A LIE ALGEBRA ATTACHED TO A PROJECTIVE VARIETY

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Abstract. Each choice of a Kähler class on a compact complex manifold defines an action of the Lie algebra $\mathfrak{sl}(2)$ on its total complex cohomology. If a nonempty set of such Kähler classes is given, then we prove that the corresponding $\mathfrak{sl}(2)$-copies generate a semisimple Lie algebra. We investigate the formal properties of the resulting representation and we work things out explicitly in the case of complex tori, hyperkähler manifolds and flag varieties. We pay special attention to the cases where this leads to a Jordan algebra structure or a graded Frobenius algebra.

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Introduction

Let X be a projective manifold of complex dimension n. If $\kappa \in \mathrm{H}^2(X)$ is an ample class, then cupping with it defines an operator e_κ in the total complex cohomology (denoted here by $\mathrm{H}(X)$) of degree 2 and the hard Lefschetz theorem asserts that for $s = 0, \dots, n$, e_κ^s maps $\mathrm{H}^{n-s}(X)$ isomorphically onto $\mathrm{H}^{n+s}(X)$. As is well-known, this is equivalent to the existence of a (unique) operator f_κ in $\mathrm{H}(X)$ of degree -2 such that the commutator $[e_\kappa, f_\kappa]$ is the operator h which on $\mathrm{H}^k(X)$ is multiplication by $k-n$. The elements e_κ, h, f_κ make up a Lie subalgebra $\mathfrak{g}_\kappa$ of $\mathfrak{gl}(\mathrm{H}(X))$ isomorphic to $\mathfrak{sl}(2)$ and the decomposition of $\mathrm{H}(X)$ as a $\mathfrak{g}_\kappa$-module into isotypical summands is just the primitive decomposition: the primitive cohomology in degree $n-s$ generates the isotypical summand associated to the irreducible representation of dimension $s+1$. As these operators respect the Hodge decomposition (in the sense that e_κ resp. f_κ has bidegree $(1,1)$ resp. $(-1,-1)$), the Hodge structure on $\mathrm{H}(X)$ is entirely determined by the Hodge structure on the primitive cohomology. However, the primitive decomposition usually depends in a nontrivial way on the choice of

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κ. This we regard as a fortunate fact, as it often leads to finding an even smaller Hodge substructure of $\mathrm{H}(X)$ that determines the one on $\mathrm{H}(X)$. To be explicit, let us define the *Néron–Severi Lie algebra* $\mathfrak{g}_{NS}(X)$ as the Lie subalgebra of $\mathfrak{gl}(\mathrm{H}(X))$ generated by the $\mathfrak{g}_\kappa$'s with κ an ample class. This Lie algebra is defined over $\mathbb{Q}$ and is evenly graded by the adjoint action of its semisimple element h (with its degree $2k$ summand acting as transformations of bidegree (k,k)). We prove in this paper that it is also semisimple. So if we regard $\mathrm{H}(X)$ as a representation of this Lie algebra, then the subspace of $\mathrm{H}(X)$ annihilated by the negative degree part of $\mathfrak{g}_{NS}(X)$ is a Hodge substructure that determines the one on $\mathrm{H}(X)$. Notice that this Hodge substructure is itself still invariant under the degree zero part of $\mathfrak{g}_{NS}(X)$ (which is a reductive Lie subalgebra). Despite its naturality, this idea appears to be new (although a note by [Verbitsky 1990], of which we were not aware of when we started this research, is suggestive in this repect).

Whereas the e_κ's commute, the corresponding f_κ's don't in general. This makes it difficult to compute the Néron–Severi Lie algebra in practice. It is often helpful when we know of a morphism from X to another projective manifold Y whose base and fibers are well-understood: for example, the fact that the associated Leray spectral sequence degenerates yields (among other things) the existence of a copy of $\mathfrak{g}_{NS}(Y)$ in $\mathfrak{g}_{NS}(X)$. This is an ingredient of our proof that the Néron–Severi Lie algebra of flag variety of a simple complex Lie group is "as big as possible" (reflected by the fact that its Hodge structure is as simple as possible): it is the Lie algebra of infinitesimal automorphisms of a naturally defined bilinear form (which is either symmetric or skew) on its cohomology.

But if the f_κ's happen to commute, then we are in a very interesting situation: the Néron–Severi Lie algebra has degrees -2, 0 and 2 only and the (complexified) Néron–Severi group acquires the structure of a Jordan algebra without preferred unit element. For abelian varieties this is a classical fact, although, as far as we know, it had not been seen from this point of view. The Néron–Severi Lie algebra appears here as a natural companion of the Mumford–Tate group: the latter helps us to find the Hodge ring as a ring of invariants, whereas the decomposition of the Hodge ring into $\mathbb{Q}$-irreducible representations of the Néron–Severi Lie algebra helps us to say more about its structure. (For example, the subring generated by the divisor classes is one such irreducible summand.)

Here are some variants of this construction: instead of working with complex projective manifolds, we could do this for compact complex manifolds that admit a Kähler metric and replace the ample classes by Kähler classes. Or we could even take all cohomology classes of degree 2 that have the Lefschetz property; clearly, the complex structure has now become irrelevant. The resulting Lie algebras are again semisimple and we call them the *Kähler Lie algebra* and the *total Lie algebra* of the manifold respectively. Examples of interest here are complex tori and hyperkähler manifolds; in both cases we get Jordan algebra structures. In a different direction, we can take for X a projective variety and take instead its complex intersection homology, even with values in a variation of polarized Hodge structure.

These examples lead us to formalize the situation by means of what we have called a *Lefschetz module*. This is essentially a graded vector space equipped with a set of commuting degree two operators that have the Lefschetz property, such that the Lie algebra generated by the corresponding $\mathfrak{sl}(2)$-triples is semisimple. So this vector space becomes a representation of a semisimple Lie algebra, and it was one of our goals to classify the representations that so arise. Although we found some rather restrictive properties, we did not succeed in this.

We now briefly describe the contents of the separate sections.

In section 1 we introduce the notion that is central to this paper, that of a Lefschetz module, and discuss its basic properties. If a Lefschetz module has a compatible Hodge structure, as is the case for the cohomology of a projective manifold, then there is also defined its Mumford–Tate group and we compare the two notions. We next define and discuss the closely related notion of a Lefschetz pair. This is followed by a partial classification of such pairs in case the associated Lie algebra is of classical type.

In section 2 we concentrate on the case when the f_κ's commute. We show that the resulting structure is essentially that of a Jordan algebra and that is why a complete classification is available. We are also led to a remarkable class of Frobenius algebras associated to each Jordan algebra, some of which we describe explicitly.

The next two sections are devoted to examples of Kähler manifolds that give rise to Lefschetz modules of Jordan type. First we compute the total Lie algebra and the Kähler Lie algebra of a complex torus. Then we turn our attention to the Néron–Severi Lie algebra of an abelian variety A and express it in terms of the endomorphism algebra of A. We find that that this Néron–Severi Lie algebra intersects $\mathrm{End}(A) \otimes \mathbb{C}$ in a Lie ideal of $\mathrm{End}(A) \otimes \mathbb{C}$ and we describe the complementary ideal.

Our treatment of hyperkähler manifolds (in section 4) follows essentially [Verbitsky 1995], a preprint that in turn is partly based on a preliminary version of the present paper. As an application we show how the Hodge structure on the cohomology algebra of a compact hyperkählerian manifold is expressed in terms of the Hodge structure on its degree two part. We also give an alternative description of the Beauville–Bogomolov quadratic form on the Néron–Severi group.

Thus the abelian varieties and the hyperkähler manifolds produce the classical Jordan algebras. The exceptional Jordan algebra can be realized topologically and we ask whether it is realizable by a Calabi-Yau threefold.

Section 5 is about filtered Lefschetz modules. The example to keep in mind here is the Leray filtration on $\mathrm{H}(X)$ defined by a surjective morphism $f : X \to Y$ of projective manifolds. We apply this to the case where f is a projective space bundle. In combination with a theorem proved in the appendix we are then able to determine the Néron–Severi Lie algebra of a flag variety. It would be interesting to do the same for the intersection homology of Schubert varieties.

In section 6 we investigate another interesting class of Lefschetz modules, which we have called *Frobenius–Lefschetz* modules. These arise as the Lefschetz submodule of the cohomology of a projective manifold generated by its unit element. The Jordan–Lefschetz modules are among them and we suspect that the remaining simple Frobenius–Lefschetz modules are "tautological representations" of orthogonal or symplectic Lie algebras. The main result (6.8) of this section supports this belief: it says that any other simple Frobenius–Lefschetz module must be a representation of an exceptional Lie algebra.

We began this work in the Fall of 1990, when both of us were at the University of Michigan in Ann Arbor. We would like to thank its Mathematics Department for providing so stimulating working conditions. One of us (Looijenga) thanks in particular Igor Dolgachev for many inspiring discussions (then and later) regarding the subject matter of this paper as well as closely related questions. Our work continued during the Spring of 1991, while Looijenga was at the University of Utah. He gratefully remembers the friendly atmosphere he encountered there. After an interruption we resumed work on this paper in the academic year 1994-95. We thank Tonny Springer for some helpful references and Bertram Kostant, Misha Verbitsky and Yuri Zarhin for useful conversations. We are also indebted to D. Huybrechts for drawing our attention to the work of Fujiki and to Gert Heckman for sending us a letter containing interesting examples from Lie theory (to which we allude in (1.13)).

1. Lefschetz modules

(1.1) We fix a field K of characteristic zero. Let $M_\bullet$ be a $\mathbb{Z}$-graded K-vector space of finite dimension and denote by $h : M \to M$ the transformation that is multiplication by k in degree k. So h determines the grading of M, and a linear transformation $u : M \to M$ has degree k if and only if $[h, u] = ku$.

We say that a linear transformation $e : M \to M$ of degree 2 has the *Lefschetz property* if for all integers $k \geq 0$, e^k maps M_{-k} isomorphically onto M_k. According to the Jacobson–Morozov lemma this is equivalent to the existence of K-linear transformation f in M of degree -2 such that $[e, f] = h$. This f is then unique and (e, h, f) is a $\mathfrak{sl}(2)$-triple: the assignment

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto e, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto h, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto f$$

defines a representation of $\mathfrak{sl}(2)$. If h and e happen to be contained in a semisimple Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(M)$, then so is f.

Now let $\mathfrak{a}$ be a finite dimensional K-vector space. We regard $\mathfrak{a}$ as a graded abelian Lie algebra which is homogeneous of degree two. We say that a graded Lie homomorphism $e : \mathfrak{a} \to \mathfrak{gl}(M)$ has the Lefschetz property if for some $a \in \mathfrak{a}$, e_a has that property. Notice that the set of $a \in \mathfrak{a}$ with the Lefschetz property is always Zariski open in $\mathfrak{a}$. For a in this open set, we have defined the operator f_a such that (e_a, h, f_a) is $\mathfrak{sl}(2)$-triple. This defines a rational map $f : \mathfrak{a} \to \mathfrak{gl}(M)$ in the sense of algebraic geometry. We let $\mathfrak{g}(\mathfrak{a}, M)$ denote the Lie subalgebra of $\mathfrak{gl}(M)$ generated by the transformations e_a, f_a. If $\mathfrak{a}$ is merely an abelian group that acts on M by operators of degree 2, then the linear extension $\mathfrak{a} \otimes K \to \mathfrak{gl}(M)$ is a Lie homomorphism and we then often write $\mathfrak{g}(\mathfrak{a}, M)$ for $\mathfrak{g}(\mathfrak{a} \otimes K, M)$. The following example shows that this Lie algebra need not act reductively in M.

Example. Consider the graded $\mathfrak{sl}(2)$-representation $M = \mathfrak{sl}(2) \oplus K^2$, where $\mathfrak{sl}(2) = Ke + Kh + Kf$ has the adjoint representation (with its usual grading) and K^2 is the trivial representation in degree zero. Define an operator e' of degree 2 in M by $e'(xe + yh + zf, u, v) = (ve, z, 0)$. Then $ee' = e'e = 0$, so that e and e' span an abelian Lie algebra $\mathfrak{a}$. Since e has the Lefschetz property, the Lie algebra $\mathfrak{g} = \mathfrak{g}(\mathfrak{a}, M)$ is defined. Now $\mathfrak{a}$ and h kill $(0, 1, 0)$, hence so does all of $\mathfrak{g}$. But the line spanned by this vector has no $\mathfrak{g}$-invariant complement. This example was chosen as to make $\mathfrak{g}$ also infinitesimally preserve a nondegenerate quadratic form on M (namely $(xe+yh+zf,u,v)\mapsto -2xz+y^2+2uv$) (so that $\mathfrak{g}$ is a nonreductive Lie subalgebra of an orthogonal Lie algebra). A smaller example without that property is the submodule $\mathfrak{sl}(2)\oplus K\oplus 0$.

Notice that $\mathfrak{g}(\mathfrak{a},M)$ is evenly graded and that the grading is induced from the action of ad_h. We say that $(\mathfrak{a},M)$ is a *Lefschetz module* if $\mathfrak{g}(\mathfrak{a},M)$ is semisimple. In case $M\neq 0$, we call greatest integer n with $M_n\neq 0$ (or equivalently, $M_{-n}\neq 0$) the *depth* of M. The collection of Lefschetz $\mathfrak{a}$-modules is closed under direct sums, tensor products and taking duals. A Lefschetz $\mathfrak{a}$-module M is irreducible as a Lefschetz module if and only if it is irreducible as a $\mathfrak{g}(\mathfrak{a},M)$-module. Since any representation of a semisimple Lie algebra is reductive, it follows that the category of Lefschetz modules of $\mathfrak{a}$ is semisimple. Notice that a Lefschetz module M always decomposes as $M=M_{\mathrm{ev}}\oplus M_{\mathrm{odd}}$, where M_{ev} (resp. M_{odd}) is the direct sum of the M_k's with k even (resp. odd).

There is also an external direct sum and tensor product: if $(\mathfrak{a}',M')$ and $(\mathfrak{a}'',M'')$ are Lefschetz modules, then we have defined Lefschetz modules

$$\begin{aligned}
(\mathfrak{a}',M')\boxplus(\mathfrak{a}'',M'') :=&(\mathfrak{a}'\times\mathfrak{a}'',M'\oplus M''),\\
&e_{(a',a'')}(m',m'')=(e_{a'}m',e_{a''}m'');\\
(\mathfrak{a}',M')\boxtimes(\mathfrak{a}'',M'') :=&(\mathfrak{a}'\times\mathfrak{a}'',M'\otimes M''),\\
&e_{(a',a'')}(m'\otimes m'')=e_{a'}m'\otimes m''+m'\otimes e_{a''}m''.
\end{aligned}$$

The associated Lie algebra is in the first case equal to $\mathfrak{g}(\mathfrak{a}',M')\times\mathfrak{g}(\mathfrak{a}'',M'')$. This is also true in the second case if both factors are nonzero.

The preceding discussion showed that when studying Lefschetz modules we may restrict ourselves to irreducible ones. The following lemma allows the further reduction of having the associated Lie algebra *simple*.

(1.2) Lemma. *Let M be an irreducible Lefschetz $\mathfrak{a}$-module and let $\mathfrak{g}(\mathfrak{a},M)=\mathfrak{g}'\times\mathfrak{g}''$ be a decomposition of Lie algebras. Then this decomposition is graded and there exist irreducible Lefschetz $\mathfrak{a}$-modules M' and M'' such that $M\cong M'\otimes M''$ as Lefschetz $\mathfrak{a}$-modules with $\mathfrak{g}'$ resp. $\mathfrak{g}''$ corresponding to $\mathfrak{g}(\mathfrak{a},M')$ resp. $\mathfrak{g}(\mathfrak{a},M'')$.*

Proof. Since the grading of $\mathfrak{g}$ is the eigen space decomposition of ad_h it is immediate that upon writing $h=(h',h'')\in\mathfrak{g}'\times\mathfrak{g}''$, $\mathfrak{g}^{(i)}$ gets a grading from $\mathrm{ad}_{h^{(i)}}$ making the decomposition a graded one.

Since M is an irreducible module of the semisimple Lie algebra $\mathfrak{g}(\mathfrak{a},M)$, it must have the form $M'\otimes M''$ with $M^{(i)}$ a $\mathfrak{g}^{(i)}$-module. This is compatible with the gradings. If the rational map $f:\mathfrak{a}\to\mathfrak{g}_{-2}=\mathfrak{g}'_{-2}\oplus\mathfrak{g}''_{-2}$ is written (f',f''), then $[h,f]=-2f$ implies $[h',f']=-2f'$ and $[h'',f'']=-2f''$. So $M^{(i)}$ a Lefschetz module of $\mathfrak{a}$ with the stated property.

(1.3) Given a Lefschetz module M of $\mathfrak{a}$, then an invariant bilinear form on M is a bilinear map $\phi:M\times M\to K$ that defines a morphism of Lefschetz modules $M\otimes M\to K$ (where $\mathfrak{a}$ acts trivially on K): so ϕ is zero on $M_k\times M_l$ unless $k+l=0$ and $\mathfrak{a}$ preserves the form ϕ infinitesimally: $\phi(e_am,m')+\phi(m,e_am')=0$

for all $m, m' \in M$ and $a \in \mathfrak{a}$. If a is a Lefschetz element, then the Jacobson–Morozov lemma implies that f_a also preserves ϕ infinitesimally. So $\mathfrak{g}(\mathfrak{a}, M)$ is then a subalgebra of $\mathfrak{aut}(M, \phi)$. If ϕ is nondegenerate and symmetric (resp. skew-symmetric), then we call (M, ϕ) an *orthogonal* (resp. *symplectic*) representation. Since a nonzero invariant bilinear form on an irreducible representation is either orthogonal or symplectic, any Lefschetz module with nondegenerate bilinear form is the perpendicular direct sum of Lefschetz modules that are irreducible orthogonal, irreducible symplectic, or the direct sum of an irreducible Lefschetz module with its dual.

(1.4) Many Lefschetz modules have the additional structure of an algebra. Let $A = \oplus_{i=0}^{2n} A_i$ be a graded-commutative algebra with $A_0 = K$. We say that A is a *Lefschetz algebra of depth* n if $A[n]$ is a Lefschetz module of depth n over A_2. Such a Lefschetz module can be endowed with an invariant $(-)^n$-symmetric bilinear form: let $\int : A \to K$ be a linear form that is an isomorphism in degree $2n$ and zero in all other degrees and define $\phi(a, b) := (-1)^q \int(ab)$ if a is homogeneous of degree $n + 2q$ or $n + 2q + 1$. If this form is nondegenerate (which is for instance the case when $A[n]$ is irreducible as a Lefschetz module), then the form $(a, b) \mapsto \int(ab)$ is also nondegenerate and so A becomes a Frobenius algebra (in the graded sense).

(1.5) Let M be a graded real vector space. A *Hodge structure of total weight* d on M consists of a bigrading on its complexification: $M \otimes \mathbb{C} = \oplus_{p,q\in\mathbb{Z}} M^{p,q}$ such that (i) $M_k \otimes \mathbb{C} = \oplus_{p+q=k+d} M^{p,q}$ for all k and (ii) complex conjugation interchanges $M^{p,q}$ and $M^{q,p}$. These data are conveniently described in terms of an action of the *Deligne torus* on M. We recall [Deligne 1979] that this is two-dimensional torus $\mathbf{S}$ defined over $\mathbb{R}$ that is obtained from $\mathrm{GL}(1)$ by restricting scalars from $\mathbb{C}$ to $\mathbb{R}$. It comes with two characters z, $\bar{z}$ that are each others complex conjugate and generate the character group. Their product is for obvious reasons called the *norm character* and is denoted Nm. There is also a natural homomorphism $w : \mathrm{GL}(1) \to \mathbf{S}$ which on the real points is given by the inclusion $\mathbb{R}^\times \subset \mathbb{C}^\times = \mathbf{S}(\mathbb{R})$. We follow Deligne's convention by letting $\mathbf{S}$ act on M so that $M^{p,q}$ becomes the eigen space of $z^{-p}\bar{z}^{-q}$ (this action is defined over $\mathbb{R}$). A positive number $t > 0$, viewed as an element of $\mathbb{C}^\times = \mathbf{S}(\mathbb{R})$, acts on M_k as multiplication by t^{-k-d}. So $w(t)$ acts on M as $t^{-d-h} (= t^{-d} \exp(-\log(t)h))$.

The action of $\sqrt{-1} \in \mathbb{C}^\times = \mathbf{S}(\mathbb{R})$ on $M_\mathbb{C}$ is on $M^{p,q}$ multiplication by $(\sqrt{-1})^{q-p}$; it is a real operator, called the *Weil operator*; we denote it by J.

Suppose we are also given a nondegenerate $(-)^d$-symmetric form $\phi : M \times M \to \mathbb{R}$ that is zero on $M^{p,q} \times M^{p',q'}$ unless $(p + p', q + q') = (d, d)$ (this is equivalent to $\phi(gm, gm') = \mathrm{Nm}(g)^{-d} \phi(m, m')$ for all $g \in \mathbf{S}(\mathbb{C})$). Let $e : M \to M$ be a real operator of bidegree $(1, 1)$ which preserves ϕ infinitesimally. Clearly, e commutes with J and it is easily checked that for $k \geq 0$, the map $H_e^k : M_{-k} \otimes \mathbb{C} \times M_{-k} \otimes \mathbb{C} \to \mathbb{C}$ defined by

$$H_e^k(m, m') := \phi(e^k m, J\overline{m'}),$$

is a Hermitian form. We say that e is a *polarization* of (M, ϕ) if for all $k \geq 0$, H_e^k is definite on $\mathrm{Ker}(e^{k+1} | M_{-k} \otimes \mathbb{C})$.

(1.6) Proposition. *Let (M, ϕ) be as above. Let $\mathfrak{a}$ be a real abelian Lie algebra with a pure weight two Hodge structure that acts morphically on (M, ϕ) (i.e.,* *the action is by mutually commuting transformations of degree 2 that preserve ϕ infinitesimally and with $\mathfrak{a}\otimes M \to M$ a morphism of Hodge structures). Assume that for some $a \in \mathfrak{a}^{1,1}$, e_a polarizes (M,ϕ). Then M is a Lefschetz module of $\mathfrak{a}$ and $\mathfrak{g}(\mathfrak{a},M)$ is a semisimple Lie algebra defined over $\mathbb{R}$ that preserves ϕ infinitesimally.*

Proof. The nondegenerateness of the Hermitian form H_a^k on $\mathrm{Ker}(e_a^{k+1}|M_{-k}\otimes\mathbb{C})$ implies that e_a^k is injective on this subspace. It is easily checked that this, together with the nondegenerateness of ϕ, implies that e_a satisfies the Lefschetz property. So $\mathfrak{g}(\mathfrak{a},M)$ is defined. If we regard ϕ as an element of $M^*\otimes M^*$ of degree zero, then the fact that ϕ is killed by e_a implies that it is killed by f_a. So $\mathfrak{g}(\mathfrak{a},M)$ preserves ϕ infinitesimally.

We next show that $\mathfrak{g}(\mathfrak{a},M)$ is reductive; since $\mathfrak{g}(\mathfrak{a},M)$ is generated by commutators, it then follows that $\mathfrak{g}(\mathfrak{a},M)$ semisimple. To this end we observe that the image of $\mathfrak{a}$ in $\mathfrak{gl}(M)$ is normalized by the Weil operator J. So the same is true for $\mathfrak{g}(\mathfrak{a},M)$. As J is semisimple, it is therefore enough to show that any subspace $N\subset M\otimes\mathbb{C}$ that is invariant under both $\mathfrak{g}(\mathfrak{a},M)$ and J is nondegenerate with respect to ϕ: then its ϕ-perpendicular space will be an invariant complement. Consider the primitive decomposition of N with respect to e_a: $N=\oplus_{k\geq 0}\mathbb{C}[e_a]P_{-k}(N)$, where $P_{-k}(N) := \mathrm{Ker}(e_a^{k+1}|N_{-k})$. This decomposition is ϕ-perpendicular and so we need to show that ϕ is nondegenerate on each summand $\mathbb{C}[e_k]P_{-k}(N)$. For this we observe that $P_{-k}(N)$ is J-invariant. Since H_a^k is definite on $P_{-k}(N)$, it follows from the definition of H_a^k that ϕ is nondegenerate on $P_{-k}(N)+e^kP_{-k}(N)$. The fact that e_k leaves ϕ infinitesimally invariant then implies that ϕ is nondegenerate on $\mathbb{C}[e_k]P_{-k}(N)$.

(1.7) We briefly explain the relation between $\mathfrak{g}(\mathfrak{a},M)$ and the Mumford–Tate group. For this we have to assume that M, its grading, $\mathfrak{a}$, and the action of $\mathfrak{a}$ on M are all defined over $\mathbb{Q}$. Then $\mathfrak{g}(\mathfrak{a},M)$ is as a Lie subalgebra of $\mathfrak{gl}(M)$ also defined over $\mathbb{Q}$. We further assume that $\mathfrak{a}$ acts by transformations of bidegree $(1,1)$. Then $\mathfrak{g}(\mathfrak{a},M)_{2k}$ acts by transformations of bidegree (k,k), in other words, for all $g\in\mathbf{S}$ and $x\in\mathfrak{g}(\mathfrak{a},M)$ we have $gxg^{-1}=\mathrm{Nm}(g)^{-h}$.

Consider the image of $\mathbf{S}$ in $\mathrm{GL}(M)\times\mathrm{GL}(1)$, where the second map is given by the norm. One defines the *Mumford–Tate group* of M, $\mathrm{MT}(M)$, as the smallest $\mathbb{Q}$-subgroup of $\mathrm{GL}(M)\times\mathrm{GL}(1)$ containing this image. It is clear that this is actually a subgroup of $(\times_k \mathrm{GL}(M_k))\times\mathrm{GL}(1)$. The projection of $\mathrm{MT}(M)$ onto the last factor is still called the *norm character* and denoted likewise. The identity

$$gxg^{-1}=\mathrm{Nm}(g)^{-h}$$

is now valid for all $g\in\mathrm{MT}(M)$ and $x\in\mathfrak{g}(\mathfrak{a},M)$. This shows in particular that the adjoint action of $\mathrm{MT}(M)$ on $\mathfrak{gl}(M)$ leaves $\mathfrak{g}(\mathfrak{a},M)$ invariant.

(1.8) This suggests to combine the Mumford–Tate group and the group associated to the Lie algebra $\mathfrak{g}(\mathfrak{a},M)$ into a single group: if $G(\mathfrak{a},M)$ denotes the closed connected subgroup of $\mathrm{GL}(M)$ with Lie algebra $\mathfrak{g}(\mathfrak{a},M)$, then $\mathrm{MT}(M)G(\mathfrak{a},M)$ is a reductive algebraic group defined over $\mathbb{Q}$. In this set up the the rôle of the Deligne torus is played by the semidirect product $\mathbf{S}\ltimes SL(2)$, where $s\in\mathbf{S}$ acts on $SL(2)$ as conjugation by the diagonal matrix $\mathrm{diag}(z(s)^{-1},\bar{z}(s))$. So the $\mathbb{R}$-homomorphism $w:\mathrm{GL}(1)\to\mathbf{S}\ltimes SL(2)$, which on the real points is given by

$t \in \mathbb{R}^\times \mapsto (t, \mathrm{diag}(t, t^{-1}))$, maps onto a central subgroup. A polarized Hodge structure of weight d on (M, ϕ) can now be thought of as a certain representation of this group on M with $w(t)$ acting as multiplication by t^{-d}.

