THE HITCHIN FIBRATION

Seminar talk based on part of Ngô Bao Châu's preprint: *Le lemme fondamental pour les algèbres de Lie* [2].

Here X is a smooth connected projective curve over a field k whose genus is denoted g. We also fix an invertible sheaf D over X. We shall eventually assume that $\deg D > 2g$ and that D is 2-divsible in $\mathrm{Pic}(X)$.

Whenever is at issue a reductive k-group, we shall assume that the characteristic of k does not divide the order of its Weyl group.

1. The BNR correspondence

Let be given $a = (a_i \in H^0(X, D^i))_{i=1}^r$. This defines a curve X_a in the total space $\mathrm{Tot}(D)$ of D as the set of $t \in \mathrm{Tot}(D)$ obeying $t^r - a_1 t^{r-1} + \cdots + (-1)^r a_r = 0$. The more precise way of giving X_a as defined by a principal ideal in $\mathcal{O}_{\mathrm{Tot}(D)}$ is as follows: let $u \in H^0(U, D^{-1})$ be a local generator of D^{-1} on an affine open subset $U \subset X$ so that the coordinate ring of $\mathrm{Tot}(D|U)$ is $\mathcal{O}(U)[u]$. If $\bar{a}_i \in \mathcal{O}(U)$ is the value of a_i on u^i, then $X_a \cap \mathrm{Tot}(D|U)$ is defined by the ideal generated by $u^r - \bar{a}_1 u^{r-1} + \cdots + (-1)^r \bar{a}_r$. We shall assume that X_a is integral (i.e., reduced and irreducible). We denote the projection $\mathrm{Tot}(D) \to D$ by π and its restriction to X_a by π_a. The latter has degree r.

Suppose L is a torsion free sheaf of rank one on X_a. Then $\pi_{a*}L$ is torsion free on X of rank r. Since X is smooth, this means that $\pi_{a*}L$ is locally free: it is a vector bundle of rank r. Its degree can be computed with the help of Riemann-Roch:

$$r\chi(X, \mathcal{O}_X) + \deg(\pi_{X_a *} L) = \chi(X, \pi_{X_a *}) = \chi(X_a, L) = \chi(X_a, \mathcal{O}_{X_a}) + \deg L$$

shows that $\deg(\pi_{X_a *} L) - \deg L$ is independent of L and hence (take $L = \mathcal{O}_{X_a}$) equal to $\deg(\pi_{X_a *} \mathcal{O}_{X_a})$. This vector bundle clearly comes with the structure of a module over the $\mathcal{O}_X$-algebra $\pi_{a*}\mathcal{O}_{X_a}$. There is however a bit more to say here. Let us first note that the bundle $\pi^* D$ over $\mathrm{Tot}(D)$ comes with a tautological section. Denote by $\tau_a \in H^0(X_a, \pi_a^* D)$ the restriction of this section to X_a. Then $\ell \in L \mapsto \tau_a \otimes \ell \in \pi_a^* D \otimes L$ is $\mathcal{O}_{X_a}$-homomorphism whose direct image under π_a yields a $\pi_{a*}\mathcal{O}_{X_a}$-homomorphism

$$\phi : \pi_{a*} L \to D \otimes \pi_{a*} L.$$

We can recover from ϕ the curve $X_a \subset \mathrm{Tot}(D)$ (or rather a general point of that curve) as its *spectral curve*: a local section σ of D (in the étale topology) defines a point of X_a precisely if it is an eigensection of ϕ in the sense that there exists a local section $\nu \neq 0$ of $\pi_{a*}L$ such that $\phi(\nu) = \sigma \otimes \nu$.

We can now state:

Theorem 1.1 (Beauville-Narasimhan-Ramanan [1] 1989)**.** *The map which assigns to a torsion free sheaf of rank one* L *on* X_a *the pair* $(\pi_{a*}L, \phi)$ *defines a bijection between the set of isomorphism classes of torsion free sheaves of rank one on* X_a *and the isomorphism classes of pairs* (V, ϕ)*, where* V *is a rank* r *vector bundle on* X *and* $\phi : V \to D \otimes V$ *is a* $\mathcal{O}_X$*-homomorphism for which* X_a *is its spectral curve.*

The set of isomorphism classes of torsion free rank one sheaves on X_a is parameterized by a scheme which contains the Picard variety $\mathrm{Pic}(X_a)$ of X_a (which parameterizes the isomorphism classes of invertible rank one sheaves on X_a) as an open subset. It is also dense (because X_a has only plane curve singularities) and therefore denoted by $\overline{\mathrm{Pic}}(X_a)$. The group structure on $\mathrm{Pic}(X_a)$ extends to an action of $\mathrm{Pic}(X_a)$ on $\overline{\mathrm{Pic}}(X_a)$. We transport this scheme structure to the other side of the BNR correspondence, so that we obtain a moduli space of pairs (V, ϕ) with spectral curve X_a and an action of $\mathrm{Pic}(X_a)$ on that space.

