THE HITCHIN FIBRATION

Seminar talk based on part of Ngô Bao Châu's preprint: *Le lemme fondamental pour les algèbres de Lie* [2].

Here $X$ is a smooth connected projective curve over a field $k$ whose genus is denoted $g$. We also fix an invertible sheaf $D$ over $X$. We shall eventually assume that $\deg D > 2g$ and that $D$ is 2-divisible in $\text{Pic}(X)$.

Whenever is at issue a reductive $k$-group, we shall assume that the characteristic of $k$ does not divide the order of its Weyl group.

1. The BNR correspondence

Let be given $\alpha = (\alpha_i \in H^0(X, D^i))_{i=1}^r$. This defines a curve $X_\alpha$ in the total space $\text{Tot}(D)$ of $D$ as the set of $t \in \text{Tot}(D)$ obeying $t^r - \alpha_1 t^{r-1} + \cdots + (-1)^r \alpha_r = 0$. The more precise way of giving $X_\alpha$ as defined by a principal ideal in $\mathcal{O}_{\text{Tot}(D)}$ is as follows: let $u \in H^0(U, D^{-1})$ be a local generator of $D^{-1}$ on an affine open subset $U \subset X$ so that the coordinate ring of $\text{Tot}(D|U)$ is $\mathcal{O}(U)[u]$. If $\bar{\alpha}_i \in \mathcal{O}(U)$ is the value of $\alpha_i$ on $U^i$, then $X_\alpha \cap \text{Tot}(D|U)$ is defined by the ideal generated by $u^r - \bar{\alpha}_1 u^{r-1} + \cdots + (-1)^r \bar{\alpha}_r$. We shall assume that $X_\alpha$ is integral (i.e., reduced and irreducible). We denote the projection $\text{Tot}(D) \to D$ by $\pi$ and its restriction to $X_\alpha$ by $\pi_\alpha$. The latter has degree $r$.

Suppose $L$ is a torsion free sheaf of rank one on $X_\alpha$. Then $\pi_\alpha^* L$ is torsion free on $X$ of rank $r$. Since $X$ is smooth, this means that $\pi_\alpha^* L$ is locally free: it is a vector bundle of rank $r$. Its degree can be computed with the help of Riemann-Roch:

$$r \chi(X, \mathcal{O}_X) + \deg(\pi_\alpha^* L) = \chi(X, \mathcal{O}_X) = \chi(X_\alpha, \mathcal{O}_{X_\alpha}) + \deg L$$

shows that $\deg(\pi_\alpha^* L) - \deg L$ is independent of $L$ and hence (take $L = \mathcal{O}_{X_\alpha}$) equal to $\deg(\pi_\alpha^* \mathcal{O}_{X_\alpha})$. This vector bundle clearly comes with the structure of a module over the $\mathcal{O}_X$-algebra $\pi_\alpha^* \mathcal{O}_{X_\alpha}$. There is however a bit more to say here. Let us first note that the bundle $\pi^* D$ over $\text{Tot}(D)$ comes with a tautological section. Denote by $\alpha \in H^0(X_\alpha, \pi_\alpha^* D)$ the restriction of this section to $X_\alpha$. Then $\ell \in L \mapsto \alpha \otimes \ell \in \pi_\alpha^* D \otimes L$ is $\mathcal{O}_{X_\alpha}$-homomorphism whose direct image under $\pi_\alpha$ yields a $\pi_\alpha^* \mathcal{O}_{X_\alpha}$-homomorphism

$$\phi: \pi_\alpha^* L \to D \otimes \pi_\alpha^* L.$$

We can recover from $\phi$ the curve $X_\alpha \subset \text{Tot}(D)$ (or rather a general point of that curve) as its *spectral curve*: a local section $\sigma$ of $D$ (in the étale topology) defines a point of $X_\alpha$ precisely if it is an eigensection of $\phi$ in the sense that there exists a local section $\nu \neq 0$ of $\pi_\alpha^* L$ such that $\phi(\nu) = \sigma \otimes \nu$.

