CELLULAR DECOMPOSITIONS OF COMPACTIFIED MODULI SPACES OF POINTED CURVES

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Abstract. To a closed connected oriented surface S of genus g and a nonempty finite subset P of S is associated a simplicial complex (the arc complex) that plays a basic rôle in understanding the mapping class group of the pair (S, P). It is known that this arc complex contains in a natural way the product of the Teichmüller space of (S, P) with an open simplex. In this paper we give an interpretation for the whole arc complex and prove that it is a quotient of a Deligne–Mumford extension of this Teichmüller space and a closed simplex. We also describe a modification of the arc complex in the spirit of Deligne–Mumford.

Introduction

Given a closed connected oriented differentiable surface S of genus g and a finite nonempty subset P of S, then the mapping class group $\Gamma(S, P)$ of this pair is the group of isotopy classes of sense preserving diffeomorphisms of S that fix P pointwise. Harer proved in a series of papers some remarkable properties of the cohomology of the $\Gamma(S, P)$ (see [4] for an overview). In this work a central rôle is played by various simplicial complexes with an action of an appropriate mapping class group that have in common the property that stabilizers of simplices look like simpler mapping class groups. The complex depends on the context, but in all cases it can for a suitable pair (S, P) be identified with a subcomplex of the *arc complex* $A(S, P)$. That complex is defined as follows: the vertices of $A(S, P)$ are ambient isotopy classes relative to P of embedded unoriented nontrivial loops and arcs in S that connect two (possibly identical) points of P and avoid all other points of P (where a loop is considered trivial if it bounds an open disk in $S - P$) and finitely many such vertices span a simplex if we can respresent them by loops and arcs that do not meet outside P. We note that there is a piecewise linear map λ from $A(S, P)$ to the simplex Δ_P spanned by P characterized by the property that it sends a vertex represented by an arc (resp. a loop) to the barycenter of the 1-simplex of Δ_P spanned by its end points (resp. the vertex of Δ_P representing the base point).

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An important property of this complex is that its interior can be identified with the product of the Teichmüller space $\mathfrak{T}(S,P)$ of the pair (S,P) (i.e., the space of isotopy classes relative to P of conformal structures on S) and the open simplex Δ_P°. We may therefore regard $A(S,P)$ as an extension of $\mathfrak{T}(S,P)\times\Delta_P^\circ$. In the applications alluded to there was no apparent need to know what this extension actually represents, and that may have been the reason that question received little attention. (An exception is the paper by Bowditch and Epstein [1] about which we shall say more below.) The situation changed with Kontsevich's work on a conjecture of Witten [6], where it became essential to interpret the part of $A(S,P)$ lying over Δ_P°. In this article Kontsevich states the answer but omits a proof. The present paper grew out the desire to supply one and one of our main results now interprets all of $A(S,P)$ in terms of the Deligne–Mumford compactification of the moduli space $\mathfrak{M}_g^P := \Gamma(S,P)\backslash\mathfrak{T}(S,P)$. For a precise statement we refer to theorem (8.6). Suffice it here to say that for every nonempty subset Q of P we describe a quotient space $K_Q\mathfrak{M}_g^P$ of the Deligne–Mumford compactification of $\mathfrak{M}_g^P$ that is contravariant in Q such that the geometric realization of the associated simplicial space over Δ_P can be identified with the orbit space $\Gamma(S,P)\backslash A(S,P)$. In particular, $\Gamma(S,P)\backslash A(S,P)$ is a quotient of the product of the Deligne–Mumford compactification and Δ_P. We suspect that the compactifications $K_Q\mathfrak{M}_g^P$ and the maps between them can be constructed in the category of projective varieties and morphisms so that $\Gamma(S,P)\backslash A(S,P)$ becomes the geometric realization of a simplicial object in this category. We state the relevant conjectures in (3.3)

An intermediate result of our proof is a combinatorial description (11.5) of (a thickened version of) the Deligne–Mumford compactification. More precisely, we equivariantly blow up $A(S,P)$ in a certain manner over its boundary (in the PL-category) to get a cell complex of which the orbit space naturally maps to $\overline{\mathfrak{M}}_g^P\times\Delta_P$ with fibers products of simplices (or finite quotients thereof). This description may be helpful in determining which of the cohomology classes that Kontsevich introduced in $\mathfrak{M}_g^P$ extend to $\overline{\mathfrak{M}}_g^P$. A paper by Milgram–Penner [7] alludes to a combinatorial construction of the Deligne–Mumford compactification (for the case that P is a singleton), but it is not clear to us whether what these authors have in mind coincides with our construction.

The article by Epstein and Bowditch mentioned above came to our attention after this paper was essentially completed. It also gives an interpretation of the arc complex, but in this it differs from ours in two respects. First, it takes the hyperbolic point of view (that gives rise to a different embedding of thickened Teichmüller space in the arc complex) and second, it is not given in terms of the Deligne–Mumford compactification. (For these reasons it is not clear to us whether it could take care of Kontsevich's assertion.) We adopted their term *arc complex* and we adapted our notation a little in order to avoid too blatant clashes with theirs.

The plan of the paper is as follows. The first seven sections are intended to have to some extent the characteristics of a review paper and were written with a nonexpert reader in mind. Yet they do contain results that we have not found in the literature. In the first section we collect facts about the Teichmüller spaces. The next two sections deal with certain extensions of them: we describe a boundary for Teichmüller spaces in the spirit of Harvey based on the Deligne–Mumford compactification and we introduce the quotients of the Deligne–Mumford compactification alluded to above. In section 4 we discuss some properties of the complex $A(S,P)$. The next two sections introduce metrized ribbon graphs and explicate the relationship between this notion and the complex $A(S,P)$. In section 7 we invoke the fundamental results of Strebel, culminating in theorem (7.5). The subsequent sections are of more technical nature. In section 8 we describe the geometric objects that are parametrized by the points of $A(S,P)$. Our first main theorem (8.6) is also stated there, but its proof is postponed to the last section. The remainder of the paper is mostly concerned with the combinatorial versions of notions related to the Deligne–Mumford compactification. In section 9 we introduce stable ribbon graphs of which we claim that it is the combinatorial analog of the notion of a stable curve. This is justified in section 10, where we show that a metrized stable ribbon graph can be obtained as the limit of a one-parameter family of ordinary metrized ribbon graphs. In the final section 11 we construct the modification $A(S,P)$ mentioned above and prove our second main theorem (11.5), namely that this modification is essentially a thickened Deligne–Mumford extension of $\mathfrak{T}(S,P)$. Once this has been established, the proof of our first main theorem is easily completed.

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Throughout this paper S stands for a compact connected oriented differentiable surface, g for its genus, and P for a finite nonempty subset of S. Therefore we often suppress (S,P) in the notation and write Γ, A, ... We assume that $S-P$ has negative Euler characteristic, which amounts to the requirement that if $g=0$, then $|P|\geq 3$.

