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Fermat varieties and the periods of some hypersurfaces

Eduard Looijenga

Abstract.

The variety of all smooth hypersurfaces of given degree and dimension has the Fermat hypersurface as a natural base point. In order to study the period map for such varieties, we first determine the integral polarized Hodge structure of the primitive cohomology of a Fermat hypersurface (as a module over the automorphism group of the hypersurface). We then focus on the degree 3 case and show that the period map for cubic fourfolds as analyzed by R. Laza and the author gives complete information about the period map for cubic hypersurfaces of lower dimension. In particular, we thus recover the results of Allcock-Carlson-Toledo on the cubic surface case.

§1. Introduction

For a smooth projective manifold X , its polarized Hodge structure is a powerful invariant, often so strong that its isomorphism type determines the isomorphism type of X . If X is given as a hypersurface in \mathbb{P}^n of degree d , say, then the interesting part of this polarized Hodge structure is the one on the primitive cohomology in degree $n-1$, but it is in general hard to characterize the structures that occur in this manner. A well-known trick is to look instead at cyclic coverings of \mathbb{P}^n of degree d branched along X : we then get another hypersurface $Y \subset \mathbb{P}^{n+1}$, still of degree d , but now of dimension n and equipped with an action of μ_d . The idea is to consider the polarized Hodge structure on the primitive part of $H^n(Y)$ with its μ_d -action instead (or pass to the substructure that remains if we pass to an eigenspace of the μ_d -action for some primitive character of μ_d). This approach at least takes into account that X is not just any complex manifold, but is given as a hypersurface in a projective space. By iterating this procedure we find a hypersurface in \mathbb{P}^{n+k} of degree d with an action of μ_d^k , or rather, of its semi-direct product with S_k . Reversing this approach, we see that once we have a good understanding of the period map for degree d hypersurfaces in \mathbb{P}^N , then there is a fair chance that we can also attain this for degree d

hypersurfaces of lower dimension. Admittedly this method has its limitations, but we will show here that it works nicely for the first interesting value for d , $d = 3$, and $N = 5$, that is, the case of cubic fourfolds, for which Voisin in her thesis proved a Torelli theorem and for which Laza [9] and the author [12] independently recently characterized the image of the period map. The goal of this paper is to show how this leads to similar characterizations for cubic threefolds and cubic surfaces, thus recovering earlier work of Allcock-Carlson Toledo [2],[3] and Looijenga-Swierstra [14]. The top-down approach that we take here appears in Swierstra's thesis [17], but his presentation is somewhat different.

The paper is organized as follows. The contents of Section 2 might be summed up by saying that it gives a detailed description of the primitive homology lattice of a Fermat variety together with all the structure that is present: the intersection form, the Hodge decomposition and the action of the automorphism group of that variety on this structure. This was certainly known in some way over \mathbb{Q} , but for the purpose of this paper we need it over \mathbb{Z} and we have not been able to find this in the literature. It may thus have an independent interest. A theorem of F. Pham on the vanishing lattices of certain singularities proved to be quite useful here.

In Section 3 we determine what happens to the geometric invariant theory (and the markings) of homogeneous forms of degree d if we add to such a form the d th power of a new variable. This is useful to know if we want to compare the resulting period maps.

The last two sections deal specifically with cubic hypersurfaces and their period maps: as explained above, we start out with the result for cubic fourfolds and descend from that to the cubic threefolds and cubic surfaces. The main results are in §5.

I thank Igor Dolgachev for help with Proposition 3.5. The observation concerning (semi)stability is due to him.

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§2. Cohomology of the Fermat varieties

Throughout this section we fix an integer $d \geq 3$. Since we later take $d = 3$, we suppress d in the notation. We start out with the Fermat hypersurface in $Y_n \subset \mathbb{P}^{n+1}$ defined by the equation $F_n(Z) := \sum_{i=0}^{n+1} Z_i^d = 0$. If we identify Y_{n-1} with the hyperplane section of Y_n defined by $Z_0 = 0$, then $\dot{Y}_n := Y_n - Y_{n-1}$ is the affine Fermat variety defined by $\sum_{i=1}^{n+1} z_i^d = -1$. Let the group μ_d^{n+2} act diagonally on \mathbb{C}^{n+2} . The projectivized action (on \mathbb{P}^{n+1}) has as kernel the scalars, that is, the diagonally embedded $\mu_d \subset \mu_d^{n+2}$. Let us fix a generator $u \in \mu_d$ and denote the corresponding generator of the i -th summand in μ_d^{n+2} by u_i , $i = 0, \dots, n+1$. So μ_d^{n+2}/μ_d is the quotient of μ_d^{n+2} by the subgroup generated by $u_0 u_1 \cdots u_{n+1}$. Permutation of coordinates defines an action of the symmetric group \mathcal{S}_{n+2} on \mathbb{C}^{n+2} and forms with μ_d^{n+2} a semidirect product acting on the same space. The associated action of $\mathcal{S}_{n+2} \ltimes \mu_d^{n+2}/\mu_d$ on \mathbb{P}^{n+1} evidently preserves Y_n . (T. Shioda, who studied the arithmetical properties of these varieties in depth, determined among other things the full automorphism group of Y_n in all characteristics, at least when $(n, d) \neq (2, 4)$; he finds that in characteristic zero—the case we are dealing with—it is not bigger than $\mathcal{S}_{n+2} \ltimes \mu_d^{n+2}/\mu_d$, [16]). The $\mathcal{S}_{n+2} \ltimes \mu_d^{n+2}/\mu_d$ -stabilizer of the hyperplane $Z_0 = 0$ in \mathbb{P}^{n+1} is $\mathcal{S}_{n+1} \ltimes \mu_d^{n+1}$ (where μ_d^{n+1} is the group generated by u_1, \dots, u_{n+1}) and so that group acts on the affine Fermat variety \dot{Y}_n .

The primitive homology of a Fermat variety

The primitive homology of Y_n , $H_n^\circ(Y)$ is by definition the kernel of $H_n(Y) \rightarrow H_n(\mathbb{P}^{n+1})$. When n is odd, this is all of $H_n(Y)$. The intersection pairing $H_n(Y) \otimes H_n(Y) \rightarrow \mathbb{Z}$ is then antisymmetric and perfect by Poincaré duality. When n is even, this is a sublattice of $H_n(Y)$ of corank one: if $\kappa \in H^2(Y_n)$ denotes the hyperplane class, then $H_n^\circ(Y)$ is the kernel of the linear form on $H_n(Y)$ defined by $\kappa^{n/2}$. The intersection pairing $H_n(Y) \otimes H_n(Y) \rightarrow \mathbb{Z}$ is symmetric and perfect by Poincaré duality. The Poincaré dual of $\kappa^{n/2}$ is as a cycle represented by a linear section and has self-intersection d . The orthogonal complement of that cycle, $H_n^\circ(Y)$, has therefore a discriminant group cyclic of order d . The classical Lefschetz theory shows that this orthogonal complement is generated by vanishing cycles. These have self-intersection ± 2 and so $H_n^\circ(Y)$ is an even lattice.

We think of \dot{Y}_n as the Milnor fiber of the isolated hypersurface singularity defined by $\sum_{i=1}^{n+1} z_i^d = 0$. It is known to have the homotopy type of a bouquet of n -spheres.

We denote by $\kappa \in H^2(Y_n)$ the hyperplane class. We have the following exact (Gysin) sequence

$$0 \rightarrow H_{n+1}(Y_n) \rightarrow H_{n-1}(Y_{n-1}) \rightarrow \tilde{H}_n(\dot{Y}_n) \rightarrow H_n(Y_n) \rightarrow H_{n-2}(Y_{n-1}) \rightarrow 0,$$

where \tilde{H} stands for reduced (co)homology.