Passage to the principal bundle. The generalization we have in mind requires us to express the foregoing as much as possible in terms of principal bundles. If V is a rank r vector bundle on X, then the associated principal bundle E/X is the bundle of local trivializations of V: a section of E over $U \subset X$ is an isomorphism $\mathcal{O}_U^r \cong V|U$. It is clear that GL_r acts on the right of E and makes it a torsor of GL_r over X. At the same time the group scheme $\mathrm{Aut}_X(V)/X$ acts on the left and commutes with the GL_r-action: $\mathrm{Aut}_X(V)$ preserves the torsor structure on E. It is in fact easy to see that this identifies $\mathrm{Aut}_X(V)$ with $\mathrm{Aut}_X(E)$. The Lie algebra scheme of $\mathrm{Aut}_X(V) = \mathrm{Aut}_X(E)$ is simply $\mathrm{End}_X(V)$. This is also the so-called adjoint bundle $\mathrm{Ad}(E) = E \times^{\mathrm{GL}_r} \mathfrak{gl}_r$ associated to E.

Twisting with torsors. In the preceding situation, let J_a/X denote the group scheme over X given by the units in the sheaf of $\mathcal{O}_X$-algebras $\pi_{a*}\mathcal{O}_{X_a}$. This group scheme is abelian and is in the generic point a rank r torus. Its Lie algebra scheme $\mathrm{Lie}(J_a)$ is the vector bundle underlying $\pi_{a*}\mathcal{O}_{X_a}$.

Now suppose V arises from the preceding construction: $V = \pi_{a*}L$ for some L on X_a. Then J_a acts on E: we have a homomorphism of group schemes over X, $J_a \to \mathrm{Aut}_X(E)$. (This might be thought of as a reduction of the structural group of V from GL_r to a torus, at least over a generic point of X in the étale topology.) On the Lie algebra level this yields the $\pi_{a*}\mathcal{O}_{X_a}$-module structure on V. The group scheme J_a acts on $\mathrm{Ad}(E) = \mathrm{End}_X(V)$ by conjugation. The $\pi_{a*}\mathcal{O}_{X_a}$-linearity of $\phi : V \to D \otimes V$ can now be expressed by saying that ϕ is J_a-invariant, when viewed as section of $D \otimes \mathrm{Ad}(E)$. This, in turn, allows us to express the $\mathrm{Pic}(X_a)$ action on pairs (E, ϕ) in a manner that does not directly involve the spectral curve X_a. To see this, let us begin with the remark that we are given a $\mathbb{G}_m$-torsor on X_a (which essentially amounts to giving an invertible sheaf on X_a), then its direct image under π_a has the structure of a J_a-torsor. Since J_a is abelian, the isomorphism classes of J_a-torsors form an abelian group $\mathrm{Pic}(J_a/X)$: the difference $[H]-[H']$ of two such is represented by $\mathrm{Iso}_{J_a}(H',H)$, the local J_a-isomorphisms from H' to H (which is indeed a J_a-torsor). We can now 'twist' the pair (E,ϕ) with a J_a-torsor H to obtain another pair (E_H,ϕ_H) as follows: we let $E_H := H\times^{J_a} E$. This is indeed a GL_r-torsor because this consruction on the left of E does not affect the right action of GL_r. Since J_a is abelian, it still acts on E_H. We have $D\otimes \mathrm{Ad}(E_H) = H\times^{J_a}(D\otimes \mathrm{Ad}(E))$. Since ϕ is a J_a-invariant section of $D\otimes\mathrm{Ad}(E)$, it determines a unique section ϕ_H of $D\otimes\mathrm{Ad}(E_H)$. It is invariant under the J_a-action. We thus have defined an action of $\mathrm{Pic}(J_a/X)$ on the set of ismorphism classes of pairs (E,ϕ).