1. Teichmüller spaces

(1.1) If T is an oriented 2-dimensional vector space, then a conformal structure on T determines an action of the circle group $U(1)$ on T and in this way T acquires the structure of a 1-dimensional complex vector space. Clearly, the converse also holds. Thus, to give the oriented surface S a conformal structure is equivalent to give its tangent bundle the structure of a complex line bundle. Such a structure comes from a (unique) complex-analytic structure on S, so that S becomes a Riemann surface. By the uniformization theorem, its universal cover will be isomorphic to the upper half plane. A conformal structure on S is given by a section of a fiber bundle whose fibre is the open convex subset in the vector space of quadratic forms on $\mathbb{R}^2$ defined by the positive ones. The C^∞-topology on this space defines a topology on the set $\mathrm{cf}(S)$ of conformal structures on S. (It also has a compatible structure of a convex set, so that $\mathrm{cf}(S)$ is contractible.)

Let $\mathrm{Diffeo}^+(S,P)$ denote the group of sense preserving diffeomorphisms that leave P pointwise fixed, and let $\mathrm{Diffeo}^0(S,P)$ denote its identity component. Its "group of connected components",

$$\Gamma := \mathrm{Diffeo}^+(S,P)/\mathrm{Diffeo}^0(S,P),$$

is the *mapping class group* of (S,P). In this definition we may replace diffeomorphism by homotopy equivalence (relative to P) or all natural choices in between such as PL-homeomorphism, quasiconformal homeomorphism or plain homeomorphism: we still get the same group. Clearly, $\mathrm{Diffeo}^+(S,P)$ acts on the space of conformal structures on S. The orbit space with respect to its identity component:

$$\mathfrak{T} := \mathrm{Diffeo}^0(S,P)\backslash \mathrm{cf}(S)$$

is called the *Teichmüller space* of (S,P). It comes with a natural action of Γ. If we substitute for $\mathrm{cf}(S)$ the bigger space of conformal structures inducing the quasiconformal structure underlying the given differentiable structure and replace Diffeo by the group of quasiconformal homeomorphisms of S, then the result is the same. For many purposes this is actually the most useful characterization.

The Fenchel-Nielsen parametrization shows that $\mathfrak{T}$ is homeomorphic to an open disk. There is even a natural Γ-invariant complex-analytic manifold structure on $\mathfrak{T}$; if $t \in \mathfrak{T}$ is represented by a Riemann surface C which underlies S, the tangent space at t is identified with $H^1(C, \theta_C(-P))$, where θ_C is the sheaf of holomorphic vector fields on C. The action of Γ on $\mathfrak{T}$ is properly discontinous and Γ has a subgroup of finite index acting freely (for instance, the kernel of the representation of Γ on $H_1(C;\mathbb{Z}/3)$). This implies that the orbit space

$$\mathfrak{M}_g^P := \Gamma\backslash\mathfrak{T}$$

is in a natural way a normal analytic space with only quotient singularities.

(1.2) We can give $\mathfrak{T}$ an interpretation as a moduli space: let us first define a *P-pointed Riemann surface* (C,x) as a Riemann surface C together with an injection $x : P \hookrightarrow C$ such that the automorphism group of the pair (C,x) is finite. Say that such an P-pointed Riemann surface (C,x) is (S,P)-*marked* if we are given an sense preserving quasiconformal homeomorphism (henceforth abbreviated as quasiconformal homeomorphism) $f : S \to C$

that extends x, with the understanding that two such homeomorphisms define the same marking if they are quasiconformal isotopic relative to P. Clearly, these markings are permuted in a simply-transitive manner by the mapping class group Γ. An isomorphism of marked P-pointed Riemann surfaces (C, x, f), (C', x, f') is given by an sense preserving quasiconformal-homeomorphism $h : C \to C'$ with $hx = x'$ such that hf is quasiconformal-isotopic to f' modulo P. Now $\mathfrak{T}(S, P)$ can be thought of as the space of isomorphism classes of (S, P)-marked Riemann surfaces. So $\mathfrak{M}_g^P := \Gamma\backslash\mathfrak{T}(S, P)$ can be identified with the set of isomorphism classes of P-pointed compact Riemann surfaces of genus g. It is a coarse moduli space that has a natural structure of a quasi-projective variety. Knudsen, Deligne and Mumford showed that there is a distinguished projective completion $\mathfrak{M}_g^P \subset \overline{\mathfrak{M}}_g^P$ by the coarse moduli space of stable P-pointed complex curves of genus g. (A *stable P-pointed complex curve* consists of a complete complex curve C with only simple crossings and an injection x of P into the nonsingular part of C such that $\mathrm{Aut}(C, x)$ is finite.) It is called the *Deligne–Mumford compactification.*

(1.3) Let $G = \Gamma/\Gamma_1$ be a finite quotient group of Γ and put

$$\mathfrak{M}_g^P[G] := \Gamma_1\backslash\mathfrak{T}.$$

Then we have a ramified G-covering $\pi_G : \mathfrak{M}_g^P[G] \to \mathfrak{M}_g^P$. The rational cohomology of $\mathfrak{M}_g^P$ is mapped by π_G^* isomorphically onto the G-invariants of the rational cohomology of $\mathfrak{M}_g^P[G]$. If Γ_1 acts without fixed point on $\mathfrak{T}$, then $\mathfrak{T}$ can be regarded as a universal covering space of $\mathfrak{M}_g^P[G]$, and as $\mathfrak{T}$ is contractible, this implies that $\mathfrak{M}_g^P[G]$ is a classifying space for Γ_1. So the group cohomology of Γ_1 is the singular cohomology of $\mathfrak{M}_g^P[G]$. We get the same statement for Γ vis-à-vis $\mathfrak{M}_g^P$, except that we must use rational coefficients:

$$H^\bullet(\mathfrak{M}_g^P; \mathbb{Q}) = H^\bullet(\Gamma; \mathbb{Q}).$$

This equality represents a bridge between algebraic geometry (the left hand side) and combinatorial group theory (the right hand side).

2. A boundary for Teichmüller space

We shall give $\mathfrak{T}$ a (noncompact) boundary with corners. This is an analogue of the Borel–Serre compactification for arithmetic groups and first appeared in a paper by W.J. Harvey [6].

We first recall that given a smooth manifold M and a closed submanifold $N \subset M$, one has defined the oriented blowing-up

$$\pi : \mathrm{Bl}_N(M) \to M.$$

This is a manifold with boundary $\pi^{-1}N$. The map is an isomorphism over $M - N$, whereas $\pi^{-1}N \to N$ can be identified with the sphere bundle associated to the normal bundle (or more intrinsically, with the bundle of rays in that bundle) with its obvious projection onto N. Notice that in case the normal bundle has the structure of a complex line bundle, $\pi^{-1}N \to N$ has the structure of a $U(1)$-bundle.