Assume first $n = 2m$ is even. Then $H_{n+1}(Y_n) = 0$ and $H_{n-2}(Y_{n-1})$ is Poincaré dual to κ^{m-1} . This means that the above sequence yields the exact sequence

$$0 \rightarrow H_{n-1}(Y_{n-1}) \rightarrow \tilde{H}_n(\dot{Y}_n) \rightarrow H_n^o(Y_n) \rightarrow 0,$$

where $H_n^o(Y_n)$ denotes the primitive homology of Y_n , i.e., the kernel of the linear form $\int \kappa^m : H_n(Y_n) \rightarrow \mathbb{Z}$. Dually, we have the exact sequence

$$0 \rightarrow H_o^n(Y_n) \rightarrow \tilde{H}^n(\dot{Y}_n) \rightarrow H^{n-1}(Y_{n-1}) \rightarrow 0,$$

where $H_o^n(Y_n) := H^n(Y)/\mathbb{Z}\kappa^m$. Poincaré duality identifies $H_n(Y_n)$ with $H^n(Y_n)$ and hence $H_n^o(Y_n)$ with $H_o^n(Y_n)$. The composite

$$j : \tilde{H}_n(\dot{Y}_n) \rightarrow H_n^o(Y_n) \cong H_o^n(Y_n) \rightarrow \tilde{H}^n(\dot{Y}_n)$$

is just the Poincaré duality map of $\tilde{H}_n(\dot{Y}_n)$. This shows in particular that $H_n^o(Y_n)$ with its intersection pairing can be identified with the Milnor lattice $\tilde{H}_n(\dot{Y}_n)$ modulo its radical. It is clear that this holds equivariantly relative to the $\mathcal{S}_{n+1} \ltimes \mu_d^{n+1}$ -action.

If n is odd, then the above discussion leads to an exact sequence $0 \rightarrow H_{n-1}^o(Y_{n-1}) \rightarrow H_n(\dot{Y}_n) \rightarrow H_n(Y_n) \rightarrow 0$ and so for all n we have the exact sequence

$$0 \rightarrow H_{n-1}^o(Y_{n-1}) \rightarrow \tilde{H}_n(\dot{Y}_n) \rightarrow H_n^o(Y_n) \rightarrow 0.$$

We have an intersection pairing on the last two nonzero terms and the map between them respects these pairings. Since the one on $H_n^o(Y_n)$ is nondegenerate, we see that the whole sequence can be reproduced from the intersection pairing on $\tilde{H}_n(\dot{Y}_n)$: $H_{n-1}^o(Y_{n-1})$ is its kernel and $H_n(\dot{Y}_n)$ is its maximal nondegenerate quotient.

The primitive Fermat lattice

Suppose L is an abelian group, $(a, b) \in L \times L \mapsto a \cdot b \in \mathbb{Z}$ a bilinear map and G a *finite abelian* group acting on L in such a manner as to preserve the pairing: $ga \cdot gb = a \cdot b$. We regard L as a $\mathbb{Z}G$ -module and extend the pairing sesquilinearly as

$$(a, b) \in L \times L \mapsto a \star b := \sum_{g \in G} (a \cdot gb)g \in \mathbb{Z}G.$$

(This means that the pairing is $\mathbb{Z}G$ -linear in the first variable and anti-linear in the second relative to the involution ‘bar’ in $\mathbb{Z}G$ defined by $g \in G \mapsto \bar{g} := g^{-1}$.) Notice that the sesquilinear form is *hermitian* ($b \star a = \overline{a \star b}$) if the given form is symmetric and is *antihermitian* ($b \star a = -\overline{a \star b}$) if it is antisymmetric.

F. Pham determined this sesquilinear extension for $L = H_n(\dot{Y}_n)$ and $G = \mu_d^{n+1}$.

Proposition 2.1 (Pham [15]). *The group $H_n(\dot{Y}_n)$ is as a $\mathbb{Z}\mu_d^{n+1}$ -module generated by a single element e_n that makes it as such isomorphic to the quotient ring*

$$R_{n+1} := \mathbb{Z}\mu_d^{n+1} / \left(\sum_{k=0}^{d-1} u_\nu^k \right)_{\nu=1, \dots, n+1}.$$

The sesquilinear extension of the intersection form is $(-1)^{n(n+1)/2} \star$, where \star is given by

$$e_n \star e_n = (1 - \overline{u_1 u_2 \cdots u_{n+1}}) \prod_{\nu=1}^{n+1} (1 - u_\nu) \in \mathbb{Z}\mu_d^{n+1}.$$

Furthermore, e_n transforms under permutation of z_1, \dots, z_{n+1} according to the sign character.

Proof. Pham actually considers the affine variety $\sum_{\nu=1}^{n+1} z_\nu^d = 1$ (rather than $= -1$), but a scalar multiplication by a root of unity of order $2d$ turns this into the statement above. Q.E.D.

We should perhaps also remark that multiplication by $(1 - u)$ identifies $\mathbb{Z}\mu_d / (\sum_{k=0}^{d-1} u^k)$ with the ideal $(1 - u)\mathbb{Z}\mu_d$. This shows that the above formula indeed defines a sesquilinear pairing.

In $\mathbb{Z}[\mu_d^{n+1}]$ the element $w := (1 - \overline{u_1 u_2 \cdots u_{n+1}}) \prod_{\nu=1}^{n+1} (1 - u_\nu)$ satisfies $\bar{w} = (-1)^n w$, which shows that the sesquilinear map in the preceding proposition is hermitian when n is even and antihermitian when n is odd (as it should, since the intersection pairing on $H_n(\dot{Y}_n)$ is in these cases symmetric resp. antisymmetric). Let us also notice that the \mathbb{Z} -rank of R_{n+1} is $(d-1)^{n+1}$.

Corollary 2.2. *The group $H_n^\circ(Y_n)$ is as a $\mathbb{Z}[\mu_d^{n+2}/\mu_d]$ -module generated by the image e'_n of e_n and thus becomes isomorphic to the quotient ring*

$$R'_{n+1} := \mathbb{Z}[\mu_d^{n+2}/\mu_d] / \left(\sum_{k=0}^{d-1} u_\nu^k, \nu = 0, \dots, n+1 \right)$$

(which also equals $R_{n+1}/(\sum_{k=0}^{d-1}(u_1 \cdots u_{n+1})^k)$ as a $\mathbb{Z}\mu_d^{n+1}$ -module). The generator e'_n transforms under permutation of the last $n+1$ variables Z_1, \dots, Z_{n+1} according to the sign character and is up to sign the only $\mathbb{Z}[\mu_d^{n+2}/\mu_d]$ -module generator with that property. The sesquilinear extension of the intersection form is $(-1)^{n(n+1)/2}\star$, where \star is given by

$$e'_n \star e'_n = \prod_{\nu=0}^{n+1} (1 - u_\nu) \in \mathbb{Z}[\mu_d^{n+2}/\mu_d].$$

Moreover, the kernel of the natural map $R_{n+1} \rightarrow R'_{n+1}$ is as a R_n -module generated $(1 - v_n) \sum_{0 \leq k \leq l \leq d-2} u_{n+1}^k v_n^l e_n$, where $v_n := u_1 \cdots u_n$; this kernel is a copy of R'_n .

Proof. The natural map $\mu_d^{n+1} \rightarrow \mu_d^{n+2}/\mu_d$ is an isomorphism of groups (we have $u_0 = (u_1 \cdots u_{n+1})^{-1}$ in μ_d^{n+2}/μ_d) and we have a corresponding ring isomorphism between their group rings. Via that identification we can write $e_n \star e_n = \prod_{\nu=0}^{n+1} (1 - u_\nu) \in \mathbb{Z}[\mu_d^{n+2}/\mu_d]$. It is immediate that this pairing factors through R'_{n+1} and it is easy to check that the resulting sesquilinear pairing is nondegenerate. The map $H_n(\dot{Y}_n) \rightarrow H_n^o(Y_n)$ has the same property (it forms the maximal nondegenerate quotient) and so via Pham's theorem we obtain an identification $R'_{n+1}e'_n \cong H_n^o(Y_n)$.

To prove the last assertion, put $v := u_1 \cdots u_{n+1} = v_n u_{n+1}$ and regard R'_{n+1} as a quotient of R_{n+1} by the ideal generated by $\sum_{k=0}^{d-1} v^k$. Since $(1 - v)(\sum_{k=0}^{d-1} v^k) = 0$, this ideal is also the R_n -submodule generated by that element. We then find that in R_{n+1}

$$\begin{aligned} \sum_{k=0}^{d-1} v^k &= \sum_{k=0}^{d-1} (v_n u_{n+1})^k = \sum_{k=0}^{d-2} (v_n u_{n+1})^k - v_n^{d-1} \sum_{k=0}^{d-2} u_{n+1}^k = \\ &= \sum_{k=0}^{d-2} u_{n+1}^k (v_n^k - v_n^{d-1}) = (1 - v_n) \sum_{0 \leq k \leq l \leq d-2} u_{n+1}^k v_n^l. \end{aligned}$$

Since R_{n+1} is as a R_n -module freely generated by $(u_{n+1}^k)_{k=0}^{d-2}$, the R_n -annihilator of this element is in fact the R_n -annihilator of $1 - v_n$, that is, $\sum_{k=0}^{d-1} v_n^k$, and so the R_n -submodule generated by it is isomorphic to $R_n/(1 - v_n)R_n = R'_n$. Q.E.D.