2. Intermezzo:The Hitchin fibration

We fix an invertible sheaf δ over X and consider the moduli stack $\mathrm{SL}_r(X,\delta)$ of rank r vector bundles V on X endowed with an isomorphism $\det V\cong\delta$. The vector bundles V which have no endomorphisms other than the scalars define an open substack $\mathrm{SL}_r(X,\delta)^\circ\subset \mathrm{SL}_r(X,\delta)$. It is smooth of dimension $(g-1)(n^2-1)$: if V represents a point of it, then the tangent space of that point can be identified with $H^1(X,\mathrm{End}^o(V))$, where $\mathrm{End}^o(V)$ stands for the $\mathcal{O}_X$-Lie algebra of traceless endomorphisms (which by Riemann-Roch has indeed dimension $(g-1)(n^2-1)$). Hence, Serre duality identifies the cotangent space $T^*_{[V]}\mathrm{SL}_r(X,\delta)$ with $H^0(X,\omega_X\otimes\mathrm{End}^o(V))$. Therefore, a covector $\phi\in T^*_{[V]}\mathrm{SL}_r(X,\delta)$ is the same thing as a *Higgs field*, i.e., a section of $\omega_X\otimes\mathrm{End}^o(V)$. The coefficients of the characteristic polynomial of such a Higgs field ϕ yield $a_i(\phi)\in H^0(X,\omega^i)$, $i=1,\dots,r$, but since ϕ has zero trace, $a_1(\phi)=0$. We thus have defined a map

$$T^*\mathrm{SL}_r(X,\delta)^\circ\to\oplus_{i=2}^r H^0(X,\omega^i).$$

With the help of Riemann-Roch, one finds that the right hand side has the same dimension as $\mathrm{SL}_r(X,\delta)^\circ$, namely $(g-1)(n^2-1)$. Now recall that a cotangent bundle of manifold comes with natural symplectic structure.

Theorem 2.1 (Hitchin [3] 1987). *The map* $T^*\mathrm{SL}_r(X,\delta)^\circ\to\oplus_{i=2}^r H^0(X,\omega^i)$ *is a morphism which defines a complete integrable system (in the algebraic setting): the fiber over a general* $a\in\oplus_{i=2}^r H^0(X,\omega^i)$ *is Lagrangian and of the same dimension as* $\mathrm{SL}_r(X,\delta)^\circ$. *Moreover, the resulting Hamiltonian action on that fiber factors through the Prym variety of the spectral cover* $X_a\to X$.

We could have instead considered the moduli stack $GL_r(X)$ of *all* rank r vector bundles V on X. Then a similar result holds. The difference between the two cases resides in the multiplicative group $\mathbb{G}_m$ and indeed, we have a corresponding result in that case, although it is not so exciting: the moduli space of $\mathbb{G}_m$-torsors on X is $\mathrm{Pic}(X)$, $T^*\mathrm{Pic}(X)$ is naturally identified with $\mathrm{Pic}(X)\times H^0(X,\omega_X)$ with the corresponding Lagrangian map being simply the projection on the second factor.

3. THE ADJOINT QUOTIENT AND THE REGULAR CENTRALIZER

We fix a reductive connected smooth k-group $\mathbb{G}$. We also fix a maximal torus $\mathbb{T} \subset \mathbb{G}$ and denote by r its rank. The normalizer of $\mathbb{T}$ in $\mathbb{G}$ acts on $\mathbb{T}$ through the Weyl group $\mathbb{W}$. Recall that we suppose that the order of $\mathbb{W}$ is invertible in k. We denote Lie algebra's of $\mathbb{G}$ and $\mathbb{T}$ by $\mathbb{g}$ and $\mathbb{t}$ respectively.

Let us also fix a Borel subgroup $\mathbb{B} \subset \mathbb{G}$ containing $\mathbb{T}$. Any pair in $\mathbb{G}$ consisting of a maximal torus and a Borel group containing that torus is conjugate to $(\mathbb{T}, \mathbb{B})$. On the other hand any inner automorphism of $\mathbb{G}$ which fixes the pair $(\mathbb{T}, \mathbb{B})$ is trivial. This means that the outer autmorphism group of $\mathbb{G}$, $\mathrm{Out}(\mathbb{G})$, is represented as the group of automorphisms of $\mathbb{G}$ that leave invariant $(\mathbb{T}, \mathbb{B})$. The group $\mathrm{Out}(\mathbb{G})$ is known to be discrete.