This construction generalizes in a straightforward manner to the case where N is a union of submanifolds that intersect multi-transversally; in that case $\mathrm{Bl}_N(M)$ is a manifold with corners and the fibres of π are products of spheres.

Now let (C, x) be a pointed stable curve of genus g. Let $\tilde{C} \to C$ be its normalization, denote by $\Sigma \subset \tilde{C}$ the pre-image of C_{sing} and consider the composite map

$$f : \mathrm{Bl}_\Sigma(\tilde{C}) \to \tilde{C} \to C.$$

For every $p \in C_{\mathrm{sing}}$, $f^{-1}(p)$ consists of two principal $U(1)$ homogeneous spaces. If we choose for every such p an anti-isomorphism of these homogeneous spaces and glue accordingly, then we get an oriented surface over C, $S \to C$, of genus g such that the pre-image of every singular point is a circle. We shall interpret the conformal structure on $f^{-1}C_{\mathrm{reg}}$ as a degenerate conformal structure on S.

The choice of the anti-isomorphism over p is the same thing as the choice of an anti-isomorphism between T_pC' and T_pC'', where C' and C'' are the local branches of C at p, given up to a positive real scalar. But this amounts to choosing a ray in the complex line $T_pC' \otimes T_pC''$. If we denote that space of rays by R_pC, then our choices are effectively parametrized by $\prod_{p \in C_{\mathrm{sing}}} R_pC$; this is a principal homogeneous space of the torus $U(1)^{C_{\mathrm{sing}}}$ which we abbreviate by $R(Z)$.

The complex lines $T_pC' \otimes T_pC''$ have an interpretation in terms of the deformation theory of C. Let us recall that there is a universal deformation

$$\big(F : (\mathcal{C}, C) \to (B, o)\,;\, x_{\mathcal{C}} : (B, o) \times P \to \mathcal{C}\big)$$

of (C, x) with base a smooth complex-analytic germ (B, o). Its universal character implies that the whole situation comes with an action of the finite group $\mathrm{Aut}(C, x)$. The $\mathrm{Aut}(C, x)$-orbit space of the base can be identified with the germ of $\overline{\mathfrak{M}}_g^P$ at the point defined by (C, x). Each singular point p of C determines a smooth divisor (D_p, o) in (B, o) that parametrizes the deformations of C that do not smooth the singularity p. So there is a natural section of (D_p, o) in the singular set of F. The derivative of F along this section defines an isomorphism of the tensor product of the relative tangent bundles of the two branches of $F^{-1}(D_p, o)$ along this section and the normal bundle of (D_p, o) in (B, o).

The divisors D_p, $p \in C_{\text{sing}}$, intersect with normal crossings so that their union D defines an oriented blowing-up:

$$\pi : \mathrm{Bl}_D(B, o) \to (B, o).$$

It follows from the preceding that a fiber $\mathrm{Bl}_D(B, o)_t$ is canonically identified with $R(\mathcal{C}_t)$ and so over $\mathrm{Bl}_D(B, o)_t$ lives canonically a family of surfaces of genus g. It is easily seen that this true over all of $\mathrm{Bl}_D(B, o)$, so that we get a family of oriented genus g surfaces

$$\mathcal{S} \to \mathrm{Bl}_D(B, o).$$

This family is P-pointed.

Let $\hat{B} \to \mathrm{Bl}_D(B, o)$ be a universal cover. Since $\mathrm{Bl}_D(B, o)$ has the torus $R(C)$ as a deformation retract, the covering group is naturally isomorphic to the fundamental group of $U(1)^{C_{\text{sing}}}$, i.e., to the free abelian group generated by C_{sing}. It is known that the fundamental group of $U(1)^{C_{\text{sing}}}$ maps injectively to the mapping class group of a fiber. So the covering transformations permute these markings freely.

It also follows that $\hat{B}$ is contractible. If $\hat{\mathcal{S}} \to \hat{B}$ is the pull-back of our family of surfaces, then is possible to mark the fibers simultaneously by means of trivialization $\hat{\mathcal{S}} \to S$ relative to the given pointing. This defines a map from $\hat{B} - \partial\hat{B}$ to $\mathfrak{T}$. That map is a homeomorphism onto an open subset of $\mathfrak{T}$. Now glue $\hat{B}$ to $\mathfrak{T}$ by means of this map. This clearly endows $\mathfrak{T}$ with a partial boundary with corners. This can be done over any point of the Deligne–Mumford compactification $\overline{\mathfrak{M}}_g^P$ and the essential uniquess of this construction ensures that the result is a manifold with corners $\hat{\mathfrak{T}}$ whose interior is $\mathfrak{T}$. By construction, $\hat{\mathfrak{T}}$ comes with a Γ-action that extends the given one on $\mathfrak{T}$. The construction also shows that Γ acts properly discontinuously on $\hat{\mathfrak{T}}$ and that there is a natural proper map $\Gamma\backslash\hat{\mathfrak{T}} \to \overline{\mathfrak{M}}_g^P$ whose fibres are finite quotients of real tori.

There is also a universal family of genus g surfaces over $\hat{\mathfrak{T}}$. As a set, $\hat{\mathfrak{T}}$ has the following moduli interpretation. Let us define a *stable conformal structure* on S as being given by a closed one-dimensional submanifold $L \subset S - P$ and a conformal structure on $S - L$ having the property that contraction of every connected component of L yields a stable P-pointed curve. The set of stable conformal structures is acted on by $\mathrm{Diffeo}^+(S, P)$ and the quotient by $\mathrm{Diffeo}^0(S, P)$ can be identified with $\hat{\mathfrak{T}}$. The following proposition is well-known and tells us when a sequence in $\hat{\mathfrak{T}}$ converges.

(2.1) Proposition. *Let $L \subset S - P$ be a compact one-dimensional submanifold such that every connected component of $S - (P \cup L)$ has negative Euler characteristic. Let $(J_n)_{n=1}^{\infty}$ be a family of conformal structures on S with the property that $(J_n|S-P)_n$ converges uniformly on compact subsets* *to a stable conformal structure J_∞ on S. If t_∞ denotes the corresponding element of $\hat{\mathfrak{T}}$ and $t_n \in \mathfrak{T}$ the image of J_n, then $(t_n)_n$ converges to t_∞.*

(2.2) In this paper, the space $\hat{\mathfrak{T}}$ will play an auxiliary rôle; we will be more concerned with a quotient $\overline{\mathfrak{T}}$ that is a kind of Stein factorization of the projection $\hat{\mathfrak{T}} \to \overline{\mathfrak{M}}_g^P$: $\overline{\mathfrak{T}}$ is obtained by collapsing every connected component of a fiber of the latter map to a point. As these connected components are affine spaces (and hence noncompact in general), the result will not locally compact. Notice that Γ still acts on $\overline{\mathfrak{T}}$, and that the orbit space $\Gamma\backslash\overline{\mathfrak{T}}$ can be identified with $\overline{\mathfrak{M}}_g^P$. So $\overline{\mathfrak{T}} \to \overline{\mathfrak{M}}_g^P$ is a Galois covering with infinite ramification.