Remark 2.3. I do not know whether $H_n^o(Y_n)$ admits a $\mathbb{Z}\mu_d^{n+1}$ -module generator which transforms under the full permutation group of Z_0, \dots, Z_{n+1} according to the sign character.

Here is a concrete description for the bilinear form defined by \star . If $K := (k_0, \dots, k_{n+1}) \in (\mathbb{Z}/d)^{n+2}$ and $I \subset \{0, \dots, n+1\}$, then abbreviate $u^K := u_0^{k_0} \dots u_{n+1}^{k_{n+1}}$ and $u_I := \prod_{i \in I} u_i$. Now $(-1)^{n(n+1)/2} u^K \cdot u^L$ is the coefficient of 1 in the expression $u^{K-L}(1-u_0) \dots (1-u_{n+1})$, where the latter is viewed as an element of the group ring $\mathbb{Z}[\mu_d^{n+2}/\mu_d]$. We thus find

$$(-1)^{n(n+1)/2} u^K \cdot u^L = \begin{cases} (-1)^{|I|} & \text{if } u^K = u^L u_I \text{ for some nonempty } I, \\ (-1)^{|I|+n} & \text{if } u_I u^K = u^L \text{ for some nonempty } I, \\ 1 + (-1)^n & \text{if } u^K = u^L, \\ 0 & \text{otherwise.} \end{cases}$$

A resolution and a rank computation

We have $R'_0 = 0$ and so repeated application of the previous corollary yields a resolution of R'_{n+1} by R_1 -modules:

$$0 \rightarrow R_1 \rightarrow R_2 \rightarrow \dots \rightarrow R_{n+1} \rightarrow R'_{n+1} \rightarrow 0.$$

It is clear that R_ν is a free R_1 -module of rank $(d-1)^{\nu-1}$ (we have $R_\nu = R_1 \otimes_{\mathbb{Z}} R_{\nu-1}$ as R_1 -modules and $\text{rk}_{\mathbb{Z}} R_{\nu-1} = (d-1)^{\nu-1}$). So

$$\begin{aligned} \text{rk}_{R_1} H_n^\circ(Y_n) &= \text{rk}_{R_1} R'_{n+1} = \\ &= \sum_{\nu=1}^{n+1} (-1)^{n+1-\nu} (d-1)^{\nu-1} = \frac{(d-1)^{n+1} + (-1)^n}{d}. \end{aligned}$$

This also yields the familiar formula for $\text{rk}_{\mathbb{Z}} H_n^\circ(Y_n)$ (namely the product of the above number with $(d-1)$). For example, in the case that will mostly concern us, $d = 3$, we find for $n = 1, 2, 3, 4$ for R_1 -ranks the values 1, 3, 5, 11 respectively and for the \mathbb{Z} -ranks twice these numbers.

Let $\zeta_d := \exp(2\pi\sqrt{-1}/d)$ and consider $\mathbb{Z}[\zeta_d] \otimes_{R_1} R'_{n+1}$, where $\mathbb{Z}[\zeta_d]$ is a R_1 -module via $u_{n+1} \mapsto \zeta_d$. Then the bilinear form on R'_{n+1} defines a sesquilinear form $R'_{n+1} \times R'_{n+1} \rightarrow \mathbb{Z}[\zeta_d]$ by

$$(a \cdot b)_1 := \sum_{i \in (\mathbb{Z}/d)} (a \cdot u_{n+1}^i b) \zeta_d^i.$$

The $\mathbb{Z}[\zeta_d]$ -module $\mathbb{Z}[\zeta_d] \otimes_{R_1} R'_{n+1}$ is generated by the u^K , with $K = (k_0, \dots, k_n) \in (\mathbb{Z}/d)^{n+1}$. For these generators our formulae yield

$$(-1)^{n(n+1)/2} (u^K \cdot u^L)_1 = \begin{cases} (-1)^{|I|} (1 - \bar{\zeta}_d) & \text{if } u^K = u^L u_I \text{ for some } I \neq \emptyset, \\ (-1)^{|I|+n} (1 - \zeta_d) & \text{if } u_I u^K = u^L \text{ for some } I \neq \emptyset, \\ (1 - \bar{\zeta}_d) + (-1)^n (1 - \zeta_d) & \text{if } u^K = u^L, \\ 0 & \text{otherwise.} \end{cases}$$

The form is hermitian or skew-hermitian according to whether n is even or odd. In the last case we can make it hermitian by multiplication with the purely imaginary number $(\zeta_d + 1)(\zeta_d - 1)^{-1}$. This suggests to introduce the hermitian forms h_+ and h_- by

$$h_{\pm}(u^K, u^L) = \begin{cases} (-1)^{|I|}(1 \mp \bar{\zeta}_d) & \text{if } u^K = u^L u_I \text{ for some } I \neq \emptyset, \\ (-1)^{|I|}(1 \mp \zeta_d) & \text{if } u_I u^K = u^L \text{ for some } I \neq \emptyset, \\ (2 \mp \zeta_d \mp \bar{\zeta}_d) = (1 \mp \zeta_d)(1 \mp \bar{\zeta}_d) & \text{if } u^K = u^L, \\ 0 & \text{otherwise,} \end{cases}$$

where we recall that $K, L \in (\mathbb{Z}/d)^{n+1}$. If n is even we have $h_+ = \pm(\cdot)_1$ and for n odd, $h_- = \pm(\zeta_d + 1)(\zeta_d - 1)^{-1}(\cdot)_1$. (Strictly speaking it is not yet clear that h_+ resp. h_- is well-defined for n odd resp. n even, but the discussion below will show that this is so.)

An extension of this result will be of interest here. Denote by $\chi_k : R_k \rightarrow \mathbb{Z}[\zeta_d]$ the character that sends u_ν to ζ_d , $\nu = 1, \dots, k$ (we assume $k \geq 1$ here). This makes $\mathbb{Z}[\zeta_d]$ an R_k -module. For a given integer $m \geq 0$, we consider the exact sequence of R_k modules

$$R_k \rightarrow R_{k+1} \rightarrow \dots \rightarrow R_{k+m} \rightarrow R'_{k+m+1} \rightarrow 0$$

and tensor it with $\chi_k : R_k \rightarrow \mathbb{Z}[\zeta_d]$. Since $R_{k+l} = R_k \otimes_{\mathbb{Z}} R_l$ is an equality of R_k -modules, we get an partial resolution of $\mathbb{Z}[\zeta_d] \otimes_{R_k} R'_{k+m+1}$ by free $\mathbb{Z}[\zeta_d]$ -modules:

$$\mathbb{Z}[\zeta_d] \rightarrow \mathbb{Z}[\zeta_d] \otimes_{\mathbb{Z}} R_1 \rightarrow \dots \rightarrow \mathbb{Z}[\zeta_d] \otimes_{\mathbb{Z}} R_m \rightarrow \mathbb{Z}[\zeta_d] \otimes_{R_k} R'_{k+m+1} \rightarrow 0.$$

The first map is induced by multiplication by $(1 - v_k) \sum_{0 \leq j \leq l \leq d-2} u_{k+1}^j v_k^l$, where $v_k = u_1 \dots u_k$. Therefore it sends $1 \in \mathbb{Z}[\zeta_d]$ to the element $(1 - \zeta_d^k)(\sum_{0 \leq j \leq l \leq d-2} \zeta_d^{kl} u_{k+1}^j)$. If d does not divide k , then this map is injective, so that we have in fact a complete resolution. Notice that it is independent of k (for $k = 1$ we have the situation considered above). It follows that then the $\mathbb{Z}[\zeta_d]$ -rank of $\mathbb{Z}[\zeta_d] \otimes_{R_k} R'_{k+m+1}$ equals $d^{-1}((d-1)^{m+2} + (-1)^{m+1})$. We express this somewhat differently as follows.