The adjoint quotient. According to Chevalley, the algebra of invariants $k[\mathbb{g}]^G$ is a polynomial algebra admitting r homogenous generators $\chi_1, \dots, \chi_r$. If we denote their degrees $e_1, \dots, e_r$, then $\sum_i e_i = \frac{1}{2}(\dim \mathbb{G} + r)$. We refer to $\operatorname{Spec} k[\mathbb{g}]^G$ as the *adjoint quotient* of $\mathbb{g}$ and denote it by $\mathbb{c}$. So $\mathbb{c}$ is an affine space for which $(\chi_1, \dots, \chi_r)$ is a coordinate system. We denote the obvious morphism $\mathbb{g} \to \mathbb{c}$ by χ. It is surjective. The restriction of χ to $\mathbb{t}$ amounts to taking the quotient by the Weyl group and indeed, this identifies $\mathbb{c}$ with $\mathbb{W}\backslash\mathbb{t}$. The discriminant of $\mathbb{t} \to \mathbb{c}$ is denoted $\mathrm{Disc}(\mathbb{c}/\mathbb{t})$. Let us point out that $\mathbb{c}$ is not in general in a natural way a vector space. It does however come with a $\mathbb{G}_m$-action inherited from scalar multiplication in $\mathbb{g}$ (and whose weights are $e_1, \dots, e_r$).

The Kostant section. This is a specific section of the morphism $\chi : \mathbb{t} \to \mathbb{c}$. In the case of GL_r it is a map that finds for every monic polynomial $t^r - a_1 t^{r-1} + \cdots + (-1)^r a_r$ of degree r a matrix T having this as a characteristic polynomial: we let T be multiplication by t in $k[t]/(t^r - a_1 t^{r-1} + \cdots + (-1)^r a_r)$ and use as basis the residue classes of t^{i-1}, $i = 1, \dots, r$. This section is $\mathbb{G}_m$-equivariant: if we replace T by λT, then we might use as basis the residue classes of $(\lambda t)^{i-1} = (\lambda T)^{i-1} 1$, $i = 1, \dots, r$, to see that the characteristic polynomial of this transformation is $t^r + (-\lambda) a_1 t^{r-1} + \cdots + (-\lambda)^r a_r$. In general, a Kostant section is obtained is obtained by choosing a principal $\mathfrak{sl}_2$-triple (e, f, h) in $\mathbb{g}$: if P denotes kernel of $\mathrm{ad}(f)$ (the space of primitive elements of $\mathbb{g}$ as a $\mathfrak{sl}_2$-representation if you like), then the affine space $e + P$ maps isomorphically onto the adjoint quotient, hence defines a section. If $p \in P$ and $\lambda \in k^\times$, then $\exp(-\lambda h) \in \mathbb{G}$ maps $\lambda^2(e + p)$ to $e + \exp(-\lambda h)\lambda^2 p)$

In other cases, the Kostant section may be homogeneous of degree two.

The regular centralizer. Recall that an element of $z \in \mathbb{g}$ is said to be *regular* if its isotropy group $\mathbb{G}_z$ in $\mathbb{G}$ has the minimal dimension r. It is known that $\mathbb{G}_z$ is abelian and that the regular elements in any fiber $\chi^{-1}(a)$ make up a single $\mathbb{G}$-orbit and that this orbit is open-dense in $\chi^{-1}(a)$. This allows us to define an abelian group scheme $\mathbb{J}$ over $\mathbb{g}$: a section of $\mathbb{J}$ over $U \subset \mathbb{c}$

is a morphism $j : \chi^{-1}U \cap \mathbb{g}^{reg} \to \mathbb{G}$ with the property that $j(z) \in \mathbb{G}_z$ and $j(Ad(g)(z)) = gj(z)g^{-1}$ for all $z \in \chi^{-1}(U)$ and $g \in G$. So if $\mathbb{I}/\mathbb{g} \subset \mathbb{G} \times \mathbb{g}/\mathbb{g}$ is the scheme of of $\mathbb{G}$-isotropy groups, then the pull-back of $\mathbb{J}$ over $\mathbb{g}^{reg}$ is naturally identified with $\mathbb{I}|\mathbb{g}^{reg}$. Since $\mathbb{I}$ is closed in $\mathbb{g} \subset \mathbb{G}$, it is affine and from the fact that $\mathbb{g} - \mathbb{g}^{reg}$ is of codimension ≥ 3, it then follows that the morphism $\chi^*\mathbb{J}|\mathbb{g}^{reg} \to \mathbb{I}$ extends to $\chi^*\mathbb{J} \to \mathbb{I}$. To sum up:

Proposition-definition 3.1. *We have defined over the adjoint quotient $\mathbb{c}$ of $\mathbb{G}$ an abelian group scheme $\mathbb{J}$ characterized by the property that a section of $\mathbb{J}$ over $U \subset \mathbb{c}$ is a morphism $j : \chi^{-1}U \to \mathbb{G}$ such that $j(z) \in \mathbb{G}_z$ and $j(Ad(g)(z)) = gj(z)g^{-1}$ for all $z \in \chi^{-1}(U)$ and $g \in G$.*