3. Quotients of Deligne–Mumford compactifications

We introduce certain quotients of $\mathfrak{M}_g^P$ that are obtained by identifying points of the boundary of its Deligne–Mumford compactification and that arise naturally in a combinatorial setting. One such quotient plays a prominent rôle in Kontsevich's proof of a conjecture of Witten [6]. Let us fix a nonempty subset Q of P. If (C, x) is an P-pointed stable curve, then the irreducible components of C that contain a point of Q make up a (not necessarily stable) Q-pointed curve $(C_Q, x|Q)$. The pairs (C, x) for which every singular point of C lies on C_Q define a Zariski open subset U_Q of $\overline{\mathfrak{M}}_g^P$. We define an equivalence relation R_Q on U_Q as follows: two P-pointed stable curves (C, x) and (C', x') representing points of U_Q are declared to be R_Q-equivalent if there exists an sense preserving homeomorphism $h : C \to C'$ such that $hx = x'$ and h restricts to an analytic isomorphism of C_Q onto C'_Q as Q-pointed curves. We denote its quotient space by $K_Q\mathfrak{M}_g^P$. The equivalence relation R_Q has a natural extension $\overline{R}_Q$ to $\overline{\mathfrak{M}}_g^P$ that is characterized by the property that if we keep both C_Q and the singular points of C on C_Q fixed, but allow C to acquire singularities outside C_Q, then we stay in the same equivalence class. So $K_Q\mathfrak{M}_g^P$ may be regarded as a quotient of $\overline{\mathfrak{M}}_g^P$.

(3.1) Lemma. *The space $K_Q\mathfrak{M}_g^P$ is compact Hausdorff. It contains $\mathfrak{M}_g^P$ as an open-dense subset.*

Proof. The last assertion of the lemma is easy and is stated for the sake of record only. The first statement is a little ambiguous since it is not clear whether we give $K_Q\mathfrak{M}_g^P$ the topology as a quotient of U_Q or of $\overline{\mathfrak{M}}_g^P$. A priori, the former could be finer than the latter, but we will show that they are the same. Now $\overline{\mathfrak{M}}_g^P$ is compact and hence so is every quotient of it. It is therefore enough for us to verify that $K_Q\mathfrak{M}_g^P$ is Hausdorff as a quotient of $\overline{\mathfrak{M}}_g^P$. This will be a consequence of the following property of the compactification $\overline{\mathfrak{M}}_g^P$.

Let $[(C_n, x_n)]_{n=1}^\infty$ be a sequence in U_Q converging to $[(C, x)]$ and suppose that all the terms of this sequence have the same topological type. Then the intersection of $C_{n,Q}$ with the union of the other irreducible components of C_n is a finite subset Z_n of the smooth part of $C_{n,Q}$ of constant cardinality. Let Z be a fixed finite set of this cardinality and choose for every n a bijection $z_n : Z \cong Z_n$. Then $(C_{n,Q}, x_n|Q \sqcup z_n)$ is a $(Q \sqcup Z)$-pointed curve, that is easily seen to be stable. If h denotes the arithmetic genus of $C_{n,Q}$, then after passing to a subsequence, $[(C_{n,Q}, x_n|Q \sqcup z_n)]_n$ will converge in $\overline{\mathfrak{M}}_h^{Q\sqcup Z}$ to some $[(C^*, y \sqcup z)]$. The property alluded to is that $(C_Q^*, y) = (C_Q, x|Q)$.

To complete the proof, let $[(C_n, x_n)]_{n=1}^\infty$ and $[(C'_n, x'_n)]_{n=1}^\infty$ be sequences in $\overline{\mathfrak{M}}_g^P$ converging to $[(C, x)]$ and $[(C', x')]$ repectively such that terms with the same index are R_Q-equivalent. We must show that $[(C, x)]$ and $[(C', x')]$ are $\overline{R}_Q$-equivalent. But this is immediate from the above mentioned property.

(3.2) Here is a simple, but perhaps instructive example. Let C be a smooth connected projective curve of genus $g \geq 2$. Then $C \times C$ parametrizes a subvariety of $\overline{\mathfrak{M}}_g^{\{0,1\}}$. A point of the diagonal, $(p, p) \in C \times C$, represents the union of C and $P^1(\mathbb{C})$ with $p \in C$ identified with $\infty \in P^1(\mathbb{C})$ and $i = 0, 1$ mapping to $i \in P^1(\mathbb{C})$. Taking the image in $K_{\{0\}}\mathfrak{M}_g^{\{0,1\}}$ means that we disregard the irreducible component C and retain $P^1(\mathbb{C})$ with its three points. So the composite map $C \times C \to K_Q\mathfrak{M}_g^P$ contracts the diagonal. As A.J. de Jong pointed out to me, this contraction can be obtained algebraically as the normalization of the image of the difference map from $C \times C$ to the Jacobian of C. The contraction can also be realized by the line bundle on $C \times C$ that is the pull-back of the canonical sheaf under the projection $(p_0, p_1) \in C^2 \to p_0 \in C$ twisted by the diagonal (a positive tensor power of that bundle is without base points).

(3.3) Notice that the $\overline{R}_Q$ gets coarser as Q gets smaller. In particular, for $Q \subset Q'$, there is a natural quotient mapping $K_{Q'}\mathfrak{M}_g^P \to K_Q\mathfrak{M}_g^P$, that is contravariant in Q.

In this connection we venture the following

Conjecture 1. *The space $K_Q\mathfrak{M}_g^P$ has the structure of a normal projective variety such that the quotient maps $\overline{\mathfrak{M}}_g^P \to K_Q\mathfrak{M}_g^P$ and $K_{Q'}\mathfrak{M}_g^P \to K_Q\mathfrak{M}_g^P$ $(Q \subset Q')$ are morphisms.*

M. Boggi (Utrecht) has verified this conjecture for $g = 0$. We actually expect the corresponding quotient to arise as the image under a certain linear system without base points. Related to this problem we propose

Conjecture 2. *The relatively dualizing sheaf of the universal stable curve of genus $g \geq 2$, $\overline{\mathfrak{M}}_g^1 \to \overline{\mathfrak{M}}_g$, is semiample, i.e., a positive tensor power of it has no base points.*

S. Wolpert [10] has shown that the natural metric on this relatively dualizing sheaf has nonnegative curvature and that this curvature is zero only in directions along the R_1-equivalence classes. Using this one can show that under the assumption of conjecture 2, a positive power of the relatively dualizing sheaf defines a morphism whose fibers are the R_1-equivalence classes. So conjecture 2 implies conjecture 1 for the case when $P = Q$ is a singleton.