We can now state:
Theorem 1.1 (Beauville-Narasimhan-Ramanan [1] 1989). The map which assigns to a torsion free sheaf of rank one \( L \) on \( X_a \) the pair \( (\pi a, L, \phi) \) defines a bijection between the set of isomorphism classes of torsion free sheaves of rank one on \( X_a \) and the isomorphism classes of pairs \( (V, \phi) \), where \( V \) is a rank \( r \) vector bundle on \( X \) and \( \phi : V \to D \otimes V \) is a \( \mathcal{O}_X \)-homomorphism for which \( X_a \) is its spectral curve.

The set of isomorphism classes of torsion free rank one sheaves on \( X_a \) is parameterized by a scheme which contains the Picard variety \( \text{Pic}(X_a) \) of \( X_a \) (which parameterizes the isomorphism classes of invertible rank one sheaves on \( X_a \)) as an open subset. It is also dense (because \( X_a \) has only curve singularities) and therefore denoted by \( \tilde{\text{Pic}}(X_a) \). The group structure on \( \text{Pic}(X_a) \) extends to an action of \( \text{Pic}(X_a) \) on \( \tilde{\text{Pic}}(X_a) \). We transport this scheme structure to the other side of the BNR correspondence, so that we obtain a moduli space of pairs \( (V, \phi) \) with spectral curve \( X_a \) and an action of \( \text{Pic}(X_a) \) on that space.

Passage to the principal bundle. The generalization we have in mind requires us to express the foregoing as much as possible in terms of principal bundles. If \( V \) is a rank \( r \) vector bundle on \( X \), then the associated principal bundle \( E/X \) is the bundle of local trivializations of \( V \): a section of \( E \) over \( U \subset X \) is an isomorphism \( \mathcal{O}_{U|} \cong V|U \). It is clear that \( GL_r \) acts on the right of \( E \) and makes it a torsor of \( GL_r \) over \( X \). At the same time the group scheme \( \text{Aut}_X(V)/X \) acts on the left and commutes with the \( GL_r \)-action: \( \text{Aut}_X(V) \) preserves the torsor structure on \( E \). It is in fact easy to see that this identifies \( \text{Aut}_X(V) \) with \( \text{Aut}_X(E) \). The Lie algebra scheme of \( \text{Aut}_X(V) = \text{Aut}_X(E) \) is simply \( \text{End}_X(V) \). This is also the so-called adjoint bundle \( \text{Ad}(E) = E \times GL_r \text{gl}_r \) associated to \( E \).

Twisting with torsors. In the preceding situation, let \( J_a/X \) denote the group scheme over \( X \) given by the units in the sheaf of \( \mathcal{O}_X \)-algebras \( \pi a \mathcal{O}_{X_a} \). This group scheme is abelian and is in the generic point a rank \( r \) torus. Its Lie algebra scheme \( \text{Lie}(J_a) \) is the vector bundle underlying \( \pi a \mathcal{O}_{X_a} \).

Now suppose \( V \) arises from the preceding construction: \( V = \pi a L \) for some \( L \) on \( X_a \). Then \( J_a \) acts on \( E \): we have a homomorphism of group schemes over \( X \), \( J_a \to \text{Aut}_X(E) \). (This might be thought of as a reduction of the structural group of \( V \) from \( GL_r \) to a torus, at least over a generic point of \( X \) in the étale topology.) On the Lie algebra level this yields the \( \pi a \mathcal{O}_{X_a} \)-module structure on \( V \). The group scheme \( J_a \) acts on \( \text{Ad}(E) = \text{End}_X(V) \) by conjugation. The \( \pi a \mathcal{O}_{X_a} \) linearity of \( \phi : V \to D \otimes V \) can now be expressed by saying that \( \phi \) is \( J_a \)-invariant, when viewed as section of \( D \otimes \text{Ad}(E) \). This, in turn, allows us to express the \( \text{Pic}(X_a) \) action on pairs \( (E, \phi) \) in a manner that does not directly involve the spectral curve \( X_a \). To see this, let us begin with the remark that we are given a \( \mathbb{G}_m \)-torsor on \( X_a \) (which essentially amounts to giving an invertible sheaf on \( X_a \)), then its direct image under \( \pi a \) has the structure of a \( J_a \)-torsor. Since \( J_a \) is abelian, the isomorphism classes
of $J_a$-torsors form an abelian group $\text{Pic}(J_a/X)$: the difference $[H] - [H']$ of two such is represented by $\text{Iso}_{J_a}(H', H)$, the local $J_a$-isomorphisms from $H'$ to $H$ (which is indeed a $J_a$-torsor). We can now ‘twist’ the pair $(\mathcal{E}, \phi)$ with a $J_a$-torsor $H$ to obtain another pair $(\mathcal{E}_H, \phi_H)$ as follows: we let $\mathcal{E}_H := H \times_J a \mathcal{E}$. This is indeed a $GL_r$-torsor because this construction on the left of $E$ does not affect the right action of $GL_r$. Since $J_a$ is abelian, it still acts on $E_H$. We have $D \otimes \text{Ad}(E_D) = H \times_J a (D \otimes \text{Ad}(E))$. Since $\phi$ is a $J_a$-invariant section of $D \otimes \text{Ad}(E)$, it determines a unique section $\phi_H$ of $D \otimes \text{Ad}(E_H)$. It is invariant under the isomorphism of pairs $(\mathcal{E}, \phi)$.