Corollary 2.4. *Let the group $\mathcal{S}_k \ltimes \mu_d^k$ act on \mathbb{P}^{m+k+1} in the obvious manner on the last k coordinates (we prefer this over the first) and denote (also) by χ_k the character $\mathcal{S}_k \ltimes \mu_d^k \rightarrow \mathbb{Z}[\zeta_d]$ that sends every generator of μ_d^k to ζ_d . When $k \notin \mathbb{Z}d$, the isomorphism class of the hermitian $\mathbb{Z}[\zeta_d]$ -module $\mathbb{Z}[\zeta_d] \otimes_{\chi_k} H_{m+k}^{\circ}(Y_{m+k})$ is independent of k with its associated hermitian form up to sign corresponding to h_+ or h_-*

according to whether $m + k$ is even or odd. In particular,

$$\begin{aligned} \mathrm{rk}_{\mathbb{Z}[\zeta_d]} \mathbb{Z}[\zeta_d] \otimes_{\chi_k} H_{m+k}^\circ(Y_{m+k}) &= \\ &= d^{-1}((d-1)^{m+2} + (-1)^{m+1}) = \mathrm{rk}_{\mathbb{Z}} H_m^\circ(Y_m) + (-1)^{m+1}. \end{aligned}$$

The verification of the type of the hermitian form is straightforward; the rest follows from the preceding discussion.

Characters and the Griffiths-De Rham description

We determine the set of characters of R'_{n+1} . If $\chi : R'_{n+1} \rightarrow \mathbb{C}$ is a ring homomorphism, then the fact that $\sum_{k=0}^{d-1} u_i^k = 0$ implies that $\chi(u_i) = \zeta_d^{k_i}$ for some nonzero $k_i \in \mathbb{Z}/d$ and so $\chi(u_0 u_1 \cdots u_{n+1}) = \zeta_d^{(\sum_i k_i)}$. The latter must be 1, and so we will have $\sum_i k_i = 0$ in \mathbb{Z}/d . Conversely, any tuple $K = (k_0, \dots, k_{n+1}) \in (\mathbb{Z}/d - \{0\})^{n+2}$ with zero sum gives via the above prescription a character χ^K of R'_{n+1} . These characters define a basis of $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{E}'_{n+1}, \mathbb{C})$ and yield the eigenline decomposition of that space under the action of the group μ_d^{n+2} . This decomposition must be orthogonal with respect to the sesquilinear form on R'_{n+1} .

Since we have an identification $\mathrm{Hom}_{\mathbb{Z}}(R'_{n+1}, \mathbb{C}) \cong H_o^n(Y_n; \mathbb{C})$, we can compare the above character basis with one obtained by means of Griffiths' De Rham description [6] of $H_o^n(Y_n; \mathbb{C})$: a basis is here given by certain rational differential forms indexed by the tuples $L = (\ell_0, \dots, \ell_{n+1})$ with $\ell_i \in \{1, \dots, d-1\}$ and with the property that d divides $\sum_{i=0}^{n+1} \ell_i$:

$$\omega_L := \mathrm{Res}_{Y_n} \mathrm{Res}_{\mathbb{P}^{n+1}} \left(\prod_{i=0}^{n+1} Z_i^{\ell_i-1} \right) \frac{dZ_0 \wedge \cdots \wedge dZ_{n+1}}{F_n^{q+1}},$$

where q is such that the expression is homogeneous of degree zero, that is, given by $q+1 = (\sum_i \ell_i)/d$. Moreover, the Griffiths theory tells us that ω_L lies in $F^{n-q} H_o^n(Y_n; \mathbb{C})$, but not in $F^{n-q+1} H_o^n(Y_n; \mathbb{C})$. For example, if $d=3$ and $n \equiv 1 \pmod{3}$, then $\omega_{(1^{n+1})}$ spans the bidegree $(n - [n/3], [n/3])$ -part of $H_o^4(Y_4)$.

It is clear that ω_L is an eigenform for the μ_d^{n+1} -action: u_i acts on it as multiplication by $\zeta^{-\ell_i}$ and so the corresponding character is χ^{-L} . Since the μ_d^{n+1} -action preserves the Hodge structure, it follows that the character basis is compatible with that structure: a small computation shows that if we represent K by an integer sequence in $\{1, \dots, d-1\}$ and denote by $|K|$ its sum, then χ^K has Hodge type $(p, n-p)$, where $p = -1 + (n+2 + |K|)/d$.

§3. Hypersurfaces with partial Fermat symmetry

The notion of a k -marking

The action of $\mathcal{S}_{n+1} \ltimes \mu_d^{n+1}$ on \mathbb{P}^{n+1} as above lifts to \mathbb{C}^{n+2} by letting the first coordinate unchanged. For $k \leq n+1$ we regard $\mathcal{S}_k \ltimes \mu_d^k$ as the subgroup of $\mathcal{S}_{n+1} \ltimes \mu_d^{n+1}$ that acts on the last k coordinates only and we denote this image of $\mathcal{S}_k \ltimes \mu_d^k$ in $\mathrm{GL}(n+2)$ by G_k . Recall that $\chi_k : G_k \rightarrow \mathbb{C}^\times$ is the character that sends each u_i to ζ_d . We also use that notation for its restriction to μ^k .

Lemma 3.1. *Any polynomial in Z_0, \dots, Z_{n+1} homogeneous degree of d and fixed by G_k is a sum of a degree d polynomial in Z_0, \dots, Z_{n+1-k} and a scalar multiple of the Fermat form $F_k := Z_{n+2-k}^d + \dots + Z_{n+1}^d$.*

The easy proof is left to the reader. We will refer to such a polynomial as a *partial Fermat form of order k* . Since such forms are parameterized by a connected variety, we have:

Lemma 3.2. *Let $Y \subset \mathbb{P}^{n+1}$ be a smooth n -fold defined by partial Fermat form of degree d and of order k . Then there is a G_k -equivariant isomorphism $H_n(Y) \cong H_n(Y_n)$ which respects the intersection pairing, and (for even n) the class of an $n/2$ -dimensional linear section.*

This suggests the following definition. First, let us agree to call an *abstract hypersurface of degree d* a variety with a complete linear system that maps that variety isomorphically onto a hypersurface of degree d (in other words, it comes with an isomorphism onto such a hypersurface, given up to projective equivalence).

Definition 3.3. Let Y be a smooth abstract hypersurface of degree d and dimension m . A k -marking of Y is given by a pair $(i : Y \hookrightarrow \hat{Y}, \phi)$, where \hat{Y} is smooth abstract hypersurface of degree d and dimension $m+k$ together with a G_k -action of the above type, i is an isomorphism of Y onto the G_k -fixed point set in \hat{Y} (as projective manifolds) and ϕ is a G_k -equivariant lattice isomorphism of $H_{m+k+1}(\hat{Y})$ onto $H_{m+k+1}(Y_{m+k})$ which, in case $m+k$ is even, takes the $(m+k)/2$ -th power of the hyperplane class of \hat{Y} to that of Y_{m+k} . We regard two such markings $(i : Y \hookrightarrow \hat{Y}, \phi)$ and $(i' : Y \hookrightarrow \hat{Y}', \phi')$ as equal if there exists a G_k -equivariant isomorphism $h : \hat{Y} \cong \hat{Y}'$ such that $i' = hi$ and $\phi'g = \phi h^*$ for some $g \in G_k$.

Notice that a k -marking induces a unitary isomorphism of the following $\mathbb{Z}[\zeta] \otimes_{G_k} H_{m+k+1}^o(\hat{Y})$ onto $\mathbb{Z}[\zeta] \otimes_{\chi_k} R'_{m+k+1}$.

Remark 3.4. The group Γ of transformations of $H_{m+k}(Y_{m+k})$ that respect the intersection product and preserve the relevant class of a

linear section is an arithmetic group of orthogonal or symplectic type (depending on the parity of n) which contains the image of G_k . The G_k -centralizer in Γ , Γ_k , is also arithmetic and permutes the k -markings of Y transitively. It is easy to see that G_k acts faithfully on R'_{m+k} . Hence it does so on $H_{m+k}(Y_{m+k})$. So $\mu_d^k \subset G_k$ may be regarded a (central) subgroup of Γ_k and the latter acts on the markings via the quotient Γ_k/μ_d^k .