(3.4) These extensions have Teichmüller counterparts: for any nonempty subset Q of P we have a Γ-equivariant quotient $K_Q\mathfrak{T}$ of $\overline{\mathfrak{T}}$ that contains $\mathfrak{T}$ and for $Q \subset Q'$ a quotient mapping $K_{Q'}\mathfrak{T} \to K_Q\mathfrak{T}$.

It is useful to have a moduli interpretation for these compactifications. We first remind the reader that one calls a complex-analytic space *weakly normal* if every continuous complex function on an open subset that is analytic outside a divisor is analytic. For curves this means that every singular point with k branches is like the union of the coordinate-axes of $\mathbb{C}^k$ at the origin.

We make two definitions: A *Q-minimal P-pointed curve of genus g* consists of a connected weakly normal curve C, a map $x : P \to C$, and a function $\epsilon : C \to \mathbb{Z}_{\geq 0}$ with finite support (the *genus defect function*) such that

(1) $x|Q$ is injective and its image is contained in $C_{\rm reg} \setminus x(P - Q)$ and meets every connected component of that space.
(2) The automorphism group of the triple (C, x, ϵ) is finite (equivalently: every connected component of $C_{\rm reg} \setminus (x(P) \cup \operatorname{supp} \epsilon)$ has negative Euler characteristic).
(3) $g = g(\hat{C}) + \sum_{z\in C}(\epsilon(z) + r(C, z) - 1)$, where $\hat{C}$ is the normalization of C and $r(C, z)$ is the number of branches of (C, z)

The above conditions imply the existence of a continuous map $f : S \to C$ that extends x such that the pre-image of a point $z \in C$ is either a connected submanifold with boundary of S of genus $\epsilon(z)$ with $r(C, z)$ boundary components if $\epsilon(z) + r(C, z) > 1$ and a singleton otherwise. If we are given such a map up to isotopy relative to P, then we say that the Q-minimal P-pointed curve is *marked* by (S, P).

If (C, x) is a stable P-pointed curve, then the following prescription leads to a Q-minimal P-pointed curve $(\bar{C}, x, \epsilon)$: contract the closure of every connected component of $C - C_Q$ to a singleton in such a way that the image is a weakly normal singularity. The result is a Q-minimal P-pointed curve with the genus defect function assigning to a point the arithmetic genus of its pre-image in C if that pre-image is a curve and 0 otherwise.

There is an obvious notion of isomorphism: two Q-minimal P-pointed curves (C, x, ϵ), (C', x', ϵ') are declared isomorphic if there exists an isomorphism $h : C \to C'$ such that $x'h = x$ and $\epsilon' h = \epsilon$. In the marked context we of course also require that h respects the markings.

(3.5) Lemma. *The isomorphism classes of (resp. marked) Q-minimal P-pointed curves of genus g are in bijective correspondence with the points of*

$K_Q\mathfrak{M}_g^P$ (resp. $K_Q\mathfrak{T}$) and the quotient map $\overline{\mathfrak{M}}_g^P \to K_Q\mathfrak{M}_g^P$ is given by the receipe $(C,x) \mapsto (\bar{C},x,\epsilon)$ given above.

Proof. We content ourselves with indicating how a Q-minimal curve (C,x,ϵ) determines an element of $K_Q\mathfrak{M}_g^P$. Extend x to a continuous map $f : S \to C$ as above. Let L be the boundary of $f^{-1}(C_{\text{sing}} \cup \operatorname{supp}\epsilon)$. Now collapse to a point every component of L as well as every component of $f^{-1}(C_{\text{sing}} \cup \operatorname{supp}\epsilon)$ that is homeomorphic to a cylinder and does not intersect P. Then $(\bar{S},\pi x)$ is a stable P-pointed pseudosurface. The map f factorizes through a map $\bar{f} : \bar{S} \to C$ and the irreducible components of $\bar{S}$ that are not contracted receive in this way a weakly normal complex structure. Extend this to a weakly normal complex structure (compatible with the given orientation) on $\bar{S}$. Then we get a stable P-pointed curve C. Its image in $K_Q\mathfrak{M}_g^P$ only depends on (C,x,ϵ).

We can form the simplicial space $K_\bullet\mathfrak{M}_g^P$. Its geometric realization is a quotient of $\overline{\mathfrak{M}}_g^P$ such that the quotient map followed by the structure map $|K_\bullet\mathfrak{M}_g^P| \to \Delta_P$ is the projection. We shall show that $|K_\bullet\mathfrak{M}_g^P|$ is homeomorphic (over Δ_P) to the semisimplicial complex $\Gamma\backslash A$ that was defined in the introduction. We look at this complex in more detail in the next section.

4. The arc complex

(4.1) We consider embedded unoriented loops and arcs α in S that connect two (possibly identical) points of P and avoid all other points of P. In case of a loop we also require that it be nontrivial in the sense that it does not bound an embedded disk in $S - P$. Let $\mathcal{A}$ denote the set of isotopy classes relative to P of these arcs and loops. We endow this set with the structure of an abstract simplicial complex by stipulating that an $(l+1)$-element subset of $\mathcal{A}$ defines an l-simplex if it is representable by arcs and loops that do not meet outside P. We denote the geometric realization of this complex by A. There is a piecewise linear map λ from A to the simplex Δ_P spanned by P characterized by the property that it sends a vertex $\langle\alpha\rangle \in \mathcal{A}$ to the barycenter of the end points of α. So if Q is a nonempty subset of P and $\Delta_Q \subset \Delta_P$ the corresponding face, then $\lambda^{-1}\Delta_Q$ is a subcomplex of A whose 0-simplices may be interpreted as the isotopy classes of embedded arcs and loops in $S-(P-Q)$ with end points in Q.

We say that the simplex $\langle\alpha_0,\dots,\alpha_l\rangle$ is *proper* if its star is finite, that is, if it is contained in finitely many simplices. This comes down to requiring that each connected component of $S - \cup_\lambda\alpha_\lambda$ is an embedded open disk that contains at most one point of P. The improper simplices make up a subcomplex $A_\infty \subset A$. We shall denote its complement $A - A_\infty$ by A°. It is clear that A has an action of Γ that preserves both A_∞ and λ.

(4.2) Lemma. *The group Γ has only a finite number of orbits in the set of simplices of $\mathcal{A}$. The dimension of a proper simplex is at least $2g-2+|P|$ and the dimension of every fiber of λ is $6g-6+2|P|$.*

Proof. The first assertion is a consequence of the fact that up to homeomorphism there are only finitely many compact surfaces with an Euler characteristic bounded from below (the details are left to the reader).