2. Intermezzo: The Hitchin fibration

We fix an invertible sheaf $\delta$ over $X$ and consider the moduli stack $\text{SL}_r(X, \delta)$ of rank $r$ vector bundles $V$ on $X$ endowed with an isomorphism $\det V \cong \delta$. The vector bundles $V$ which have no endomorphisms other than the scalars define an open substack $\text{SL}_r(X, \delta)^o \subset \text{SL}_r(X, \delta)$. It is smooth of dimension $(g - 1)(n^2 - 1)$: if $V$ represents a point of it, then the tangent space of that point can be identified with $H^1(X, \text{End}^o(V))$, where $\text{End}^o(V)$ stands for the $\mathcal{O}_X$-Lie algebra of traceless endomorphisms (which by Riemann-Roch has indeed dimension $(g - 1)(n^2 - 1)$). Hence, Serre duality identifies the cotangent space $T^*_V \text{SL}_r(X, \delta)$ with $H^0(X, \omega_X \otimes \text{End}^o(V))$. Therefore, a covector $\phi \in T^*_V \text{SL}_r(X, \delta)$ is the same thing as a Higgs field, i.e., a section of $\omega_X \otimes \text{End}^o(V)$. The coefficients of the characteristic polynomial of such a Higgs field $\phi$ yield $a_i(\phi) \in H^0(X, \omega^i)$, $i = 1, \ldots, r$, but since $\phi$ has zero trace, $a_1(\phi) = 0$. We thus have defined a map

$$T^* \text{SL}_r(X, \delta)^o \to \bigoplus_{i=2}^r H^0(X, \omega^i).$$

With the help of Riemann-Roch, one finds that the right hand side has the same dimension as $\text{SL}_r(X, \delta)^o$, namely $(g - 1)(n^2 - 1)$. Now recall that a cotangent bundle of manifold comes with natural symplectic structure.

**Theorem 2.1** (Hitchin [3] 1987). The map $T^* \text{SL}_r(X, \delta)^o \to \bigoplus_{i=2}^r H^0(X, \omega^i)$ is a morphism which defines a complete integrable system (in the algebraic setting): the fiber over a general $a \in \bigoplus_{i=2}^r H^0(X, \omega^i)$ is Lagrangian and of the same dimension as $\text{SL}_r(X, \delta)^o$. Moreover, the resulting Hamiltonian action on that fiber factors through the Prym variety of the spectral cover $X_a \to X$.

We could have instead considered the moduli stack $\text{GL}_r(X)$ of all rank $r$ vector bundles $V$ on $X$. Then a similar result holds. The difference between the two cases resides in the multiplicative group $\mathbb{G}_m$ and indeed, we have a corresponding result in that case, although it is not so exciting: the moduli space of $\mathbb{G}_m$-torsors on $X$ is $\text{Pic}(X)$, $T^* \text{Pic}(X)$ is naturally identified with $\text{Pic}(X) \times H^0(X, \omega_X)$ with the corresponding Lagrangian map being simply the projection on the second factor.
3. THE ADJOINT QUOTIENT AND THE REGULAR CENTRALIZER

We fix a reductive connected smooth $k$-group $G$. We also fix a maximal torus $T \subset G$ and denote by $r$ its rank. The normalizer of $T$ in $G$ acts on $T$ through the Weyl group $W$. Recall that we suppose that the order of $W$ is invertible in $k$. We denote Lie algebra’s of $G$ and $T$ by $g$ and $\mathfrak{t}$ respectively.