A period map is defined by assigning to Y the primitive Hodge structure of \hat{Y} with its G_k -action. A k -marking of Y makes this correspond to a G_k -invariant Hodge structure on $\text{Hom}_{\mathbb{Z}}(R'_{n+1}, \mathbb{C})$. In our examples the most interesting part of that structure is its restriction to the $\bar{\chi}_k$ -character space. That is why we payed special attention to $\mathbb{Z}[\zeta_d] \otimes_{\chi_k} R'_{n+1}$.

Moduli spaces of partial Fermat varieties

If S_{n+2}^d is an abbreviation for the $\text{SL}(n+2)$ -representation on the space degree d polynomials in Z_0, \dots, Z_{n+1} , then Lemma 3.1 asserts that

$$(S_{n+2}^d)^{G_k} = S_{n+2-k}^d \oplus \mathbb{C} \cdot F_k.$$

Notice that the G_k -centralizer of $\text{SL}(n+2)$, $\text{SL}(n+2)^{G_k}$, can be identified with $\text{SL}(n+2-k) \times \mathbb{C}^\times$, where the second factor is the space of diagonal matrices of the form $(\lambda I_{n+2-k}, \mu I_k)$ with determinant $\lambda^{n+2-k} \mu^k = 1$. Since this factor affects the coefficient of F_k , it follows that we get a map of orbit categorical spaces $\text{SL}(n+2-k) \backslash S_{n+2-k}^d \rightarrow \text{SL}(n+2) \backslash S_{n+2}^d$ whose image is also the image of $\text{SL}(n+2)^{G_k}$ (it takes the origin of S_{n+2-k}^d to the orbit, or rather its closure, of F_k). It is the morphism of (affine) quasicones defined by $\mathbb{C}[S_{n+2}^d]^{\text{SL}(n+2)} \rightarrow \mathbb{C}[S_{n+2-k}^d]^{\text{SL}(n+2-k)}$.

We denote by $St_{n+2}^d \subset sSt_{n+2}^d \subset S_{n+2}^d$ the stable and the semistable loci. The former contains the locus Sm_{n+2}^d which parametrizes the smooth n -folds.

Proposition 3.5. *The morphism of quasicones*

$$\text{SL}(n+2-k) \backslash S_{n+2-k}^d \rightarrow \text{SL}(n+2) \backslash S_{n+2}^d$$

is finite and is a normalization of its image (equivalently, the image of $\mathbb{C}[S_{n+2}^d]^{\text{SL}(n+2)} \rightarrow \mathbb{C}[S_{n+2-k}^d]^{\text{SL}(n+2-k)}$ is an integral extension of $\mathbb{C}[S_{n+2-k}^d]^{\text{SL}(n+2-k)}$ in the latter's field of fractions). It sends stable (resp. semistable) orbits to stable (resp. semistable) orbits.

Proof. Observe that the morphism in question is generically injective and that its domain and range are normal. We prove that when

$k = 1$, it sends (semi)stable points to (semi)stable points; the proposition then follows with induction on k .

For this we invoke the well-known simplex criterion. Assign a monomial of degree d in m variables the vector in $\mathbb{Z}^m = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_m$ whose components are the exponents of the monomial. This makes the monomials in S_m^d correspond to the integral vectors in the simplex Δ_m spanned by de_1, \dots, de_m . Then an element of $F \in S_m^d$ is semistable resp. stable if and only if the convex hull of the integral vectors in Δ_m representing monomials appearing in F contains the barycenter $\frac{d}{m} \sum_i e_i$ of Δ_m resp. contains the barycenter in its interior. Since the convex set attached to $F + X_{m+1}^d$ is the cone (with vertex e_{m+1}) over the convex set attached to F , the assertion follows easily. Q.E.D.

This allows us to regard a projective equivalence class of stable $(n - k)$ -folds of degree d as a projective equivalence class of stable n -folds of degree d with a G_k -action of the type described above. More precisely, it associates to a $(n - k)$ -fold of degree d an n -fold of degree d that is given up to an action by $G_k \cdot \mathbb{C}^\times$.

Remark 3.6. By definition a k -marked m -fold of degree d defines a marked $(m + k)$ -fold of that same degree up to G_k -symmetry. This notion is useful if we happen to have a good understanding of the period map for the latter. In that case it is desirable to characterize via the period map the locus of k -marked m -folds in the moduli space of marked $(m + k)$ -folds, for instance as the locus where the Hodge structure has a G_k -symmetry. This is particularly so when the period map is a local isomorphism. If we assume n even, then this only happens for cubic fourfolds and quartic surfaces but it at least helps us to get hold of the period maps for cubic threefolds, cubic surfaces and quartic curves. The period map for quartic surfaces is essentially that of K3 surfaces of degree four and the derived period map for quartic curves was treated by Kondō [7] (see also Looijenga [11] and Artebani [4]). Given its share of attention, we shall here focus on the cubic case.

§4. A period map for a cubic hypersurface of dimension ≤ 4

We now take $n = 4, d = 3$. So G_k stands for the subgroup $\mathcal{S}_k \ltimes \mu_3^k \subset \mathrm{GL}(6)$ acting on the last k coordinates. We write \mathcal{E} for $R_1 = \mathbb{Z}[u]/(u^2 + u + 1)$ and sometimes refer to its elements as *Eisenstein integers* (it is the ring of integers in the corresponding quadratic extension of \mathbb{Q}). We also write Λ for the lattice $H_4(Y_4)$ with its action of G_5 .

The lattice Λ is unimodular, odd and of signature $(21, 2)$. The square of the hyperplane class, denoted $\eta \in \Lambda$, has the property that $\eta \cdot \eta = 3$ and that its orthogonal complement, denoted $\Lambda_o \subset \Lambda$, is even (the self-product of every vector is even). Moreover, $\mathbb{Z}\eta$ is the fixed point sublattice of the action of G_5 . We put $V := \mathrm{Hom}(\Lambda_o, \mathbb{C})$ and notice that V has an underlying \mathbb{Q} -structure with inner product. The group of lattice automorphisms of Λ that fix η is denoted by Γ .

The period domains

If $Y \subset \mathbb{P}^5$ is any smooth cubic hypersurface, then the Hodge numbers in the middle dimension are $h^{3,1}(Y) = h^{1,3}(Y) = 1$ and $h^{2,2}(Y) = 21$. The Griffiths theory shows that a generator of $H^{3,1}(Y)$ is determined by an equation $F \in \mathbb{C}[Z_0, \dots, Z_5]$ for Y , namely

$$\mathrm{Res}_Y \mathrm{Res}_{\mathbb{P}^5} \frac{dZ_0 \wedge \dots \wedge dZ_5}{F(Z_0, \dots, Z_5)^2}.$$

We denote this class $\omega(F) \in H^4(Y; \mathbb{C})$. It satisfies the familiar equality $\omega(F) \cdot \omega(F) = 0$ and the inequality $\omega(F) \cdot \bar{\omega}(F) < 0$.

If $Y = Y_4$ and take for F its Fermat equation F_6 , then we see that $\omega(F_6)$ transforms under G_5 as its numerator $dZ_0 \wedge \dots \wedge dZ_5$: it is invariant under \mathcal{S}_5 and u_i acts as multiplication by ζ_3 . We denote this character χ and its restriction to G_k by χ_k . We further denote by V_k the χ_k -eigenspace in V .

Lemma 4.1. *The χ_k -eigenspace V_k is for $k = 1, 2, 3$ a subspace on which the Hermitian form defined by $h(\omega_1, \omega_2) := \omega_1 \cdot \bar{\omega}_2$ is nondegenerate and of hyperbolic signature $(10, 1)$, $(4, 1)$, $(1, 1)$ respectively.*

Proof. It is clear that any linear map $f : H_4^o(Y_4) \cong \mathcal{E}'_5 \rightarrow \mathbb{C}$ which transforms according to the character χ is $f(1)$ times the ring homomorphism defined by $u_i \mapsto \zeta_3$. So the χ -eigenspace of G_5 in $V = H_4^o(Y_4; \mathbb{C})$ is one-dimensional (and hence spanned by $\omega(F_6)$). The $\bar{\chi}$ -eigenspace is then spanned by $\bar{\omega}(F_6)$. We know from Hodge theory that $\omega(F_6)$ and $\bar{\omega}(F_6)$ span a maximal negative definite subspace of V for h . This implies that for $k \leq 3$, V_k is nondegenerate and of hyperbolic signature. It remains to compute the dimension of V_k . This is straightforward. Q.E.D.