Let $a=\langle\alpha_0,\dots,\alpha_l\rangle$ be an l-simplex of A and let $Q\subset P$ the set of points of P that are end point of some α_λ. This means that l maps the relative interior of a to the relative interior of Δ_Q. If a is a proper simplex, then the formula for the Euler characteristic gives

$$2-2g=|Q|-(l+1)+d,$$

where d is the number of connected components of $S-\cup_\lambda\alpha_\lambda$. Since every connected component contains at most one point of $P-Q$, we have $d\geq |P|-|Q|$. It follows that $l\geq 2g-2+|P|$. If a is maximal in the preimage of Δ_Q, then every connected component of $S-\cup_\lambda\alpha_\lambda$ either is an open disk that contains precisely one point of $P-Q$ and is bounded by a single member of a or contains no point of $P-Q$ and is bounded by three members of a. A straightforward computation shows that then $d=\frac{2}{3}(l+1+|P|-|Q|)$. Substituting this in the formula for the Euler characteristic gives $l=6g-7+3|Q|+2(|P|-|Q|)=6g-6+2|P|+\dim\Delta_Q$.

Example. We take for S the torus $\mathbb{R}^2/\mathbb{Z}^2$ and for P the origin. An element of $\mathcal{A}$ is uniquely represented by a circle that is also a subgroup of S. Such a subgroup is the image of a line in $\mathbb{R}^2$ through the origin and another point of $\mathbb{Z}^2$. In this way we obtain an identification of $\mathcal{A}$ with the rational projective line $\mathbf{P}^1(\mathbb{Q})$. The two circles defined by the relatively prime pairs of integers (x_0,x_1) and (y_0,y_1) define a 1-simplex iff they do not intersect outside the origin. This is the case iff $x_0y_1-x_1y_0=\pm1$, or equivalently, iff $x=(x_0,x_1)$ and $y=(y_0,y_1)$ make up a basis of $\mathbb{Z}^2$. Then this 1-simplex is adjacent to exactly two 2-simplices, namely those defined by $\{x,y,x+y\}$ and $\{x,y,x-y\}$. A simplex is proper iff it is of dimension >0. The geometric realization of A can be pictured in the upper half plane (with the vertex at ∞ missing) as a hyperbolic tesselation associated to a subgroup of the modular group of index two.

Let $b\mathcal{A}$ denote the barycentric subdivision of $\mathcal{A}$. So a vertex of $b\mathcal{A}$ is the barycenter of a simplex a of $\mathcal{A}$ and a k-simplex of $b\mathcal{A}$ is spanned by the barycenters of a strictly increasing chain $a_0<a_1<\cdots<a_k$ of simplices of $b\mathcal{A}$. Let $\mathcal{A}_{\mathrm{pr}}$ denote the full subcomplex of $b\mathcal{A}$ whose vertices are the barycenters of proper simplices. Clearly, its geometric realization A_{pr} can be viewed as a subset of A°. In the previous example we have drawn A_{pr} with dotted lines.

(4.3) Proposition. *The fibres of $\lambda|A_{\mathrm{pr}}$ have dimension $4g-4+|P|$ and there is a natural Γ-equivariant deformation retraction of A° resp. $A-A_{\mathrm{pr}}$*

FIG. 1 THE ARC COMPLEX OF A ONCE-POINTED TORUS

onto $A_{\rm pr}$ *resp.* A_∞ *that preserves the pre-image of every relatively open face of* Δ_P *under* λ.

Proof. A k-simplex of $\mathcal{A}_{\rm pr}$ is represented by a chain $a_0 < a_1 < \cdots < a_k$ of simplices of $\mathcal{A}$ with a_0 proper. According to the previous lemma $\dim a_0 \geq 2g - 2 + |P|$ and $\dim a_k \leq 6g - 6 + 2|P| + \dim \Delta_Q$, where $Q \subset P$ is the smallest subset of P such that λ maps a_k in Δ_Q. So $k \leq (6g - 6 + 2|P| + \dim \Delta_Q) - (2g - 2 + |P|) = 4g - 4 + |P| + \dim \Delta_Q$. It is easily verified that this value is attained.

The proof of the remaining assertions is a standard argument in the theory of simplicial complexes, but let us give it nevertheless, say for $A_{\rm pr} \subset A^\circ$. If $x \in A^\circ = |b\mathcal{A}| - |b\mathcal{A}_\infty|$, then we can write $x = \sum_{i=0}^k x_i a_i$ with $a_0 < a_1 < \cdots < a_k$, $x_i > 0$, and a_k proper. Let r be the first index such that a_r is proper. Then

$$x' := \sum_{i=r}^{k} (\sum_{j=r}^{k} x_j)^{-1} x_i a_i \in A_{\rm pr}$$

and $x(t) := (1-t)x + tx'$ defines a deformation retraction of A° onto $A_{\rm pr}$.

Our goal is to construct a Γ-equivariant homeomorphism of A onto $|K_\bullet \mathfrak{T}|$ that commutes with the given projections onto Δ_P. For this we first need to discuss ribbon graphs.

5. RIBBON GRAPHS

(5.1) A *ribbon graph* is a nonempty finite graph in that may have loops and multiple bonds, but no isolated points (in other words, a semi-simplicial complex of pure dimension 1), such that for every vertex we are given a cyclic order of its outgoing edges.

A finite graph embedded in an oriented surface acquires naturally such a structure. Conversely, a ribbon graph can be embedded in a compact oriented surface with boundary of which it is a deformation retract (see fig. 2 below).

Fig. 2 Ambient surface of a ribbon graph

Contracting each boundary component yields a closed oriented surface. This surface can be obtained in a purely combinatorial way as follows. Let G be a ribbon graph. Denote by $X(G)$ its set of oriented edges (so that each edge determines two distinct elements of $X(G)$). Reversal of orientation defines a fixed point free involution σ_1 in $X(G)$. For $e \in X(G)$, let v be its vertex of origin, and denote by $\sigma_0(e) \in X(G)$ the outgoing edge of v that succedes e with respect to the given cyclic order. This defines a permutation σ_0 of $X(G)$. We define the permutation σ_∞ by the equality $\sigma_\infty \sigma_1 \sigma_0 = 1$.

Fig. 3 The operations σ_i

Denote the orbit space of σ_i in $X(G)$ by $X_i(G)$. For $i = 0$ resp. $i = 1$ it can be identified with the set of vertices resp. of (unoriented) edges of G; the elements of $X_\infty(G)$ are called *boundary cycles.* So G can be reconstructed from $X(G)$ equipped with the permutations σ_0 and σ_1. (Indeed, any nonempty finite set equipped with a fixed point free involution and another permutation determines a ribbon graph.)