The image of $\mathbb{t}^{reg}$ in $\mathbb{c}$ is just $\mathbb{c} - \mathrm{Disc}(\mathbb{c}/\mathbb{t})$. Since the elements of $\mathbb{t}^{reg}$ have $\mathbb{T}$ as their common stabilizer in $\mathbb{G}$, it follows that $\mathbb{J}|\mathbb{c} - \mathrm{Disc}(\mathbb{c}/\mathbb{t})$ can also be obtained as $\mathbb{T} \times^{W} \mathbb{t}^{reg} \to \mathbb{c} - \mathrm{Disc}(\mathbb{c}/\mathbb{t})$. There is a corresponding description of the Lie algebra scheme $\mathrm{Lie}(\mathbb{J})$ over all of $\mathbb{c}$:

$$\mathrm{Lie}(\mathbb{J}) = \left(\pi_*(\mathcal{O}_{\mathbb{t}} \otimes \mathbb{t})\right)^{\mathbb{W}}.$$

Observe that the outer automorphism group $\mathrm{Out}(G)$ naturally acts on $\mathbb{c}$ and the group scheme $\mathbb{J}$ over it.

4. THE HITCHIN SPACE

We fix a group scheme G over X that is a *quasi-split* form of $\mathbb{G}$: G is locally trivial for the étale topology with fiber $\mathbb{G}$. One way to obtain such a form of $\mathbb{G}$ over X is to choose a finite subgroup $\mathbb{O} \subset \mathrm{Out}(\mathbb{G})$ and a connected étale $\mathbb{O}$-covering $\hat{X} \to X$. If we identify $\mathbb{O}$ with a group of automorphisms of $\mathbb{G}$ which fix $(\mathbb{T}, \mathbb{B})$, then $G := \mathbb{G} \times^{\mathbb{O}} \hat{X}$ is of that type. Conversely, any such G is so obtained: the local automorphisms of G/X define a scheme $\mathrm{Aut}_X(G)$ that is étale over X and although not necessarily of finite type (when $\mathbb{G}$ is a torus a closed fiber is the automorphism group of the character group of a torus), the identity component $\mathrm{Aut}_X(G)^0$ of its total space is. The Stein factorizarion of $\mathrm{Aut}_X(G)^0 \to X$ has as an intermediate factor a connected étale Galois covering $\hat{X} \to X$, whose Galois group can be identified with a finite subgroup of $\mathrm{Out}(\mathbb{G})$. The two constructions are easily seen to give each others inverse up to isomorphism.

As long as a $\mathbb{O}$-equivariant construction on the (split) group scheme $\mathbb{G}_X$ yields a corresponding result for G, there is no loss in generality in assuming that we are in the split case. That is why we assume in the rest of this note that $G = \mathbb{G} \times X \to X$.

Relative adjoint quotient and regular centralizer. We introduce for G twisted relative versions of the absolute case: we have the Lie algebra $\mathfrak{g}$ of G defined as $\mathcal{O}_X \otimes \mathbb{g}$ and likewise $\mathfrak{t} = \mathcal{O}_X \otimes \mathbb{t}$. We also have defined a sheaf with $\mathbb{G}_m$-action $\mathfrak{c}$ over X whose sections over $U \subset X$ are the morphisms $U \to \mathbb{c}$. We put $\mathfrak{g}_D := D \otimes \mathfrak{g}$, $\mathfrak{t}_D := D \otimes \mathfrak{t}$ and $\mathfrak{c}_D := D \times^{\mathbb{G}_m} \mathbb{c}$ (to be understood as that the sections of $\mathfrak{c}_D$ over $U \subset X$ are the $\mathbb{G}_m$-equivariant morphisms $\mathrm{Tot}(D^{-1}|U) \to \mathbb{c}$). The latter is not an abelian sheaf, but observe that the basic characters $\chi_1, \dots, \chi_r$ identify c_D with the $\mathcal{O}_X$-module $\oplus_{i=1}^r D^{e_i}$. In particular, its set of global sections can be identified with $\oplus_{i=1}^r H^0(X, D^i)$. Although there is no obvious vector space structure on this set of sections, it is more than just a set, for the underlying structure of an affine variety with $\mathbb{G}_m$-action is naturally defined; we shall denote it by $\mathcal{A}$. We have a natural sheaf morphism $\mathfrak{g}_D \to c_D$, whose restriction to t_D is passing to the W-orbit sheaf. Over $\mathrm{Tot}(c_D)$ we have defined an abelian scheme J_D, the analogue of the regular centralizer.