The corresponding action of $\mathbf{S}$ on $\mathfrak{sl}(2)$ is given by $s \mapsto (-z(s) - \bar{z}(s))\,\mathrm{ad}_h$, so that for the resulting Hodge structure on $\mathfrak{sl}(2)$, the bidegrees of e, h, f are $(1,1)$, $(0,0)$, $(-1,-1)$ respectively. In this spirit one can also enhance the notion of a set of Shimura data, as defined by [Deligne 1979].

(1.9) It is high time to give the examples that motivated the preceding definitions. Let X be a compact Kählerian manifold of dimension n. We take for M its shifted total complex cohomology $\mathrm{H}(X)[n]$ and we let ϕ be defined by

$$\phi(\alpha, \beta) := (-1)^q \int_X \alpha \cup \beta$$

if α is homogeneous of degree $n + 2q$ or $n + 2q + 1$. (We shall always suppose that $\mathrm{H}(X)$ is equipped with this form and $\mathfrak{aut}\,\mathrm{H}(X)$ will stand for the Lie algebra of endomorphisms of $\mathrm{H}(X)$ that preserve this form infinitesimally.) The fundamental theorems of Hodge theory tell us that M comes with a Hodge structure of total weight n and that ϕ together with cupping with a Kähler class defines a polarization of M. So by proposition (1.6) $\mathrm{H}(X)[n]$ is a Lefschetz module over $\mathrm{H}^2(X)$. The corresponding semisimple Lie subalgebra of $\mathfrak{aut}\,\mathrm{H}(X)$ will be called the *total Lie algebra* of M and be denoted $\mathfrak{g}_{\mathrm{tot}}(X)$; it is defined over $\mathbb{Q}$. It is equivalent to say that the cohomology algebra of X is a Lefschetz algebra. Clearly, $\mathfrak{g}_{\mathrm{tot}}(X)$ is independent of the complex structure. For example, if X is a product of an even number of circles, then $\mathfrak{g}_{\mathrm{tot}}(X)$ is defined.

If we want to take the complex structure into account, then it is more natural to regard $\mathrm{H}(X)[n]$ as module over $\mathrm{H}^{1,1}(X)$. This is by (1.6) also a Lefschetz module structure. We shall refer to the associated Lie algebra as the *Kähler Lie algebra* of X and denote it by $\mathfrak{g}_K(X)$. For a complex projective manifold we can restrict further and take for $\mathfrak{a}$ the Néron–Severi group $\mathrm{NS}(X)$. We call the corresponding semisimple Lie algebra the *Néron–Severi Lie algebra* of X (denoted $\mathfrak{g}_{NS}(X)$). It is defined over $\mathbb{Q}$. Notice that the Néron–Severi Lie algebra and the Mumford–Tate group behave in opposite ways under specialization: the former gets bigger, whereas the latter gets smaller.

If one of these Lie algebras $\mathfrak{g}_*(X)$ is defined over a subfield $K \subset \mathbb{C}$, then we often write $\mathfrak{g}_*(X; K)$ for the corresponding Lie algebra of K-points.

(1.10) Here is another example. Let V be a complex vector space and $W \subset \mathrm{GL}(V)$ a finite complex reflection group acting effectively (that is, $V^W = \{0\}$). This group acts naturally in the symmetric algebra of $\mathrm{Sym}(V)$. According to a theorem of Chevalley, the subalgebra of invariants, $\mathrm{Sym}(V)^W$ is a polynomial algebra on $\dim(V)$ homogeneous generators. Let I be the ideal generated by the invariants of positive degree. Then the quotient $\mathrm{Sym}(V)/I$ is a graded complete intersection algebra of finite dimension. As a W-representation it is isomorphic to the regular representation. In case W is a Weyl group, then after doubling the degrees, $\mathrm{Sym}(V)/I$ has the interpretation of the cohomology algebra of a flag variety. From this we see that for a suitable regrading, $\mathrm{Sym}(V)/I$ is a Lefschetz representation for the obvious action of V. (We do not know whether this is true for an arbitrary reflection group.) We shall determine the Lie algebra $\mathfrak{g}(V, \mathrm{Sym}(V)/I)$ in (5.8).

(1.11) The Néron–Severi Lie algebra can also be defined when X is an irreducible projective variety: take for M the total intersection cohomology $\mathrm{IH}(X)$ with the same shift in the grading. There is in general no such thing as a cup product on this graded vector space, but the cohomology ring of X acts on M via the natural map $H(X) \to IH(X)$, and according to [Saito] polarizations have the Hodge–Lefschetz property. This even extends to the case where we take intersection cohomology with values in a local system defined on a Zariski open-dense subset that underlies a polarized variation of Hodge structure. There is an invariant form ϕ defined as in the previous example. The natural setting here is that of polarizable Hodge modules.

We return to general properties of Lefschetz modules.

(1.12) Proposition. *The Lie algebra $\mathfrak{g}(\mathfrak{a}, M)$ is a Lefschetz module of $\mathfrak{a}$ via the adjoint action. If $a \in \mathfrak{a}$ is such that f_a is defined, then the Lie subalgebra $\mathfrak{g}(\mathfrak{a}, M)_{\geq 0}$ (resp. $\mathfrak{g}(\mathfrak{a}, M)_{\leq 0}$) is generated by $\mathfrak{g}(\mathfrak{a}, M)_0$ and e_a (resp. f_a). We have*

$$U\mathfrak{g}(\mathfrak{a}, M) = U(\mathfrak{g}(\mathfrak{a}, M)_{>0}).U(\mathfrak{g}(\mathfrak{a}, M)_0).U(\mathfrak{g}(\mathfrak{a}, M)_{<0}).$$

Proof. The first statement is clear and the second follows from this. The third is clear also.

We define the *primitive subspace* of M as the set of vectors killed by $\mathfrak{g}(\mathfrak{a}, M)_{<0}$. It is a graded $\mathfrak{g}(\mathfrak{a}, M)_0$-subrepresentation of M that we denote by $\mathrm{Prim}(M)$. Since $\mathfrak{g}(\mathfrak{a}, M)_{<0}$ is nilpotent, $\mathrm{Prim}(M) \neq 0$. The previous proposition yields:

(1.13) Corollary. *In the situation of the previous proposition, the primitive subspace $\mathrm{Prim}(M)$ is the maximal $\mathfrak{g}(\mathfrak{a}, M)_0$-invariant subspace contained in $\mathrm{Ker}(f_a)$. Hence M is irreducible as a $\mathfrak{g}(\mathfrak{a}, M)$-representation if and only if $\mathrm{Prim}(M)$ is irreducible as a $\mathfrak{g}(\mathfrak{a}, M)_0$-representation (and if $M \neq 0$, then $\mathrm{Prim}(M)$ is the summand of M of lowest degree).*

Proposition (1.12) suggests to shift, in Tannakian spirit, the emphasis from modules to Lie algebras. For suppose that we are given a semisimple Lie algebra $\mathfrak{g}$, a simple element $h \in \mathfrak{g}$ (in the sense of appearing as the middle element of an $\mathfrak{sl}(2)$-triple) and an abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ such that

(i) the adjoint representation of $\mathfrak{g}$ makes $\mathfrak{g}$ a Lefschetz module over $\mathfrak{a}$, i.e., there is a rational map $f : \mathfrak{a} \to \mathfrak{g}_{-2}$ so that for e in the domain of f, we have an $\mathfrak{sl}(2)$-triple (e, h, f_e) and

(ii) $\mathfrak{g}$ is as a Lie algebra generated by $\mathfrak{a}$ and the image of f.

If M is a finite dimensional representation of $\mathfrak{g}$, then h determines a grading of M and every $e \in \mathfrak{a}$ in the domain of f has the Lefschetz property in M with respect to this grading. So M is then a Lefschetz module of $\mathfrak{a}$. Since $\mathfrak{g}$ is generated by $\mathfrak{a}$ and the image of f, it follows that $\mathfrak{g}(\mathfrak{a}, M)$ is just the image of $\mathfrak{g}$ in $\mathfrak{gl}(M)$. One sees that every Lefschetz module arises way. So this reduces the classification of Lefschetz modules to classifying triples $(\mathfrak{g}, h, \mathfrak{a})$ as above. We shall call such a triple a *Lefschetz triple* and its first two items, $(\mathfrak{g}, h)$, a *Lefschetz pair*. If we are given a Lefschetz pair $(\mathfrak{g}, h)$, then we say that an associated Lefschetz triple $(\mathfrak{g}, h, \mathfrak{a})$ is *saturated* if $\mathfrak{a}$ is maximal for this property. The stabilizer G_h of h in the adjoint group G permutes these. At some point we hoped that the number of orbits would be finite, but Gert Heckman recently showed us examples for which there are curves of orbits.

(1.14) Let $(\mathfrak{g}, h)$ be a Lefschetz pair. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that contains h. It is clear that then $\mathfrak{h} \subset \mathfrak{g}_0$. Let $R \subset \mathfrak{h}^*$ denote the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$ and let R_k be the set of $\alpha \in R$ such that $\alpha(h) = k$, or equivalently, $\mathfrak{g}^\alpha \subset \mathfrak{g}_k$ (remember that only even values of k occur). The subset R_0 is a closed root subsystem of R; it is the set of roots of $\mathfrak{h}$ in $\mathfrak{g}_0$. Choose a root basis $B \subset R$ such that $\alpha(h) \geq 0$ for all $\alpha \in B$ (we then say that $\mathfrak{h}$ and B are *adapted to* h). Since h is the semisimple element of a $\mathfrak{sl}(2)$-triple, the elements of B take on h values in $\{0, 1, 2\}$ [Bourbaki], Ch. VIII, §11, Prop. 5. Since these are also even, we get a decomposition $B = B_0 \sqcup B_2$. Clearly B_0 will be a root basis of R_0. According to [Bourbaki], Ch. VIII, §11, Prop. 8 all $\mathfrak{sl}(2)$-triples with h as semisimple element are conjugate under the stabilizer G_h of h in the adjoint group G of $\mathfrak{g}$. So if (e, h, f) is such a triple and M is any representation of $\mathfrak{g}$, then the isomorphism class of M as a representation of this $\mathfrak{sl}(2)$-copy only depends on $(\mathfrak{g}, h)$. We call it the $\mathfrak{sl}(2)$-*type* of M. The following property narrows down the possible subsets $B_2 \subset B$. Let $V(k)$ denote the standard irreducible representation of $\mathfrak{sl}(2)$ of dimension $k+1$ $(k = 1, 2, \dots)$.

(1.15) Proposition. *Let $(\mathfrak{g}, h)$ be a Lefschetz pair and let M be an irreducible representation of $\mathfrak{g}$ of depth n. Then the dimensions of the irreducible $\mathfrak{sl}(2)$-representations that occur in the $\mathfrak{sl}(2)$-type of M make up an arithmetic progression with increment 2. In other words, there exists an integer r with $0 \leq r \leq \lfloor \frac{1}{2} n \rfloor$ such that $\dim M_{-n} < \dim M_{-n+2} < \cdots < \dim M_{-n+2r} = \dim M_{-n+2r+2} = \cdots = \dim M_{n-2r} < \dim M_{n-2r+2} < \cdots < \dim M_n$. Moreover, $r > 0$ unless (i) the image of $\mathfrak{g}$ in $\mathfrak{gl}(M)$ is reduced to $\mathfrak{sl}(2)$ with $M \cong V(n)$ or (ii) the $\mathfrak{sl}(2)$-type of M consists of a number of copies of $V(1)$.*

Proof. Denote this set of dimensions by I. The irreducibility of M implies that the elements of I all have the same parity. Suppose I is not an arithmetic progression with increment 2. Then there exists an integer k such that M contains $V(k)$ (the standard irreducible $\mathfrak{sl}(2)$-representation of dimension $k+1$) and $V(k+2l)$ for some $l \geq 2$, but not $V(k+2)$. Let (e, h, f) be an $\mathfrak{sl}(2)$-triple in $\mathfrak{g}$ containing h and decompose M as $M = M' \oplus M''$ with M' resp. M'' the sum of the irreducible subrepresentations of $\mathfrak{sl}(2)$ of dim $\leq k+1$ resp. $\geq k+5$. Any linear transformation in M of degree two that commutes with e must preserve this decomposition, because any $K[e]$-linear homomorphism

$$V(n) \cong K[e]/(e^{n+1})[n] \to K[e]/(e^{m+1})[m] \cong V(m)$$

of degree two is zero if $|n - m| > 2$. If $(\mathfrak{g}, h, \mathfrak{a})$ is a Lefschetz triple with $e \in \mathfrak{a}$, then this applies in particular to any $e' \in \mathfrak{a}$. If e' has the Lefschetz property, then $f_{e'}$ will also preserve this decomposition (since $f_{e'}$ is unique) and hence $\mathfrak{g}$ will. This contradicts our assumption that M is irreducible.

The second statement is proved in a similar way: suppose $M \cong V(n) \otimes P$ for some nonzero vector space P with $n \geq 2$. If e' is any element of $\mathfrak{a}$, then the fact that e' and e commute, implies that e' acts as $e \otimes \sigma$ for some $\sigma \in \mathfrak{gl}(P)$. In this way, $\mathfrak{a}$ maps onto subspace $\bar{\mathfrak{a}}$ of $\mathfrak{gl}(P)$ that contains the identity of P. The elements of $e \otimes \bar{\mathfrak{a}}$ commute in $V(n) \otimes P$ and hence the elements of $\bar{\mathfrak{a}}$ commute in P (here we use that $n \geq 2$).

If $\dim P = 1$, then we see that the image of $\mathfrak{a}$ in $\mathfrak{gl}(M)$ consists of multiples of e. This implies that the image of $\mathfrak{g}$ in $\mathfrak{gl}(M)$ is a copy of $\mathfrak{sl}(2)$ and that $M \cong V(n)$. We now show that $\dim P \geq 2$ is impossible. For this we may assume that K is algebraically closed. Then the commutative Lie algebra $\bar{\mathfrak{a}}$ leaves invariant a line $L \subset P$. So every $e' \in \mathfrak{a}$ leaves invariant $V(n) \otimes L$. If e' has the Lefschetz property in M, then it also has that property in $V(n) \otimes L$, and so the associated operator f' leaves $V(n) \otimes L$ invariant. It follows that $\mathfrak{g}$ leaves $V(n) \otimes L$ invariant. This again contradicts the irreducibility of M.

The question which subset $B_2 \subset B$ defines the semisimple element of an $\mathfrak{sl}(2)$-triple is not difficult to answer for the classical Lie algebras (see [Springer-St] and the discussion below) and for the exceptional Lie algebras this was tabulated by [Dynkin 1952a]. (The weighted Dynkin diagrams in these tables that matter here are only those that have all weights 0 or 2.) The preceding proposition leads us to discard more possibilities, but we have not seriously studied the interesting question whether what is thus left actually occurs.

(1.16) Let us carry out this procedure in case $\mathfrak{g}$ is a classical Lie algebra.

So we assume that $\mathfrak{g}$ is simple and classical and we let V be a standard representation of $\mathfrak{g}$ (of dimension $l+1, 2l+1, 2l, 2l$ in case $\mathfrak{g}$ is of type A_l, B_l, C_l, D_l respectively). The element h induces a grading on V. The degrees that occur all have the same parity; we refer to this as the *parity* of V. Let us recall that the $\mathfrak{sl}(2)$-invariant bilinear forms on $V(k)$ are generated by a nonzero $(-)^k$-symmetric form. So a finite dimensional $\mathfrak{sl}(2)$-representation of even parity always admits a nondegenerate invariant symmetric form, whereas it admits a nondegenerate invariant skew-symmetric form if and only if all multiplicities are even. In the case of odd parity it is just the other way around.

The $\mathfrak{sl}(2)$-triple (e, h, f) in $\mathfrak{g}$ determines a primitive decomposition of V. According to (1.15) the set of positive integers $i \geq 0$ for which $V(i)$ appears in the $\mathfrak{sl}(2)$-module V is of the form $\{n, n-2, \dots, n-2r\}$ (with $n - 2r \geq 0$). So if we put $k := \lfloor n/2 \rfloor$, then the dimensions $d_t := \dim V_{-n+2t}$ $(i = 0, 1, \dots, k)$ satisfy

$$(*) \qquad 1 \leq d_0 < d_1 < \cdots < d_r = d_{r+1} = \cdots = d_k.$$

By the remark above, the d_i's must all be even in the orthogonal cases with odd parity and in the symplectic cases with even parity.

We choose an $\mathfrak{h}$-invariant basis of V indexed as in [Bourbaki]: $(e_1, \dots, e_{l+1})$ in case A_l, $(e_1, \dots, e_l, e_0, e_{-l}, \dots, e_{-1})$ in case B_l and in the cases C_l and D_l, $(e_1, \dots, e_l, e_{-l} \dots, e_{-1},)$. The same shall apply to our labeling of the simple roots $(\alpha_1, \alpha_2, \dots \alpha_l)$ (as recalled below).

Case A_l. We let $\alpha_i(\mathrm{diag}(\lambda_1, \dots, \lambda_{l+1})) = \lambda_i - \lambda_{i+1}$. We find that the elements of B_2 are the simple roots with index $d_0, d_0 + d_1, \dots, d_0 + \cdots + d_k, d_0 + \cdots +$

$d_{k-1}+2d_k,\dots,d_0+2(d_1+\cdots+d_{k-1}+d_k)$ (then $l=2(d_0+\cdots+d_k)$) or $d_0,d_0+d_1,\dots,d_0+\cdots+d_k,d_0+\cdots+2d_{k-1}+d_k,\dots,d_0+2(d_1+\cdots+d_{k-1})+d_k$ (then $l=2(d_0+\cdots+d_{k-1})+d_k$). So B_2 is symmetric with respect to the natural involution of B. Notice that no two elements of B_2 will be adjacent in the Dynkin diagram: if that would be the case, then $d_i=1$ for some $i>0$ and (*) shows that this is impossible.

Case B_l. Since $\dim V=2l+1$, its parity must be even. This means that n is even and d_k is odd; we have $l=d_0\cdots+d_{k-1}+\frac{1}{2}(d_k-1)$. We let

$$\alpha_i(\mathrm{diag}(\lambda_1,\dots\lambda_l,0,-\lambda_{-l},\dots,-\lambda_{-1}))=\begin{cases}\lambda_i-\lambda_{i+1} & \text{if } i=1,\dots,l-1;\\ \lambda_l & \text{if } i=l.\end{cases}$$

We find that the elements of B_2 are the simple roots with index $d_0,d_0+d_1,\dots,d_0+\cdots+d_{k-1}$. As in the previous case we see that no two elements of B_2 will be adjacent in the Dynkin diagram.

Case C_l. We let

$$\alpha_i(\mathrm{diag}(\lambda_1,\dots,\lambda_l,-\lambda_{-l},\dots,-\lambda_{-1}))=\begin{cases}\lambda_i-\lambda_{i+1} & \text{if } i=1,\dots,l-1;\\ 2\lambda_l & \text{if } i=l.\end{cases}$$

In the case of odd parity, n is odd and $l=d_0+\cdots+d_k$. The elements of B_2 are the simple roots with index $d_0,d_0+d_1,\dots,d_0+\cdots+d_k=l$.

In the case of even parity n and $d_0,\dots,d_k$ are even. We have $l=d_0+\cdots+d_{k-1}+\frac{1}{2}d_k$ and the elements of B_2 are simple roots with index $d_0,d_0+d_1,\dots,d_0+\cdots+d_{k-1}$.

In both cases no two elements of B_2 will be adjacent in the Dynkin diagram.

Case D_l, $l\geq 4$. We let

$$\alpha_i(\mathrm{diag}(\lambda_1,\dots,\lambda_l,-\lambda_{-l},\dots,-\lambda_{-1}))=\begin{cases}\lambda_i-\lambda_{i+1} & \text{if } i=1,\dots,l-1;\\ \lambda_{l-1}+\lambda_l & \text{if } i=l.\end{cases}$$

In the case of odd parity, n is odd and all d_i's are even. We have $l=d_0+\cdots+d_{k-1}+d_k$ and the elements of B_2 are simple roots with index $d_0,d_0+d_1,\dots,d_0+\cdots+d_k$. We claim that no two such elements are adjacent in the Dynkin diagram. For this can only happen when $d_k=2$. In view of (*), this implies that $d_i=2$ for all i, in other words, $V\cong V(n)\oplus V(n)$. But since $n>1$, this is excluded by (1.15).

In the case of even parity, n and d_k are even. We have $l=d_0+\cdots+d_{k-1}+\frac{1}{2}d_k$. Suppose first that $d_k\geq 4$. Then the elements of B_2 are the simple roots with index $d_0,d_0+d_1,\dots,d_0+\cdots+d_{k-1}$. In that case $\alpha_l,\alpha_{l-1}\in B_0$ and $\alpha_1\in B_2$ if and only if $d_0=1$. Clearly no two elements of B_2 are adjacent.

If $d_k=2$, then by (*), $d_0=1$ and $d_2=\cdots=d_k=2$, in other words, $V\cong V(2k)\oplus V(2k-2)$. Then $l=2k$ is even and the elements of B_2 are in position $1,3,5,\dots,2k-3,2k-1,2k$. No two elements of B_2 are adjacent.

For future reference we record:

(1.17) Corollary. *If $\mathfrak{g}$ is simple and classical, then no two elements of B_2 are connected in the Dynkin diagram.*

2. Jordan–Lefschetz modules

In this section we discuss a particularly nice class of Lefschetz triples. We introduce them via the following proposition.

(2.1) Proposition. *Let M be a Lefschetz $\mathfrak{a}$-module. Then the following two properties are equivalent*

(i) *The graded Lie algebra $\mathfrak{g}(\mathfrak{a}, M)$ has degrees -2, 0 and 2 only.*
(ii) *The operators f_a with $a \in \mathfrak{a}$ in the domain of f, mutually commute.*

Proof. We only prove the nontrivial implication $(ii) \Rightarrow (i)$. Write $\mathfrak{g}$ for $\mathfrak{g}(\mathfrak{a}, M)$.

If $a \in \mathfrak{a}$ is such that f_a is defined, then regard $\mathfrak{g}$ as an $\mathfrak{sl}(2)$-module via the $\mathfrak{sl}(2)$-triple (e_a, h, f_a) and let $V(a) \subset \mathfrak{g}$ be sum of the irreducible summands of dimension 1 and 3. Notice that $V(a)$ contains the image of $\mathfrak{a}$. So the intersection V of the $V(a)$'s also contains the image of $\mathfrak{a}$. We have $V = V_{-2} \oplus V_0 \oplus V_2$ where V_{2k} can be characterized as the subspace of $\mathfrak{g}_{2k}$ that is annihilated by all the operators $\mathrm{ad}^{-k+2}(e_a)$, or alternatively, by all the operators $\mathrm{ad}^{k+2}(f_a)$. Since the operators e_a resp. f_a mutually commute it follows that V is invariant under these operators. Hence $V = \mathfrak{g}$ and so $\mathfrak{g}$ has degrees -2, 0 and 2 only.

As we are interested in the graded Lie algebras that so arise, we make the following definition. Say that a Lefschetz pair $(\mathfrak{g}, h)$ is a *Jordan–Lefschetz pair* if $(\mathfrak{g}, h, \mathfrak{g}_2)$ is a Lefschetz triple. It is clear that such a Lefschetz triple is saturated.

(2.2) Proposition. *If $(\mathfrak{g}, h)$ is a Jordan–Lefschetz pair, then $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$ and $U\mathfrak{g} = U\mathfrak{g}_2.U\mathfrak{g}_0.U\mathfrak{g}_{-2}$.*

Proof. If $e \in \mathfrak{g}_2$ is a Lefschetz element, then $[e, \mathfrak{g}_2] = 0$. The first assertion now follows from the primitive decomposition under ad_e. The second is a consequence of this.

(2.3) Corollary. *Let $(\mathfrak{g}, h)$ be a Jordan–Lefschetz pair and let M be a finite dimensional representation of $\mathfrak{g}$ and regard M as a Lefschetz module of $\mathfrak{g}_2$. Then M is generated as a $U\mathfrak{g}_2$-module by* $\mathrm{Prim}(M)$.

Our terminology is easily explained: according to [Springer], §2.21, a Jordan–Lefschetz pair defines a "Jordan algebra without unit element". These have been classified. Let us give a quick proof of this classification. It is based on two lemma's. In what follows, $(\mathfrak{g}, h)$ is a Jordan–Lefschetz pair with $\mathfrak{g}$ simple and and $\mathfrak{h} \subset \mathfrak{g}$ and B are adapted.

(2.4) Lemma. *The subset B_2 is a singleton.*

Proof. If not, then there is a positive root in which β and β' appear with positive coefficient. This root will lie in R_{2k} for some $k \geq 2$, a set which is supposed to be empty.

It follows that h is twice a fundamental coweight.

The following lemma concerns a general property of root systems.

(2.5) Lemma. *Let $\beta \in B$ and let $R(\beta)$ be the set of positive roots that have β-coefficient one. Then $R(\beta) + R(\beta)$ does not contain a root if and only if $R(\beta)$ contains the highest root.*

Proof. It is clear that if $R(\beta)$ contains the highest root, then $R(\beta) + R(\beta)$ cannot contain a root. If on the other hand the highest root has β-coefficient ≥ 2, then it follows from [Bourbaki], Ch. VI,§1, Prop. 19 that there exists a sequence of roots $\alpha_1, \dots, \alpha_r$ such that $\alpha_1 \in B$, $\alpha_{i+1} - \alpha_i \in B$ for $i = 1, \dots, r-1$ and α_r is the highest root. If α_i is the last root in this sequence for which the coefficient of β is one, then α_i and β are elements of $R(\beta)$ whose sum is a root.

(2.6) Corollary. *Let β be the unique element of B_2. Then the pair $(B, B - \{\beta\})$ is of the following type:*

$(A_{2m-1}, A_{m-1} + A_{m-1})$ *$(m \geq 1)$,*
(B_m, B_{m-1}) *$(m \geq 2)$,*
(C_m, A_{m-1}) *$(m \geq 2)$,*
(D_m, D_{m-1}) *$(m \geq 5)$,*
(D_{2m}, A_{2m-1}) *$(m \geq 2)$ or*
(E_7, E_6).

Conversely, if the (characteristic zero) field K is algebraically closed, then every item of this list determines an isomorphism class of Jordan–Lefschetz pairs.

Proof. The pairs (B, β) with the property of the previous lemma are those in this list plus the following: in case A_l we can take l and β arbitrary, in case D_{2l+1} any end of the Dynkin diagram, and B of type E_6 with β the end of the branch of length 3. These addional possibilities disappear if we want h to be a simple element (that is, $h = [e, f]$ for certain $e \in \mathfrak{g}_2$ and $f \in \mathfrak{g}_{-2}$): for the classical cases A_l and D_{2l+1} this follows from the analysis following (1.16). For E_6 this follows from table 18 of [Dynkin]. On the other hand, case $3''$ of table 19 of this reference shows that in case E_7 this element h is simple. The discussion below will show that all the other listed cases occur as well.