Let us also fix a Borel subgroup $B \subset G$ containing $T$. Any pair in $G$ consisting of a maximal torus and a Borel group containing that torus is conjugate to $(T, B)$. On the other hand any inner automorphism of $G$ which fixes the pair $(T, B)$ is trivial. This means that the outer automorphism group of $G$, $\text{Out}(G)$, is represented as the group of automorphisms of $G$ that leave invariant $(T, B)$. The group $\text{Out}(G)$ is known to be discrete.

**The adjoint quotient.** According to Chevalley, the algebra of invariants $k[g]^G$ is a polynomial algebra admitting $r$ homogenous generators $\chi_1, \ldots, \chi_r$. If we denote their degrees $e_1, \ldots, e_r$, then $\sum_i e_i = \frac{1}{2}(\dim G + r)$. We refer to $\text{Spec} k[g]^G$ as the adjoint quotient of $g$ and denote it by $c$. So $c$ is an affine space for which $(\chi_1, \ldots, \chi_r)$ is a coordinate system. We denote the obvious morphism $g \to c$ by $\chi$. It is surjective. The restriction of $\chi$ to $\mathfrak{t}$ amounts to taking the quotient by the Weyl group and indeed, this identifies $c$ with $W/\mathfrak{t}$. The discriminant of $t \to c$ is denoted $\text{Disc}(c/t)$. Let us point out that $c$ is not in general in a natural way a vector space. It does however come with a $\mathbb{G}_m$-action inherited from scalar multiplication in $g$ (and whose weights are $e_1, \ldots, e_r$).

**The Kostant section.** This is a specific section of the morphism $\chi : \mathfrak{t} \to c$. In the case of $\text{GL}_r$ it is a map that finds for every monic polynomial $t^r - a_1t^{r-1} + \cdots + (-1)^r a_r$ of degree $r$ a matrix $T$ having this as a characteristic polynomial: we let $T$ be multiplication by $t$ in $k[t]/(t^r - a_1t^{r-1} + \cdots + (-1)^r a_r)$ and use as basis the residue classes of $t^{i-1}, i = 1, \ldots, r$. This section is $\mathbb{G}_m$-equivariant: if we replace $T$ by $\lambda T$, then we might use as basis the residue classes of $(\lambda t)^{i-1} = (\lambda T)^{i-1}, i = 1, \ldots, r$, to see that the characteristic polynomial of this transformation is $t^r + (-\lambda) a_1 t^{r-1} + \cdots + (-\lambda)^r a_r$. In general, a Kostant section is obtained by choosing a principal $sl_2$-triple $(e, f, h)$ in $g$: if $P$ denotes kernel of $\text{ad}(f)$ (the space of primitive elements of $g$ as a $sl_2$-representation if you like), then the affine space $e + P$ maps isomorphically onto the adjoint quotient, hence defines a section. If $p \in P$ and $\lambda \in k^\times$, then $\exp(-\lambda h) \in G$ maps $\lambda^2(e + p)$ to $e + \exp(-\lambda h)\lambda^2 p$.

In other cases, the Kostant section may be homogeneous of degree two.

**The regular centralizer.** Recall that an element of $z \in g$ is said to be regular if its isotropy group $G_z$ in $G$ has the minimal dimension $r$. It is known that $G_z$ is abelian and that the regular elements in any fiber $\chi^{-1}(a)$ make up a single $G$-orbit and that this orbit is open-dense in $\chi^{-1}(a)$. This allows us to define an abelian group scheme $J$ over $g$: a section of $J$ over $U \subset c$
is a morphism \( j : \chi^{-1}U \cap g_{\reg} \to G \) with the property that \( j(z) \in G_z \) and \( j(\Ad(g)(z)) = gj(z)g^{-1} \) for all \( z \in \chi^{-1}(U) \) and \( g \in G \). So if \( \mathbb{I}/\mathfrak{g} \subset G \times g/\mathfrak{g} \) is the scheme of of \( G \)-isotropy groups, then the pull-back of \( \mathfrak{g} \) over \( g_{\reg} \) is naturally identified with \( \mathbb{I}/\mathfrak{g}_{\reg} \). Since \( \mathbb{I} \) is closed in \( G \subset G \), it is affine and from the fact that \( g - g_{\reg} \) is of codimension \( \geq 3 \), it then follows that the morphism \( \chi^*\mathfrak{g}_{\reg} \to \mathbb{I} \) extends to \( \chi^*\mathfrak{g} \to \mathbb{I} \). To sum up:

**Proposition-definition 3.1.** We have defined over the adjoint quotient \( \mathfrak{c} \) of \( G \) an abelian group scheme \( \mathfrak{j} \) characterized by the property that a section of \( \mathfrak{j} \) over \( U \subset \mathfrak{c} \) is a morphism \( j : \chi^{-1}U \to G \) such that \( j(z) \in G_z \) and \( j(\Ad(g)(z)) = gj(z)g^{-1} \) for all \( z \in \chi^{-1}(U) \) and \( g \in G \).

The image of \( t_{\reg} \) in \( \mathfrak{c} \) is just \( \mathfrak{c} = \operatorname{Disc}(\mathfrak{c} / \mathfrak{t}) \). Since the elements of \( t_{\reg} \) have \( \mathfrak{t} \) as their common stabilizer in \( G \), it follows that \( \mathfrak{j} |_{\mathfrak{c}} = \operatorname{Disc}(\mathfrak{c} / \mathfrak{t}) \) can also be obtained as \( \mathfrak{t} \times W t_{\reg} \to \mathfrak{c} = \operatorname{Disc}(\mathfrak{c} / \mathfrak{t}) \). There is a corresponding description of the Lie algebra scheme \( \operatorname{Lie}(\mathfrak{j}) \) over all of \( \mathfrak{c} \):

\[
\operatorname{Lie}(\mathfrak{j}) = \left( \pi_*(\mathcal{O}_{\mathfrak{c}} \otimes \mathfrak{t}) \right)^{\mathfrak{w}}.
\]

Observe that the outer automorphism group \( \operatorname{Out}(G) \) naturally acts on \( \mathfrak{c} \) and the group scheme \( \mathfrak{j} \) over it.

4. The Hitchin space

We fix a group scheme \( G \) over \( X \) that is a *quasi-split* form of \( G \): \( G \) is locally trivial for the étale topology with fiber \( G \). One way to obtain such a form of \( G \) over \( X \) is to choose a finite subgroup \( \mathcal{O} \subset \operatorname{Out}(G) \) and a connected étale \( \mathcal{O} \)-covering \( \hat{X} \to X \). If we identify \( \mathcal{O} \) with a group of automorphisms of \( G \) which fix \( (\mathfrak{t}, \mathbb{B}) \), then \( G := G \times \mathcal{O} \hat{X} \) is of that type. Conversely, any such \( G \) is so obtained: the local automorphisms of \( G / X \) define a scheme \( \operatorname{Aut}_X(G) \) that is étale over \( X \) and although not necessarily of finite type (when \( G \) is a torus a closed fiber is the automorphism group of the character group of a torus), the identity component \( \operatorname{Aut}_X(G)^0 \) of its total space is. The Stein factorization of \( \operatorname{Aut}_X(G)^0 \to X \) has as an intermediate factor a connected étale Galois covering \( \hat{X} \to X \), whose Galois group can be identified with a finite subgroup of \( \operatorname{Out}(G) \). The two constructions are easily seen to give each others inverse up to isomorphism.

As long as a \( \mathcal{O} \)-equivariant construction on the (split) group scheme \( G_X \) yields a corresponding result for \( G \), there is no loss in generality in assuming that we are in the split case. That is why we assume in the rest of this note that \( G = G \times X \to X \).