G**-torsors and how to twist them.** Let E/X be a G-torsor (which in the case that we consider here is the same thing as a G-principal bundle over X). Then $\mathrm{Aut}^0_X(E)$ has a Lie algebra scheme over X that is also obtained from smashing E with the adjoint representation of G and hence is denoted $\mathrm{Ad}(E)$. There is a natural sheaf morphism $\mathrm{Ad}(E) \to c$. We will be mostly concerned with $D \otimes \mathrm{Ad}(E)$, for which we have a natural sheaf morphism $\chi_E : D \otimes \mathrm{Ad}(E) \to c_D$.

Any $\phi \in H^0(X, D \otimes \mathrm{Ad}(E))$ can be composed with $\chi_{E,D}$ to produce a section of c_D: $\chi_E\phi \in H^0(X, c_D) = \mathcal{A}$. Write a for this element and regard it as a morphism $a : X \to \mathrm{Tot}(c_D)$. We denote by J_a/X the pull-back of J along a. This is now an abelian scheme over X. The characterizing property of $\mathbb{J}$ shows that we also have a homomorphism of group schemes $J_a \to \mathrm{Aut}^0_X(E)$ whose induced action in $D \otimes \mathrm{Ad}(E)$ fixes ϕ. (If the image of ϕ contains a regular element, then it is easy to see that J_a is in fact the the stabilizer of ϕ in $\mathrm{Aut}^0_X(E)$.)

Now let H/X be a J_a-torsor. Then we can twist the pair (E, ϕ) in much the same way as we did for GL_r: we put $E_H := H \times^{J_a} E$ and ϕ_H is the section of $D \otimes \mathrm{Ad}(E_H) = H \times^{J_a} (D \otimes \mathrm{Ad}(E))$ that its locally represented by (h, ϕ), where $h \in H$ is arbitrary. The associated section $\chi_{E_H}(\phi_H)$ of c_D coincides with $\chi_E(\phi) = a$. In particular, J_a acts on E_H and stabilizes ϕ_H. Most of this is summed up by:

Proposition-definition 4.1. *The* Hitchin space *of* G *(relative to* D*) is the moduli stack of Hitchin pairs* (E, ϕ) *as above: here* E *is a* G*-torsor and* ϕ *a section of* $D \otimes \mathrm{Ad}(E)$. *We denote it by* $\mathcal{M}$. *We have a natural morphism (the* Hitchin morphism*)*

$$\mathcal{M} \to \mathcal{A} = H^0(X, c_D), \quad (E, \phi) \mapsto \chi_E(\phi).$$

The evaluation map $ev : X \times \mathcal{A} \to \mathrm{Tot}(c_D)$ *is smooth and surjective. If* $J_{\mathcal{A}}$ *denotes the pull-back of* J_D *over* $X \times \mathcal{A}$*, then for each* $a \in \mathcal{A}(\bar{k})$*,* J_a *acts on any Hitchin pair representing a point of the fiber* $\mathcal{M}_a$*. And if* $\pi_{\mathcal{A}} : X \times \mathcal{A} \to \mathcal{A}$ *denotes the projection, then the relative Picard scheme* $\mathcal{P}/\mathcal{A} = \mathrm{Pic}(J_{\mathcal{A}}/\mathcal{A})$ *acts naturally on the* $\mathcal{A}$*-scheme* $\mathcal{M}$*.*

Proof. The only assertion not yet discussed is that $ev : X \times \mathcal{A} \to \mathrm{Tot}(c_D)$ is smooth and surjective. This however follows from the fact that $\mathrm{Tot}(c_D)$ is the total space of a direct sum of very ample line bundles over X, having $\mathcal{A}$ as its space of sections. □

Let $\mathcal{M}^{\mathrm{reg}} \subset \mathcal{M}$ represent the set of Hitchin pairs (E, ϕ) for which ϕ takes values in the regular elements of $D \otimes \mathrm{Ad}(E)$ (this makes sense because the set of regular element in $\mathfrak{g}$ is invariant under scalar multiplication). This is clearly an open subset of $\mathcal{M}$. The following proposition generalizes the BDR-correspondence.

Proposition 4.2. *The open subset $\mathcal{M}^{\mathrm{reg}}$ maps onto $\mathcal{A}$ and is a torsor of the relative Picard scheme $\mathcal{P}/\mathcal{A}$.*

Sketch of proof. The surjectivity of $\mathcal{M}^{\mathrm{reg}} \to \mathcal{A}$ is established by means of a Kostant section.