(2.7) We continue with the Jordan–Lefschetz pair $(\mathfrak{g}, h)$ and let $\mathfrak{h}$, B and $\beta \in B$ be as above. But in the remainder of this section we assume that K is algebraically closed. Let $\varpi \in \mathfrak{h}^*$ be the fundamental weight that is zero on $B^\vee - \{\beta^\vee\}$ and 1 on $\beta^\vee$. We say that an irreducible representation M of $\mathfrak{g}$ is a *Jordan–Lefschetz module of level* k (where k is a positive integer) if its lowest weight is $-k\varpi$; for $k = 1$, we shall also call it a *fundamental* Jordan–Lefschetz module. Notice that M then has depth $k\varpi(h)$. A more intrinsic description of these modules is the following: given a Jordan–Lefschetz pair $(\mathfrak{g}, h)$, then an irreducible representation M of $\mathfrak{g}$ is a Jordan–Lefschetz module if and only if it has a nonzero vector stabilized by $\mathfrak{g}_{-2} + \mathfrak{g}_0$.

(2.8) Lemma. *Let M be an irreducible representation of the Jordan–Lefschetz pair $(\mathfrak{g}, h)$ and let n be its depth as a Lefschetz module. Then M is a Jordan–Lefschetz module if and only if $\dim M_{-n} = 1$. If these equivalent conditions are fulfilled, then the natural map $\mathfrak{g}_2 \otimes M_{-n} \to M_{-n+2}$ is an isomorphism.*

Proof. Let $\lambda \in \mathfrak{h}^*$ be the lowest weight of M. Then the lowest weight space $M^{-\lambda}$ is contained in M_{-n}. Hence $\dim M_{-n} = 1$ is equivalent to $M^{-\lambda} = M_{-n}$. The latter is equivalent to: $\mathfrak{g}_{-2}+\mathfrak{g}_0$ stabilizes $M^{-\lambda}$, which in turn is equivalent to λ vanishing on B_0. But this means that λ is a multiple of ϖ, i.e., that M is Jordan–Lefschetz.

Suppose the two conditions satisfied. The fact that $\mathfrak{g}_{-2}+\mathfrak{g}_0$ is the $\mathfrak{g}$-stabilizer of M_{-n} implies that $\mathfrak{g}^2\otimes M_{-n}\to M_{-n+2}$ is injective. Since M is as a $U\mathfrak{g}_2$-module generated by M_{-n}, it is also surjective.

(2.9) Let us describe the fundamental Jordan–Lefschetz representations in the classical cases (i.e., those that are not of type (E_7,E_6)). We only do this over the complex numbers.

Case $(A_{2m-1},A_{m-1}+A_{m-1})$: $(\mathfrak{sl}(2m),\mathfrak{sl}(m)\times\mathfrak{sl}(m))$. Let V be a vector space of dimension $2m$, $V=V_{-1}\oplus V_1$ a direct sum decomposition into subspaces of dimension m. We take $\mathfrak{g}:=\mathfrak{sl}(V)$ and let $h\in\mathfrak{sl}(V)$ be ±1 on $V_{\pm1}$. Then $\mathfrak{g}_0$ maps isomorphically onto $\mathfrak{sl}(V_{-1})\times\mathfrak{sl}(V_1)\times\mathbb{C}h$ and $\mathfrak{g}_2\cong\operatorname{Hom}(V_{-1},V_1)$ resp. $\mathfrak{g}_{-2}\cong\operatorname{Hom}(V_1,V_{-1})$. The corresponding fundamental representation is $M:=\wedge^m V$ with lowest degree piece $M_{-m}=\wedge^m V_{-1}$.

Case (B_m,B_{m-1}) *or* (D_m,D_{m-1}): $(\mathfrak{so}(n),\mathfrak{so}(n-2))$ with $m=2n+1$ resp. $m=2n$ $(n\geq2)$. Let V be a vector space of dimension n equipped with a nondegenerate symmetric bilinear form and let $V_{\pm2}$ be isotropic lines in V such that $V_{-2}\oplus V_2$ is nondegenerate. Let V_0 be the orthogonal complement of $V_{-2}\oplus V_2$ in V. We take $\mathfrak{g}=\mathfrak{so}(V)$ and let $h\in\mathfrak{so}(V)$ be the element with the eigen space decomposition $V_{-2}\oplus V_0\oplus V_2$. Then $\mathfrak{g}_0=\mathfrak{so}(V_0)\times\mathfrak{gl}(V_2)$ and $\mathfrak{g}_{\pm2}$ projects isomorphically to $\operatorname{Hom}(V_0,V_{\pm2})$. We take $M=V$.

Case (C_m,A_{m-1}): $(\mathfrak{sp}(2m),\mathfrak{sl}(m))$ $(m\geq2)$. Let V be a vector space of dimension $2m$ equipped with a nondegenerate symplectic form and let $V=V_{-1}\oplus V_1$ be a decomposition of V into totally isotropic subspaces of dimension m. We take $\mathfrak{g}=\mathfrak{sp}(V)$ and let $h\in\mathfrak{sp}(V)$ be the element with the eigen space decomposition $V_{-1}\oplus V_1$. Then $\mathfrak{g}_0$ maps isomorphically to $\mathfrak{gl}(V_1)$ and $\mathfrak{g}_{\pm2}$ is naturally isomorphic to the space of symmetric elements in $(V_{\pm1})^{\otimes2}$. We take for M the primitive quotient of $\wedge^m V$ (i.e., the quotient by $\wedge^{m-2}V\wedge\omega$, where $\omega\in\wedge^2V$ is the dual of the symplectic form).

Case (D_{2m},A_{2m-1}): $(\mathfrak{so}(4m),\mathfrak{sl}(2m))$ $(m\geq2)$. Let V be a vector space of dimension $4m$ equipped with a nondegenerate symmetric bilinear form and let $V=V_{-1}\oplus V_1$ be a decomposition of V into totally isotropic subspaces of dimension $2m$. We take $\mathfrak{g}=\mathfrak{so}(V)$ and let $h\in\mathfrak{so}(V)$ be the element with the eigen space decomposition $V_{-1}\oplus V_1$. Then $\mathfrak{g}_0$ maps isomorphically to $\mathfrak{gl}(V_1)$ and $\mathfrak{g}_{\pm2}$ maps onto the skew elements in $(V_{\pm1})^{\otimes2}$. Consider the spinor representation $\wedge^\bullet V_1$. (We recall that this factors through the representation of the Clifford algebra of V on $\wedge^\bullet V_1$ for which $v\in V_1$ acts as wedging with v and $v\in V_{-1}$ acts as the interior product under the obvious isomorphism $V_{-1}\cong V_1^*$.) It splits into a direct sum of subrepresentations $\wedge^{\mathrm{ev}}V_1$ and $\wedge^{\mathrm{odd}}V_1$. They are irreducible and nonisomorphic and correspond to the case when β is an end of the Dynkin diagram connected with a branch point. For even m both are orthogonal and for odd m both are symplectic. We take $M=\wedge^{\mathrm{ev}}V_1[m]$.

(2.10) The Jordan–Lefschetz modules give rise to Frobenius algebras with remarkable properties. Let $(\mathfrak{g},h)$ be a Jordan pair and let M be a Jordan–Lefschetz module of $(\mathfrak{g},h)$ of depth n. Then M is a monic module over the commutative algebra $U\mathfrak{g}_2$. So if $I \subset U\mathfrak{g}_2$ denote the annihilator of M, then I defines the origin in $\mathrm{Spec}(U\mathfrak{g}_2) = \mathfrak{g}_2^*$ with local algebra $A := U\mathfrak{g}_2/I$. The latter is an evenly graded Lefschetz algebra that has M as a free graded module of rank one. The next proposition shows that it has a lot of automorphisms. Let $\mathfrak{g}_0'$ be the Lie subalgebra of $\mathfrak{g}_0$ that kills $1 \in A_0$ (or equivalently, kills M_{-n}); this Lie algebra is complementary to the span of h in $\mathfrak{g}_0$.

(2.11) Proposition. *The Lie algebra $\mathfrak{g}_0'$ acts on A as derivations and so the associated Lie subgroup of* $\mathrm{GL}(A)$ *is a group of algebra automorphisms of A. Moreover, the Lie subgroup $G_0 \subset \mathrm{GL}(A)$ associated to $\mathfrak{g}_0$ has a dense orbit in A_2 consisting of Lefschetz elements.*

Proof. If $u \in \mathfrak{g}_0$ and $e \in \mathfrak{g}_2$, then for all $x \in A$, $u(ex) = [u, e]x + e(ux)$. Since A is generated by $\mathfrak{g}_2$, it follows with induction that u acts as a derivation if (and only if) u kills 1. The last assertion follows from a well-known result [Bourbaki], Ch. VIII, §11, Prop. 6.

(2.12) A decomposition of $\mathfrak{g}$ into simple components corresponds to a decomposition of A as a tensor product of algebras, so little is lost in assuming that $\mathfrak{g}$ is simple. The form $\int$ defined in (1.4) makes of A a Frobenius algebra with soccle A_{2n}; at the same time $\int$ defines a generator of A_{2n} that serves as the identity element for another algebra structure defined by the action of $U\mathfrak{g}_{-2}$. Here is a description of the algebras associated to the fundamental Jordan–Lefschetz modules in all cases.

Case $(A_{2m-1}, A_{m-1} + A_{m-1})$: Let W, W' be vector spaces of dimension m and let $A := \oplus_{k=0}^m \wedge^k W \otimes \wedge^k W'$. This is just the subalgebra of the graded algebra $\wedge^\bullet W \otimes \wedge^\bullet W'$ generated by $W \otimes W'$. It is the fundamental Frobenius algebra associated to $(\mathfrak{sl}(W \oplus W'), h = (1_W, -1_{W'}))$. (To see the relation with the description given in (2.9), take $V_{-1} = (W')^*$ and $V_1 = W$ and observe that a choice of a generator of $\wedge^m V_{-1}$ identifies $\wedge^{m-k} V_{-1}$ with $\wedge^k W'$ and hence $\wedge^m(V_1 \oplus V_{-1})$ with A.)

Here is a presentation of this algebra. Choose bases $(w_1, \dots, w_m)$ of W and $(w_1', \dots, w_m')$ of W'. Then $x_{ij} := w_i \otimes w_j'$, $(i, j = 1, \dots, m)$ generate A as an algebra and a set of defining relations is $x_{ij}x_{kl} + x_{il}x_{kj} = 0$.

Case (D_{2m}, A_{2m-1}): Let W be a vector space of dimension $2m$, then let A be the subalgebra of $\wedge^\bullet W$ generated by $\wedge^2 W$. This is the fundamental Frobenius algebra associated to $\mathfrak{so}(W \oplus W^*), h = (1_W, -1_{W^*})$).

A presentation of A is a follows $(w_1, \dots, w_m)$ is a basis of W, then generators for A are $\omega_{ij} := w_i \wedge w_j$ $(1 \le i, j \le m)$, subject to the linear relations $\omega_{ij} + \omega_{ji} = 0$ and the quadratic relations $\omega_{ij}\omega_{kl} + \omega_{il}\omega_{kj} = 0$. So this is a quotient of the algebra of the previous case.

Case (C_m, A_{m-1}): Let W be a vector space of dimension m. Then a model for the fundamental Frobenius algebra associated to $\mathfrak{sp}(W \oplus W^*), h = (1_W, -1_W))$ is the subalgebra A of $\wedge^\bullet W \otimes \wedge^\bullet W$ generated by the symmetric elements $w \otimes w$, $w \in W$. If $(w_1, \dots, w_m)$ is a basis of W, then generators for A are $u_{ij} := w_i \otimes w_j + w_j \otimes w_i$ $(1 \le i, j \le m)$. A defining set of relations is $u_{ij} = u_{ji}$ and $u_{ij}u_{jk} + u_{jj}u_{ik} = 0$.

Cases (B_n, B_{n-1}) *and* (D_n, D_{n-1})*:* Let $(W, q : W \to K)$ be a vector space of dimension m endowed with a nondegenerate quadratic form. Consider the graded vector space $A := K \oplus W \oplus K\mu$, where K has degree zero, W has degree 2 and μ is a generator of a one-dimensional K-vector space that has degree 4. Equip A with the quadratic form $\tilde{q}(x+w+x'\mu) = q(w)-xx'$. Make it also a graded algebra by letting $1 \in K$ be the identity element and letting for $w, w' \in W$, $w.w' = b(w,w')\mu$, where b is the bilinear form associated to q. Then A is the fundamental Frobenius algebra associated to $(\mathfrak{so}(A,\tilde{q}), h = (-2, 0_W, 2))$. If $(w_1, \dots, w_m)$ is a q-orthogonal basis of W then a presentation of A has generators $w_1, \dots, w_m$ and relations $w_i w_j = 0$ $(i \neq j)$ and $q(w_i)^{-1} w_i w_i = q(w_j)^{-1} w_j w_j$.

Case (E_7, E_6): Let W be a finite dimensional K-vector space and let be given a cubic form $c : \mathrm{Sym}^3(W) \to K$ that does not factor through proper linear quotient of W. The latter condition implies that the associated homomorphism $\tilde{c} : \mathrm{Sym}^2 W \to W^*$ is surjective. Consider the graded vector space $A := K \oplus W \oplus W^* \oplus K\mu$ with respective summands in degree 0, 2, 4, 6 (here μ is a generator of a one dimensional vector space) and use $\tilde{c}$ and the obvious pairing $W \times W^* \to K\mu$ to give A the structure of a commutative graded K-algebra. We require that multiplication makes $A[3]$ a Jordan–Lefschetz module over A_2. This forces $\dim W = 27$ and after possibly passing to an algebraic closure of K, c will be unique up to a linear transformation. The group G_c of $g \in \mathrm{GL}(W)$ that leave c invariant is of type E_6 and the associated Lie algebra $\mathfrak{g}$ is of type E_7. (This Lie algebra can be characterized as the Lie algebra of linear transformations of A that leaves invariant a certain quartic form on A.) If we regard A as a quotient of the symmetric algebra of W, then we checked by means of the program [LiE] that the relations are again quadratic: the ideal $I \subset \mathrm{Sym}(W)$ that defines A is generated by $\ker(\tilde{c})$ (the latter is an irreducible representation of G_c whose highest weight is twice that of W).

The Jordan–Lefschetz algebras of higher level can be expressed in terms of a fundamental one:

(2.13) Proposition. *Let A be a fundamental Jordan–Lefschetz algebra for a Jordan–Lefschetz pair $(\mathfrak{g}, h)$. Let k be a positive integer and give $\mathrm{Sym}^k(A)$ the structure of an algebra by identifying it with the algebra of symmetric invariants in $A^{\otimes k}$. Then the subalgebra $A(k)$ of $\mathrm{Sym}^k(A)$ generated by A_2 is a Jordan–Lefschetz algebra of level k for $(\mathfrak{g}, h)$.*

Proof. We regard A as an fundamental representation of $\mathfrak{g}$ with lowest weight space spanned by its unit element $1 \in A$. Then the irreducible representation of $\mathfrak{g}$ with lowest weight k times the one of A is contained in $A^{\otimes k}$ as the $U\mathfrak{g}$-submodule generated by the unit element $1 \otimes \cdots \otimes 1$. But this is also the $U\mathfrak{g}_2$-submodule generated by this element. In other words, this is the subalgebra of $A^{\otimes k}$ generated by the elements $a \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes a \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes a$, with $a \in A_2$.

In section 4 we will be concerned with the algebras of higher level in the cases (B_n, B_{n-1}) and (D_n, D_{n-1}), and that is why we want to describe them here in more explicit terms. Let (W, q) and $(A, \tilde{q})$ be as under the relevant case above.

(2.14) Proposition. *Fix an integer $k \geq 1$ and let I_k be the ideal in $\mathrm{Sym}(W)$ generated by the $k+1$-st powers of q-isotropic vectors (i.e., the w^{k+1} for which $q(w) = 0$). Then $A(k) = \mathrm{Sym}(W)/I_k$. Moreover, the subalgebra of $\mathrm{Sym}(W)/I_k$ of $\mathfrak{so}(W)$-invariants is $K(u)/(u^{k+1})$, where $u \in \mathrm{Sym}^2(W)$ represents the inverse form of q on the dual of W. The soccle of $A(k)$ is spanned by the image of u^k and if* *we identify W with its image in $A(k)$, then for $i = 0, \dots, k$, there exists a nonzero constant c_i such that $x^{2i}u^{k-i} = c_i q(x)^i u^k$ for all $x \in W$.*

Proof. Let $W(d)$ denote the image of $\mathrm{Sym}^d(W)$ in the graded algebra $\mathrm{Sym}(W)/(u)$. Then $W(d)$ is an irreducible representation of $\mathfrak{so}(W)$ that can be identified with the subrepresentation of $\mathrm{Sym}^d(W)$ which is linearly spanned the pure dth powers of q-isotropic vectors in W. In particular, I_k is the ideal in $\mathrm{Sym}(W)$ generated by $W(k+1)$. It is easy to see that

$$\mathrm{Sym}^d(W) = \oplus_{i=0}^{\lfloor d/2 \rfloor} u^i W(d-2i),$$

as graded $\mathfrak{so}(W)$-representations. A Clebsch–Gordan rule asserts that for $p \geq q \geq 0$, the image of $W(p) \otimes W(q) \to \mathrm{Sym}^{p+q} W$ under the multiplication map is $\oplus_{i=0}^{q} u^i W(p+q-2i)$. So the cokernel can be identified with $u^{q+1}\mathrm{Sym}^{p-q-2}(W)$. This remains so if we replace $W(q)$ by $\mathrm{Sym}^q(W)$. It follows that

$$\begin{aligned} \mathrm{Sym}(W)/I_k =& \mathrm{Sym}^0(W) \oplus \cdots \oplus \mathrm{Sym}^{k-1}(W) \oplus \mathrm{Sym}^k(W) \\ & \oplus u\,\mathrm{Sym}^{k-1}(W) \oplus u^2\,\mathrm{Sym}^{k-1}(W) \oplus \cdots \oplus u^k\,\mathrm{Sym}^0(W). \end{aligned}$$

If $\tilde{u} \in \mathrm{Sym}^2(A)$ denotes the symmetric tensor dual to $\tilde{q}$, then $A(k)$ is the image of $\mathrm{Sym}^k(A)$ in $\mathrm{Sym}(A)/(\tilde{u})$. If we write $A = Kt \oplus W \oplus K\mu$, with $\deg t = 0$, then $\mathrm{Sym}(A) = K[t,\mu]\,\mathrm{Sym}\,W$ with $\tilde{u}$ corresponding to $t\mu + u$. Hence we have the following identity of graded $\mathfrak{so}(W)$-representations:

$$A(k) = \oplus_{i=0}^{k} t^{k-i}\,\mathrm{Sym}^i(W) \;\oplus\; \oplus_{i=1}^{k} \mu^i\,\mathrm{Sym}^{k-i}(W).$$

We know that there is a surjective graded algebra homomorphism $\mathrm{Sym}(W) \to A(k)$. This homomorphism is $\mathfrak{so}(W)$-equivariant and hence contains $W(k+1)$ in its kernel. We therefore have a graded, $\mathfrak{so}(W)$-equivariant, algebra epimorphism $\mathrm{Sym}(W)/I_k \to A(k)$. The $\mathfrak{so}(W)$-decompositions that we found for both source and target show that this must be an isomorphism.

It is clear that the soccle is spanned by u^k. Let $i \in \{0, 1, \dots, k\}$. If $x \in W$ is nonisotropic, then $x^{2i}u^{k-i} = c(x)q(x)^i u^k$ for some $c(x) \in \mathbb{C}$. Since $\mathrm{SO}(W)$ acts on $A(k)$ by algebra automorphisms, c is constant on $\mathrm{SO}(W)$-orbits. It is also clear that c is constant on $\mathbb{C}^\times$-orbits. So c is constant on the set of all nonisotropic vectors and therefore constant everywhere. Since the linear span of the $2i$th powers of elements of W contains u^i, this constant must be nonzero.

Part of the last assertion of this proposition appears in a geometric setting as theorem (4.7) in [Fujiki]. Our formulation was suggested by remark (4.12) of that paper. The special case $x^{2k} = c_k q^k(x) u^k$, can be regarded as a characterization of q up to proportionality in terms of the algebra structure on $A(k)$.

3. Geometric examples of Jordan type I: complex tori

In this section we describe the total Lie algebra of a complex torus, the Kähler Lie algebra of a complex torus, and the Néron–Severi Lie algebra of an abelian variety.

(3.1) We begin with determining the total Lie algebra of a complex torus. As observed before this is independent of its complex structure. We first recall the construction of the spinor representations.

Let V be a real finite dimensional vector space. For $\alpha \in V^*$, left exterior product with α defines a map $e_\alpha : \wedge^\bullet V^* \to \wedge^{\bullet+1} V^*$ of degree one. Dually, contraction with $a \in V$ defines a map $i_a : \wedge^\bullet V^* \to \wedge^{\bullet-1} V^*$ of degree minus one. Both are derivations of odd degree: $i_a(\omega \wedge \omega') = i_a(\omega) \wedge \omega' + (-)^{\deg \omega} \omega \wedge i_a(\omega')$ and similarly for e_α. The anti-commutator $i_a e_\alpha + e_\alpha i_a$ is simply multiplication by $\alpha(a)$. Iterated use of this identity shows that for $a, b \in V$ and $\alpha, \beta \in V^*$, we have

$$\begin{aligned} [i_a i_b, e_\alpha e_\beta] = &- \alpha(b) e_\beta i_a + \alpha(a) e_\beta i_b + \beta(b) e_\alpha i_a - \beta(a) e_\alpha i_b \\ &+ (-\alpha(a)\beta(b) + \beta(a)\alpha(b)) \mathbf{1}_{\wedge^\bullet V^*}. \end{aligned} \tag{**}$$

We rewrite this as follows: if we denote by $\sigma(a \wedge b, \alpha \wedge \beta) \in \mathfrak{gl}(V^*)$ the transformation

$$\xi \mapsto -\alpha(b)\xi(a)\beta + \alpha(a)\xi(b)\beta + \beta(b)\xi(a)\alpha - \beta(a)\xi(b)\alpha,$$

and write $\tilde\sigma(a \wedge b, \alpha \wedge \beta)$ for its extension as a degree zero derivation in $\wedge^\bullet V^*$, then the righthand side of eq. (**) is simply the sum of $\tilde\sigma(a \wedge b, \alpha \wedge \beta)$ and the scalar operator that is multiplication by $-\frac{1}{2}\operatorname{Tr}(\sigma(a \wedge b, \alpha \wedge \beta))$.

With the help of this formula we determine the Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(\wedge^\bullet V^*)$ generated by these operators. In fact, by means of a Clifford construction we shall identify it with the Lie subalgebra $\mathfrak{so}(V \oplus V^*)$ of $\mathfrak{gl}(V \oplus V^*)$ of infinitesimal automorphisms of the quadratic form $q(x, \xi) = \xi(x)$.

The semi-simple element $u := (-\mathbf{1}_V, +\mathbf{1}_{V^*}) \in \mathfrak{so}(V \oplus V^*)$ defines a grading of the latter with degrees 2, 0 and -2. We define a Lie algebra isomorphism

$$\psi_0 : \mathfrak{gl}(V^*) \to \mathfrak{so}(V \oplus V^*)_0; \quad \psi_0(\sigma)(x, \xi) = (-\sigma^*(x), \sigma(\xi)).$$

We also have isomorphisms of abelian Lie algebras

$$\begin{aligned} \psi_2 : \wedge^2 V^* \to \mathfrak{so}(V \oplus V^*)_2; &\quad \psi_2(\alpha \wedge \beta)(x, \xi) = (0, \alpha(x)\beta - \beta(x)\alpha), \\ \psi_{-2} : \wedge^2 V \to \mathfrak{so}(V \oplus V^*)_{-2}; &\quad \psi_{-2}(a \wedge b)(x, \xi) = (\xi(a)b - \xi(b)a, 0). \end{aligned}$$

(3.2) Proposition. *The maps $\psi_2(\alpha \wedge \beta) \mapsto e_\alpha e_\beta$, $\psi_{-2}(a \wedge b) \mapsto i_a i_b$ extend to a graded Lie algebra isomorphism of $\mathfrak{so}(V \oplus V^*)$ onto $\mathfrak{g}$ that maps $\psi_0(\sigma)$ to $\tilde\sigma - \frac{1}{2}\operatorname{Tr}(\sigma)\mathbf{1}_{\wedge^\bullet V^*}$, where $\tilde\sigma$ denotes the extension of σ as a degree zero derivation of $\wedge^\bullet V^*$.*

Proof. In view of formula (**) it suffices to show that $[\psi_{-2}(a \wedge b), \psi_2(\alpha \wedge \beta)] = \psi_0(\sigma(a \wedge b, \alpha \wedge \beta))$. This is straightforward.

The pair $(\mathfrak{so}(V \oplus V^*), u)$ is a Jordan–Lefschetz pair (it is a real form of case (D_{2n}, A_{2n-1})).

Now let X be a real torus of even dimension $2n$. We identify the universal cover of X with $\mathrm{H}_1(X; \mathbb{R})$. We will write V for this real vector space (of dimension $2n$) so that $\mathrm{H}(X; \mathbb{R}) = \wedge^\bullet V^*$. The rational homology defines a rational structure $V_\mathbb{Q} \subset V$. Let $\kappa \in \wedge^2 V^*$ be nondegenerate. If $\alpha_{\pm 1}, \dots, \alpha_{\pm n}$ is a basis of V^* such that $\kappa = \sum_{k=1}^{n} \alpha_k \wedge \alpha_{-k}$ then $e_\kappa = \sum_{k=1}^{n} e_{\alpha_k} e_{\alpha_{-k}}$. If $a_{\pm 1}, \dots a_{\pm n}$ is the dual basis, then we see from eq. (**) that

$$[e_\kappa, \sum_{k=1}^{n} i_{a_{-k}} i_{a_k}] = -n + \sum_{k=1}^{n} (e_{\alpha_k} i_{a_k} + e_{\alpha_{-k}} i_{a_{-k}}).$$

Since this element acts on $\wedge^l V^*$ as multiplication by $-n+l$, it follows that f_κ is defined and equal to $\sum_{k=1}^{n} i_{a_{-k}} i_{a_k}$. The nondegenerate 2-forms make up a nonempty open subset of $\wedge^2 V^*$ and therefore span that space. The corresponding 2-vectors form an open subset of $\wedge^2 V$ and so $\mathfrak{g}_{\mathrm{tot}}(X)$ is generated by $\mathfrak{g}_2 \oplus \mathfrak{g}_{-2}$ as a Lie algebra. Combining this with the above computation gives:

(3.3) Proposition. *There is a natural identification* $(\mathfrak{g}_{\mathrm{tot}}(X;\mathbb{R}), h) \cong (\mathfrak{so}(V^* \oplus V), u)$*; this is a real form of the case* (D_{2n}, A_{2n-1})*. Furthermore,* $\mathrm{H}^{\mathrm{ev}}(X)[n]$ *is a semispinorial representation of* $g_{\mathrm{tot}}(X;\mathbb{R})$ *and a fundamental Jordan–Lefschetz module of* $\mathrm{H}^2(X,\mathbb{R})$.