We introduce the period domain

$$\Omega := \{\omega \in V : \omega \cdot \omega = 0, \omega \cdot \bar{\omega} < 0\}$$

and put $\Omega_k := \Omega \cap V_k$. The following lemma is a formal consequence of Lemma 4.1 and we therefore omit its proof.

Lemma 4.2. *The projectivization $\mathbb{P}\Omega \subset \mathbb{P}(V)$ has two connected components that are interchanged by complex conjugation; each of these is a 20-dimensional symmetric domain for the orthogonal group $O(\Lambda_o \otimes \mathbb{R})$. The group Γ is arithmetic in $O(\Lambda_o \otimes \mathbb{R})$ so that $\Gamma \backslash \Omega$ has the structure of a Shimura variety.*

For $k = 1, 2, 3$, $\mathbb{P}\Omega_k$ is a complex ball of dimension 10, 4, 1 respectively and a symmetric domain for the special hyperbolic unitary group $SU(V_k)$. The centralizer of G_k in Γ (which we shall denote by Γ_k) is arithmetic in the centralizer of $\mathcal{S}_k \ltimes \mu_3^k$ in $O(\Lambda_o \otimes \mathbb{R})$ and acts as such (hence properly discontinuously) in $\mathbb{P}\Omega_k$ so that $\Gamma_k \backslash \mathbb{P}\Omega_k$ has the structure of a Shimura variety.

The definition of the period map

We first characterize a partial Fermat cubic's symmetry in cohomological terms.

Proposition 4.3. *Let Y be a cubic fourfold endowed with an action of $\mathcal{S}_k \ltimes \mu_3^k$ such that the fixed point set of $u_1 \in \mu_3^k \subset \mathcal{S}_k \ltimes \mu_3^k$ in $H_o^4(Z)$ is trivial. Then Y with its $\mathcal{S}_k \ltimes \mu_3^k$ -action is projectively equivalent to a partial Fermat cubic of order k (with $\mathcal{S}_k \ltimes \mu_3^k$ acting as G_k).*

Proof. Since $\text{Pic}(Y)$ is generated by the hyperplane class κ_Y , $\mathcal{S}_k \ltimes \mu_3^k$ acts projectively. We first show that u_1 fixes a hyperplane $H_1 \subset \mathbb{P}^5$ pointwise. To this end we compute its Lefschetz number. Our assumption implies that the characteristic polynomial of u_1 in $H_o^4(Y; \mathbb{C})$ is $11(T^2 + T + 1)$ so that its trace is -11 . Since the powers $1, \kappa_Z, \dots, \kappa_Z^4$ of the hyperplane class are fixed by u_i , it follows that the Lefschetz number of u_i equals $-11 + 5 = -6 < 0$. So by Lefschetz theorem, the Euler characteristic of the fixed point set of u_i is negative. An eigenspace of u_1 in \mathbb{P}^5 of dimension d meets Y in linear section Y_1 of dimension $\geq d - 1$. If $d \leq 3$, Y_1 is a finite set, a line, a cubic curve or a cubic surface. Since it is also smooth, it follows that Y_1 has positive Euler characteristic. Hence u_1 has an eigen space H_1 of dimension 4. Since u_i is conjugate to u_1 in $\mathcal{S}_k \ltimes \mu_3^k$, it fixes a hyperplane $H_i \subset \mathbb{P}^5$ pointwise.

The u_i pairwise commute and act differently in $H^4(Y)$, and so the hyperplanes H_i are linearly independent. They are permuted by \mathcal{S}_k via its tautological action on the index set. It then easily follows that we can

find coordinates $[W_0 : \cdots : W_5]$ for \mathbb{P}^5 such that \mathcal{S}_k acts as permutation group of the last k coordinates (here W_j has the index $6 - j$) and u_1 acts as $u_1[W_0 : \cdots : W_5] = [W_0 : \cdots : W_4 : \zeta_3 W_5]$. Q.E.D.

It follows from Proposition 4.3 that the h in the definition of a k -marking of a cubic $(4 - k)$ -fold will be unique up to a composition with an element of G_k . We also observe that these markings are permuted simply transitively by Γ_k/G_k . The pairs (F, ϕ) with $F \in Sm_{6-k}^3$ and ϕ a marking of the associated smooth cubic define an unramified Γ_k/G_k -covering $\widetilde{Sm}_{6-k}^3 \rightarrow Sm_{6-k}^3$.

The projective automorphism group of a smooth cubic 4-fold acts faithfully on its set of markings and so the marked smooth cubic $(4 - k)$ -folds in \mathbb{P}^{5-k} . The scalars $\mathbb{C}^\times \subset GL(6 - k)$ act on the first component via their third power. Hence the action of $GL(6 - k)$ on Sm_{6-k}^3 lift to one on \widetilde{Sm}_{6-k}^3 and the latter factors through a free action of $GL(6 - k)/\mu_3$.

If Y is given in \mathbb{P}^5 by an equation F , then a marking ϕ of Y defines an element Ω , so that we have a map

$$\widetilde{\Psi} : SL(6 - k) \backslash \widetilde{Sm}_{6-k}^3 \rightarrow G_k \backslash \Omega_k.$$

Here is $\Omega_0 = \Omega$ and $G_0 = 1$, whereas for $k > 0$, G_k acts on Ω_k via χ_k . Without the marking we only get an element of $\Gamma \backslash \Omega$: we have a map

$$\Psi_k : SL(6 - k) \backslash Sm_{6-k}^3 \rightarrow \Gamma_k \backslash \Omega_k.$$

Domain and range of Ψ_k come with a \mathbb{C}^\times -action: we let $\lambda \in \mathbb{C}^\times$ act on P_{6-k} by scalar multiplication and on V by inverse scalar multiplication (which by duality corresponds to ordinary scalar multiplication on the homology $H_4(Y_o; \mathbb{C})$). These determine proper \mathbb{C}^\times -actions in the orbit space $SL(6 - k) \backslash \widetilde{Sm}_{6-k}^3$ and in $\Gamma_k \backslash \Omega_k$. These actions are not always faithful, so let us refer to the action of the image groups (which are copies of \mathbb{C}^\times) as the *reduced \mathbb{C}^\times -action*.

Proposition 4.4. *The period map Ψ_k is equivariant for the reduced \mathbb{C}^\times -actions on domain and range.*

Proof. Let us do the case $k = 0$ first. If $\lambda \in \mathbb{C}^\times$ and $G \in Sm_6^3$, then $\Psi(\lambda G) = \lambda^2 \Psi(G)$, where we bear in mind that \mathbb{C}^\times acts on Ω by inverse scalar multiplication. So Ψ has degree 2. But $SL(6)$ meets the scalars in the 6th roots of unity and these act in S_6^3 through ± 1 . On the other hand Γ does not contain -1 , for that would mean that minus the orthogonal reflection in η preserves Λ and this is clearly not the case. So Ψ is equivariant for the reduced \mathbb{C}^\times -actions.

Let us now do the cases $k = 1, 2, 3$. If $F \in Sm_{6-k}^3$ and $F_k = Z_{7-k}^3 + \cdots + Z_6^3$, then $(\lambda^{-1}I_{6-k}, I_k) \in \mathrm{GL}(6)$ takes the cubic defined by $F + F_k$ to the cubic defined by $\lambda^3 F + F_k$ and this isomorphism takes $\omega(F + F_k)$ to $\lambda^{6-k}\omega(\lambda^3 F + F_k)$. So $\Psi_k(\lambda^3 F) = \lambda^{6-k}\Psi_k(F)$. This shows that Ψ_k is in an algebraic sense homogeneous of degree $(6-k)/3$. However, the center of $\mathrm{SL}(6-k)$, which consists of the $(6-k)$ th roots of unity, acts on S_{6-k}^3 through $\zeta \in \mu_3^{6-k} \mapsto \zeta^{-3} \in \mu_3^{6-k}$ (for S_{6-k}^3 is the third symmetric power of the dual of the tautological representation of $\mathrm{SL}(6-k)$). This means that when 3 is relatively prime to $6-k$, i.e., when $k = 1, 2$, we have a factorization through $\mu_3^{6-k} \backslash S_{6-k}^3$. On the other hand, the center of Γ_k is μ_3 . It follows that in all these cases, Ψ_k is equivariant for the reduced \mathbb{C}^\times -actions. Q.E.D.