In order to prove the second assertion, let us begin with making the following observation. Let $\mathbb{I} \subset \mathbb{G}$ be the centralizer of a regular element of $\mathfrak{g}$. We know that $\mathbb{I}$ is then a connected abelian group of dimension r. We regard $\mathbb{G}$ as a homogeneous space over $\mathbb{I} \times \mathbb{G}$ with $\mathbb{G}$ acting by right translations and $\mathbb{I}$ by left translations. This makes $\mathbb{I}$ an automorphism group of $\mathbb{G}$, when the latter is regarded as a torsor over itself. If $\mathbb{E}$ is a variety with $\mathbb{I} \times \mathbb{G}$-action isomorphic to $\mathbb{G}$, then two $\mathbb{I} \times \mathbb{G}$-equivariant isomorphisms $\mathbb{G} \cong \mathbb{E}$ will differ by an automorphism of $\mathbb{G}$ which centralizes $\mathbb{I}$, hence is contained in $\mathbb{I}$ in $\mathrm{Aut}(\mathbb{G})$. In other words, the set of such isomorphisms is a $\mathbb{I}$-torsor.

Now let (E, ϕ) and (E', ϕ') be Hitchin pairs representing points of $\mathcal{M}^{\mathrm{reg}}$ over the same point a of $\mathcal{A}$. Then we have morphisms $J_a \to \mathrm{Aut}(E, \phi)$ and $J_a \to \mathrm{Aut}(E', \phi')$. For U open in X, consider the set of isomorphisms $E|U \cong E'|U$ with $J_a|U$ action. This defines a presheaf whose associated sheaf is a J_a-torsor. One then verifies that (E', ϕ') is obtained from (E, ϕ) by a twist with this torsor. □

Some dimension computations. We compute the dimensions of some of the spaces that appear in the definition of the Hitchin space. *From now on, we assume that* $\deg D > 2g$.

If E is a general G-torsor, then $\mathrm{Ad}(E)$ has no nonzero global sections. Kodaira-Spencer theory tells us that the moduli stack of G-torsors is smooth at such a E with tangent space identified with $H^1(X, \mathrm{Ad}(E))$. A multiple of the Killing form identifies the vector bundle $\mathrm{Ad}(E)$ with its dual. This implies that $\mathrm{Ad}(E)$ has degree zero and so if we invoke Riemann-Roch, we find

$$\dim H^1(X, \mathrm{Ad}(E)) = (g-1) \dim \mathbb{G}.$$

Given this E, then Riemann-Roch also gives

$$\dim H^0(X, D \otimes \mathrm{Ad}(E)) = (1 - g + \deg D) \dim \mathbb{G},$$

and so we find that

$$\dim \mathcal{M} = \deg D \dim \mathbb{G}.$$

We verify that this is also $\dim \mathcal{A} + \dim \mathrm{Pic}(J_a)$ for a general $a \in \mathcal{A}(\bar{k})$.

Since $\mathcal{A} \cong H^0(X, D^{e_i})$, it follows from Riemann-Roch that

$$\dim \mathcal{A} = r_0 + r(1-g) + \sum_i e_i \deg D = r_0 + r(1-g) + \tfrac{1}{2}(\dim \mathbb{G} + r) \deg D,$$

where r_0 is the number of i's with $e_i = 1$ (this is also the dimension of the center of $\mathbb{G}$.

Lemma 4.3. *The Lie algebra scheme* $\mathrm{Lie}(J_D)$ *on* $\mathrm{Tot}(c_D)$ *can be identified with the pull-back of* $D^{1-e_1} \oplus \cdots \oplus D^{1-e_r}$.

Proof. Recall that $\mathrm{Lie}(\mathbb{J}) = \big(\pi_*(\mathcal{O}_{\mathfrak{t}} \otimes \mathfrak{t})\big)^{\mathbb{W}}$. A suitable multiple of the Killing form identifies $\mathfrak{t}$ with $\mathfrak{t}^*$ as a $\mathbb{W}$-module and so $\mathrm{Lie}(\mathbb{J}) = \big(\pi_*(\mathcal{O}_{\mathfrak{t}} \otimes \mathfrak{t}^*)\big)^{\mathbb{W}}$. The latter can also be written $\big(\pi_*(\Omega_{\mathfrak{t}})\big)^{\mathbb{W}}$ and this, in turn, can be identified with $\Omega_{\mathfrak{c}}$, because both have $d\chi_1, \ldots, d\chi_r$ as a basis. In the present setting this yields the identification

$$\big(\pi_*(\mathcal{O}_{\mathrm{Tot}(t_D)} \otimes \mathfrak{t}^* \otimes D^{-1})\big)^{W} \cong \big(\pi_{D*}\Omega_{t_D/X}\big)^{W} \cong \Omega_{c_D/X}$$