We now assume that X comes with a complex structure. We shall determine its Kähler Lie algebra. Let V^* resp. $\bar{V}^*$ denote the $\mathbb{R}$-dual of V equipped with the complex structure J^* resp. $-J^*$. The quadratic form q defined above is invariant under the complex structure $(J, -J^*)$ and so extends to a Hermitian form on $V \oplus \bar{V}^*$. Let $\mathfrak{su}(V \oplus V^*)$ be the Lie algebra of the corresponding special unitary group. Since $u \in \mathfrak{su}(V \oplus \bar{V}^*)$, it inherits a grading with degrees -2, 0 and 2. In fact, $(\mathfrak{su}(V \oplus \bar{V}^*), u)$ is a Jordan–Lefschetz pair. It is a real form of case $(A_{2n-1}, A_{n-1} + A_{n-1})$.

The J^*-invariant elements of $\wedge^2_{\mathbb{R}} V^*$ are precisely the real $(1,1)$-forms. So they make up the span of the Kähler classes of X. Now a straightforward verification shows that ψ maps the J-invariants of $\wedge^2_{\mathbb{R}} V$ resp. J^*-invariants of $\wedge^2_{\mathbb{R}} V^*$ onto $\mathfrak{su}(V \oplus \bar{V}^*)_{-2}$ resp. $\mathfrak{su}(V \oplus \bar{V}^*)_2$. We have:

(3.4) Proposition. *There is a natural identification* $(\mathfrak{g}_K(X;\mathbb{R}), h) \cong (\mathfrak{su}(V \oplus \bar{V}^*), u)$*; this is a real form of the case* $(A_{2n-1}, A_{n-1} + A_{n-1})$*. Furthermore, the subspace* $\oplus_k \mathrm{H}^{k,k}(X)[n]$ *is a fundamental Jordan–Lefschetz module of* $\mathfrak{g}_K(X)$.

Proof. The first assertion is clear from the preceding discussion. The choice of a generator of $\wedge^n_{\mathbb{C}} V$ determines an isomorphism $\wedge^{n-k}_{\mathbb{C}} V \cong \wedge^k_{\mathbb{C}} V^*$. This yields a graded isomorphism

$$\wedge^n_{\mathbb{C}}(V \oplus \bar{V}^*) = \oplus_k \wedge^{n-k}_{\mathbb{C}} V \otimes \wedge^k_{\mathbb{C}} \bar{V}^* \cong \oplus_k \wedge^k_{\mathbb{C}} V^* \otimes \wedge^k_{\mathbb{C}} \bar{V}^*[n] \cong \oplus_k \mathrm{H}^{k,k}(X)[n].$$

The last assertion follows from this.

We next determine the Néron–Severi Lie algebra (or rather the Lie algebra of its rational points) of an abelian variety X. We adhere to the convention to denote the $\mathbb{Q}$-algebra $\mathrm{End}(X) \otimes \mathbb{Q}$ by $\mathrm{End}^0(X)$ and we write $V_{\mathbb{Q}}$ for $\mathrm{H}_1(X;\mathbb{Q})$ and V for $\mathrm{H}_1(X;\mathbb{R})$. We think of $\mathrm{End}^0(X)$ as a subalgebra of $\mathrm{End}(V_{\mathbb{Q}})$; if $J : V \to V$, $J^2 = -\mathbf{1}_V$, is the complex structure determined by the one of X, then $\mathrm{End}^0(X)$ is centralizer of J in $\mathrm{End}(V)$ intersected with $\mathrm{End}(V_{\mathbb{Q}})$. Likewise, $\mathrm{NS}(X) \otimes \mathbb{Q}$ may be identified with the J-invariants in $\wedge^2 V^*$ intersected with $\wedge^2 V^*_{\mathbb{Q}}$.

If $\kappa \in \mathrm{NS}(X)$ is a polarization, then taking adjoints with respect to this form defines an (anti-)involution † in $\mathrm{End}(V)$:

$$\kappa(\sigma v, w) = \kappa(v, \sigma^\dagger w),$$

which preserves $\mathrm{End}^0(X)$; it is called the *Rosati involution* defined by κ. A well-known fact can be stated as follows:

(3.5) Proposition. *If $\lambda \in \mathrm{NS}(X)$ then the restriction of $[e_\lambda, f_\kappa] \in \mathfrak{gl}(\wedge^\bullet V^*)$ to V^* is in $\mathrm{End}^0(X)$ (acting on V^* contragradiently) and is invariant under †; this defines an isomorphism of $\mathrm{NS}(X)$ onto the †-invariants in $\mathrm{End}^0(X)$.*

Proof. For any bilinear form $\lambda : V \times V \to \mathbb{R}$ there is a unique $\sigma \in \mathrm{End}(V)$ such that $\lambda(a, b) = \kappa(\sigma_\lambda a, b)$. The condition that λ be anti-symmetric is equivalent to that σ_λ be †-invariant; the condition that λ be J-invariant to that σ_λ be J-equivariant. If λ is skew-symmetric and is regarded as an element of $\wedge^2 V^*$, then eq. (**) shows that $[e_\lambda, f_\kappa]|V^*$ is equal to $\pm\sigma_\lambda^*$ plus a scalar operator. The proposition follows.

(3.6) Let us write $\mathrm{End}^0(X)^\pm$ for the ± 1-eigen space of † in $\mathrm{End}^0(X)$. Since † is an anti-involution, $\mathrm{End}^0(X)^-$ is a Lie subalgebra of $\mathrm{End}^0(X)$ and $\mathrm{End}^0(X)^+$ is a module of this Lie algebra. The group of units $(\mathrm{End}(X) \otimes \mathbb{R})^\times$ acts on $\mathrm{NS}(X) \otimes \mathbb{R}$ and it is well-known that the polarisations are contained in a single orbit. So the Rosati involutions are all conjugate under $(\mathrm{End}(X) \otimes \mathbb{R})^\times$. Let $\mathfrak{u}(X)$ denote the set of elements in $\mathrm{End}^0(X)$ that are anti-invariant with respect to all Rosati involutions. This is clearly a Lie ideal in $\mathrm{End}^0(X)$.

(3.7) Proposition. *The Néron–Severi Lie algebra of the abelian variety X is of Jordan type. Its degree 2 summand is canonically isomorphic to $\mathrm{NS}(X) \otimes \mathbb{Q}$. Its degree 0 summand can be identified with the Lie ideal of $\mathrm{End}^0(X)$ that is generated by $\mathrm{End}^0(X)^+$ and this isomorphism makes h correspond to a scalar operator in $\mathrm{End}^0(X)$. Moreover, $\mathrm{End}^0(X) = \mathfrak{g}_{NS}(X; \mathbb{Q})_0 \times \mathfrak{u}(X)$.*

Proof. We only prove the last two assertions, as the others just sum up the preceding discussion. For a Jordan pair $(\mathfrak{g}, h)$, $[\mathfrak{g}_2, \mathfrak{g}_{-2}]$ generates $\mathfrak{g}_0$ and so (3.5) implies that $\mathfrak{g}_{NS}(X)_0$ is the Lie algebra generated by the elements invariant under some Rosati involution. As the Rosati involutions are dense in a single $(\mathrm{End}(X) \otimes \mathbb{R})^\times$-conjugacy class, a standard argument shows that $\mathfrak{g}_{NS}(X, \mathbb{R})_0$ must be the Lie ideal in $\mathrm{End}(X) \otimes \mathbb{R}$ generated by $(\mathrm{End}(X) \otimes \mathbb{R})^+$.

If $\sigma \in \mathrm{End}^0(X)^-$ and $\tau \in \mathrm{End}^0(X)^+$, then $\mathrm{Tr}(\sigma\tau) = \mathrm{Tr}((\sigma\tau)^\dagger) = \mathrm{Tr}(-\tau\sigma) = -\mathrm{Tr}(\sigma\tau)$ (here Tr denotes the $\mathbb{Q}$-trace). This shows that $\mathrm{End}^0(X)^-$ is the the orthoplement of $\mathrm{End}^0(X)^+$ in $\mathrm{End}^0(X)$ with respect to the trace form. So $\mathfrak{u}(X)$ is the orthogonal complement of the span of the elements fixed by some Rosati involution. This orthoplement is an ideal as well and hence equal to $\mathfrak{g}_{NS}(X)_0$.

(3.8) In order to convert this into a more explicit statement, we first make a standard reduction.

The Néron–Severi Lie algebra of X only depends on the isogeny type. So by the Poincaré's complete reducibility theorem we may without loss of generality assume that the abelian variety is of the form

$$X = X_1^{m_1} \times \cdots \times X_k^{m_k}$$

with $X_1,\dots,X_k$ simple, pairwise non-isogenous abelian varieties. Since the Néron–Severi group of X is just the direct sum of the Néron–Severi groups of its isotypical factors, the same is true for the Néron–Severi Lie algebra:

$$\mathfrak{g}_{NS}(X)=\mathfrak{g}_{NS}(X_1^{m_1})\times\cdots\times\mathfrak{g}_{NS}(X_k^{m_k}).$$

We therefore assume that X is a power A^m of a simple abelian variety A.

We first concentrate on A. (For a discussion and the proofs of the properties that we are going to use we refer to [Lange-Birk].) Since A is simple, $\mathrm{End}^0(A)$ is a skew field over $\mathbb{Q}$. We shall write F for it and denote the center of F by K. Then $\mathrm{H}_1(A;\mathbb{Q})$ is in a natural way a K-vector space. We fix a polarization κ so that there is defined a corresponding Rosati involution †. This involution is positive in the sense that for every nonzero $g\in F$, the action of $g^\dagger g$ on the $\mathrm{H}_1(A;\mathbb{Q})$ has positive trace over $\mathbb{Q}$. The involution that † induces in K is independent of †: for any embedding of K in the complex field it is given by complex conjugation. We therefore denote it by $^-$. The subfield $K_0\subset K$ fixed by $^-$ is totally real; so if $\mathbf{e}_0$ stands for the set of embeddings of K_0 in $\mathbb{R}$, then $K_0\otimes_{\mathbb{Q}}\mathbb{R}$ is as a real vector space canonically isomorphic to $\mathbb{R}^{\mathbf{e}_0}$. The cardinality e_0 of $\mathbf{e}_0$ is the degree of K_0 over $\mathbb{Q}$. The isomorphism between $\mathrm{NS}(A)\otimes\mathbb{Q}$ and the †-invariants F^+ in F gives $\mathrm{NS}(A)\otimes\mathbb{Q}$ the structure of a K_0-vector space as well. This is also independent of $\dagger$. There are four cases:

The case of totally real multiplication. Then $F=K=K_0$, in particular, the involution is trivial on F.

The case of totally indefinite quaternion multiplication. Here $K=K_0$ and F is a K_0-form of $\mathrm{End}(2)$: there is an $\mathbb{R}$-algebra isomorphism $F\otimes_{\mathbb{Q}}\mathbb{R}\cong\mathrm{End}(2,\mathbb{R})^{\mathbf{e}_0}$ such that the involution corresponds to the transpose in every summand. In particular, F^+ is a K_0-form of the space of binary quadratic forms.

The case of totally definite quaternion multiplication. Here also $K=K_0$ and F is a K_0-form of the quaternion algebra $\mathbb{H}$ over K_0: there is an $\mathbb{R}$-algebra isomorphism $F\otimes_{\mathbb{Q}}\mathbb{R}\to\mathbb{H}^{\mathbf{e}_0}$ such that † corresponds to quaternion conjugation in every summand. Since the self-conjugate elements in $\mathbb{H}$ are the reals, it follows that $F^+=K_0$.

The case of totally complex multiplication. The field K has no real embedding (so is a totally imaginary quadratic extension of K_0) and F is a K-form of $\mathrm{End}(d)$: there is an $\mathbb{R}$-algebra isomorphism $F\otimes_{\mathbb{Q}}\mathbb{R}\to\mathrm{End}(d,\mathbb{C})^{\mathbf{e}_0}$ such that the involution corresponds to the conjugate transpose in every summand. So F^+ is a K_0-form of the space of Hermitian $d\times d$-matrices.

According to Albert all these cases occur. Recall that an isogeny type of a polarized abelian variety is given by rational vector space $V_{\mathbb{Q}}$, a nondegenerate symplectic form κ on $V_{\mathbb{Q}}$ and a complex structure J on the realification V of $V_{\mathbb{Q}}$ such that κ is J-invariant and $\kappa(a,Ja)>0$ for all nonzero $a\in V$. In order to realize the above cases, we fix a free finitely generated left F-module W and a nondegenerate skewhermitian form $\phi:W\times W\to F$ (i.e., $\phi(b,a)=-\phi(a,b)^\dagger$ and ϕ F-linear in the first variable). Such a ϕ can be brought into a standard form: if the involution is trivial ($F=K=K_0$), then $\dim_F V$ must be even, say $2r$, and there exists a basis $(e_{\pm1},\dots,e_{\pm r})$ such that $\phi(a,b)=\sum_{i=1}^r a_ib_{-i}-a_{-i}b_i$; if it is not, then there exists a basis $(e_1,\dots,e_r)$ of W and nonzero $u_i\in F$ with $u_i^\dagger=-u_i$

$(i = 1, \dots, r)$ such that $\phi(a, b) = \sum_{i=1}^{r} a_i u_i b_i^\dagger$. We take for $V_\mathbb{Q}$ the $\mathbb{Q}$-vector space underlying W and let $\kappa := \mathrm{Tr}_{F/\mathbb{Q}}\, \phi : V_\mathbb{Q} \times V_\mathbb{Q} \to \mathbb{Q}$. Then one can find an complex structure J on V which commutes with F, preserves κ and is such that $\kappa(a, Ja) > 0$ for all nonzero $a \in V$. Under some mild restrictions (given in [Shimura], §4, Thm. 5 ff.), one can also arrange that the centralizer of J in $\mathrm{End}(V_\mathbb{Q})$ is no more than F; this means that F appears as the endomorphism algebra tensorized with $\mathbb{Q}$ of any abelian variety associated to $(V_\mathbb{Q}, J)$.

Let us define a K_0-Lie subalgebra of $\mathrm{End}(2m, F)$ by:

$$\mathfrak{s}\mathfrak{k}\mathfrak{u}(2m, F, \dagger) = \{\begin{pmatrix} A & B \\ C & -{}^tA^\dagger \end{pmatrix} | A, B, C \in \mathrm{End}(m, F); B = {}^tB^\dagger, C = {}^tC^\dagger\}.$$

This is the Lie algebra of infinitesimal automorphisms of the skew-hermitian form

$$\sum_{k=1}^{m} (z_k w_{-k}^\dagger - z_{-k} w_k^\dagger).$$

It is a reductive K_0-Lie algebra whose center is the space of scalars $\lambda \in K$ with $\lambda^\dagger = -\lambda$. So $\mathfrak{s}\mathfrak{k}\mathfrak{u}(2m, F, \dagger)$ is semisimple unless we are in the case of totally complex multiplication. We grade this Lie algebra by means of the semisimple element

$$u_m := \begin{pmatrix} -\mathbf{1}_m & 0 \\ 0 & \mathbf{1}_m \end{pmatrix} \in \mathfrak{s}\mathfrak{k}\mathfrak{u}(2m, F, \dagger)$$

so that A, B and C parametrize the summands of degree 0, -2 and 2 respectively. Let $\mathfrak{g}(2m, F, \dagger)$ denote the K_0-Lie subalgebra of $\mathfrak{s}\mathfrak{k}\mathfrak{u}(2m, F, \dagger)$ generated by the summands of degree 2 and -2; let $\mathfrak{u}(m, F, \dagger)$ denote the union of $\mathrm{GL}(m, F)$-conjugacy classes in $\mathrm{End}(m, F)$ made up of anti-invariants with respect to the involution $A \mapsto {}^tA^\dagger$ and identify $\mathfrak{u}(m, F, \dagger)$ with a subspace of $\mathfrak{s}\mathfrak{k}\mathfrak{u}(2m, F, \dagger)_0$ in an obvious way.

The proof of the following lemma is left to the reader.

(3.9) Lemma. *The pair $(\mathfrak{g}(2m, F, \dagger), u_m)$ is a Jordan pair. The space $\mathfrak{u}(m, F, \dagger)$ is a Lie ideal in $\mathfrak{s}\mathfrak{k}\mathfrak{u}(2m, F, \dagger)$ supplementary to $\mathfrak{g}(2m, F, \dagger)$. It is trivial except in the following cases:*

(1) *$m = 1$ and F is totally definite quaternion: then $\mathfrak{g}(2, F, \dagger) \cong \mathfrak{sl}(2, K_0)$ and $\mathfrak{u}(1, F)$ can be identified with the pure quaternions in F (i.e., the †-anti-invariants in F) or*
(2) *K is totally complex: then $\mathfrak{g}(2m, F, \dagger)$ consists of the matrices for which A has its K-trace in K_0, whereas $\mathfrak{u}(m, F)$ can be identified with the purely imaginary scalars in K (i.e., the $\lambda \in K$ with $\bar{\lambda} = -\lambda$).*

Notice that in the exceptional cases the connected Lie subgroup of $\mathrm{GL}(m, F \otimes_\mathbb{Q} \mathbb{R})$ with Lie algebra $\mathfrak{u}(m, F, \dagger) \otimes_\mathbb{Q} \mathbb{R}$ is a product of e_0 copies of $U(1)$ resp. $SU(2)$ and hence compact.

(3.10) Theorem. *The graded Lie algebra $\mathfrak{g}_{NS}(A^m;\mathbb{Q}) \times \mathfrak{u}(A^m)$ is in a natural way a product of graded K_0-Lie algebras and as such it is isomorphic (factor by factor) to $\mathfrak{g}(2m, F, \dagger) \times \mathfrak{u}(m, F, \dagger) = \mathfrak{stu}(2m, F, \dagger)$.*

Proof. We make the identification $\mathrm{H}_1(A^m;\mathbb{Q}) = \mathrm{H}_1(A;\mathbb{Q})^m$. This identifies the algebra $\mathrm{End}^0(A^m)$ with $\mathrm{End}(m, F)$. We polarize each summand by means of κ. Then the sum of these polarizations is a polarization of A^m and the correponding Rosati involution in $\mathrm{End}(m, F)$ is given by $\sigma \mapsto {}^t\sigma^\dagger$. Hence $\mathrm{NS}(A^m) \otimes \mathbb{Q}$ can be identified with the space of †-hermitian matrices in $\mathrm{End}(m, F)$. This identifies $\mathfrak{g}_{NS}(A^m;\mathbb{Q})_2$ resp. $\mathfrak{g}_{NS}(A^m;\mathbb{Q})_{-2}$ with $\mathfrak{g}(2m, F, \dagger)_2$ resp. $\mathfrak{g}(2m, F, \dagger)_{-2}$. Since the Néron–Severi Lie algebra is generated by these summands, it follows that this extends to an isomorphism of $\mathfrak{g}_{NS}(A^m;\mathbb{Q})$ onto $\mathfrak{g}(2m, F, \dagger)$. The rest is easy.

We describe the situation in each of the four cases:

Totally real multiplication. Then $\mathfrak{g}_{NS}(A^m;\mathbb{Q})$ is a K_0-form of $\mathfrak{sp}(2m)$ and we have $\mathfrak{g}_{NS}(A^m;\mathbb{Q})_0 = \mathrm{End}(A^m) \otimes \mathbb{Q} \cong \mathrm{End}(m, K_0)$.

This is a K_0-form of the case (C_m, A_{m-1}).

Totally indefinite quaternion multiplication. Then $\mathfrak{g}_{NS}(A^m;\mathbb{Q})$ is a K_0-form of $\mathfrak{sp}(4m)$ and $\mathfrak{g}_{NS}(A^m;\mathbb{Q})_0 = \mathrm{End}(A^m) \otimes \mathbb{Q} \cong \mathfrak{gl}(2m; K_0)$.

This is a K_0-form of the case (C_{2m}, A_{2m-1}).

Totally definite quaternion multiplication. The Lie algebra $\mathfrak{g}_{NS}(A;\mathbb{Q})$ is isomorphic to $\mathfrak{sl}(2, K_0)$ and so $\mathfrak{g}_{NS}(A;\mathbb{Q})_0 \cong K_0$, whereas $\mathrm{End}(A)$ is a quaternion K_0-algebra. For $m \geq 2$, $\mathfrak{g}_{NS}(A^m;\mathbb{Q})$ is a K_0-form of $\mathfrak{so}(4m)$ and $\mathfrak{g}_{NS}(A;\mathbb{Q})_0 = \mathrm{End}(A^m) \otimes \mathbb{Q} \cong \mathrm{End}(2m, K_0)$.

This is a K_0-form of the case (D_{2m}, A_{2m-1}).

Totally complex multiplication. The Lie algebra $\mathfrak{g}_{NS}(A^m;\mathbb{Q})$ is a K_0-form of $\mathfrak{sl}(2md)$. The inclusion $\mathfrak{g}_{NS}(A^m;\mathbb{Q})_0 \subset \mathrm{End}(A^m) \otimes \mathbb{Q}$ corresponds to $\mathfrak{sl}(md, K) \times K_0\mathbf{1}_m \subset \mathfrak{gl}(md, K)$.

This is a K_0-form of the case $(A_{2md-1}, A_{md-1} + A_{md-1})$.

In all these cases, the subalgebra of $\mathrm{H}(A)$ generated by the Néron–Severi group is a Jordan–Lefschetz algebra, or rather a tensor product of such: if $A(k)$ denotes the Jordan–Lefschetz K_0-algebra of level k associated to K_0-Jordan pair $(\mathfrak{g}_{NS}(A^m;\mathbb{Q}), h)$, then the subalgebra of $\mathrm{H}(A;\mathbb{R})$ generated by the Néron–Severi group can be identified with the tensor product of the $\mathbb{R}$-algebras $\mathbb{R}\otimes_\sigma A(k)$, where σ runs over $\mathbf{e}_0$ and k can be calculated from the equality $ke_0.\,\mathrm{depth}\, A(1) = \dim_{\mathbb{C}} A^m$. In the case of complex multiplication, k will be divisible by d.

(3.11) It is clear that the algebra of Hodge classes $\mathrm{H}_{\mathrm{Hdg}}(X) \subset \mathrm{H}(X)$ (i.e., the complex span of the rational part of $\oplus_k \mathrm{H}^{k,k}(X)$) is $\mathfrak{g}_{NS}(X)$-invariant and contains the subalgebra generated by the Néron–Severi classes as a $\mathfrak{g}_{NS}(X)$-submodule. So the Hodge conjecture for X is basically concerned with the primitive subspace $\mathrm{Prim}(\mathrm{H}_{\mathrm{Hdg}}(X))$ (in the sense of (1.13)) in positive cohomological degree. In this way the Néron–Severi Lie algebra neatly supplements the Mumford–Tate group in helping us (at least in principle) to understand the Hodge algebra: the latter characterizes $\mathrm{H}_{\mathrm{Hdg}}(X)$ as the ring of invariants of the Mumford–Tate group, whereas the decomposition of $\mathrm{H}_{\mathrm{Hdg}}(X)$ into $\mathbb{Q}$-irreducible representations of $\mathfrak{g}_{NS}(X)$ tells us among other things which classes do not come from divisors. To illustrate the point, let us observe that $\mathfrak{u}(X)$ kills the Néron–Severi group, hence kills the subalgebra of $\mathrm{H}_{\mathrm{Hdg}}(X)$ generated by this group. So as soon as $\mathfrak{u}(X)$ acts nontrivially on $\mathrm{H}_{\mathrm{Hdg}}(X)$, X will have Hodge classes that are not in this subalgebra. This often happens when $\mathfrak{u}(X) \neq 0$ [Moonen-Zar].

4. Geometric examples of Jordan type II: hyperkählerian manifolds

(4.1) In the previous section we saw that complex tori and abelian varieties furnish examples of the Jordan–Lefschetz algebras that are of type $(A_{2m-1}, A_{m-1}+A_{m-1})$, (C_m, A_{m-1}) and (D_{2m}, A_{2m-1}). The classical Jordan–Lefschetz algebras that remain are those of type (B_m, B_{m-1}) and (D_m, D_{m-1}) and the purpose of this section is provide geometric examples of them. One way to get such examples is to take a compact Kähler surface X: then $\mathrm{H}^{\mathrm{ev}}(X;\mathbb{R})[2]$ is a fundamental Jordan–Lefschetz module of $\mathrm{H}^2(X;\mathbb{R})$ of the desired type. The corresponding Lie algebra is $\mathfrak{so}(\phi)$, where ϕ is the form defined in (1.9) and its degree zero part is $\mathfrak{so}(\mathrm{H}^2(X;\mathbb{R})) \times \mathbb{R}h$. A more interesting class of examples was found by M. Verbitsky, and this is what we will discuss in what follows.

Let $\mathbb{H}$ be a quaternion algebra over $\mathbb{R}$. We denote its trace by $\mathrm{Tr} : \mathbb{H} \to \mathbb{R}$ so that $\frac{1}{4}\,\mathrm{Tr}$ is the projection onto the real subfield. Elements of the kernel of Tr are called *pure quaternions*; they make up a Lie algebra that we denote by $\mathbb{H}_0$. The *pure part* of $a \in \mathbb{H}$ is its projection in $\mathbb{H}_0$: $a_0 := a - \frac{1}{4}\,\mathrm{Tr}(a)$.

We have also defined the norm $\mathrm{Nm} : \mathbb{H} \to \mathbb{R}$, $\mathrm{Nm}(a) = a\bar{a}$ (where $^-$ is the natural anti-involution that is -1 on $\mathbb{H}_0$). The set $\mathbb{H}_1$ of elements of norm 1 is a Lie subgroup of the group of units $\mathbb{H}^\times$ of $\mathbb{H}$ and has $\mathbb{H}_0$ as its Lie algebra. It is isomorphic to $SU(2)$. The unit sphere in $\mathbb{H}_0$, $\mathbb{H}_0 \cap \mathbb{H}_1$, is precisely the set of square roots of -1 in $\mathbb{H}$ and so effectively parametrizes the field homomorphisms $\mathbb{C} \to \mathbb{H}$. (Such a field homomorphism can also be thought of as (defining) an $\mathbb{R}$-homomorphism of algebraic groups $\mathbf{S}(\mathbb{R}) = \mathbb{C}^\times \to \mathbb{H}^\times$.) It is a conjugacy class of $\mathbb{H}^\times$.

Let T be a left $\mathbb{H}$-module of finite rank $m \geq 1$, equipped with positive definite real inner product $\langle\, , \,\rangle : T \times T \to \mathbb{R}$ that is $\mathbb{H}$-invariant (this gives rise to $\mathbb{H}$-Hermitian form). We write V for its $\mathbb{R}$-dual $\mathrm{Hom}(T,\mathbb{R})$ and we let $\mathbb{H}$ act on the latter on the right.

