type D_r. If $p_2, \dots, p_{r+1}$ are pairwise distinct and no line through two of them contains p_1, then D is ample and the pair (S, ω) is as desired. This is also true in the degenerate cases, provided we blow down the (-2) curves that then arise.

A marking will be relative the Manin lattice $\Lambda_{1,r+1}$ with the distinguished pair $\mathbf{d} = (d_1, d_2)$ of elements $d_1 := 2\ell - e_1 - \cdots - e_r$ and $d_2 := \ell - e_{r+1}$. Then

$$(\alpha_0 := \ell - e_1 - e_2 - e_3, \alpha_2 = e_2 - e_3, \dots, \alpha_{r-1} := e_{r-1} - e_r)$$

is a basis of the orthogonal complement $Q_{\mathbf{d}}$ of $\mathbb{Z}d_1 + \mathbb{Z}d_2$ in $\Lambda_{1,r+1}$. This consists of roots which generate a root system of type D_r. Its Weyl group $W_{\mathbf{d}}$ can be identified with the group of elements of $O(\Lambda_{1,r+2})$ which fix each d_i and so the markings are simply transitively permuted by $W_{\mathbf{d}}$. If we put $V_{\mathbf{d}} := \mathrm{Hom}(Q_{\mathbf{d}}, \mathbb{C})$, then Theorem 4.1 and the subsequent discussion holds in this case as well with A_r replaced by D_r.

The E_r**-case.** That case was treated already.

The multiplicative analogue. The preceding constructions admit a multiplicative version in which D is its multiplicative analogue: it is obtained from a D as above by a small perturbation which preserves the irreducible components, but brings them in general position: so in the A_r case we let $\overline{D}_1, \overline{D}_2, \overline{D}_3$ be three distinct lines with no point in common and for the D_r series, L is not a tangent of Q (hence meets Q in two distinct points). We treated already the multiplicative E_r case for $r = 6, 7, 8$. Then we find an analogue of Theorem 2.5: put $T_{\mathbf{d}} := \mathrm{Hom}(Q_{\mathbf{d}}, \mathbb{C}^\times)$.

Theorem 4.2. *If we put* $T_{\mathbf{d}} := \mathrm{Hom}(Q_{\mathbf{d}}, \mathbb{C}^\times)$*, then the orbit space* $T^\circ_{\mathbf{d}}/\{\pm 1\}.W_{\mathbf{d}}$ *parametrizes pairs* (S, D) *with* S *a rational surface,* D *an anticanonical divisor as above.*

We can in fact, proceed as we did did for Del Pezzo case: fix the curve D of the above multiplicative type and a point in each connected component of D_{reg}, except for the one contained in D_1. Also fix a line bundle ν on D which has 3-degree $(1-r,0,0)$ in the A_r-case and bidegree $(0,4-r)$ in the D_r case. Then define a $Q_{\mathbf{d}} \otimes \mathbb{C}^\times$-torsor $\mathcal{T}^\circ_{\mathbf{d}}$ in $D^{r+2}_{1,reg)}$ resp. $D^{r+1}_{1,reg)}$ with $W_{\mathbf{d}}$-action in much the same way as we did for Del Pezzo case such that $\mathcal{T}^\circ_{\mathbf{d}}/W_r$ carries a family of such pairs which extends to a family with Kleinian singularities over $\mathcal{T}_{\mathbf{d}}/W_{\mathbf{d}}$.

The monodromy representation on the affine fibers is an action of the fundamental group of $\tilde{T}^\circ_{\mathbf{d}}/W_{\mathbf{d}}$ on $H_2(S-D)$. The latter is, just as we have seen in the cuspidal case, an extension

$$0 \to \mathbb{Z}\delta \to H_2(S-D) \xrightarrow{j} Q_{\mathbf{d}} \to 0.$$

A priori this action has its image in the subgroup of automorphisms of this extension that is a semi-direct product of $Q_{\mathbf{d}}$ (with $\nu \in Q_{\mathbf{d}}$ sending $a \in H_2(S-D)$ to $a + (j(a)\cdot\nu)\delta)$ and $W_{\mathbf{d}}$. (Note that $Q_{\mathbf{d}}$ is also the fundamental group of $\tilde{T}_{\mathbf{d}}$.) This inclusion turns out to be an identity. The group in question is an affine Weyl group.

The elliptic version for Del Pezzo surfaces. Let S be a Del Pezzo surface of degree d and $D \subset S$ a smooth anticanonical divisor. Then $D\cdot D = d$ and if S is obtained as an iterated blow up of a projective plane: $\pi : S \to P_{p_1p_2\cdots p_r}$, then $\pi(p_1),\dots,\pi(p_r)$ are pairwise distinct and π maps D isomorphically onto a cubic curve passing through these points. According to Lemma 2.1 we have an exact sequence

$$0 \to H_1(D) \to H_2(S-D) \to H_2(S) \xrightarrow{\cdot i_*[D]} \mathbb{Z} \to 0.$$

Via the geometric marking ϕ associated to the iterated blow up description, we then obtain the short exact sequence

$$0 \to H_1(D) \to H_2(S-D) \to Q_r \to 0.$$

Let us now fix a basis (δ_1,δ_2) for $H_1(D)$ which is oriented in the sense that if η is the unique differential on D with $\int_{\delta_1}\eta = 1$, then $\tau := \int_{\delta_2}\eta$ lies in the upper half plane H_+. The elliptic curve $E_\tau := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ can be identified with D up to translation. We prefer to think of E_τ as the group of translations of D.

There is also a unique meromorphic 2-form ω on S with divisor $-D$ and residue η. In a way similar to the cuspidal and nodal cases, the cohomology class $[\omega] : H_2(S-D) \to \mathbb{C}$ gives rise to homomorphism $\chi : Q_r \to E_\tau)$ that has the property that no root lies in its kernel. Notice that the normal bundle $\nu_{D/S}$ of D in S is of degree $d = 9-d$.