The more customary form of the period map for cubic 4-folds is the projectivization of Ψ ,

$$\mathbb{P}\Psi : \mathrm{SL}(6) \backslash \mathbb{P}Sm_6^3 \rightarrow \Gamma \backslash \mathbb{P}\Omega.$$

A deep theorem of Claire Voisin asserts that this map is an open embedding (another proof is given in [12]). The openness, which amounts to a local Torelli theorem, is easy and was known before; the hard part is the injectivity. The last property is more directly stated as follows:

Theorem 4.5 (Voisin [18]). *Let Y and Y' be smooth cubic 4-folds. Then any lattice isomorphism $H^4(Y') \cong H^4(Y)$ that sends hyperplane class to hyperplane class and respects the Hodge decomposition (briefly, a Hodge isometry), is induced by a unique projective isomorphism $Y \cong Y'$.*

If we combine this with Proposition 4.4, then it follows that Ψ itself is also an open embedding. According to 3.5, Ψ_k is a restriction of Ψ and so the same is true for Ψ_k . Our reason for considering this slight refinement of the classical period map is that it maps automorphic forms on $\mathbb{P}\Omega_k$ to $\mathbb{C}[P_{6-k}]^{\mathrm{SL}(n-k)}$. For a Γ_k -automorphic form on Ω_k of weight d is simply a Γ_k -invariant analytic function $f : \Omega_k \rightarrow \mathbb{C}$ that is homogenous of degree d relative to the \mathbb{C}^\times -action inverse to scalar multiplication (so $-d$ relative to scalar multiplication) that is subject to certain growth conditions (which are in fact empty unless $k = 3$). It is clear that Ψ_k takes such a function to one on Sm_{6-k}^3 which is complex-analytic, homogeneous and $\mathrm{SL}(6-k)$ -invariant. We will see that the latter is always the restriction of a $\mathrm{SL}(6-k)$ -invariant homogeneous polynomial on S_{6-k}^3 .

§5. The fundamental arithmetic arrangement

We recall from [12] the following notion.

Definition 5.1. A vector $v \in \Lambda_o$ is called *special* if $v \cdot v = 6$ and $v - \eta$ is divisible by 3 and is called *nodal* if $v \cdot v = 2$. A hyperplane in V is called *special* resp. *nodal* if it is perpendicular to such a vector.

A simple argument in lattice theory shows that the 3-divisibility of $v - \eta$ can be expressed in terms of Λ_o only: it is equivalent to requiring that the inner product of v with any element of Λ_o lies in $3\mathbb{Z}$. Notice that if $-v + \eta = 3e$, then $e \cdot e = e \cdot \eta = 1$.

We denote the collection of special resp. nodal hyperplanes by \mathcal{H}_s resp. \mathcal{H}_n . It is shown in [12] (the proof is not difficult) that each of these is a single Γ -orbit. It is clear that every such hyperplane has signature $(19, 2)$ and hence will meet Ω . The union of these two collections of hyperplanes is locally finite in Ω . So the complement of the union of these hyperplanes in Ω is an open subset $\Omega^\circ \subset \Omega$. The following theorem is independently due to Laza [8], [9] and the author [12].

Theorem 5.2. *The map $\tilde{\Psi}$ maps $\mathrm{SL}(6) \backslash \widetilde{Sm}_6^3$ isomorphically onto the arrangement complement Ω° , and hence Ψ maps $\mathrm{SL}(6) \backslash Sm_6^3$ isomorphically onto $\Gamma \backslash \Omega^\circ$.*

This immediately leads to corresponding results for cubics of lower dimension. The case of cubic surfaces is due to Allcock-Carlson-Toledo [2], and the case of cubic threefolds was obtained independently by Allcock-Carlson-Toledo [3] and Looijenga-Swierstra [13] (the case of cubic curves is of course classical). The uniform approach that we take here was developed essentially (although somewhat differently) in the thesis of Swierstra [17] and provides among other things a new proof for the case of cubic surfaces.

Let us write Ω_k° for $\Omega^\circ \cap V_k$.

Corollary 5.3. *For $k \geq 1$, the period map $\tilde{\Psi}_k$ defines an isomorphism of $\mathrm{SL}(6-k) \backslash \widetilde{Sm}_{6-k}^3$ onto $\mu_3 \backslash \Omega_k^\circ$ and (hence) drops to an isomorphism Ψ_k of $\mathrm{SL}(6-k) \backslash Sm_{6-k}^3$ onto $\Gamma_k \backslash \Omega_k^\circ$.*

Proof. It is clear that $\tilde{\Psi}_k$ takes values in Ω_k° .

If the marked the cubic forms $F, F' \in Sm_{6-k}^3$ have the same image in Ω_k° , then by Voisin's Torelli theorem there is a $g \in \mathrm{SL}(6)$ that takes $F + F_k$ to $F' + F_k$ such that the induced isomorphism $Y(F + F_k) \cong Y(F' + F_k)$ is compatible with the markings. The latter is in particular equivariant for the actions of G_k on the cohomology of these 4-folds. Since the projective automorphism group of a smooth cubic fourfold is

faithfully represented on its cohomology, it follows that the isomorphism in question is already G_k -equivariant. This means that the marked cubic forms lie in the same $\mathrm{SL}(6-k)$ -orbit.

As for surjectivity, if $\omega \in \Omega_k^\circ$, then by Theorem 5.2, ω is represented by a marked cubic form (F, ϕ) . Since ω transforms via G_k under the character χ_k , it follows that F is G_k -invariant. Hence the last k variables of F separate as multiple of F_k . So (F, ϕ) naturally underlies a marked cubic $(4-k)$ -fold. Q.E.D.

Denote by $Sf_6^3 \subset S_6^3$ the space of cubic forms with isolated singularities with finite local monodromy groups. According to Arnol'd the singularities with that property are analytically isomorphic to singularities with a local equation of the form $f(z_1, z_2, z_3) + z_4^2 + z_5^2$, where f defines a Kleinian singularity. They are indexed by the Weyl group labels A_* , D_* , E_* , as these describe the local monodromy groups. It is known that the period map extends to $Sf_6^3 \subset S_6^3$. According to Yokoyama [19] and Laza [8] such a cubic form is stable: $Sf_6^3 \subset St_6^3$.

In the cited references Laza and the author also show:

Theorem 5.4. *Denote by Ω^f the complement in Ω of the union of the hyperplanes in \mathcal{H}_s . Then the period map Ψ extends to an isomorphism from $\mathrm{SL}(6) \backslash Sf_6^3$ onto $\Gamma \backslash \Omega^f$.*

We use the theory developed in [10] to refine and generalize the above theorem.

Write Ω_k^f for $\Omega^f \cap \Omega_k$. It is a priori clear that the period map induces a map $\Psi_k : \mathrm{SL}(6-k) \backslash Sf_{6-k}^3 \rightarrow \Gamma_k \backslash \Omega_k^f$. This implies that Ω_k is not contained in a special hyperplane. Denote by $A_d(\Omega_k^f)^{\Gamma_k}$ the space of Γ_k -automorphic forms of weight d on $\mathbb{P}\Omega_k^f$, that is, the space of Γ_k -invariant complex-meromorphic functions on $\mathbb{P}\Omega_k$ of weight d that are regular on $\mathbb{P}\Omega_k^f$, subject to a growth condition when $k = 3$ (in which case $\mathbb{P}\Omega_3^f = \mathbb{P}\Omega_3$ is the unit disk). Here d may be any integer. The direct sum of these spaces forms a graded \mathbb{C} -algebra $A_\bullet(\Omega_k^f)^{\Gamma_k}$. In what follows we exclude the somewhat special classical case $k = 3$ (for which the period map is one between one dimensional varieties).