Every $J \in \mathbb{H}_0 \cap \mathbb{H}_1$ gives T the structure of a complex vector space of dimension $2m$. Since the inner product is $\mathbb{H}$-invariant, $H_J(x,y) := \langle x,y\rangle - \sqrt{-1}\langle Jx, y\rangle$ is a J-Hermitian form on V. Its imaginary part is antisymmetric and thus determines a 2-form $\kappa_J \in \wedge^2 V$. Wedging with κ_J defines an operator in $\wedge V[2m]$ that we denote by e_J. It has the Lefschetz property: the corresponding degree -2 operator f_J is characterized by $f_J = \star e_J \star^{-1}$. This makes sense for any nonzero element $a \in \mathbb{H}_0$: e_a has the Lefschetz property and $f_a = \mathrm{Nm}(a)^{-1} \star e_a \star^{-1}$. It is clear that the f_a's commute. We denote the Lie algebra generated by these elements by $\mathfrak{g}(V)$. This applies in particular to the left $\mathbb{H}$-module that underlies $\mathbb{H}$ itself (with inner product given by the norm). So there is defined a Lie algebra $\mathfrak{g}(\mathbb{H})$. This is actually the universal case, because an orthogonal splitting of T into m $\mathbb{H}$-lines allows us to identify T with an orthogonal direct sum $\mathbb{H}\oplus \cdots \oplus \mathbb{H}$. This induces a graded algebra isomorphism $\wedge^\bullet V \cong (\wedge^\bullet_{\mathbb{R}}\mathbb{H}) \otimes \cdots \otimes (\wedge^\bullet_{\mathbb{R}}\mathbb{H})$ that is compatible with the actions of e_a and h. The Lie algebra $\mathfrak{g}(V)$ now appears as $\mathfrak{g}(\mathbb{H})$ acting on the m-fold tensor power of its defining representation. In particular, we find an isomorphism of graded Lie algebras $\mathfrak{g}(\mathbb{H}) \cong \mathfrak{g}(V)$ that extends the identifications between the actions of e_a, $a \in \mathbb{H}_0$.

The following lemma gives more information about the nature of this Lie algebra.

(4.2) Lemma.

(i) $(\mathfrak{g}(\mathbb{H}), h)$ *is a Jordan–Lefschetz pair with* $\mathfrak{g}(\mathbb{H})_2$ *canonically isomorphic to the vector space underlying* $\mathbb{H}_0$.

(ii) *We have a natural isomorphism* $\mathfrak{g}(\mathbb{H})_0 \cong \mathbb{H}_0 \times \mathbb{R}h$, *where* $\mathbb{H}_0$ *is regarded as the Lie algebra of* $\mathbb{H}_1$. *The given action of* $\mathbb{H}_1$ *on* $\wedge^\bullet V$ *integrates the action of this summand.*

(iii) $(\mathfrak{g}(\mathbb{H}), h)$ *is isomorphic to the orthogonal Lie algebra defined by the form* $x_1x_5 + x_2^2 + x_3^2 + x_4^2$ *with* h *corresponding to* $diag(-1, 0, 0, 0, 1)$.

(iv) *The subalgebra* $M \subset \wedge^\bullet V$ *generated by the* κ_J*'s is invariant under the star operator and* $\mathfrak{g}(\mathbb{H})$, *and* $M[2m]$ *becomes a Jordan–Lefschetz module of* $(\mathfrak{g}(\mathbb{H}), h)$ *of level* m.

Proof. In view of the preceding discussion, we may assume that $m = 1$. Besides the established fact that the e_a's and the f_a's commute among each other, one verifies that

$$[e_a, f_b] = -(ab^{-1})_0 + \frac{1}{4}\operatorname{Tr}(ab^{-1})h,$$

where h defines the grading and $(ab^{-1})_0 \in \mathbb{H}_0$ operates on $\wedge^\bullet V$ on the right via the $\mathbb{H}$-module structure on V. One further checks that e_a and f_a commute with the action of $\mathbb{H}$. The assertions then follow in a straightforward manner, but let us nevertheless make some remarks that make the verification rather simple and give a clearer picture as well. We keep assuming that $m = 1$.

The star operator $\star$ in $\wedge^\bullet V$ defines an involution in $\wedge^2 V$ whose eigen spaces we denote $(\wedge^2 V)^\pm$. There are two interesting symmetric bilinear forms on $\wedge^\bullet V$. One, denoted ϕ, is characterized by

$$\phi(x, y) := \int x \wedge y \quad \text{if } \deg(x) \geq 2,$$

where $\int : \wedge^\bullet V \to \wedge^4 V \cong \mathbb{R}$ is the obvious projection. It has the property that the e_u's and h leave this form infinitesimally invariant, so that $\mathfrak{g} \subset \mathfrak{so}(\phi)$. The other form is the natural extension of the inner product: $\langle x, y\rangle = \int(x \wedge \star y)$. So on $(\wedge^2 V)^+$ both are equal, whereas on $(\wedge^2 V)^-$ they are opposite, and these eigen spaces are perpendicular for both forms. The κ_J's make up the unit sphere in the space of selfdual 2-forms $(\wedge^2 V)^+$. Since the wedge product of a selfdual form and an antiselfdual form is zero, we find that each e_u annihilates $(\wedge^2 V)^-$, so that $\mathfrak{g}(\mathbb{H})$ acts trivially on this space. Hence $\mathfrak{g}(\mathbb{H})$ acts on $\wedge^{\mathrm{ev}} V$ via $M = \mathbb{R} \oplus (\wedge^2 V)^+ \oplus \wedge^4 V$. One checks that the e_J's and f_J's generate all of $\mathfrak{so}(M, \phi)$, so that we have a surjective Lie homomorphism $\mathfrak{g}(\mathbb{H}) \to \mathfrak{so}(M, \phi)$ (a quick way to see this is to invoke (2.1) and our classification of Jordan–Lefschetz pairs: the former implies that the image is a Jordan–Lefschetz pair with 3-dimensional degree two summand and latter shows that such a pair must be of type B_2). This homomorphism is in fact an isomorphism. The action on $\wedge^{\mathrm{odd}} V = V \oplus \star V$ can be understood as follows: V is after complexification no longer irreducible as a $\mathbb{H}$-module: we can write

$V\otimes\mathbb{C}=P\oplus\bar{P}$ where P is a $\mathbb{H}$-invariant complex plane. Such a plane is totally isotropic with respect to the complexified inner product. The group $\mathbb{H}_1$ acts on P faithfully with image $SU(P)$. Now $P\oplus\star\bar{P}$ is $\mathfrak{g}(\mathbb{H})\otimes\mathbb{C}$-invariant and comes with a natural symplectic form. This symplectic form is infinitesimally preserved by the e_a's so that we have a Lie homomorphism $\mathfrak{g}(\mathbb{H})\otimes\mathbb{C}\to\mathfrak{sp}(P\oplus\star\bar{P})$. This is an isomorphism (defining a spinor representation of $\mathfrak{g}(\mathbb{H})$).

(4.3) We denote by $U(m,\mathbb{H})$ the group of quaternionic transformations in $\mathbb{H}^m$ that preserve the standard positive definite hermitian form $\sum_{i=1}^r u_i\bar{v}_i$. This is a maximal compact subgroup of a (standard) complex symplectic group of genus $2m$ (and is often denoted by $Sp(m)$, but we avoid that notation since it may lead to confusion with the way we refer to the symplectic groups). The centralizer of $U(m,\mathbb{H})$ in $\mathrm{End}(4m)$ is the algebra of quaternions.

A connected Riemann manifold of dimension $4m$ is called a *hyperkähler manifold* if its holonomy group is equivalent to the standard embedding of $U(m,\mathbb{H})$ in $\mathrm{GL}(4m,\mathbb{R})$. Let (X,g) be such a manifold. The centralizer of the holonomy group at $p\in X$ in $\mathrm{End}(T_pX)$ is a quaternion algebra $\mathbb{H}(p)$ that gives T_pX the structure of a (left) vector space over the skew field $\mathbb{H}(p)$. The subalgebras $\mathbb{H}(p)$, $p\in X$, make up a subbundle of $\mathrm{End}(TX)$. Since these subalgebras have trivial holonomy, the corresponding bundle is flat for the Levi-Civita connection and has trivial monodromy. This allows us to identify each fiber with the algebra of flat sections of this bundle. Although that subalgebra depends on (X,g), we will somewhat ambiguously denote it by $\mathbb{H}$. So TX is naturally endowed with an action of $\mathbb{H}$. The metric is an eigen tensor of this action with character the norm.

The name hyperkähler manifold is explained by the following fact. Any $J\in\mathbb{H}_0\cap\mathbb{H}_1$ (notation is as above) defines an almost-complex structure on X. This almost-complex structure flat with respect to the Levi-Civita connection and (hence) integrable. It combines with the given metric on X to a Kähler structure on X; we denote the Kähler form by κ_J.

From now on we assume that X is also compact. Then $\mathrm{H}(X;\mathbb{R})$ can be identified with the space of harmonic forms on X. Since $\mathbb{H}$ preserves the harmonic forms $\mathbb{H}$ acts on $\mathrm{H}(X;\mathbb{R})$. The Hodge structure on $\mathrm{H}(X;\mathbb{R})$ is defined by the restriction of this action to $\mathbf{S}(\mathbb{R})=\mathbb{C}^\times$ under the homomorphism $\mathbb{C}^\times\to\mathbb{H}^\times$ that sends $\sqrt{-1}$ to J. So J acts as the Weil operator (which on $\mathrm{H}^{p,q}(X,J)$ is multiplication by $\sqrt{-1}^{-p+q}$).

The assignment $J\mapsto\kappa_J$ extends linearly to $\mathbb{H}_0$ and this extension is an isomorphism of $\mathbb{H}_0$ onto a space of harmonic 2-forms, of which the nonzero elements are Kähler classes. We denote the image of this map by $\mathfrak{a}$. We think of $\mathfrak{a}$ as 3-plane in $\mathrm{H}(X;\mathbb{R})$ and call it the *characteristic 3-plane* of the metric. Its nonzero elements have the Lefschetz property, so that is defined the Lie algebra $\mathfrak{g}(\mathfrak{a},\mathrm{H}(X;\mathbb{R}))$.

(4.4) Proposition.

(i) *There is a unique isomorphism $\mathfrak{g}(\mathbb{H})\cong\mathfrak{g}(\mathfrak{a},\mathrm{H}(X;\mathbb{R}))$ (of graded Lie algebras) that extends the identifications between the actions of e_a $a\in\mathbb{H}$ and the semisimple element h defining the grading.*

(ii) *Under the isomorphism of (i), the action of the Lie group $\mathbb{H}_1$ on the space of harmonic forms integrates the action of the semisimple part $\mathfrak{g}(\mathbb{H})'_0$ of*

$\mathfrak{g}(\mathbb{H})_0$ on $\mathrm{H}(X)$. This action preserves the algebra structure on $\mathrm{H}(X)$ so that $\mathfrak{g}(\mathbb{H})_0'$ acts on $\mathrm{H}(X)$ by derivations.

(iii) *The subalgebra $A_\mathfrak{a} \subset \mathrm{H}(X;\mathbb{R})$ generated by $\mathfrak{a}$ is invariant under the star operator and $\mathfrak{g}(\mathbb{H})$ and $A_\mathfrak{a}[2m]$ is a Jordan–Lefschetz module of $\mathfrak{g}(\mathbb{H})$ of level m.*

Proof. Part (i) follows from the observation that for every $x \in X$, the harmonic forms define a subspace of the exterior algebra of the cotangent space of X at x that is invariant under both $\star$ and cupping with alefschetz operator κ_a with $a \in \mathbb{H}_0$ nonzero.

For (ii) we remember that every $J \in \mathbb{H}_1 \cap \mathbb{H}_0$ acts as a Weil operator. So $\cos\theta + J\sin\theta$ acts on $\mathrm{H}^{p,q}(X,J)$ as multiplication by $\exp(-p+q)\theta$, hence acts as an algebra automorphism. This proves that $\mathbb{H}_1$ acts by algebra automorphism.

As for (iii), note that $A_\mathfrak{a}$ is additively spanned by the subalgebras $\mathbb{R}[a]$, with $a \in \mathfrak{a}$ nonzero. As such a subalgebra is invariant under $\star$, so is $A_\mathfrak{a}$. Hence $A_\mathfrak{a}$ is also invariant under f_a. The rest of the assertion is clear.

Property (ii) was observed by [Fujiki], the rest of this statement is due to [Verbitsky 1990].

[Beauville] shows that once X admits one Riemann metric with $U(m,\mathbb{H})$ as holonomy group, then it admits many of them. As we have seen above any such metric defines a characteristic 3-plane in $\mathrm{H}^2(X;\mathbb{R})$. Using a theorem of S.T. Yau, he proves among other things the following:

(i) There is a nonempty open subset of the 3-plane Grassmannian of $\mathrm{H}^2(X,\mathbb{R})$ parametrizing characteristic planes.

(ii) There is a nonzero symmetric bilinear form q_0 on $\mathrm{H}^2(X,\mathbb{R})$ with the property that for every characteristic 3-plane H defined by the metric g, its g-orthogonal complement $H^\perp$ coincides with its q_0-orthogonal complement and q_0 and g define proportional forms on H (with positive ratio) and $H^\perp$ (with negative ratio).

So q_0 is unique up to positive factor and is nondegenerate of signature $(3, b_2(X)-3)$.

This will imply the following analogue of (4.4), which is also mostly due to [Verbitsky 1995].

(4.5) Proposition. *We then have:*

(i) *The pair $(\mathfrak{g}_{\rm tot}(X;\mathbb{R}), h)$ is of Jordan–Lefschetz type of type (B,B) or (D,D) with $\mathfrak{g}_{\rm tot}(X;\mathbb{R})$ isomorphic to $\mathfrak{so}(4, b_2(X)-2)$.*

(ii) *We have natural identifications $\mathfrak{g}_{\rm tot}(X;\mathbb{R})_2 \cong \mathrm{H}^2(X;\mathbb{R})$ and $\mathfrak{g}_{\rm tot}(X;\mathbb{R})_0 \cong \mathfrak{so}(q_0)\times\mathbb{R}h$. The semisimple part of $\mathfrak{g}_{\rm tot}(X;\mathbb{R})_0$, $\mathfrak{g}_{\rm tot}(X;\mathbb{R})_0' \cong \mathfrak{so}(q_0)$, acts on $\mathrm{H}(X;\mathbb{R})$ by derivations.*

(iii) *The subalgebra A of $\mathrm{H}(X;\mathbb{R})$ generated by $\mathrm{H}^2(X;\mathbb{R})$ is invariant under $\mathfrak{g}_{\rm tot}(X;\mathbb{R})$ and $A[2m]$ is a Jordan–Lefschetz module of $\mathfrak{g}_{\rm tot}(X;\mathbb{R})$ of level m.*

Proof. According to the previous proposition, the operators f_a and f_b commute when a and b are nonzero elements of a characteristic 3-plane. By (i) this is therefore the case for a and b in a nonempty open subset of $\mathrm{H}^2(X;\mathbb{R})$. Since the expression

$[f_a, f_b]$ is rationally dependent on its arguments, it follows that f_a and f_b commute whenever both are defined.

Let $\mathfrak{g}_2$ resp. $\mathfrak{g}_{-2}$ denote the abelian Lie subalgebras of $\mathfrak{gl}(\mathrm{H}(X;\mathbb{R}))$ spanned by the e_a's resp. f_a's and let let $\mathfrak{g}_0$ be the Lie subalgebra generated by $[\mathfrak{g}_2, \mathfrak{g}_{-2}]$.

Claim 1. $\mathfrak{g}_0 = \mathfrak{g}_0' \times \mathbb{R}h$ (with $\mathfrak{g}_0' = [\mathfrak{g}_0, \mathfrak{g}_0]$) and $\mathfrak{g}_0'$ consists of derivations of $\mathrm{H}(X;\mathbb{R})$. Moreover, the image of $\mathfrak{g}_0'$ in $\mathfrak{gl}(\mathrm{H}^2(X,\mathbb{R}))$ is $\mathfrak{so}(q_0)$.

Proof. Notice that by the above (rationality) argument, $\mathfrak{g}_0$ is already generated by the brackets $[e_a, f_b]$ with a, b nonzero and contained in a characteristic 3-plane. If we fix such a 3-plane $\mathfrak{a}$, then the Lie subalgebra of $\mathfrak{g}_0$ generated by the $[e_a, f_b]$, a, b nonzero elements of $\mathfrak{a}$, is $\mathfrak{g}(\mathfrak{a}, \mathrm{H}(X;\mathbb{R}))_0' \times \mathbb{R}h$ with $\mathfrak{g}(\mathfrak{a}, \mathrm{H}(X;\mathbb{R}))_0'$ acting as infinitesimal algebra automorphisms of $\mathrm{H}(X)$, that is, as derivations. Moreover $\mathfrak{g}(\mathfrak{a}, \mathrm{H}(X;\mathbb{R}))_0'$ leaves $\mathfrak{q}_0$ invariant. So $\mathfrak{g}_0'$ acts as derivations and maps naturally to $\mathfrak{so}(q_0)$. This last homomorphism is surjective, because $\mathfrak{so}(q_0)$ is generated by its elements that kill the q_0-orthogonal complement of a characteristic 3-plane.

Claim 2. $\mathrm{ad}_{\mathfrak{g}_0}$ leaves $\mathfrak{g}_2$ and $\mathfrak{g}_{-2}$ invariant.

Proof. Let $u \in \mathfrak{g}_0'$. Since u is a derivation, we have for every $a \in \mathrm{H}^2(X;\mathbb{R})$ and $z \in \mathrm{H}(X;\mathbb{R})$, that $[u, e_a](z) = u(a.z) - a.u(z) = u(a).z = e_{u(a)}(z)$. So ad_u leaves the space of operators e_a invariant. If $G_0' \subset \mathrm{GL}(\mathrm{H}(X;\mathbb{R}))$ denote the connected closed subgroup with Lie algebra $\mathfrak{g}_0'$, then for every $g \in G_0'$ we have $g e_a g^{-1} = e_{g(a)}$ and $g h g^{-1} = h$. Hence if f_a is defined, then $g f_a g^{-1} = f_{g(a)}$. It follows that $\mathrm{Ad}_{G_0'}$ leaves $\mathfrak{g}_{-2}$ invariant. The same assertion then holds for $\mathrm{ad}_{\mathfrak{g}_0'}$.

We conclude that $\mathfrak{g} := \mathfrak{g}_{-2} + \mathfrak{g}_0 + \mathfrak{g}_{-2}$ is a Lie subalgebra of $\mathfrak{gl}(\mathrm{H}(X;\mathbb{R}))$ and hence equal to $\mathfrak{g}_{\mathrm{tot}}(X;\mathbb{R})$. Clearly, $(\mathfrak{g}, h)$ is a Jordan–Lefschetz pair.

Claim 3. A is an irreducible $\mathfrak{g}$-submodule that has 1 as lowest weight vector and $\mathfrak{g}$ acts faithfully on A. Moreover, $\mathfrak{g}_0'$ maps isomorphically onto $\mathfrak{so}(q_0)$.

Proof. From $U\mathfrak{g} = U\mathfrak{g}_2.U\mathfrak{g}_0.U\mathfrak{g}_{-2}$ and and the fact that $\mathfrak{g}_0 + \mathfrak{g}_{-2}$ stabilizes the unit element it follows that A is $\mathfrak{g}$-invariant and has 1 as lowest weight vector. It is clear that the kernel of the homomorphism $\mathfrak{g} \to \mathfrak{gl}(M)$ is contained in $\mathfrak{g}_0'$. This kernel acts trivially on $\mathrm{H}^2(X;\mathbb{R})$, hence has zero Lie bracket with any e_a (here we use that $\mathfrak{g}_0'$ acts by derivations). Applying the Jacobson–Morozov theorem to e_a acting on $\mathfrak{g}$, we also find that the kernel has zero Lie bracket with f_a, when defined. So the kernel has zero Lie bracket with $\mathfrak{g}_2$, $\mathfrak{g}_{-2}$ and hence also with $\mathfrak{g}_0 = [\mathfrak{g}_2, \mathfrak{g}_{-2}]$. Since $\mathfrak{g}$ is semisimple, this implies that the kernel is trivial.

We have already seen that the map $\mathfrak{g}_0' \to \mathfrak{so}(q_0)$ is surjective. Since $\mathfrak{g}_0'$ acts by derivations, its action on A is completely determined by its restriction to $\mathrm{H}^2(M,\mathbb{R})$. Hence it is also injective.

The theorem now follows easily.

(4.6) Corollary. *The product map* $\mathrm{H}^2(X) \times \mathrm{H}(X;\mathbb{Q}) \to \mathrm{H}(X;\mathbb{Q})$ *and the Hodge structure on* $\mathrm{H}^2(X)$ *determine the Hodge structure on all of* $\mathrm{H}(X)$.

Proof. Let $\tilde{G}$ be the closed connected $\mathbb{Q}$-subgroup of $\mathrm{GL}(\mathrm{H}(X))$ with Lie algebra $\mathfrak{g}_{tot}(X)_0'$. Then its image G in $\mathrm{GL}(\mathrm{H}^2(X))$ is an orthogonal group of rank ≥ 3 and the projection $\tilde{G} \to G$ has finite kernel. The Hodge structure on $\mathrm{H}^2(X)$ is given by a real representation of the Deligne torus $\mathbf{S}$ on $\mathrm{H}^2(X)$. This defines an $\mathbb{R}$-morphism of algebraic groups $\mathbf{S} \to G$. Similarly, the Hodge structure on $\mathrm{H}(X)$ is given by an

$\mathbb{R}$-morphism of algebraic groups $\mathbf{S} \to \tilde{G}$. The latter must be a lift of the former. But clearly such a lift is unique.

(4.7) *Example.* Let T be a complex torus of complex dimension 2 and let m be an positive integer. As Beauville explains, the $(m+1)$-fold symmetric product $S^{m+1}(T)$ of T admits a natural nonsingular resolution $T^{[m+1]} \to S^{m+1}(T)$ (the Hilbert scheme paramatrizing finite subschemes of T of length $m+1$). Composition of this resolution with the "sum map" $S^{m+1}(T) \to T$ gives a fibration $T^{[m+1]} \to T$ which is locally trivial in the étale sense. Let K_m be the fiber over the origin. This fiber is simply connected and if $m \geq 2$, then $\mathrm{H}^2(K_m)$ (with its Hodge structure) is canonically isomorphic to the direct sum of $\mathrm{H}^2(T)$ and the span of the class of the exceptional divisor restricted to K_m. So in that case, $\dim \mathrm{H}^2(K_m) = 7$. Beauville shows that for a generic choice of T, K_m admits a hyperkähler metric. Hence (4.5) implies that $\mathfrak{g}_{\mathrm{tot}}(K_m;\mathbb{R}) \cong \mathfrak{so}(4,5)$. The cohomology of K_2 is computed in [Salamon]. We can interpret his result as saying that as a $\mathfrak{g}_{\mathrm{tot}}(K_m;\mathbb{R})$-module, $\mathrm{H}(X)$ is the orthogonal direct sum of the subalgebra generated by $\mathrm{H}^2(K_2)$, a trivial representation of dimension 80 and the spinor representation (of dimension 16). Their Hodge polynomials are $1 + (s^2 + 5st + t^2) + (s^4 + 5s^3t + 16s^2t^2 + 5st^3 + t^4) + (s^4t^2 + 5s^3t^3 + s^2t^4) + s^4t^4$, $80s^2t^2$ and $4(s^2t + st^2) + 4(s^3t^2 + s^2t^3)$ respectively.

We have already looked at algebras such as A in (2.14). Invoking (2.14) we find that if $u \in A_4 \subset \mathrm{H}^4(X)$ represents the dual of the Beauville–Bogomolov form q_0, then $a \mapsto \int_X a^{2i}u^{m-i}$ is proportional to q_0^i. The next theorem, which for $i = m, m-1$ is due to Fujiki, gives a natural choice for u. (The case $i = m$ was recently rediscovered by Bogomolov [Bogomolov].)

(4.8) Theorem. *Let $p \in \mathrm{H}^4(X;\mathbb{R})$ be the image of the first Pontryagin class of X under orthogonal projection of $\mathrm{H}(X;\mathbb{R})$ onto A. Then for $i = 0,\dots,m$ there exists a nonzero constant c_i such that*

$$q_0(x)^i = c_i \int_X x^{2i} p^{m-i}$$

for all $x \in \mathrm{H}^2(X)$.

We first show:

(4.9) Proposition. *The Lie algebra $\mathfrak{g}_{\mathrm{tot}}(X;\mathbb{R})_0'$ acts trivially on the subalgebra $P(X)$ generated by the Pontryagin classes.*

Proof. Let z be an element in this subalgebra of degree $4k$. Choose a metric g with holonomy group $\cong U(m,\mathbb{H})$. This determines an action of $\mathbb{H}$ on $\mathrm{H}(X;\mathbb{R})$ and a characteristic 3-plane $\mathfrak{a} \subset \mathrm{H}^2(X;\mathbb{R})$. If $J \in \mathbb{H}_0 \cap \mathbb{H}_1$, then with respect to the complex structure defined by J, $P^{4k}(X)$ consists of classes of bidegree $(2k,2k)$. Equivalently: the Weil operator J leaves $P(X)$ invariant. As this is true for all $J \in \mathbb{H}_0 \cap \mathbb{H}_1$, it follows that $P(X)$ is invariant under all of $\mathbb{H}_1$. Hence $P(X)$ is killed by the Lie algebra $\mathbb{H}_0$. Since these Lie algebras generate the semisimple part of $\mathfrak{g}_{\mathrm{tot}}(X;\mathbb{R})_0$, the proposition follows.

Proof of (4.8). Denote by $\pi : \mathrm{H}(X;\mathbb{R}) \to M$ the ϕ-orthogonal projection onto M. So if $a \in \mathrm{H}^2(X;\mathbb{R})$, then $\pi(a) \in M$ is characterized by the property that for all

$b \in \mathrm{H}(X;\mathbb{R})$ we have $\int_X ab = \int_X \pi(a)b$. This is a $\mathfrak{g}_{\mathrm{tot}}(A;\mathbb{R})$-equivariant projection. Since $p_1(X)$ is $\mathfrak{g}_{\mathrm{tot}}(M;\mathbb{R})$-invariant, $p = \pi(p_1(X))$ must be a multiple of u and so in view of (2.14) all we need to see is that $p \neq 0$. But this follows from a theorem of [Chen-Ogiue] which states that for some Kähler class κ, $\int_X \kappa^{2m-2} p_1(X)$ is positive.

The proof shows that the displayed identity in (4.8) is also valid if we replace p^{m-i} by a polynomial in the Pontryagin classes, except that the constant may be zero. We do not know whether in the case $i > 1$ there exists a polynomial for which this constant is nonzero.