**Relative adjoint quotient and regular centralizer.** We introduce for \( G \) twisted relative versions of the absolute case: we have the Lie algebra \( g \) of \( G \) defined as \( \mathcal{O}_X \otimes g \) and likewise \( \mathfrak{t} = \mathcal{O}_X \otimes \mathfrak{t} \). We also have defined a sheaf with \( G_m \)-action \( \mathfrak{c} \) over \( X \) whose sections over \( U \subset X \) are the morphisms \( U \to \mathfrak{c} \). We put \( g_D := D \otimes g \), \( t_D := D \otimes \mathfrak{t} \) and \( c_D := D \times G_m \to \mathfrak{c} \) (to be understood as that the sections of \( c_D \) over \( U \subset X \) are the \( G_m \)-equivariant
morphisms $\text{Tot}(D^{-1}|U) \to c$). The latter is not an abelian sheaf, but observe that the basic characters $\chi_1, \ldots, \chi_r$ identify $c_D$ with the $O_X$-module $\oplus_{i=1}^r D^1$. In particular, its set of global sections can be identified with $\oplus_{i=1}^r H^0(X, D^i)$. Although there is no obvious vector space structure on this set of sections, it is more than just a set, for the underlying structure of an affine variety with $\mathbb{G}_m$-action is naturally defined; we shall denote it by $\mathcal{A}$. We have a natural sheaf morphism $g_D \to c_D$, whose restriction to $t_D$ is passing to the $W$-orbit sheaf. Over $\text{Tot}(c_D)$ we have defined an abelian scheme $J_D$, the analogue of the regular centralizer.

**G-torsors and how to twist them.** Let $E/X$ be a $G$-torsor (which in the case that we consider here is the same thing as a $G$-principal bundle over $X$). Then $\text{Aut}_X^G(E)$ has a Lie algebra scheme over $X$ that is also obtained from smashing $E$ with the adjoint representation of $G$ and hence is denoted $\text{Ad}(E)$. There is a natural sheaf morphism $\text{Ad}(E) \to c$. We will be mostly concerned with $D \otimes \text{Ad}(E)$, for which we have a natural sheaf morphism $\chi_E : D \otimes \text{Ad}(E) \to c_D$.

Any $\phi \in H^0(X, D \otimes \text{Ad}(E))$ can be composed with $\chi_{E,D}$ to produce a section of $c_D$: $\chi_E \phi \in H^0(X, c_D) = \mathcal{A}$. Write $a$ for this element and regard it as a morphism $a : X \to \text{Tot}(c_D)$. We denote by $J_a/X$ the pull-back of $J$ along $a$. This is now an abelian scheme over $X$. The characterizing property of $J$ shows that we also have a homomorphism of group schemes $J_a \to \text{Aut}_X^G(E)$ whose induced action in $D \otimes \text{Ad}(E)$ fixes $\phi$. (If the image of $\phi$ contains a regular element, then it is easy to see that $J_a$ is in fact the the stabilizer of $\phi$ in $\text{Aut}_X^G(E)$.)

Now let $H/X$ be a $J_a$-torsor. Then we can twist the pair $(E, \phi)$ in much the same way as we did for $\text{GL}_r$: we put $E_H := H \times^J_a E$ and $\phi_H$ is the section of $D \otimes \text{Ad}(E_H) = H \times^J_a (D \otimes \text{Ad}(E))$ that its locally represented by $(h, \phi)$, where $h \in H$ is arbitrary. The associated section $\chi_{E,H}(\phi_H)$ of $c_D$ coincides with $\chi_E(\phi) = a$. In particular, $J_a$ acts on $E_H$ and stabilizes $\phi_H$. Most of this is summed up by:

**Proposition-definition 4.1.** The Hitchin space of $G$ (relative to $D$) is the moduli stack of Hitchin pairs $(E, \phi)$ as above: here $E$ is a $G$-torsor and $\phi$ a section of $D \otimes \text{Ad}(E)$. We denote it by $\mathcal{M}$. We have a natural morphism (the Hitchin morphism)

$$\mathcal{M} \to \mathcal{A} = H^0(X, c_D), \quad (E, \phi) \mapsto \chi_E(\phi).$$

The evaluation map $ev : X \times \mathcal{A} \to \text{Tot}(c_D)$ is smooth and surjective. If $J_A$ denotes the pull-back of $J_D$ over $X \times \mathcal{A}$, then for each $a \in \mathcal{A}(\overline{k})$, $J_a$ acts on any Hitchin pair representing a point of the fiber $\mathcal{M}_a$. And if $\pi_A : X \times \mathcal{A} \to \mathcal{A}$ denotes the projection, then the relative Picard scheme $\mathcal{P}/\mathcal{A} = \text{Pic}(J_A/\mathcal{A})$ acts naturally on the $\mathcal{A}$-scheme $\mathcal{M}$.