(5.2) For every oriented edge e, let K_e be the semi-infinite rectangle $|e| \times \mathbb{R}_{\geq 0}$ with the obvious orientation. Its one-point compactification is denoted by K_e^*. We define a complex $S(G)$ as a quotient of the disjoint union of the K_e^*'s that identifies the base of K_e^* with the base of $K_{\sigma_1(e)}^*$ and the righthand edge of K_e^* with the lefthand edge of $K_{\sigma_\infty(e)}^*$. We call the image of K_e^* in this quotient the *tile* defined by e and we sometimes make the slight abuse of notation by denoting that tile also by K_e^*.

Clearly, G appears here as a subspace of $S(G)$, see the picture below. The added points (arising from the onepoint compacfifications) are naturally indexed by the set $X_\infty(G)$ and we shall therefore regard $X_\infty(G)$ as a subset of $S(G)$. We refer to its elements as the *cusps* of $S(G)$. In what follows a special rôle is played by the points that are cusps or are vertices of G of valency ≤ 2. We shall call such points *distinguished.*

FIG. 4 COMBINATORIAL CONSTRUCTION OF THE AMBIENT SURFACE

It is not difficult to see that $S(G)$ is a compact surface. The orientations of the tiles determines one of $S(G)$ and this orientation is compatible with the ribbon graph structure of $X(G)$. The surface also has a quasiconformal structure.

The symmetry axis of K_e^* that joins the midpoint of its base and the point at infinity maps to a curve in $S(G)$ (an interval or a loop depending on whether e has an 'end'). The union of these curves make up another ribbon graph, called the *dual* of G, and denoted by G^*. Combinatorially it is obtained by passing from $(X(G);\sigma_0,\sigma_1)$ to $(X(G);\sigma_\infty,\sigma_1)$. Notice that there is a natural identification of $S(G^*)$ with $S(G)$. The edges of G^* are in bijective correspondence with those of G.

Remark. The permutations $\sigma_0, \sigma_1, \sigma_\infty$ associated to a ribbon graph Γ arise as monodromies in the following manner. Let S_0 be the topological sphere obtained from $([-1,1]\times\mathbb{R}_{\geq 0})^*$ by identifying points on its boundary according to reflection in the symmetry axis. It is clear that there is a natural finite quotient map $S(G)\to S_0$. This map is a ramified covering with as branch locus the images of $(1,0),(0,0),\infty$. The pre-image of the two-cell is naturally identifyable with $X\times(-1,1)\times\mathbb{R}_{\geq 0}$ and the monodromy of $S(G)\to S_0$ around these points acts on the preimage as $\sigma_0,\sigma_1,\sigma_\infty$ (acting on the first factor).

6. Metrized ribbon graphs

(6.1) A *metric* on a ribbon graph is G simply a map from its edges to $\mathbb{R}_{>0}$. If this map has in addition the property that the total length of the graph is 1, then we call it a *unital metric*.

A *conformal structure* on G is a metric on every connected component of G, given up to a factor of proportionality. This is of course equivalent to a unital metric on every connected component of G. We denote the space of conformal structures on G by $\mathrm{cf}(G)$. So for connected G, $\mathrm{cf}(G)$ may be identified with the open simplex spanned by the set of edges of G.

(6.2) Let G be a ribbon graph G with metric $l : X_1(G)\to\mathbb{R}_{>0}$. If we equip $|e|\times\mathbb{R}_{\geq 0}$ with the product metric, then $S(G)-X_\infty(G)$ inherits from these metrics a complete piecewise Euclidean metric. The complement of the vertex set of $S(G)$ has a unique smooth structure for which this metric is Riemannian on that set. It is easy to check that its underlying conformal structure extends across the vertices, so that now $S(G)$ acquires a conformal structure. We denote the Riemann surface thus obtained by $C(G,l)$. This Riemann surface comes with a meromomorphic quadratic differential q_l whose absolute value gives the metric: if we identify the interior of the tile associated to e as a metric space in the obvious way with the Euclidean rectangle $\{z\in\mathbb{C} : \Im(z)>0, |\Re(z)|<\frac{1}{2}l(e)\}$, then this is a complex-analytic chart and the quadratic differential is given by $dz\otimes dz$. One finds that q_l has a pole of order two at each cusp and a zero at of order $k-2$ at each k-valent vertex of G (so a pole of order one at a univalent vertex). This implies that successive outgoing oriented edges at a k-valent vertex make an angle of $2\pi/k$. There are no other singularities of q_l. Observe that as a piecewise-linear complex valued quadratic differential on $S(G)$, q_l embodies all the extra structure: the smooth structure, the metric and (hence) the complex-analytic structure.

Notice that the conformal structure on $S(G)$ only depends on the conformal structure on G subordinate to l. Hence we can always assume that l is unital on every connected component of G.

If v is a bivalent vertex of G, then "forgetting" that vertex yields a metrized ribbon graph whose associated Riemann surface can be identified with $C(G,l)$.

(6.3) A *P-pointed ribbon graph* is a ribbon graph G together with an injection $x : P \hookrightarrow X_\infty(G) \sqcup X_0(G)$ whose image contains all the distinguished points. Notice that in that case every connected component of $S(G) - x(P)$ has negative Euler characteristic: this is because $S(G) - X_\infty(G)$ admits G as a deformation retract and every connected graph that is contractible (resp. a homotopy circle) has at least two (resp. one) vertices of valency at most 2.

Let (G, x) be a P-pointed ribbon graph. If s is an edge of G that is neither isolated nor a loop, then contracting that edge (to a vertex) yields a ribbon graph G/s. It inherits a P-pointing iff at most one of its vertices is in the image of P. The corresponding surface $S(G/s)$ is obtained as a quotient of $S(G)$ by collapsing the two tiles defined by s according to the level sets of r. We call this an *edge collapse*.

If s is a non-isolated loop, and for some orientation e of s, e is by itself a boundary cycle, then it is still true that G/s is a ribbon graph. In this case, G/s inherits a P-pointing if and only if the vertex of s is not in the image of P. The surface $S(G/s)$ is then obtained by collapsing the correponding tile to a point (a *total collapse*) and by applying an edge collapse to the opposite tile.

In either case the quotient map $S(G) \to S(G/s)$ has in its homotopy class relative to P a unique isotopy class relative to P of quasiconformal homeomorphisms.

We can apply these two procedures successively to a collection Z of edges of G if and only if every connected component of the corresponding subgraph $G_Z \subset G$ is either

(1) a tree with at most one marked vertex, or
(2) a homotopy circle without marked vertices that contains an entire boundary cycle of G.