(4.10) We have now seen that all the classical Jordan–Lefschetz algebras arise geometrically. The question comes up whether the same is true for the exceptional case of type E_7 (that corresponds to a Jordan algebra of dimension 27). In the topological setting the answer is yes: if the 27-dimensional vector space W in (2.11) is equipped with an integral structure $W_{\mathbb{Z}}$ such that the cubic form only takes integral even values (this is indeed possible), then it follows from theorems of Wall and Jupp that there is a simply connected closed oriented 6-manifold X for which the integral cohomology ring is isomorphic to the corresponding integral algebra $A_{\mathbb{Z}}$ (see [Okonek-VdV] for a general discussion). Now $H^{\mathrm{ev}}(X;\mathbb{Z})$, does not change if we take a connected sum of X with a number of copies of $S^3 \times S^3$. We wonder whether such a manifold admits a complex structure. Since all our interesting examples have trivial canonical bundle we are inclined to make this question more specific by asking:

Question. Does there exist a Calabi-Yau 3-fold with Picard group of rank 27 such that its Néron–Severi Lie algebra is of type E_7?

We notice that the mirror dual family of such a Calabi-Yau manifold will, if it exists, have its period mapping take values in a Hermitian domain of type E_7.

5. Filtered Lefschetz modules

(5.1) We recall a version of the Jacobson–Morozov lemma. If e is a nilpotent transformation in a vector space M, then there is a unique nonincreasing filtration $W^\bullet$ preserved by e such that e has the Lefschetz property in $\mathrm{Gr}^\bullet_W(M)$. Any $\mathfrak{sl}(2)$-triple (e,h,f) containing e descends to an $\mathfrak{sl}(2)$-triple in $\mathrm{Gr}_W(M)$ and splits the filtration (so the k-eigen space of h is a supplement of W^{k+1} in W^k). We shall refer to $W^\bullet$ as the *Lefschetz filtration* of e.

(5.2) Lemma. *Let $\mathfrak{g}$ be a reductive Lie algebra, $\mathfrak{s} \subset \mathfrak{g}$ a commutative subalgebra consisting of semisimple elements and $\chi \in \mathfrak{s}^*$ a character of $\mathfrak{s}$ in $\mathfrak{g}$. Then for every nilpotent $e \in \mathfrak{g}^\chi$ there exists a $f \in \mathfrak{g}^{-\chi}$ such that $(e,[e,f],f)$ is an $\mathfrak{sl}(2)$-triple.*

Proof. Choose an $\mathfrak{sl}(2)$-triple (e,h',f') containing e. Let f'' be the $\mathfrak{g}^{-\chi}$-component of f'. Then $h := [e,f'']$ is the $\mathfrak{g}^0$-component of h' and so $[h,e]$ is the $\mathfrak{g}^\chi$-component of $[h',e] = 2e$ and hence equal to $2e$. Since h is in the image of $\mathrm{ad}(e)$, the pair (e,h) is by [Bourbaki], Ch. VIII, §11, Lemme 6, extendable to a $\mathfrak{sl}(2)$-triple (e,h,f). This remains an $\mathfrak{sl}(2)$-triple if we replace f by its $\mathfrak{g}^{-\chi}$-component and so the lemma follows.

Let $(\mathfrak{a},M)$ be a Lefschetz module and $\mathrm{hor}^\bullet M$ a nonincreasing filtration on the graded vector space underlying M (so $\mathrm{hor}^k M = \oplus_l \mathrm{hor}^k M_l$) which is preserved by

$\mathfrak{a}$. We shall refer to this filtration as the *horizontal filtration*. Then the associated *vertical filtration* is defined by $\mathrm{ver}_k M := \sum_r \mathrm{hor}^{r-k} M_r$. This filtration is nondecreasing; we call the corresponding grading of $\mathrm{Gr}_{\mathrm{ver}} M$ the *vertical grading*. The notations $\mathrm{Gr}_{\mathrm{hor}} M$ and $\mathrm{Gr}^{\mathrm{ver}} M$ refer to the same vector space but with different gradings: $\mathrm{Gr}^k_{\mathrm{hor}} M_r = \mathrm{Gr}^{\mathrm{ver}}_{r-k} M_r$ has horizontal degree k and vertical degree $r-k$. The notation $\mathrm{Gr}\, M$ refers to their common bigraded structure.

Suppose now that some $a \in \mathfrak{a}$ preserves the vertical grading (i.e., $e_a(\mathrm{hor}^k M) \subset \mathrm{hor}^{k+2} M$ for all k) and has the Lefschetz property in $\mathrm{Gr}_{\mathrm{hor}} M$: e_a^k sends $\mathrm{Gr}^{-k}_{\mathrm{hor}} M$ isomorphically onto $\mathrm{Gr}^k_{\mathrm{hor}} M$. It is then immediate that the horizontal filtration is the Lefschetz filtration of the transformation e_a in M. If we apply (5.2) to $e := e_a$ and $\mathfrak{s} := \mathbb{C}h$, we find an $\mathfrak{sl}(2)$-triple $(e_a, h_{\mathrm{hor}}, f_a)$ in $\mathfrak{g}(\mathfrak{a}, M)$ with f_a of total degree -2 and h_{hor} of total degree 0. So f_a will map $\mathrm{hor}^k M_r$ to $\mathrm{hor}^{k-2} M_{r-2}$. This shows that f_a preserves the vertical filtration. It also follows that $h_{\mathrm{hor}} = [e_a, f_a]$ has this property. It is clear that the eigen spaces of h_{hor} split the horizontal filtration. This element commutes with h, so if we put $h_{\mathrm{ver}} := h - h_{\mathrm{hor}}$, then the eigen spaces of the commuting pair $(h_{\mathrm{hor}}, h_{\mathrm{ver}})$ define a bigrading of M that identifies M with $\mathrm{Gr}\, M$. The eigen spaces of $(h_{\mathrm{hor}}, h_{\mathrm{ver}})$ under the adjoint representation also define a bigrading of $\mathfrak{g}(\mathfrak{a}, M)$. The image of $\mathfrak{a}$ in $\mathfrak{g}(\mathfrak{a}, M)$ need not be bigraded.

(5.3) Proposition. *Let $\mathfrak{a}_{\mathrm{hor}} \subset \mathfrak{a}$ be the set of $a \in \mathfrak{a}$ that preserve the vertical grading and suppose that $\mathrm{Gr}_{\mathrm{hor}} M$ is a Lefschetz module of $\mathfrak{a}_{\mathrm{hor}}$. Then we can write $h = h_{\mathrm{hor}} + h_{\mathrm{ver}}$ with h_{hor} and h_{ver} semisimple elements of $\mathfrak{g}(\mathfrak{a}, M)$ that have integral eigen values and commute with each other (so for the resulting bigrading of M, $M_{k,l}$ gets identified with $\mathrm{Gr}^k_{\mathrm{hor}} M_{k+l}$).*

The span of the components of $\mathfrak{a}_{\mathrm{hor}}$ of lowest horizontal degree 2 make up an abelian subalgebra $\mathfrak{a}_{2,0}$ of $\mathfrak{g}(\mathfrak{a}, M)_{2,0}$ that has the Lefschetz property in M with respect to the horizontal grading. Moreover, $\mathfrak{g}(\mathfrak{a}_{2,0}, M_{\mathrm{hor}})$ is a subalgebra of $\mathfrak{g}(\mathfrak{a}, M)_{\bullet,0}$ that maps isomorphically onto $\mathfrak{g}(\mathfrak{a}_{\mathrm{hor}}, \mathrm{Gr}_{\mathrm{hor}} M)$.

If in addition, $\mathrm{Gr}^{\mathrm{ver}} M$ is a Lefschetz module of $\mathfrak{a}$, then the span of the components of $\mathfrak{a}$ of highest vertical degree 2 make up an abelian subalgebra $\mathfrak{a}_{0,2}$ of $\mathfrak{g}(\mathfrak{a}, M)_{0,2}$ that has the Lefschetz property in M with respect to the vertical grading. Moreover, $\mathfrak{g}(\mathfrak{a}_{0,2}, M_{\mathrm{ver}})$ is a subalgebra of $\mathfrak{g}(\mathfrak{a}, M)_{0,\bullet}$ that maps isomorphically onto $\mathfrak{g}(\mathfrak{a}, \mathrm{Gr}^{\mathrm{ver}} M)$. The obvious map $\mathfrak{g}(\mathfrak{a}_{2,0}, M_{\mathrm{hor}}) \times \mathfrak{g}(\mathfrak{a}_{0,2}, M_{\mathrm{ver}}) \to \mathfrak{g}(\mathfrak{a}, M)$ is then an injective homomorphism of Lie algebras.

Proof. Everything follows from the preceding or is obvious except the very last statement. We claim that any "horizontal" $\mathfrak{sl}(2)$-triple (e', h', e') acting in $\mathrm{Gr}\, M$ commutes with any "vertical" $\mathfrak{sl}(2)$-triple (e'', h'', e'') acting in $\mathrm{Gr}\, M$. This just follows from the fact that (e', h') commutes with (e'', h'') and the fact that in either case the last member is a rational expression in the first two. It follows that $\mathfrak{g}(\mathfrak{a}_{\mathrm{hor}}, \mathrm{Gr}_{\mathrm{hor}} M)$ and $\mathfrak{g}(\mathfrak{a}, \mathrm{Gr}^{\mathrm{ver}} M)$ commute. The same is therefore true for their bigraded lifts $\mathfrak{g}(\mathfrak{a}_{2,0}, M)$ and $g(\mathfrak{a}_{0,2}, M)$ in $\mathfrak{g}(\mathfrak{a}, M)$. To see that $\mathfrak{g}(\mathfrak{a}_{2,0}, M) \cap \mathfrak{g}(\mathfrak{a}_{0,2}, M) = 0$, note that this intersection has bidegree $(0,0)$ and is normal in either of them. If it were nonzero, then it would contain a simple factor of $\mathfrak{g}(\mathfrak{a}_{2,0}, M)$ of bidegree $(0,0)$. But this is impossible since $\mathfrak{g}(\mathfrak{a}_{2,0}, M)$ is (as a Lie algebra) generated by its degree ± 2 summands. The proposition follows.

Let $f : X \to Y$ be a fibration of projective manifolds which is topologically locally trivial and let n and m be the complex dimensions of X and Y repectively, so that $d := n - m$ is the complex fiber dimension. Following [Deligne 1968] the Leray spectral sequence of f degenerates: if $L^\bullet$ denotes the Leray filtration of $\mathrm{H}^\bullet(X)$, then $\mathrm{Gr}_L^k \mathrm{H}^r(X) \cong \mathrm{H}^k(Y, R^{r-k} f_*\mathbb{C})$.

(5.4) Proposition. *If h denotes the basic semisimple element of $\mathfrak{g}_{NS}(X)$, then we can write $h = h_{\mathrm{hor}} + h_{\mathrm{ver}}$ with h_{hor} and h_{ver} semisimple elements of $\mathfrak{g}_{NS}(X)$ that have integral eigen values and commute with each other so that for the resulting bigradings of $\mathrm{H}(X)$ and $\mathfrak{g}_{NS}(X)$ have the following properties: $\mathrm{H}(X)_{k,l}$ gets identified with $\mathrm{H}^{k+m}(Y, R^{l+d} f_*\mathbb{C})$ and*

$$\mathfrak{g}(\mathrm{NS}(Y), \mathrm{H}^\bullet(Y, Rf_*\mathbb{C})[m]) \times \mathfrak{g}(\mathrm{NS}(X/Y), \mathrm{H}(Y, R^\bullet f_*\mathbb{C}[d]))$$

lifts (uniquely) to a bigraded Lie subalgebra of $\mathfrak{g}_{NS}(X)$. In case the graded local system $R^\bullet f_*\mathbb{C}$ is trivial (which is for instance the case when Y is simply connected), then this amounts to a lift of $\mathfrak{g}_{NS}(Y) \times \mathfrak{g}_{NS}(X_y)$ to $\mathfrak{g}_{NS}(X)$, where X_y is a general fiber of f.

Proof. We prove the proposition by verifying the hypotheses of the previous proposition for $M := \mathrm{H}^\bullet(X)[n]$ with as horizontal filtration the Leray filtration appropriately shifted: $\mathrm{hor}^k M_r := L^{d+k}\,\mathrm{H}^{n+r}(M)$.

First recall that $R^l f_*\mathbb{C}$ underlies a variation of Hodge structure of weight l. If $\xi \in \mathrm{NS}(X)$ is ample relative f, then its image in $\mathrm{H}^0(Y, R^2 f_*\mathbb{C})$ has the Lefschetz property in the graded local system $R^\bullet f_*\mathbb{C}$ and induces a polarization in each summand. This implies that ξ has the Lefschetz property in $\mathrm{H}^k(Y, R^\bullet f_*\mathbb{C}[d])$ and satisfies the hypotheses of (1.6). So $\mathrm{H}^k(Y, R^\bullet f_*\mathbb{C})[d]$ is a Lefschetz module over $\mathrm{NS}(X/Y)$.

If $\eta \in \mathrm{NS}(Y)$ is a polarization, then cupping with η^k defines an isomorphism $\mathrm{H}^{m-k}(Y, R^l f_*\mathbb{C}) \to \mathrm{H}^{m+k}(Y, R^l f_*\mathbb{C})$ [Zucker], [Saito] and $\eta\cup$ has the Lefschetz property in $\mathrm{H}^\bullet(Y, R^l f_*\mathbb{C})[m]$. We apply (1.6) again and find that all the hypotheses of (5.3) are fulfilled.

The last remark uses the fact that for a general $y \in Y$, $\mathrm{NS}(X/Y)$ restricts isomorphically to $\mathrm{NS}(X_y)$ so that $\mathfrak{g}(\mathrm{NS}(X/Y), \mathrm{H}(Y, R^\bullet f_*\mathbb{C}[d]))$ can be identified with $\mathfrak{g}_{NS}(X_y)$.

Remark. This splitting of the Leray filtration is certainly a splitting that is invariant under the action of $\mathrm{NS}(Y)$. We do not know however whether it can be chosen to be a splitting of $\mathrm{H}(Y)$-modules.

(5.5) The conditions X and Y nonsingular and f topologically locally trivial can all be eliminated in the context of Hodge modules: if $f : X \to Y$ is a morphism of projective varieties, and E is a polarized Hodge module on X of pure weight, then according to [Saito] the Leray spectral sequence for f_*E degenerates and the Leray filtration satisfies all the hypotheses of (5.3) (here no shifting is necessary). If the map from a space Z to a fixed singleton is denoted a_Z, then we find that $\mathfrak{g}(\mathrm{NS}(X), a_{X*}(E))$ contains graded Lie subalgebras isomorphic to $\mathfrak{g}(\mathrm{NS}(Y), a_{Y*}\,\mathrm{H}^\bullet(f_*E))$ and $\mathfrak{g}(\mathrm{NS}(X/Y), a_{Y*}\,\mathrm{H}^\bullet(f_*E))$ that centralize each other. (Here the cohomology is taken in the sense of Hodge modules; for a representing complex of constructible sheaves, this amounts to taking perverse cohomology.)

(5.6) Proposition. *Let $f : X \to Y$ be a (topologically locally trivial) fibration of projective manifolds with fiber $\mathbb{P}^d$. Then $\mathfrak{g}_{NS}(X)$ contains a Lie subalgebra isomorphic to $\mathfrak{g}_{NS}(Y) \times \mathfrak{sl}(2)$. If this subalgebra is equal to $\mathfrak{g}_{NS}(X)$, then the characteristic classes of this bundle are trivial, i.e.,* $\mathrm{H}(X) = \mathrm{H}(Y) \otimes \mathrm{H}(\mathbb{P}^n)$ *as* $\mathrm{H}(Y)$-*algebras.*

Proof. As an algebra, $\mathrm{H}(X)$ is a simple integral extension of $\mathrm{H}(Y)$: $\mathrm{H}(X) = \mathrm{H}(Y)[\xi]/(P)$ with P a monic polynomial of degree $d+1$: $P = \xi^{d+1} + c_1\xi^d + \cdots + c_{d+1}$. We make P unique by requiring that $c_1 = 0$; then $c_2, \dots, c_{d+1}$ are the characteristic classes of our bundle. Here ξ is any element of $\mathrm{NS}(X) \otimes \mathbb{Q}$ that spans a supplement of $\mathrm{NS}(Y) \otimes \mathbb{Q}$ in $\mathrm{NS}(X) \otimes \mathbb{Q}$. Take a bigrading on $\mathrm{H}(X)$ as in the previous proposition so that we get an embedding of $\mathfrak{g}_{NS}(Y) \times \mathfrak{sl}(2)$ in $\mathfrak{g}_{NS}(X)$. If it is surjective, then take for ξ a nonzero element of $\mathrm{NS}(X)$ that projects in $\mathrm{H}^2(X)_{(-\dim Y, -d+2)}$. Cupping with ξ then corresponds to a nonzero element of the $\mathfrak{sl}(2)$ factor. So we have $\xi^{d+1} = 0$. Hence all the c_i's are zero.

(5.7) Theorem. *Let $f : X \to Y$ be a $\mathbb{P}^d$-bundle involving projective manifolds. Assume that the Néron–Severi Lie algebra of Y is maximal: $\mathfrak{g}_{NS}(Y) = \mathfrak{aut}(\mathrm{H}(Y))$, and that $\mathrm{H}(Y)$ is not an inner product space of dimension 4. Then either the characteristic classes of f are trivial and $\mathfrak{g}_{NS}(X) \cong \mathfrak{g}_{NS}(Y) \times \mathfrak{sl}(2)$ or the Néron–Severi Lie algebra of X is maximal.*

Proof. Suppose not all the characteristic classes are trivial and let $\xi \in \mathrm{NS}(X) \otimes \mathbb{Q}$ be such that $\mathrm{H}(X) = \mathrm{H}(Y)[\xi]/(P)$ with $P = \xi^{d+1} + c_2\xi^{d-1} + \cdots + c_{d+1}$ as in the proof of (5.6). Then $\mathfrak{g}_{NS}(X)$ contains a copy of $\mathfrak{g}_{NS}(Y) \times \mathfrak{sl}(2)$, but ξ, viewed as an element of $\mathfrak{g}_{NS}(Y)$, commutes with neither factor. So the simple component of $\mathfrak{g}_{NS}(X)$ that contains ξ contains $\mathfrak{g}_{NS}(Y) \times \mathfrak{sl}(2)$ as well. It remains to apply the theorem of the appendix.

(5.8) Theorem. *Let X be the flag space of a simple complex algebraic group. Then its Néron–Severi Lie algebra is equal to $\mathfrak{aut}(\mathrm{H}(X))$.*

Proof. In case the group is of rank one, then $X = \mathbb{P}^1$ and then the assertion is clear. So assume that the rank is ≥ 2. Then X admits an iterated fibered structure

$$X = X_s \to X_{s-1} \to \cdots \to X_0$$

with X_0 a singleton, $X_t \to X_{t-1}$ a projective space bundle with positive fiber dimension, and $s \geq 2$. Iterated application of (5.6) yields an embedding of $(\mathfrak{sl}(2))^s$ in $\mathfrak{g}_{NS}(X)$. In view of the theorem of the appendix it is therefore enough to show that $\mathfrak{g}_{NS}(X)$ is simple. Were that not the case then we could write nontrivially $\mathfrak{g}_{NS}(X) = \mathfrak{s}_1 \times \mathfrak{s}_2$ with $\mathfrak{s}_i$ nonzero semisimple. In that case, let H_i be the $\mathfrak{s}_i$-submodule of $\mathrm{H}(X)$ generated by the unit element. As $\mathrm{H}(X)$ is generated by $\mathrm{NS}(X)$ it follows that $\mathrm{H}(X) = H_1 \otimes H_2$ as algebras. However, such a decomposition is precluded by Borel's description of $\mathrm{H}(X)$. According to this theory, there is up to scalar a unique quadratic form on $\mathrm{NS}(X) \otimes \mathbb{C}$ that becomes a relation for $\mathrm{H}^4(X)$. This form is nondegenerate, and so $\mathrm{H}(X)$ cannot be the tensor product of two graded subalgebras (see also (1.2)).

(5.9) The Borel description of $\mathrm{H}(X)$ actually shows that as an algebra, $\mathrm{H}(X)$ is isomorphic to $\mathrm{Sym}(V)/I$, where V is the complexified weight lattice of G and I

is the ideal generated by the Weyl group invariant homogeneous forms of positive degree. In other words, we are in the case of the example discussed in (1.10) and the above theorem gives a complete description of $\mathfrak{g}(V, \mathrm{Sym}(V)/I)$ in case W is an irreducible Weyl group.

Another interesting class of projective manifolds with the property that their rational cohohomology is generated by the Néron–Severi group are the Knudsen–Mumford moduli spaces $\overline{\mathcal{M}}_0^n$ of stable n-pointed curves of genus zero. It it likely that here also the Néron–Severi Lie algebra of $\overline{\mathcal{M}}_0^n$ equals $\mathfrak{aut}(\mathrm{H}(\overline{CalM}_0^n))$ (compare theorem (6.8) below).

6. Frobenius–Lefschetz modules

(6.1) We say that a Lefschetz module $(M, \mathfrak{a})$ of depth n is *Frobenius* if it satisfies the following three properties:

(1) $\mathrm{Prim}\, M_{-n}$ is of dimension one (and so M is irreducible),
(2) the map $\mathfrak{a} \otimes M_{-n} \to M_{-n+2}$ is an isomorphism,
(3^d) M is generated as a $U\mathfrak{a}$-module by M_{-n}.

If only the first two conditions are satisfied we say that M is a *quasi-Frobenius* of depth n and if instead of (3), we have

(3') the $U\mathfrak{a}$-module generated by M_{-n} contains M_{-n+2k} for $k \leq d$,

then we say that M is *Frobenius up to order d.*

Observe that if A is Lefschetz algebra of depth n, then $A[n]$ is a Frobenius–Lefschetz module of A_2 if and only if A is generated by A_2. Moreover, any Frobenius–Lefschetz module is of this form.

We also note that a Jordan–Lefschetz module is Frobenius. The property of being quasi-Frobenius is a useful one, as it turns out to be a rather strong approximation to being Frobenius, that (somewhat in contrast to the latter) is generally easy to verify in practice. Another reason of our interest in this notion is that it occurs naturally in geometric examples:

(6.2) Proposition. *Let X be a connected compact Kähler (resp. complex projective) manifold. Then the $\mathfrak{g}_K(X)$-submodule (resp. $\mathfrak{g}_{NS}(X)$-submodule) of $\mathrm{H}(X)$ generated by $\mathrm{H}^0(X)$ is quasi-Frobenius as a Lefschetz module of $\mathrm{H}^{1,1}(X)$ (resp. $\mathrm{NS}(X)$).*

Proof. In the Kähler case, it is clear that this submodule is contained in $\oplus_k \mathrm{H}^{k,k}(X)$. So its intersection with $\mathrm{H}^2(X)$ is $\mathrm{H}^{1,1}(X)$ and the proposition follows. The Néron–Severi case is proved similarly.

The following proposition explains our terminology for it shows that a Lefschetz module $(M, \mathfrak{a})$ is Frobenius if and only if M is Frobenius (= Gorenstein) as a $U\mathfrak{a}$-module.

(6.3) Proposition. *Let $(\mathfrak{a}, M)$ be a Frobenius–Lefschetz module of depth n. Then there exists a nondegenerate $\mathfrak{g}(\mathfrak{a}, M)$-invariant $(-)^n$-symmetric bilinear form on M.*

Proof. Since $\dim M_{-n} = 1$, we also have $\dim M_n = 1$. The choice of a generator $u \in M_{-n}$ identifies M with a graded quotient R of the symmetric algebra $U\mathfrak{a}$ (with a shift of degree). Pick a nonzero linear form $\int : R_{2n} \to \mathbb{C}$, and define a graded bilinear form $\langle\, ,\, \rangle : M \times M \to \mathbb{C}$ by $\langle au, bu\rangle = (-1)^k \int(abu)$ if $a \in R^{2k}$ $b \in R^{2n-2k}$. This form is symmetric or skew according to whether n is even or odd. We claim that it is $\mathfrak{g}(\mathfrak{a}, M)$-invariant. For if we regard $\langle\, ,\, \rangle$ as an element of the Lefschetz module $(M \otimes M)^*$, then it is of degree zero and annihilated by $\mathfrak{a}$. So if f_a $(a \in \mathfrak{a})$ is defined, then $\langle\, ,\, \rangle$ is primitive for the $\mathfrak{sl}(2)$-triple (e_a, h, f_a) and hence annihilated by f_a. This proves our claim. Since M is irreducible, $\langle\, ,\, \rangle$ must be nondegenerate.

(6.4) *Example.* The following example is not just instructive, it also will be used in a proof (of theorem (6.8)). We give $V(2l) = K[e]/(e^{2l+1})[2l]$ the unique $\mathfrak{sl}(2)$-invariant quadratic form for which the inner product of 1 and e^{2l} is $(-1)^l$, so the inner product of e^p and e^q is $(-1)^{l+p}$ if $q = l - p \in \{0, \dots, l\}$ and zero otherwise. Let $V := V(2k) \oplus V(2k-2)$, $k \geq 2$, regarded as the orthogonal direct sum of $K[e]$-modules. Any endomorphism of V that commutes with e is $K[e]$-linear and so representable by a 2×2 matrix with coefficients in $K[e]$. We readily compute that the intersection of the centralizer of e with $\mathfrak{so}(V)_2$ is the set of matrices of the form

$$\begin{pmatrix} ae & be^2 \\ b & ce \end{pmatrix}$$

with a, b, c scalars. These do not mutually commute, so any subspace $\mathfrak{a} \subset \mathfrak{so}(V)_2$ that defines Lefschetz module structure on V and contains e is of dim ≤ 2. Hence $\mathfrak{a}$ will be spanned by e and an element e' that can be represented by a matrix of the above type with $c = -a$. If $a \neq 0 = b$, then it is easy to see $\mathfrak{g}(\mathfrak{a}, V) \cong \mathfrak{sl}(2) \times \mathfrak{sl}(2)$. Suppose therefore that $b \neq 0$. We normalize e' to make $b = 1$. Then e and e' satisfy: $(e')^2 = (a^2+1)e^2$, $e^{2k+1} = 0$ and $e^{2k-1}(e' - ae) = 0$. This suggests to take as a new basis for $\mathfrak{a}$ $a_\pm := e' \pm \sqrt{a^2+1}e$. Then the relations become: $a_+a_- = a_+^{2k+1} = a_-^{2k+1} = 0$ plus a relation of degree $2k$. It is then clear that as a $\mathbb{C}[a_-, a_+]$-module, V is generated by V_{-2k}. (One can now show that $\mathfrak{g}(\mathfrak{a}, V) = \mathfrak{so}(V)$, but we will not need this in what follows.)