It is an easy fact, and proved in [12], that if a positive semidefinite sublattice of Λ_o is spanned by special vectors, then its rank is ≤ 2 . Together with the inequality $\dim \Omega_k > 5$ this implies, according to [10], the following lemma (which for $k = 0$ is in fact a step in our proof of Theorem 5.4).

Lemma 5.5. *For $k = 0, 1, 2$ the \mathbb{C} -algebra $A_d(\Omega_k^f)^{\Gamma_k}$ is normal with finitely many generators in positive degree and the analytic map*

$$\Gamma_k \backslash \Omega_k^f \rightarrow \text{Spec}(A_d(\Omega_k^f)^{\Gamma_k})$$

is an open embedding with complement a subcone of codimension > 1 .

So $\Gamma_k \backslash \Omega_k^f$ has the structure of a quasi-affine variety with \mathbb{C}^\times -action and $A_d(\Omega_k^f)^{\Gamma_k}$ can be understood as its (graded) algebra of regular functions.

Corollary 5.6. *For $k = 0, 1, 2$ the period map defines an isomorphism $\Psi_k : Sf_{6-k}^3 \rightarrow \Gamma_k \backslash \Omega_k^f$ and identifies the \mathbb{C} -algebra $A_\bullet(\Omega_k^f)^{\Gamma_k}$ with the \mathbb{C} -algebra $\mathbb{C}[S_{6-k}^3]^{\text{SL}(6-k)}$, but rescales the degrees by a factor $(6-k)/3$.*

Proof. The map $\text{SL}(6-k) \backslash Sf_{6-k}^3 \rightarrow \Gamma_k \backslash \Omega_k^f$ is a birational, finite morphism between normal varieties. So it is an isomorphism. We have that the complement of $\text{SL}(6-k) \backslash Sf_{6-k}^3$ in $\text{SL}(6-k) \backslash S_{6-k}^3 = \text{Spec}(\mathbb{C}[S_{6-k}^3]^{\text{SL}(6-k)})$ is of codimension > 1 . Similarly, the complement of $\Gamma_k \backslash \Omega_k^f$ in $\text{Spec}(A_\bullet(\Omega_k^f)^{\Gamma_k})$ is of codimension > 1 . Hence $A_\bullet(\Omega_k^f)^{\Gamma_k}$ and $\mathbb{C}[S_{6-k}^3]^{\text{SL}(6-k)}$ can be characterized as the algebra of regular functions on these varieties and so the second statement also follows. The rescaling of the grading by the factor $(6-k)/3$ is explained by the following equality: if $F \in S_{6-k}^3$, then if we put $\tilde{z}_i := t^{1/3} z_i$ we have

$$\begin{aligned} \omega(tF + F_k) &= \frac{dz_0 \wedge \cdots \wedge dz_5}{(tF(z_0, \dots, z_{5-k}) + F_k(z_{6-k}, \dots, z_5))^2} = \\ t^{-(6-k)/3} \frac{dz_0 \wedge \cdots \wedge dz_{5-k} \wedge d\tilde{z}_{6-k} \wedge \cdots \wedge d\tilde{z}_5}{(F(z_0, \dots, z_{5-k}) + F_k(\tilde{z}_{6-k}, \dots, \tilde{z}_5))^2} &= t^{-(6-k)/3} \omega(F + F_k). \end{aligned}$$

Q.E.D.

The next proposition implies that in the case of cubic surfaces, the GIT compactification coincides with the Baily-Borel compactifications, a theorem that is due to Allcock-Carlson-Toledo [2]:

Proposition 5.7. *The domain Ω_2 does not meet a special hyperplane so that we have an isomorphism $\Psi_2 : Sf_4^3 \rightarrow \Gamma_2 \backslash \Omega_2$ and identifies the \mathbb{C} -algebra $A_\bullet(\Omega_2)^{\Gamma_2}$ with the \mathbb{C} -algebra $\mathbb{C}[S_4^3]^{\text{SL}(4)}$.*

Sketch of proof. There is only one strictly semistable closed orbit of cubic surfaces, namely the one with three A_2 singularities, represented by $Z_0^3 - Z_1 Z_2 Z_3 = 0$. It corresponds to a cubic fourfold with three singularities of type \tilde{E}_6 . The period map goes near these singularities

to the Baily-Borel boundary. This means that we actually get a proper map from $\mathrm{SL}(4) \backslash Sf_4^3$ to $G_k \backslash \Omega_2$. This can only happen if $\Omega_2 = \Omega^f$. In other words, Ω_2 does not meet a special hyperplane. Q.E.D.

Remark 5.8. In view of Corollary 5.6, the preceding proposition is in fact a statement about lattices: it says that there is no special vector $v \in \Lambda_o$ such that $u_4(v) = u_5(v)$ and the span v and $u_5(v)$ is a positive definite lattice of rank two (we recall that u_k is the $(k+1)$ th generator of μ_3 ⁶ and hence acts on Λ_o).

References

- [1] D. Allcock: *The moduli space of cubic threefolds*, J. Alg. Geom. 12 (2003), 201–223.
- [2] D. Allcock, J.A. Carlson, D. Toledo: *The complex hyperbolic geometry of the moduli space of cubic surfaces*, J. Algebraic Geom. 11 (2002), 659–724.
- [3] D. Allcock, J.A. Carlson, D. Toledo: *The moduli space of cubic threefolds as a ball quotient*, to appear in the Mem. Amer. Math. Soc. (available at [math.AG/0608287](#)).
- [4] M. Artebani: *A compactification of \mathcal{M}_3 via K3 surfaces*, available at [math.AG/0511031](#)
- [5] I. Dolgachev, B. van Geemen, S. Kondō: *A complex ball uniformization of the moduli space of cubic surfaces via periods of K3 surfaces*, J. Reine Angew. Math. 588 (2005), 99–148.
- [6] Ph. Griffiths: *On the periods of certain rational integrals. I, II*, Ann. of Math. (2) 90 (1969), 460–495; *ibid.* (2) 90 (1969) 496–541.
- [7] S. Kondō: *A complex hyperbolic structure for the moduli space of curves of genus three*, J. Reine Angew. Math. 525 (2000), 219–232.
- [8] R. Laza: *The moduli space of cubic fourfolds*, J. Algebraic Geom. 18 (2009), 511–545, available at [arXiv:0704.3256\[math.AG\]](#).
- [9] R. Laza: *The moduli space of cubic fourfolds via the period map*, to appear in the Ann. of Math., available at [arXiv:0705.0949\[math.AG\]](#).
- [10] E. Looijenga: *Compactifications defined by arrangements I, II*, Duke Math. J. 118 (2003), 151–187; *ibid.* 119 (2003), 527–588.
- [11] E. Looijenga: *Invariants of plane curves as automorphic forms*, Algebraic Geometry, 107–120, Contemp. Math. 422, Amer. Math. Soc.
- [12] E. Looijenga: *The period map for cubic fourfolds*, available at [arXiv:0705.0951\[math.AG\]](#).
- [13] E. Looijenga, R. Swierstra: *On period maps that are open embeddings*, J. Reine Angew. Math. 617 (2008), 169–192.
- [14] E. Looijenga, R. Swierstra: *The period map for cubic threefolds*, Compos. Math. 143 (2007), 1037–1049.
- [15] F. Pham: *Formules de Picard-Lefschetz généralisées et ramification des intégrales*, Bull. Soc. Math. France 93 (1965), 333–367.
- [16] T. Shioda: *Arithmetic and geometry of Fermat curves*, Algebraic Geometry Seminar (Singapore, 1987), 95–102, World Sci. Publishing, Singapore.
- [17] R. Swierstra: *Moduli spaces of cubic hypersurfaces through a period map*, Thesis (2008), vi+77 pp. Utrecht.
- [18] C. Voisin: *Théorème de Torelli pour les cubiques de \mathbb{P}^5* , Invent. Math. 86 (1986), 577–601. *Erratum*, *ibid.* 172 (2008), 455–458.
- [19] M. Yokoyama: *Stability of cubic hypersurfaces of dimension 3 and 4*, Preprint Nagoya University.

*Mathematisch Instituut
Universiteit Utrecht
P.O. Box 80.010, NL-3508 TA Utrecht
Nederland
E-mail address: E.J.N.Looijenga@uu.nl*