**Proof:** The only assertion not yet discussed is that $ev : X \times \mathcal{A} \to \text{Tot}(c_D)$ is smooth and surjective. This however follows from the fact that $\text{Tot}(c_D)$ is
the total space of a direct sum of very ample line bundles over \( X \), having \( A \) as its space of sections.

Let \( \mathcal{M}^{\text{reg}} \subset \mathcal{M} \) represent the set of Hitchin pairs \((E, \phi)\) for which \( \phi \) takes values in the regular elements of \( D \otimes \text{Ad}(E) \) (this makes sense because the set of regular element in \( g \) is invariant under scalar multiplication). This is clearly an open subset of \( \mathcal{M} \). The following proposition generalizes the BDR-correspondence.

**Proposition 4.2.** The open subset \( \mathcal{M}^{\text{reg}} \) maps onto \( A \) and is a torsor of the relative Picard scheme \( \mathcal{P}/A \).

*Sketch of proof.* The surjectivity of \( \mathcal{M}^{\text{reg}} \to A \) is established by means of a Kostant section.

In order to prove the second assertion, let us begin with making the following observation. Let \( I \subset G \) be the centralizer of a regular element of \( g \). We know that \( I \) is then a connected abelian group of dimension \( r \). We regard \( G \) as a homogeneous space over \( I \times G \) with \( G \) acting by right translations and \( I \) by left translations. This makes \( I \) an automorphism group of \( G \), when the latter is regarded as a torsor over itself. If \( E \) is a variety with \( I \times G \)-action isomorphic to \( G \), then two \( I \times G \)-equivariant isomorphisms \( G \cong E \) will differ by an automorphism of \( G \) which centralizes \( I \), hence is contained in \( I \) in \( \text{Aut}(G) \). In other words, the set of such isomorphisms is a \( I \)-torsor.

Now let \((E, \phi)\) and \((E', \phi')\) be Hitchin pairs representing points of \( \mathcal{M}^{\text{reg}} \) over the same point \( a \) of \( A \). Then we have morphisms \( J_a \to \text{Aut}(E, \phi) \) and \( J_a \to \text{Aut}(E', \phi') \). For \( U \) open in \( X \), consider the set of isomorphisms \( E|_U \cong E'|_U \) with \( J_a|_U \) action. This defines a presheaf whose associated sheaf is a \( J_a \)-torsor. One then verifies that \((E', \phi')\) is obtained from \((E, \phi)\) by a twist with this torsor.

**Some dimension computations.** We compute the dimensions of some of the spaces that appear in the definition of the Hitchin space. From now on, we assume that \( \deg D > 2g \).

If \( E \) is a general \( G \)-torsor, then \( \text{Ad}(E) \) has no nonzero global sections. Kodaira-Spencer theory tells us that the moduli stack of \( G \)-torsors is smooth at such a \( E \) with tangent space identified with \( H^1(X, \text{Ad}(E)) \). A multiple of the Killing form identifies the vector bundle \( \text{Ad}(E) \) with its dual. This implies that \( \text{Ad}(E) \) has degree zero and so if we invoke Riemann-Roch, we find

\[
\dim H^1(X, \text{Ad}(E)) = (g - 1) \dim G.
\]

Given this \( E \), then Riemann-Roch also gives

\[
\dim H^0(X, D \otimes \text{Ad}(E)) = (1 - g + \deg D) \dim G,
\]

and so we find that

\[
\dim \mathcal{M} = \deg D \dim G.
\]

We verify that this is also \( \dim A + \dim \text{Pic}(J_a) \) for a general \( a \in A(\bar{k}) \).
Since $A \cong H^0(X, D^e_1)$, it follows from Riemann-Roch that
\[ \dim A = r_0 + r(1 - g) + \sum_i e_i \deg D = r_0 + r(1 - g) + \frac{1}{2}(\dim G + r) \deg D, \]
where $r_0$ is the number of $i$'s with $e_i = 1$ (this is also the dimension of the center of $G$).

**Lemma 4.3.** The Lie algebra scheme $\text{Lie}(J_D)$ on $\text{Tot}(c_D)$ can be identified with the pull-back of $D^{1-e_1} \oplus \cdots \oplus D^{1-e_r}$.