The left hand side can be written as $\big(\pi_*(\mathcal{O}_{\mathrm{Tot}(t_D)} \otimes \mathfrak{t})\big)^{W} \otimes D^{-1}$, which we the recognize as $\mathrm{Lie}(J_D) \otimes D^{-1}$. The right hand side is isomorphic to the pull-back of $D^{-e_1} \oplus \cdots \oplus D^{-e_r}$. □

Corollary 4.4. *We have* $\dim \mathrm{Pic}(J_a) = -r_0 + (g-1)r + \frac{1}{2}(\dim \mathbb{G} - r) \deg D$ *for all* $a \in \mathcal{A}$.

Proof. The general theory tells us that the tangent space of $\mathrm{Pic}(J_a)$ at the identity element can be identified with $H^1(X, \mathrm{Lie}(J_a))$. The previous lemma identifies this with $\oplus_{i=1}^r H^1(X, D^{1-e_i})$. The assertion now follows from an application of Riemann-Roch. □

So $\dim \mathrm{Pic}(J_a) + \dim \mathcal{A} = \dim \mathbb{G} \deg D$ is indeed equal to $\dim \mathcal{M}$.

5. Cameral covers

Over $\mathrm{Tot}(c_D)$ we have the W-covering $\mathrm{Tot}(t_D)$ whose total space is also smooth. Let us denote by $\widetilde{X \times \mathcal{A}}$ the pull-back of this cover along the evaluation map:

$$\begin{array}{ccc} \widetilde{X \times \mathcal{A}} & \longrightarrow & \mathrm{Tot}(t_D) \\ \pi_{X\times\mathcal{A}} \downarrow & & \downarrow \pi_D \\ X \times \mathcal{A} & \xrightarrow{ev} & \mathrm{Tot}(c_D). \end{array}$$

The vertical maps are W-covers and the horizontal maps are smooth surjective. In particular, $\widetilde{X \times \mathcal{A}}$ is smooth and connected. We may refer to the left vertical map as the *universal cameral cover*. We can recover the Lie algebra of the abelian group scheme $J_{\mathcal{A}}$ over $X \times \mathcal{A}$ as

$$\mathrm{Lie}(J_{\mathcal{A}}) = (\pi_{X\times\mathcal{A}*}\mathcal{O}_{\widetilde{X\times\mathcal{A}}} \otimes \mathfrak{t})^{W}.$$

Now consider the projection $\widetilde{X \times \mathcal{A}} \to \mathcal{A}$. For $a \in \mathcal{A}(\bar{k})$, the above diagram restricts to

$$\begin{array}{ccc} \widetilde{X}_a & \longrightarrow & \mathrm{Tot}(t_D) \\ \downarrow & & \downarrow \\ X & \xrightarrow{a} & \mathrm{Tot}(c_D). \end{array}$$

The left vertical map is called the *cameral cover* attached to a. It is clear that $\tilde{X}_a$ is a complete intersection in the smooth $\widetilde{X \times \mathcal{A}}$. So the generic fiber is smooth and by a Lefschetz type theorem connected. A form of Zariski's main theorem implies that then in fact all fibers $\tilde{X}_a$ are connected. A simple local computation also shows that $\tilde{X}_a$ is smooth precisely when a is transversal to $\mathrm{Disc}(t_D/c_D)$ in the sense that $a^* \,\mathrm{Disc}(t_D/c_D)$ is reduced. The corresponding locus in $\mathcal{A}$ is therefore the projection in $\mathcal{A}$ of $ev^{-1}(\mathrm{Tot}(c_D) - \mathrm{Disc}(t_D/c_D))$ and hence open-dense. We denote it by $\mathcal{A}^{\diamondsuit}$.

If $a(X) \not\subset \mathrm{Disc}(t_D/c_D)$, then $\tilde{X}_a$ is reduced (and vice versa); we denote the locus in $\mathcal{A}$ defined by that property by $\mathcal{A}^{\heartsuit}$. It is also open in $\mathcal{A}(\bar{k})$ and clearly

$$\mathcal{A}^{\diamondsuit} \subset \mathcal{A}^{\heartsuit} \subset \mathcal{A}(\bar{k}).$$

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