Now assume that $k \geq 3$ and consider the semispinorial representation W of $\mathfrak{so}(V)$. Let us first recall how W is obtained. The inner product is nondegenerate on the plane V_0. Let F_0 and F_0' be complementary isotropic lines in V_0 and put $F := F_0 + \sum_{i>0} V_{2i}$ and $F' := F_0' + \sum_{i<0} V_{2i}$. These are complementary isotropic subspaces of V. One of the semispinorial representations of $\mathfrak{so}(V)$ can be realized on $W := \wedge^{\mathrm{ev}} F[n]$, with $n = k^2$. Now W has as its lowest degree summand a line in degree $-n$ spanned by $1 \in \wedge^0 F$. The action of $\mathfrak{a}$ on $\wedge^{\mathrm{ev}} F$ is as follows: if $f \in F_0$ and $f' \in F_0'$ are such that their inner product is 1, then we have

$$a \cdot (x_1 \wedge \cdots \wedge x_{2l}) = -f \wedge a(f') \wedge x_1 \wedge \cdots \wedge x_{2l} + \sum_{i=1}^{2l} x_1 \wedge \cdots \wedge a(x_i) \wedge \cdots \wedge x_{2l},$$

where $a \in \mathfrak{a}$ and $x_1, \dots, x_{2l} \in F$. A calculation shows that $a_\pm(f') = \pm c a_\pm(f)$, with c a nonzero constant. This enables us to determine the $\mathbb{C}[a_-, a_+]$-submodule of W generated by $1 \in W_{-n}$: we find that in degree $\leq -n+6$ it is the span of 1, $f \wedge a_\pm(f)$, $f \wedge a_\pm^2(f)$, $a_+(f) \wedge a_-(f)$, $f \wedge a_\pm^3(f) + a_\pm(f) \wedge a_\pm^2(f)$, $a_\pm(f) \wedge a_\mp^2(f)$. This contains the summands of degree $\leq -n+4$ of W, but not W_{-n+6}: since

$n = k^2 \geq 9$, $\dim W_{-n+6} \geq 5$, whereas the previous list has only four elements of degree $-n+6$. So W is Frobenius up to order 2, but not up to order 3.

(6.5) Let $(\mathfrak{g}, h, \mathfrak{a})$ be a Lefschetz triple, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ containing h and B a root basis of $R = R(\mathfrak{g}, \mathfrak{h})$ whose members are ≥ 0 on h. Let M be an irreducible representation of $\mathfrak{g}$ with $-\lambda \in \mathfrak{h}^*$ as lowest weight (with respect to B). We give a simple criterion for M to be quasi-Frobenius–Lefschetz module.

The lowest weight subspace of M, $M^{-\lambda}$, is a line. The $\mathfrak{g}$-stabilizer of this line is the standard parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_X$ of $\mathfrak{g}$, where X is the set of $\alpha \in B$ with $\lambda(\alpha^\vee) = 0$. In other words, $\mathfrak{p}$ is the direct sum of $\mathfrak{h}$ and the root spaces of roots β with $\lambda(\beta^\vee) \leq 0$. We endow M with the grading defined by h and denote the grade by a subscript. It is clear that the depth n of M is equal to $\lambda(h)$. So $M^{-\lambda} \subset M_{-n}$. The map that assigns to $x \in \mathfrak{g}$ the homomorphism $x|M^{-\lambda} \in \operatorname{Hom}(M^{-\lambda}, M)$ induces an injective map $\mathfrak{g}/\mathfrak{p} \to \operatorname{Hom}(M^{-\lambda}, M/M^{-\lambda})$. In particular, if $\mathfrak{p}_2$ denotes the degree two part of $\mathfrak{p}$:

$$\mathfrak{p}_2 := \sum_{\alpha \in R_2, \lambda(\alpha^\vee)=0} \mathfrak{g}^\alpha,$$

then we have an injective map $\mathfrak{g}_2/\mathfrak{p}_2 \to \operatorname{Hom}(M^{-\lambda}, M_{-n+2})$. We conclude:

(6.6) Lemma. *Suppose* $\dim M_n = 1$. *Then M is quasi-Frobenius if and only if $\mathfrak{a}$ is a supplement of $\mathfrak{p}_2$ in $\mathfrak{g}_2$.*

(6.7) If $\mathfrak{g}$ is a Lie algebra, then call a representation M of $\mathfrak{g}$ *tautological* if $\mathfrak{g}$ maps isomorphically onto the Lie algebra of infinitesimal isometries of a nondegenerate $(\pm)$-symmetric form on M. So $\mathfrak{g}$ is then a symplectic or orthogonal Lie algebra and if $\mathfrak{g} \not\cong \mathfrak{so}(4)$, then M is a fundamental representation of $\mathfrak{g}$ with highest weight at an end of the Dynkin diagram.

We have not found an example of a simple Frobenius–Lefschetz module that is not of Jordan–Lefschetz type or tautological. The following theorem says that such an example must involve an exceptional Lie algebra.

(6.8) Theorem. *A simple Frobenius–Lefschetz module that is not a Jordan-Lefschetz module and whose Lie algebra is simple and of classical type, is tautological.*

(6.9) Before we begin the proof, we derive some general properties of quasi-Frobenius–Lefschetz modules. So from now on, M is a quasi-Frobenius–Lefschetz module and we retain the notation introduced above. Since $\mathfrak{p}$ contains $\mathfrak{g}_0$, we have $X \supset B_0$. We denote $X \cap B_2$ by $B_2^{\mathfrak{p}}$ and we let $B_2^{\mathfrak{a}} := B_2 \setminus X = B_2 - B_2^{\mathfrak{p}}$.

For $\beta \in B_2$, we denote by $R_2(\beta)$ the set of roots in R_2 that have coefficient one on β and put

$$\mathfrak{g}_2(\beta) := \sum_{\gamma \in R_2(\beta)} \mathfrak{g}^\gamma.$$

Since R_2 is the disjoint union of the $R_2(\beta)$'s, $\mathfrak{g}_2$ is the direct sum of the $\mathfrak{g}_2(\beta)$'s. Each $\mathfrak{g}_2(\beta)$ is a $\mathfrak{g}_0$-invariant subspace of $\mathfrak{g}$. Notice that $\mathfrak{p}_2 = \sum_{\beta \in B_2^{\mathfrak{p}}} \mathfrak{g}_2(\beta)$ so that

$$\bar{\mathfrak{a}} := \sum_{\beta \in B_2^{\mathfrak{a}}} \mathfrak{g}_2(\beta).$$

is a $\mathfrak{g}_0$-invariant supplement of $\mathfrak{p}_2$ in $\mathfrak{g}_2$. By (6.6), $\mathfrak{a}$ is the graph of a linear map $\phi : \bar{\mathfrak{a}} \to \mathfrak{p}_2$.

(6.10) Lemma. *For every $\beta' \in B_2^{\mathfrak{p}}$ there exists a $\beta \in B_2^{\mathfrak{a}}$ such that the map $\mathfrak{g}_2(\beta) \to \mathfrak{g}_2(\beta')$ induced by ϕ is nonzero.*

Proof. Suppose not. Then $\mathfrak{a}$ centralizes the fundamental coweight p_β corresponding to β. We show that then $[f(e), p_\beta] = 0$, whenever $f(e)$ is defined. This will imply that $\mathfrak{a} \cup f(\mathfrak{a})$ is in the centralizer of p_β and thus contradict the fact that $\mathfrak{a} \cup f(\mathfrak{a})$ generates $\mathfrak{g}$ as a Lie algebra.

Write $f(e) = \sum_k f_k$ with $[h_\beta, f_k] = k f_k$. By homogeneity, (e, h, f_0) is then also an $\mathfrak{sl}(2)$-triple and by uniqueness of $f(e)$, we then have $f(e) = f_0$.

Virtually all the properties that we shall derive about quasi-Frobenius–Lefschetz modules come from the following lemma.

(6.11) Lemma. *The spaces $\bar{\mathfrak{a}}$ and $\phi(\bar{\mathfrak{a}})$ are abelian subalgebras and $[X, \phi(Y)] = [Y, \phi(X)]$ for all $X, Y \in \bar{\mathfrak{a}}$. (In particular, $[X, \phi(Y)] = 0$ if $X \in \mathfrak{g}_2(\beta)$, $Y \in \mathfrak{g}_2(\beta')$ with $\beta, \beta' \in B_2^{\mathfrak{a}}$ distinct.)*

Proof. Let γ and γ' be distinct roots of R_2 that have a $B_2^{\mathfrak{a}}$-coefficient equal to one and let $X \in \mathfrak{g}^\gamma$, $Y \in \mathfrak{g}^{\gamma'}$. Since $\mathfrak{a}$ is abelian, we have

$$\begin{aligned} 0 &= [X + \phi(X), Y + \phi(Y)] \\ &= [X, Y] + ([X, \phi(Y)] + [\phi(X), Y]) + [\phi(X), \phi(Y)]. \end{aligned}$$

The three groups of terms belong to direct sums of root spaces that do not intersect and so each of them must be zero.

(6.12) Corollary. *Let $\beta \in B_2^{\mathfrak{a}}$, C its connected component in $B - B_2^{\mathfrak{p}}$. Then:*

(i) *β is the unique element of $B_2 \cap C$ and the coefficient of β in the highest root with support in C is 1,*
(ii) *if $\beta' \in B_2^{\mathfrak{p}}$ is connected with C and $\beta_1 \in B_2^{\mathfrak{a}}$, then ϕ induces a nonzero map $\mathfrak{g}_2(\beta_1) \to \mathfrak{g}_2(\beta')$ if and only if $\beta_1 = \beta$,*
(iii) *if $B_2^{\mathfrak{p}} \neq \emptyset$, then β is an end of B or not connected with B_0.*

Proof. (i) If $C \cap B_2$ contains an element distinct from β, then let β' be such an element that is not separated from β by another member B_2. Then $\beta' \in B_2^{\mathfrak{a}}$. We then can find a $\gamma \in R_2(\beta)$ such that $\beta' + \gamma$ is a root. But this contradicts the fact that $\mathfrak{g}_2(\beta)$ and $\mathfrak{g}_2(\beta')$ commute.

The fact that $\mathfrak{g}_2(\beta)$ is abelian implies that the sum of no two elements of $R_2(\beta)$ is a root. So the coefficient of β in the highest root with support in C is 1 (see (2.5)).

(ii) Let $\beta' \in B_2^{\mathfrak{p}}$ and $\beta_1 \in B_2^{\mathfrak{a}}$ be as in the statement. Choose $\gamma_1 \in R_2(\beta_1)$ and $\gamma' \in R_2(\beta')$ such that the map $\mathfrak{g}^{\gamma_1} \to \mathfrak{g}^{\gamma'}$ induced by ϕ is nonzero and choose $\gamma \in R_2(\beta)$ such that $\gamma + \gamma'$ is a root. So if $X^\gamma \in \mathfrak{g}^\gamma$ and $X^{\gamma_1} \in \mathfrak{g}^{\gamma_1}$ are generators, then the $\mathfrak{g}^{\gamma+\gamma'}$ component of $[X_\gamma, \phi(X_{\gamma_1})]$ is nonzero. By (6.11), this implies that $\beta_1 = \beta$.

(iii) For this assertion we prove that if $C - \{\beta\}$ is connected with $\beta' \in B_2^{\mathfrak{p}}$, then $C - \{\beta\}$ is connected and separates β' from β. Since the Dynkin diagram is a tree this amounts to showing that the union Y of connected components of $C - \{\beta\}$ that do not separate β and β' is empty.

Suppose this is not the case: $Y \neq \emptyset$. Since β' is separated from Y by β, all the roots of $R_2(\beta')$ will have zero coefficient on Y. According to (ii) there exist a $\gamma_1 \in R_2(\beta)$ and a $\gamma' \in R_2(\beta')$ such that $\phi(X^{\gamma_1})$ (with $X^{\gamma_1} \in \mathfrak{g}^{\gamma_1}$) has nonzero $\mathfrak{g}^{\gamma'}$-component. Choose $\gamma \in R_2(\beta)$ such that $\gamma + \gamma'$ is a root and $\gamma - \gamma_1$ has a nonzero coefficient on an element of Y. Let $X^\gamma \in \mathfrak{g}^\gamma$ be a generator and consider the identity

$$[X^\gamma, \phi(X^{\gamma_1})] = [X^{\gamma_1}, \phi(X^\gamma)].$$

The $\mathfrak{g}^{\gamma+\gamma'}$-component of the lefthand side is clearly nonzero. So this is also the case for the righthand side. This implies that $-\gamma_1 + \gamma + \gamma' \in R_2(\beta')$. This is therefore a root whose Y-coefficients are zero. However, these coefficients are those of $-\gamma_1 + \gamma$ and so we arrive at a contradiction.

(6.13) Corollary. *If $B_2^{\mathfrak{p}} = \emptyset$, then B_2 is a singleton, $\mathfrak{a} = \bar{\mathfrak{a}} = \mathfrak{g}_2$ and M is a Jordan–Lefschetz module.*

Proof. The first part of previous corollary implies that $B_2 = B_2^{\mathfrak{a}}$ is a singleton, $\{\beta\}$, say. Then $\bar{\mathfrak{a}}$ is the sum of the root spaces $\mathfrak{g}^\gamma$ with $\gamma \in R_2$. Since $\bar{\mathfrak{a}}$ is abelian, the sum of two elements of R_2 is never a root. Hence $R_2 \cup R_0$ contains all the positive roots. This implies that $\bar{\mathfrak{a}} = \mathfrak{g}_2$. Since the lowest weight of M is a negative multiple of the fundamental weight at the vertex labeled by β, it is a Jordan–Lefschetz module.

Corollary (6.12) does not exploit (6.11) to the fullest. For instance, we have:

(6.14) Lemma. *Suppose that C is connected with some $\beta' \in B_2^{\mathfrak{p}}$ and assume that $C \cup \{\beta'\} - \{\beta\}$ is of type A_l, $l \geq 2$. Then C is a string and β has no greater root length then β'.*

Proof. Let us number the roots of $C \cup \{\beta'\} - \{\beta\}$ in order: $\beta', \alpha_1, \dots, \alpha_{l-1}$ and let β be connected with α_k, $1 \leq k \leq l-1$.

According to (6.12-ii), ϕ induces a nonzero map $\phi_{\beta',\beta} : \mathfrak{g}_2(\beta) \to \mathfrak{g}_2(\beta')$ such that $[X, \phi_{\beta,\beta'}(Y)]$ is symmetric in $(X,Y) \in \mathfrak{g}(\beta) \times \mathfrak{g}(\beta)$. The simple roots $\{\alpha_1, \dots, \alpha_{l-1}\}$ define a Lie subalgebra $\mathfrak{s} \subset \mathfrak{g}$ isomorphic to $\mathfrak{sl}(l)$. We denote its fundamental weights by $\varpi_i \in (\mathfrak{h} \cap \mathfrak{s})^*$ $(i = 1, \dots, l-1)$. Any root γ' of $R_2(\beta')$ will have the form $\gamma + \alpha_1 \cdots + \alpha_i$ (including the case $i = 0$: $\gamma = \gamma'$), such that $\gamma \in R_2(\beta')$ has all its C-coefficients zero. We take $\gamma' \in R_2(\beta')$ such that $\phi_{\beta',\beta}$ has nonzero component on $\mathfrak{g}^{\gamma'}$. Then

$$V := \mathfrak{g}^\gamma + \mathfrak{g}^{\gamma+\alpha_1} + \cdots + \mathfrak{g}^{\gamma+\alpha_1+\cdots+\alpha_{l-1}}.$$

is the $\mathfrak{s}$-subrepresentation of $\mathfrak{g}_2(\beta')$ generated by $\mathfrak{g}^{\gamma'}$. It is irreducible with highest weight ϖ_1 and is therefore a standard representation of $\mathfrak{s}$.

Suppose that $k \notin \{1, l-1\}$. In view of the classification of Dynkin diagrams, β has then the same root length as β'.

Claim 1: The set of roots $R_2(\beta)$ is the orbit of β under the action of the Weyl subgroup W defined by the subroot system $\{\alpha_1 \dots, \alpha_{l-1}\}$ of R.

Proof. Notice that $R_2(\beta)$ consists of positive roots that are linear combinations of $\beta, \alpha_1, \dots, \alpha_{l-1}$ with the coefficient of β being 1. Each W-orbit in $R_2(\beta)$ contains a root $\delta = \beta + r_1\alpha_1 + \cdots + r_{l-1}\alpha_{l-1}$ in the closed W-chamber opposite the fundamental one: $\delta(\alpha_i^\vee) \leq 0$ for $i = 1, \dots, l-1$. This means that $2r_i \leq r_{i-1} + r_{i+1}$ for $i \neq k$

(where we put $r_0 = r_l = 0$) and $2r_k \le r_{k-1} + r_{k+1} + 1$. It is not difficult to verify that this can only happen when all r_i's are zero, i.e., when $\delta = \beta$. So $R_2(\beta) = W\beta$.

Claim 2: $\mathfrak{g}(\beta)$ is as a $\mathfrak{s}$-representation isomorphic to $\wedge^k V$.

Proof. According to the previous claim the weights of $\mathfrak{g}(\beta)$ with respect to $\mathfrak{h}$ are in a single W-orbit. This implies that $\mathfrak{g}(\beta)$ is irreducible as a $\mathfrak{s}$-representation. Since β defines the minus the fundamental weight $-\varpi_k$ of the root system generated by $\{\alpha_1, \dots, \alpha_{l-1}\}$, this representation must be equivalent to $\wedge^k V$.

We identify $\mathfrak{g}(\beta)$ with $\wedge^k V$ so that ϕ induces a nonzero linear map $\psi : \wedge^k V \to V$. The Lie bracket defines a bilinear $\mathfrak{s}$-equivariant map from $V \times \wedge^k V$ to a representation space of $\mathfrak{s}$ and so we may think of it as a projection onto $\mathfrak{s}$-subrepresentation of $V \otimes \wedge^k V$. This subrepresentation is nonzero since for instance the root spaces $\mathfrak{g}^{\gamma+\alpha_1+\cdots+\alpha_k}$ and $\mathfrak{g}^\beta$ have nonzero Lie bracket. Moreover, $[\psi(x), y]$ is symmetric in (x, y). We show that this is impossible.

We observe that $\wedge^k V \otimes V$ decomposes into two irreducible representations: one is $\wedge^{k+1} V$ and the other is the space U spanned by the elements $x_1 \otimes x_1 \wedge \cdots \wedge x_k$. Suppose first that the image of the Lie bracket has a nonzero projection on $\wedge^{k+1} V$. Then

$$\psi(x_1 \wedge \cdots \wedge x_k) \wedge y_1 \wedge \cdots \wedge y_k = \psi(y_1 \wedge \cdots \wedge y_k) \wedge x_1 \wedge \cdots \wedge x_k$$

for all $x_i, y_i \in V$. This identity shows that if $\psi(y_1 \wedge \cdots \wedge y_k) \ne 0$, then each x_i must be in the span of $\psi(x_1 \wedge \cdots \wedge x_k), y_1, \cdots, y_k$. Since ψ is not identically zero, we deduce (by letting $y_1, \dots, y_k$ vary) that each x_i is proportional to $\psi(x_1 \wedge \cdots \wedge x_k)$ or that $k = l - 1$. The last case is excluded and the first case implies that $k = 1$, which is excluded as well. So the bracket map has image in U. The $\mathfrak{s}$-equivariant projection $\pi_U : V \otimes \wedge^k V \to U$ is given by

$$\pi_U(x_0 \otimes (x_1 \wedge \cdots \wedge x_k)) =$$
$$= \frac{k}{k+1} x_0 \otimes (x_1 \wedge \cdots \wedge x_k) - \frac{1}{k+1} \sum_{i=1}^{k} (-1)^i x_i \otimes (x_0 \wedge x_1 \wedge \cdots \wedge \widehat{x}_i \wedge \cdots \wedge x_k).$$

The same reasoning as above shows that if $\pi_U(\psi(x) \otimes y)$ is symmetric in (x, y) arguments, then $\psi = 0$. This proves the first assertion of the lemma.

The proof of the second assertion uses a similar argument. Suppose that the root length of β is greater than that of β'. In view of the classification this can only happen when β is connected with α_{l-1} so that $C \cup \{\beta'\}$ is of type C_{l+1}.

Claim 3: $\mathfrak{g}(\beta)$ is as a $\mathfrak{s}$-representation isomorphic to the space $\mathrm{Sym}^2(V^*)$ of quadratic forms on V.

Proof. Note that W has two orbits in $R_2(\beta)$: the orbit of the long root β and the orbit of the short root $\alpha_{l-1} + \beta$. The $\mathfrak{s}$-representation generated by $\mathfrak{g}^\beta$ has $\mathfrak{g}^\beta$ as lowest weight space. The lowest weight is $-2\varpi_{l-1}$ and the corresponding representation is therefore $\mathrm{Sym}^2(V^*)$. The weights of this representation are just the elements of $R_2(\beta)$ and so the claim follows.

We finish the argument as before. The contraction mapping $V \otimes \mathrm{Sym}^2(V^*) \to V^*$ is equivariant and surjective and its kernel U' is an irreducible representation of $\mathfrak{s}$.

We first show that the Lie bracket cannot induce a nonzero mapping $V \otimes \mathrm{Sym}^2 V^* \to V^*$. For then $\psi : \mathrm{Sym}^2(V^*) \to V$ is a nonzero linear map such that the expression $\xi(\psi(\eta^2))\xi$ is symmetric in $\xi, \eta \in V^*$. Since $\mathrm{Sym}^2(V^*)$ is spanned by the squares, there exists a $\eta \in V^*$ with $\psi(\eta^2) \neq 0$. It then follows that η is proportional to ξ for almost all ξ, which is absurd since $\dim V \geq 2$.

To finish the argument we now suppose that the Lie bracket induces a nonzero mapping $V \otimes \mathrm{Sym}^2 V^* \to U'$. The equivariant section of the above contraction map assigns to $\xi \in V^*$ the symmetrization $\mathrm{sym}(\mathbf{1}_V \otimes \xi) \in V \otimes \mathrm{Sym}^2(V^*)$. So the equivariant projection $V \otimes \mathrm{Sym}^2 V^* \to U'$ is given by $v \otimes \xi^2 \mapsto v \otimes \xi^2 - \mathrm{sym}(\xi(v)\mathbf{1}_V \otimes \xi)$. This means that the expression

$$\xi(x)\xi(y)\psi(\eta^2) - \frac{1}{2}\xi(\psi(\eta^2))(\xi(x)y + \xi(y)x), \quad \xi, \eta \in V^*,\ x, y \in V,$$

is symmetric in ξ and η. Let us first take $x = y$ and $\eta(x) = 0$. Then $\xi(x)^2\psi(\eta^2) - \xi(\psi(\eta^2))\xi(x)x = 0$. By taking $\xi(x) \neq 0$, we see that $\psi(\eta^2)$ and x must be proportional. As this is true for all $x \in V$ with $\eta(x) = 0$, it follows that $\eta(\psi(\eta^2)) = 0$. Next let η such that $\psi(\eta^2) \neq 0$ and take $y = \psi(\eta^2)$. So $\eta(y) = 0$. Writing out the symmetry property and comparing x-coefficients yields $-\frac{1}{2}\xi(y)^2 = 0$. Since ξ is generic, it follows that $y = 0$, which gives a contradiction.

(6.15) Proposition. *Let $(\mathfrak{a}, M)$ be an irreducible Lefschetz module of depth n with $\dim M_{-n} = 1$. Suppose that an irreducible representation of $\mathfrak{g}(\mathfrak{a}, M)$ whose highest weight is $k \geq 1$ times that of M, is Frobenius up to order l, with $1 \leq l \leq k$. Then $(M, \mathfrak{a})$ is quasi-Frobenius and $M_{-n+2i} = \bar{\mathfrak{a}}^i M_{-n}$ for $i \leq l$.*

Proof. Let $u \in M_{-n}$ be nonzero and let $M(k)$ be the $\mathfrak{g}(\mathfrak{a}, M)$-subrepresentation of $\mathrm{Sym}^k(M)$ generated by u^k. Then $M(k)$ is irreducible and has highest weight k times that of M. It is also a Lefschetz $\mathfrak{a}$-module of depth kn with u^k spanning $M(k)_{-kn}$. By assumption, $M(k)$ is Frobenius up to order l. In particular, $\mathfrak{a}(u^k) = \bar{\mathfrak{a}}(u^k) = M(k)_{-kn+2}$. In view of the fact that $M(k)_{-kn+2} = u^{k-1}M_{-n+2}$, it follows that $\mathfrak{a}M_{-n} = \bar{\mathfrak{a}}M_{-n} = M_{-n+2}$, so that M is quasi-Frobenius also. Hence for $i \leq l$, the map

$$\mathrm{Sym}^i(\bar{\mathfrak{a}}) \otimes u^k \to \mathrm{Sym}^i(M_{-n+2})u^{k-i}, \quad a_1 a_2 \cdots a_i \otimes u^k \mapsto a_1(u) \cdots a_i(u) u^{k-i}$$

is an isomorphism. There is an obvious projection of $\mathrm{Sym}^k(M)_{-kn+2i}$ onto its subspace $\mathrm{Sym}^i(M_{-n+2})u^{k-i}$ which makes the above map factor as

$$\mathrm{Sym}^i(\bar{\mathfrak{a}}) \otimes u^k \to M(k)_{-kn+2i} \to \mathrm{Sym}^i(M_{-n+2})u^{k-i},$$

with the first map given by the $\bar{\mathfrak{a}}$-action on $M(k)$. The image of that first map is just $\bar{\mathfrak{a}}^i(u^k)$ and hence $\dim \bar{\mathfrak{a}}^i(u^k) = \dim \mathrm{Sym}^i(\bar{\mathfrak{a}}) = \dim \mathrm{Sym}^i(\mathfrak{a})$. Since $M(k)$ is quasi-Frobenius up to order $l \geq i$, we also have $\dim M(k)_{-kn+2i} \leq \dim \mathrm{Sym}^i(\mathfrak{a})$. It follows that $M(k)_{-kn+2i} = \bar{\mathfrak{a}}^i(u^k)$. Taking the projection in the summand $M_{-n+2i}u^{k-i}$, then gives that $\bar{\mathfrak{a}}^i(u) = M_{-n+2i}$.

(6.16) *Example (6.4) continued.* In this case $\bar{\mathfrak{a}}$ is the intersection of $\mathfrak{so}(V)$ with $\mathrm{Hom}(V_{-2}, F_0) + \mathrm{Hom}(F_0', V_2)$. We may identify $\bar{\mathfrak{a}}$ with $F_0 \wedge V_2$ and if do so, then its action on W is given by the wedge product. From this it is immediate that $\bar{\mathfrak{a}}^2$ acts trivially on W. So the previous proposition implies that any irreducible representation of $\mathfrak{so}(V)$ with lowest weight a multiple of that of W is not Frobenius.

Proof of (6.8). In view of (6.13), the assumption that $(M, \mathfrak{a})$ is not a Jordan–Lefschetz module implies that $B_2^{\mathfrak{p}}$ is nonempty. According to (1.17), B_2 is totally disconnected. So it follows from (6.12) that the elements of $B_2^{\mathfrak{a}}$ are ends of the Dynkin diagram. Let the numbers n, k, r and $1 \leq d_0 < d_1 < \cdots < d_r = d_{r+1} = \cdots = d_k$ have the same meaning as in (1.16).

Case A_l. Since $B_2^{\mathfrak{a}}$ consists of ends, we must have $d_0 = 1$. Hence M has lowest weight of the form $-p\varpi_1 - q\varpi_l$, with p, q nonnegative integers. According to (6.3), M is self-dual, so that $p = q$ and $B_2^{\mathfrak{a}} = \{\alpha_1, \alpha_l\}$.