**Proof.** Recall that $\text{Lie}(J) = (\pi_*(\mathcal{O}_E \otimes t))^W$. A suitable multiple of the Killing form identifies $t$ with $t^*$ as a $W$-module and so $\text{Lie}(J) = (\pi_*(\mathcal{O}_E \otimes t^*))^W$. The latter can also be written $(\pi_*(\Omega_L))^W$ and this, in turn, can be identified with $\Omega_c$, because both have $d\chi_1, \ldots, d\chi_r$ as a basis. In the present setting this yields the identification
\[ (\pi_*(\mathcal{O}_{\text{Tot}(tD)} \otimes t^* \otimes D^{-1}))^W \cong (\pi_*\Omega_{tD/X})^W \cong \Omega_{cD/X}. \]
The left hand side can be written as $(\pi_*(\mathcal{O}_{\text{Tot}(tD)} \otimes t^*))^W \otimes D^{-1}$, which we recognize as $\text{Lie}(J_D) \otimes D^{-1}$. The right hand side is isomorphic to the pull-back of $D^{-e_1} \oplus \cdots \oplus D^{-e_r}$. $\square$

**Corollary 4.4.** We have $\dim \text{Pic}(J_a) = -r_0 + (g - 1)r + \frac{1}{2}(\dim G - r) \deg D$ for all $a \in A$.

**Proof.** The general theory tells us that the tangent space of $\text{Pic}(J_a)$ at the identity element can be identified with $H^1(X, \text{Lie}(J_a))$. The previous lemma identifies this with $\oplus_{i=1}^r H^1(X, D^{1-e_i})$. The assertion now follows from an application of Riemann-Roch. $\square$

So $\dim \text{Pic}(J_a) + \dim A = \dim G \deg D$ is indeed equal to $\dim \mathcal{M}$.

5. CAMERAL COVERS

Over $\text{Tot}(c_D)$ we have the $W$-covering $\text{Tot}(t_D)$ whose total space is also smooth. Let us denote by $\widetilde{X \times A}$ the pull-back of this cover along the evaluation map:
\[ \begin{array}{ccc}
\widetilde{X \times A} & \longrightarrow & \text{Tot}(t_D) \\
\pi_{\widetilde{X \times A}} & \downarrow & \downarrow \pi_D \\
X \times A & \xrightarrow{\text{ev}} & \text{Tot}(c_D).
\end{array} \]
The vertical maps are $W$-covers and the horizontal maps are smooth surjective. In particular, $\widetilde{X \times A}$ is smooth and connected. We may refer to the left vertical map as the *universal cameral cover*. We can recover the Lie algebra of the abelian group scheme $J_A$ over $X \times A$ as
\[ \text{Lie}(J_A) = (\pi_{\widetilde{X \times A}}^* \mathcal{O}_{\widetilde{X \times A}} \otimes t)^W. \]
Now consider the projection $\tilde{X} \times A \to A$. For $a \in A(\bar{k})$, the above diagram restricts to

\[
\begin{array}{ccc}
\tilde{X}_a & \to & \text{Tot}(t_D) \\
\| & & \downarrow \\
X & \xrightarrow{a} & \text{Tot}(c_D).
\end{array}
\]

The left vertical map is called the \textit{cameral cover} attached to $a$. It is clear that $\tilde{X}_a$ is a complete intersection in the smooth $\tilde{X} \times A$. So the generic fiber is smooth and by a Lefschetz type theorem connected. A form of Zariski’s main theorem implies that then in fact all fibers $\tilde{X}_a$ are connected. A simple local computation also shows that $\tilde{X}_a$ is smooth precisely when $a$ is transversal to $\text{Disc}(t_D/c_D)$ in the sense that $a^* \text{Disc}(t_D/c_D)$ is reduced. The corresponding locus in $A$ is therefore the projection in $A$ of $\text{ev}^{-1}(\text{Tot}(c_D) - \text{Disc}(t_D/c_D))$ and hence open-dense. We denote it by $A^{\triangleright}$.

If $a(X) \not\subset \text{Disc}(t_D/c_D)$, then $\tilde{X}_a$ is reduced (and vice versa); we denote the locus in $A$ defined by that property by $A^{\lozenge}$. It is also open in $A(\bar{k})$ and clearly

$$A^{\triangleright} \subset A^{\lozenge} \subset A(\bar{k}).$$

**References**