Notice that for $p = q = 1$ we get the adjoint representation of $\mathfrak{g} = \mathfrak{sl}(V)$. This representation is not quasi-Frobenius: its lowest degree summand is $\mathfrak{sl}(V)_{-2n} = \mathrm{Hom}(V_n, V_{-n})$ and so every element of $\mathfrak{a}(\mathfrak{sl}(V)_{-2n})$ must be in $\mathrm{Hom}(V_{n-2}, V_{-n})$, which is a proper subspace of $\mathfrak{sl}(V)_{-2n+2} = \mathrm{Hom}(V_n, V_{-n+2}) \oplus \mathrm{Hom}(V_{n-2}, V_{-n})$.

So by (6.15), M cannot be quasi-Frobenius either and therefore this case does not occur.

Case B_l. Then n is even, d_k is odd and $l = d_0 + \cdots + d_{k-1} + \frac{1}{2}(d_k - 1)$. The elements of B_2 are in position $d_0, d_0 + d_1, \ldots, d_0 + \cdots + d_{k-1}$. So in this case $d_0 = 1$ and $B_2^{\mathfrak{a}} = \{\alpha_1\}$, the simple root at the tautological vertex of B.

Case C_l, odd parity. Then n is odd and $l = d_0 + \cdots + d_k$. The elements of B_2 are in position $d_0, d_0 + d_1, \ldots, d_0 + \cdots + d_k = l$. The last element is the large simple root, so by (6.14) cannot belong to $B_2^{\mathfrak{a}}$. Hence $d_0 = 1$ and $B_2^{\mathfrak{a}} = \{\alpha_1\}$, the simple root at the standard tautological vertex of B.

Case C_l, even parity. Then n and $d_0, \ldots, d_k$ are even and $l = d_0 + \cdots + d_{k-1} + \frac{1}{2} d_k$. The elements of B_2 are in position $d_0, d_0 + d_1, \ldots, d_0 + \cdots + d_{k-1}$. Neither the first nor the last root are among them, so this case cannot occur.

Case D_l, odd parity. Then n is odd, all d_i's are even and $l = d_0 + \cdots + d_{k-1} + d_k$. The elements of B_2 are in position $d_0, d_0 + d_1, \ldots, d_0 + \cdots + d_k$ and $d_k \geq 4$. Now $\alpha_{l-d_k}, \alpha_l \in B_2$, whereas $\alpha_i \in B_0$ for the intermediate indices $i = l - d_k + 1, \ldots, l-1$. Since $d_k \geq 4$, it follows from (6.14), that we cannot have $\alpha_l \in B_2^{\mathfrak{a}}$. So $d_0 = 1$ and $B_2^{\mathfrak{a}} = \{\alpha_1\}$, the simple root at the tautological vertex of B.

Case D_l, even parity. Then n and d_k are even and $l = d_0 + \cdots + d_{k-1} + \frac{1}{2} d_k$. If $d_k \geq 4$, then the elements of B_2 are in position $d_0, d_0 + d_1, \ldots, d_0 + \cdots + d_{k-1}$. So α_{l-1} and α_l do not belong to this set. It follows from (6.12) that $d_0 = 1$ and $B_2^{\mathfrak{a}} = \{\alpha_1\}$, the simple root at the tautological vertex of B.

Suppose now $d_k = 2$. Then according to (1.16) $d_0 = 1$ and $d_1 = \cdots = d_k = 2$, in other words, $V \cong V(2k) \oplus V(2k-2)$ $(k \geq 2)$. If $k = 2$, then we are in the D_4-case with all ends belonging to B_2 and the center in B_0. According to (6.12) this can only be if $B_2^{\mathfrak{a}}$ is a singleton. So let us assume that $k \geq 3$; we are then in the case of example (6.4) and its continuation (6.16). According to (6.12), $B_2^{\mathfrak{a}}$ cannot contain both α_{l-1} and α_l. Suppose it contains one of them, say α_l. If $B_2^{\mathfrak{a}}$ also contains α_1, then $\bar{\mathfrak{a}}$ is the sum of the root spaces corresponding to the four roots $\alpha_1, \alpha_1 + \alpha_2, \alpha_l, \alpha_l + \alpha_{l-2}$. But this contradicts our finding in (6.4) that $\dim \mathfrak{a} \leq 2$. We also cannot have $B_2^{\mathfrak{a}} = \{\alpha_l\}$: if that were the case, then M has lowest weight $-p\varpi_l$ ($p>0$). The representation with lowest weight $-\varpi_l$ is a semispinorial representation W as discussed (6.4). So this possibility is excluded by (6.4) (in case $p=1$) and (6.16) (in case $p>1$).

We conclude that we are in the orthogonal or symplectic case and that $B_2^{\mathfrak{a}}=\{\alpha_1\}$ always. So M has lowest weight $-p\varpi_1$, with p a positive integer. To finish the argument, we must show that $p=1$. For this we invoke (6.15), with V taking the rôle of M. Notice that in all cases the assumption $B_2^{\mathfrak{p}}\neq\emptyset$ implies that $n>2$. The lowest degree part of V, V_{-n}, is one dimensional and $\bar{\mathfrak{a}}\subset \mathrm{Hom}(V_{-n},V_{-n+2})+\mathrm{Hom}(V_{n-2},V_n)$. So $\bar{\mathfrak{a}}V_n=V_{-n+2}$, but $\bar{\mathfrak{a}}V_{-n+2}=0$, whereas $V_{-n+4}\neq 0$ (here we use that $n>2$). So (6.15) implies that $p=1$.

7. Appendix: a property of the orthogonal and symplectic Lie algebras

The purpose of this appendix is to prove the following theorem.

Theorem. *Let U_i ($i=1,\dots,k$) be finite dimensional complex vector spaces ($k\geq 2$) endowed with a nondegenerate form (symmetric or skew) and assume that no U_i is an inner product space of dimension 2. If we give $U_1\otimes\cdots\otimes U_k$ the product form*

$$\langle u_1\otimes\cdots\otimes u_k, v_1\otimes\cdots\otimes v_k\rangle := \langle u_1,v_1\rangle\cdots\langle u_k,v_k\rangle,$$

then every simple Lie subalgebra of $\mathfrak{aut}(U_1\otimes\cdots\otimes U_k)$ that contains $\mathfrak{aut}(U_i)$ or contains a copy of $\mathfrak{sl}(2)$ in $\mathfrak{aut}(U_i)$ acting irreducibly on U_i ($i=1,\dots k$) is equal to $\mathfrak{aut}(U_1\otimes\cdots\otimes U_k)$.

Before we begin the proof we make some preliminary observations and recall two results of Dynkin.

As before, $V(d)$ denotes the standard irreducible $\mathfrak{sl}(2)$-module of dimension $d+1$, which we regard as the d-fold symmetric product of the tautological representation of $\mathfrak{sl}(2)$.

(7.1) Lemma. *The decomposition of $\mathfrak{gl}(V(d))$ into irreducible $\mathfrak{sl}(2)$-submodules is*

$$\mathfrak{gl}(V(d))=\oplus_{i=0}^{d-1}\mathfrak{gl}^{(i)}(V(d)),$$

where $\mathfrak{gl}^{(i)}(V(d))$ is the $\mathfrak{sl}(2)$-submodule generated by e^i. Furthermore, $\mathfrak{gl}^{(0)}(V(d))$ consists of the scalars, $\mathfrak{gl}^{(1)}(V(d))$ can be identified with the image of $\mathfrak{sl}(2)$ in $\mathfrak{gl}(V(d))$ and $\mathfrak{gl}^{(\mathrm{odd})}(V(d))=\mathfrak{aut}(V(d))$.

Proof. Let W be an irreducible $\mathfrak{sl}(2)$-submodule of $\mathfrak{gl}(V(d))$ of dimension $m+1$. If $T\in W$ is a highest weight vector, then $[e,T]=0$ and $[h,T]=mT$. Since $V(d)$ is a monic $\mathbb{C}[e]$-module, it follows that T is a polynomial in e (of degree $\leq d$, of course). Since $[h,e^i]=2ie^i$ it follows that m is even and that T is proportional to $e^{\frac{1}{2}m}$. On the other hand is clear that for $i=0,\dots,m$, e^i is coprimitive of weight $2i$ and hence generates an irreducible $\mathfrak{sl}(2)$-submodule of $\mathfrak{gl}(V(d))$ of dimension $2i+1$. This proves the first part of the lemma. The identity $\langle e^ix,y\rangle=(-)^i\langle x,e^iy\rangle$ shows that $e^i\in\mathfrak{aut}(V(d))$ if and only if i is odd.

(7.2) Lemma. *In the situation of the previous lemma, we have for $i = 0, \dots, m$ that $f^i \in \mathfrak{gl}^{(i)}(V(d))$. Furthermore, $u_i := \mathrm{ad}_f^i e^i$ resp. $h_i := [e^i, f^i]$ is a semisimple element in $\mathfrak{gl}^{(i)}(V))$ resp. $\mathfrak{aut}(V(d))$ which commutes with $h = [e, f]$. For $d \geq 3$ and $2 \leq i \leq d$, we have $h_i \notin \mathfrak{sl}(2)$.*

Proof. There is an inner automorphism of $\mathfrak{sl}(2)$ that sends e to f and so the first statement follows. For the second, regard $V(d)$ as the space of homogeneous polynomials of degree d in two variables x, y and let e resp. f act as $x\partial/\partial y$ resp. $y\partial/\partial x$. The calculation is then straightforward: $x^k y^l$ is an eigen vector of h_i with eigen value $c_{k,l} := k(k-1)\cdots(k-1+d)(l+1)(l+2)\cdots(l+d) - (k+1)(k+2)\cdots(k+d)l(l-1)\cdots(l-1+d)$. We have $c_{k,l} = -c_{k,l}$ and so $h_i \in \mathfrak{aut}(V(d))$. A simple verification shows that under the given constraints, h_i is not proportional to h and hence not in $\mathfrak{sl}(2)$. It is clear that $[h, u_i] = 0$. Hence u_i preserves each eigen space of h and so u_i is semisimple.

We will also need two theorems due to Dynkin [Dynkin 1952b].

(7.3) Theorem. *(Dynkin) Let $\mathfrak{s}$ be a Lie subalgebra of $\mathfrak{aut}(V(d))$ that contains $\mathfrak{sl}(2)$ as a proper subalgebra. Assume $d \neq 6$. Then $\mathfrak{s} = \mathfrak{aut}(V(d))$.*

(7.4) Theorem. *(Dynkin) Let U and V be vector spaces with a nondegenerate form (symmetric or skew), neither of which is an inner product spaces of dimension 4. Then every semisimple Lie subalgebra $\mathfrak{g}$ of $\mathfrak{aut}(U \otimes V)$ that strictly contains $\mathfrak{aut}(U) \times \mathfrak{aut}(V)$ is equal to $\mathfrak{aut}(U \otimes V)$.*

The reason for excluding 4-dimensional inner product spaces is that such a space is of the form $V = W_1 \otimes W_2$ with W_i a symplectic plane and $\mathfrak{aut}(V) = \mathfrak{aut}(W_1) \times \mathfrak{aut}(W_2)$. In that case $\mathfrak{aut}(U \otimes W_1) \times \mathfrak{aut}(W_2)$ is a semisimple Lie subalgebra of $\mathfrak{aut}(U \otimes V)$ that strictly contains $\mathfrak{aut}(U) \times \mathfrak{aut}(V)$ (at least, if $\dim U \geq 3$). However this algebra is not simple and as we shall see the exceptions disappear if $\mathfrak{g}$ is assumed to be simple.

Let U be vector space of finite dimension ≥ 2 with a nondegenerate ϵ-symmetric form and denote its Lie algebra of infinitesimal automorphisms by $\mathfrak{aut}(U)$. Let $\mathfrak{g}_\pm(U)$ be the set of $x \in \mathfrak{sl}(U)$ satisfying $\langle xu, u'\rangle = \pm\langle u, xu'\rangle$ and let $\mathfrak{g}_0(U)$ denote the scalar operators in $\mathfrak{gl}(U)$.

(7.5) Lemma. *Suppose that U is not an inner product space of dimension two. Then $\mathfrak{gl}_-(U) = \mathfrak{aut}(U)$ and $\mathfrak{gl}(U) = \mathfrak{g}_-(U) \oplus \mathfrak{g}_0(U) \oplus \mathfrak{g}_+(U)$ is an $\mathfrak{aut}(U)$-invariant decomposition. The summands are irreducible, except when U is an inner product space of dimension 4. If $\dim U > 2$, then $[\mathfrak{g}_+(U), \mathfrak{g}_+(U)] = \mathfrak{g}_-(U)$ and for $\epsilon = \pm$, there exists $Y \in \mathfrak{g}_\epsilon(U)$ such that $Y^2 \in \mathfrak{g}_+(U) + \mathfrak{g}_0(U)$ is not a scalar and has nonzero trace.*

Proof. The first statements are well-known. If $\dim U > 2$, then $[\mathfrak{g}_+(U), \mathfrak{g}_+(U)]$ is a nontrivial subspace of $\mathfrak{g}_-(U)$ and hence equal to it (since $\mathfrak{g}_-(U)$ is irreducible). The last statement is an easy exercise.

The following lemma describes the exceptional case of theorem (7.3). This is (at least implicit) in the tables and in any case, the proof is straightforward.

(7.6) Lemma. *Let $\mathfrak{s} := \mathfrak{gl}^{(1)}(V(6)) + \mathfrak{gl}^{(5)}(V(6))$. This is a simple Lie subalgebra of $\mathfrak{gl}(V(6))$ of type G_2 and any semisimple Lie algebra of $\mathfrak{aut}(V(6))$ that strictly contains $\mathfrak{sl}(2)$ contains $\mathfrak{s}$. The subspace $\mathfrak{g}_+(V(6))$ is an irreducible representation of $\mathfrak{s}$ (of dimension 27).*

We now treat the essential part of the case $k = 2$ of the theorem. The proof is however typical for the way we prove it in general.

(7.7) Proposition. *Let d, d' be positive integers and let $\mathfrak{g}$ be a semisimple Lie subalgebra of $\mathfrak{aut}(V(d) \otimes V(d'))$ which contains $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$, but is not contained in $\mathfrak{aut}(V(d)) \times \mathfrak{aut}(V(d'))$. Then $\mathfrak{g}$ contains $\mathfrak{aut}(V(d)) \times \mathfrak{aut}(V(d'))$.*

Proof. We abbreviate $V := V(d)$ and $V' := V(d')$.

The irreducible $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$-submodules of $\mathfrak{gl}(V \otimes V')$ are $\mathfrak{gl}^{(k)}(V) \otimes \mathfrak{gl}^{(l)}(V')$, where $0 \le k \le d$ and $0 \le l \le d'$. The submodule $\mathfrak{aut}(V \otimes V')$ is the sum of the $\mathfrak{gl}^{(k)}(V) \otimes \mathfrak{gl}^{(l)}(V')$ with $k + l$ odd. We are given that $\mathfrak{g}$ contains the summands indexed by the pairs $(1, 0)$, $(0, 1)$ and some (i, j) with i and j both positive and with odd sum. Suppose i is even and j is odd.

We first prove that $\mathfrak{g} \supset \mathfrak{aut}(V) \otimes \mathbf{1}$. According to (7.2), $u_j \in \mathfrak{gl}^{(j)}(V')$ is nonzero semisimple with integral eigen values and so $\mathrm{Tr}(u^2) \neq 0$. Consider $h_i \otimes u_j^2 = [e^i \otimes u_j, f^i \otimes u_j] \in \mathfrak{g}$. This element must have a nonzero component in some $\mathfrak{gl}^{(k)}(V) \otimes \mathbf{1}$ with k odd and $\neq 1$.

If $d \neq 6$, then theorem (7.3) implies that $\mathfrak{aut}(V) \otimes \mathbf{1} \subset \mathfrak{g}$. If $d = 6$, then it follows that $\mathfrak{g}$ contains $\mathfrak{s} \otimes \mathbf{1}$, with $\mathfrak{s}$ a Lie algebra of type G_2 as described in (7.6). Since i is even and positive, it follows from (7.6) that $\mathfrak{g}$ contains $\mathfrak{g}_+(V) \otimes \mathfrak{gl}^{(j)}(V')$. If $X, Y \in \mathfrak{g}_+(V)$, then $[X, Y] \otimes h_j^2 = [X \otimes h_j, Y \otimes h_j] \in \mathfrak{g}$ and we find as before that $[X, Y] \otimes \mathbf{1} \in \mathfrak{g}$. The elements $[X, Y]$, $X, Y \in \mathfrak{g}_+(V)$, span $\mathfrak{g}_-(V)$ by (7.5) and so also in this case $\mathfrak{g} \supset \mathfrak{aut}(V) \otimes \mathbf{1}$.

We next prove that $\mathbf{1} \otimes \mathfrak{aut}(V') \subset \mathfrak{g}$. If $d' = 1$, there is nothing to show, so suppose $d' \ge 2$. In passing we have shown that $\mathfrak{g}$ contains the summand $\mathfrak{gl}^{(k)}(V) \otimes u_j^2$ with k odd. It is easily verified that for $d' \ge 2$, u_j^2 is not a scalar and so $\mathfrak{g}$ contains a summand $\mathfrak{gl}^{(k)}(V) \otimes \mathfrak{gl}^{(l)}(V')$ with $l > 0$ (and necessarily even). So the same argument as above (with (i, j) replaced by (k, l)) shows that $\mathbf{1} \otimes \mathfrak{aut}(V') \subset \mathfrak{g}$.

(7.8) Proposition. *Let d be a positive integer and U a finite dimensional vector space with a nondegenerate ϵ-symmetric form that is not an inner product space of dimension two. If $\mathfrak{g}$ is a semisimple Lie subalgebra of $\mathfrak{aut}(U \otimes V(d))$ which contains $\mathfrak{aut}(U) \times \mathfrak{sl}(2)$, but is not contained in $\mathfrak{aut}(U) \times \mathfrak{aut}(V(d))$, then $\mathfrak{g}$ contains $\mathfrak{aut}(U) \times \mathfrak{aut}(V(d))$.*

The proof of this proposition is analogous to the proof of the proposition preceding it (relying sometimes on (7.5) instead of (7.2)) and so we omit it.

Proof of the theorem. As any 4-dimensional inner product space is the tensor product of two sumplectic planes, we may assume that no U_i is of that type. Then for $k = 2$ the assertion follows from the conjunction of (7.7), (7.8) and theorem (7.4). We proceed with induction on k and assume $k \ge 3$.

In case $k = 3$ and all factors U_1, U_2, U_3 symplectic planes, then

$$\mathfrak{aut}(U_1 \otimes U_2 \otimes U_3) = \mathfrak{sl}(U_1) \times \mathfrak{sl}(U_2) \times \mathfrak{sl}(U_3) + \mathfrak{sl}(U_1) \otimes \mathfrak{sl}(U_2) \otimes \mathfrak{sl}(U_3).$$

Since the last summand is irreducible as a $\mathfrak{sl}(U_1) \times \mathfrak{sl}(U_2) \times \mathfrak{sl}(U_3)$-module and contains a nonzero element of $\mathfrak{g}$, the theorem is then immediate.

Assume therefore that we are not in this situation. First note that $\mathfrak{aut}(U_1 \otimes \cdots \otimes U_k)$ is the direct sum of the summands $\mathfrak{g}_{\epsilon_1}(U_1) \otimes \cdots \otimes \mathfrak{g}_{\epsilon_k}(U_k)$ with $\epsilon_i \in \{-, 0, +\}$ with the value $-$ being taken an odd number of times (if U_i is a symplectic plane, then ϵ_i cannot take the value $+$). By assumption $\mathfrak{g}$ contains a nonzero element in a summand $\mathfrak{g}_{\epsilon_1}(U_1) \otimes \cdots \otimes \mathfrak{g}_{\epsilon_k}(U_k)$ with at least two nonzero ϵ_i's.

If $\epsilon_i = 0$ for some i, then let J denote the set of indices j with $\epsilon_j \neq 0$. We wish to apply our induction hypothesis to $U := \otimes_{j \in J} U_j$. For this, we need to know that $\prod_{j \in J} \mathfrak{aut}(U_j)$ is contained in a simple component of $\mathfrak{g} \cap \mathfrak{aut}(U)$. The latter is certainly reductive. The simple component of $\mathfrak{g} \cap \mathfrak{aut}(U)$ that contains $\mathfrak{aut}(U_j)$ $(j \in J)$ must also contain $\otimes_{l \in J} \mathfrak{g}_{\epsilon_l}(U_l)$ (since no nonzero element of $\mathfrak{g}_{\epsilon_j}(U_j)$ commutes with the elements of $\mathfrak{aut}(U_j)$) and so the desired property holds. Our induction hypothesis therfore applies and we conclude that $\mathfrak{g}$ contains $\mathfrak{aut}(U)$. Now U cannot be the tensor product of two symplectic planes $U_i \otimes U_j$, for in that case $\mathfrak{aut}(U) = \mathfrak{sl}(U_i) \times \mathfrak{sl}(U_j)$, so that $\epsilon_i = 0$ or $\epsilon_j = 0$. Hence U will not be a 4-dimensional inner product space and we may therefore apply the induction hypothesis once more to the tensor product of U and the U_i with $\epsilon_i = 0$ and conclude that $\mathfrak{g} = \mathfrak{aut}(U_1 \otimes \cdots \otimes U_k)$.

We next deal with the case when $\epsilon_i \neq 0$ for all i. Suppose first that not all factors U_i are symplectic planes. Since at least one ϵ_i is $-$, we can by renumbering assume that $\epsilon_1 = -$ and U_2 is not a symplectic plane. We then show that $\epsilon_3, \dots, \epsilon_k$ can be made zero while keeping ϵ_1, ϵ_2 nonzero, so that this takes us to the case considered above. With the help of (7.2) and (7.5) we can find $X_1, X_2 \in \mathfrak{g}_-(U_1)$ with $[X_1, X_2] \neq 0$ and $Y_j \in \mathfrak{g}_\epsilon(U_i)$ $(j = 2, \dots k)$ such that Y_2^2 is not scalar, Y_j^2 is not traceless for $j \geq 3$, and $X_i \otimes Y_2 \otimes \cdots \otimes Y_k \in \mathfrak{g}$ $(i = 1, 2)$. Then $[X_1, X_2] \otimes Y_2^2 \otimes \cdots \otimes Y_k^2$ is an element of $\mathfrak{g}$ whose component in $\mathfrak{g}_-(U_1) \otimes \mathfrak{g}_+(U_2) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$ is nonzero.

It remains to do the case when all factors U_i are symplectic planes (and so all ϵ_i's are $-$). We then choose $X_i \in \mathfrak{sl}(U_i)$ for $i = 1, 2, 3$ and $Y_j \in \mathfrak{sl}(U_j)$ for $j = 1, \dots, k$ such that $\mathrm{Tr}(Y_j^2) \neq 0$ for $j \geq 4$ and $Z := [X_1 \otimes X_2 \otimes X_3, Y_1 \otimes Y_2 \otimes Y_3] \notin \mathfrak{sl}(U_1) \times \mathfrak{sl}(U_2) \times \mathfrak{sl}(U_3)$. Then

$$[X_1 \otimes X_2 \otimes X_3 \otimes Y_4 \otimes \cdots \otimes Y_k, Y_1 \otimes \cdots \otimes Y_k] = Z \otimes Y_4^2 \cdots Y_k^2$$

is an element of $\mathfrak{g}$ with a nonzero component in $\mathfrak{sl}(U_1) \otimes \mathfrak{sl}(U_2) \otimes \mathfrak{sl}(U_3) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$. Therefore, $\mathfrak{g}$ contains $\mathfrak{aut}(U_1 \otimes U_2 \otimes U_3)$ and the induction proceeds.

References

[Beauville] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Diff. Geom. **18** (1983), 755–782.

[Bogomolov] F.A. Bogomolov, *On the cohomology ring of a simple hyperkähler manifold*, GAFA **6** (1996), 612–618.

[Bourbaki] N. Bourbaki, *Groupes et Algèbres de Lie*, Hermann, Paris, 1975.

[Chen-Ogiue] B.-Y. Chen and K. Ogiue, *Some characterizations of complex space forms in terms of Chern classes*, Q. J. of Math. **26** (1975), 459–464.

[Deligne 1968] P. Deligne, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. **35** (1968), 107–126.

[Deligne 1979] P. Deligne, *Variétés de Shimura*, Automorphic Forms, Representations, and L-functions, Part 2 (A. Borel and W. Casselman, eds.), Proc. Symp. in Pure Math., vol. 33, Amer. Math. Soc., Providence, RI, 1979, pp. 247–290.

[Dynkin 1952a] E.B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Transl., Series 2, vol. 6, Amer. Math. Soc., Providence, RI, pp. 111–244.

[Dynkin 1952b] E.B. Dynkin, *The maximal subgroups of the classical groups*, Amer. Math. Soc. Transl., Series 2, vol. 6, Amer. Math. Soc., Providence, RI, pp. 245–378.

[Fujiki] A. Fujiki, *On the De Rham cohomology group of a compact Kähler symplectic manifold*, Adv. Studies in Math. (T. Oda, ed.), Algebraic Geometry, Sendai 1985, vol. 10, 1987, pp. 105–165.

[Lange-Birk] H. Lange and Ch. Birkenhage, *Complex Abelian Varieties*, Springer, Berlin and New York, 1992.

[LiE] M.A.A. van Leeuwen, A.M. Cohen and B. Lisser, *Lie, a Package for Lie Group Computations*, Computer Algebra Nederland, Amsterdam, 1992.

[Moonen-Zar] B.J.J. Moonen and Yu.G. Zarhin, *Hodge classes and tate classes on simple abelian fourfolds*, Duke Math J. **77** (1995), 553–581.

[Okonek-VdV] Ch. Okonek and A. Van de Ven, *Cubic forms and complex 3-folds*, l'Ens. Math. **41** (1995), 297–333.

[Saito] M. Saito, *Mixed Hodge modules*, Publ. Res. Math. Sci. **26** (1990), 221–333.

[Salamon] S.M. Salamon, *On the cohomology of Kähler and hyper-kähler manifolds*, Topology **35** (1996), 137–155.

[Shimura] G. Shimura, *On analytic families of polarized abelian varieties and automorphic functions*, Ann. of Math. **78** (1963), 149–192.

[Springer] T.A. Springer, *Jordan algebras and algebraic groups*, Springer, Berlin and New York, 1973.

[Springer-St] T.A. Springer and R. Steinberg, *Conjugacy classes*, Seminar on Algebraic Groups and Related Finite Groups (A. Borel et al., eds.), Lecture Notes in Mathematics, vol. 131, Springer, Berlin and New York, 1970.

[Verbitsky 1990] M. Verbitsky, *On the action of a Lie algebra* $\mathfrak{so}(5)$ *on the cohomology of a hyperkähler manifold*, Func. An. and Appl. **24** (1990), 70–71.

[Verbitsky 1995] M. Verbitsky, *Cohomology of compact hyperkähler manifolds*, alg. geom. eprint 9501001, announcement published in GAFA **6** (1996), 601–611.

[Zucker] S. Zucker, *Locally homogeneous variations of Hodge structure*, l'Ens. Math. **27** (1981), 243–276.

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