We then say that Z is *negligible*. So if Z is negligible and G/G_Z is the semi-simplicial complex obtained by collapsing every connected component of G_Z to a point, then G/G_Z has still the structure of a ribbon graph pointed by P and the corresponding surface $S(G/G_Z)$ can be obtained by means of a succession of edge collapses and contractions of the tiles labeled by the oriented edges in Z. The quotient map $S(G) \to S(G/G_Z)$ determines an isotopy class relative P of sense preserving quasiconformal homeomorphisms $S(G) \to S(G/G_Z)$.

An *near-metric* on G is a function $l : X_1(G) \to \mathbb{R}_{\geq 0}$ whose zero set Z is negligible. It is clear that l then factorizes over a metrized ribbon graph G/G_Z with metric (still denoted) l and we define $C(G, l)$ simply as $C(G/G_Z, l)$. We have a corresponding notion of an *near-conformal structure*.

We shall denote the space of unital near-conformal structures on (G, x) by $\mathrm{ncf}(G, x)$. It is clear that for a negligible $Z \subset X_1(G)$, we have a natural embedding of $\mathrm{ncf}(G/G_Z, x)$ in $\mathrm{ncf}(G, x)$.

(6.4) We now assume that G is a connected ribbon graph. Over $\mathrm{cf}(G)$ lives a "tautological" topologically trivial family of metrized graphs and a corresponding family of Riemann surfaces. We extend the latter as a family of pseudosurfaces; in section 8 we give each of its fibers the structure of a weakly normal curve. (A *pseudosurface* is a surface with isoloted singularities, precisely, a space that can be triangulated in such a way that the link of each vertex is a combinatorial one-manifold.)

The family appears as a quotient of the projection $S(G) \times a(G) \to a(G)$ and is defined as follows. Any edge s of G determines by definition a vertex of $a(G)$. The codimension-one face opposite this vertex is identified with $a(G/s)$ and for each orientation e of s, we apply an edge collapse to $K_e^* \times a(G/s)$ relative to its projection onto $a(G/s)$. Likewise, every boundary cycle β of G determines a face $a(G/G_\beta)$ of $a(G)$ and we perform a total collapse on the tiles $K_e^* \times a(G/G_\beta)$ relative to $a(G/G_\beta)$ with $e \in \beta$. The result is a semisimplicial space $\mathcal{C}(G)$ that comes with a projection $\pi_G : \mathcal{C}(G) \to a(G)$.

Over $l \in \mathrm{cf}(G)$ the fiber is the surface $S(G)$; it has a conformal structure that makes it canonically isomorphic to $C(G, l)$. That last fact is still true in case $l \in \mathrm{ncf}(G)$. The fiber $\mathcal{C}(G)_l$ over an arbitrary $l \in a(G)$ is gotten as follows. Let $Z \subset X_1(G)$ be the zero set of l and let $S(G)_Z$ be the quotient of $S(G)$ obtained by performing for every oriented edge e of Z a contraction or an edge collapse on K_e^*, depending on whether or not the boundary cycle of G generated by e is contained in G_Z. Then $\mathcal{C}(G)_l$ can be identified with $S(G)_Z$. We will see in section 8 that $S(G)_Z$ is a pseudosurface and that $\mathcal{C}(G)_l$ has a natural conformal structure on its smooth part given by quadratic differential. (This conformal structure determines a unique complex-analytic structure such that $\mathcal{C}(G)_l$ is weakly normal.)

(6.5) We conclude this discussion with a few remarks.

Every vertex or cusp of G determines a section of $\mathcal{C}(G) \to a(G)$. Those that are indexed by P are disjoint over $\mathrm{ncf}(G)$.

One can show that the complement of the sections defined by the vertices and cusps has a natural smooth structure. (To see this, use an atlas naturally indexed by the elements of $X_1(G) \sqcup X_\infty(G)$.) The conformal structures along the the fibers vary differentiably on this open subset.

7. Moduli spaces

(7.1) We say that a ribbon graph G is (S, P)-*marked* (or briefly, *marked*) if we are given a given isotopy class relative to P of sense preserving quasiconformal homeomorphisms $f : S \cong S(G)$ such that $f|P$ defines a P-pointing of G: f maps P to $X_\infty(G) \sqcup X_0(G)$ and its image contains the distinguished points. It is clear that G permutes the markings.

(7.2) We claim that a marked ribbon graph is the same thing as a proper simplex of $\mathcal{A}$. Let $f : S \cong S(G)$ be a marking. Regard the dual ribbon graph G^* as lying on $S(G)$. Then the pre-image of every edge of G^* under f connects two points of P and therefore the collection of these determines a simplex $a(G, f)$ of $\mathcal{A}$. A connected component of $S - G^*$ is given by a vertex of G; it contains one or no point of P depending on whether this vertex is marked by P. If the vertex is unmarked it has valency $k \geq 3$ and the connected component is a k-gon. So distinct edges of G^* yield distinct vertices of $a(G, f)$ and $a(G, f)$ is a proper simplex. We also notice that the space of unital metrics $\mathrm{cf}(G)$ may be identified with the relative interior of $a(G, f)$; we shall therefore denote that relative interior by $\mathrm{cf}(G, f)$.

Conversely, if $a = \langle \alpha_0, \ldots, \alpha_l \rangle$ is a proper simplex of A, then the union of the α_i's define a ribbon graph G_a on S with vertex set contained in P. It is easily seen that the inclusion $G_a \subset S$ extends to a quasiconformal homeomorphism $S(G_a) \to S$ such that $X_\infty(G_a)$ is mapped in P. If we identify $S(G_a^*)$ with $S(G, S)$, then we see that G_a has in a natural way the structure of a marked ribbon graph.

We remark that $\mathrm{cf}(G, f)$ has maximal dimension iff all vertices of G are trivalent (so that P maps bijectively onto the set boundary cycles of G).

(7.3) Lemma. *Let a be a proper simplex of A as above with associated marked ribbon graph (G, f). Let $Z \subset X_1(G)$ be a set of edges of G and let $a(G/G_Z)$ be the codimension $|Z|$ face of a opposite the face defined by Z. Then Z is negligible if and only if $a(G/G_Z)$ is proper and in that case $S(G/G_Z)$ inherits an marking (denoted f/Z).*

Proof. It is enough to show this in case Z has only one element and this we leave to the reader.

So given a marking f, then the space of unital near-metrics $\mathrm{ncf}(G, f|P)$ may be identified with $|a(G, f)| \cap A^\circ$. We denote the latter by $\mathrm{ncf}(G, f)$. The restriction of $\lambda : A \to \Delta_P$ to $\mathrm{ncf}(G, f)$ has the following simple description: for $p \in P$ $f(p)$ is either a boundary cycle or a vertex; if l is a unital near-metric on G, then in the first case the barycentric coordinate $\lambda_p(l)$ is half the l-length of that boundary cycle, whereas in the second case $\lambda_p(l) = 0$.

Remember that every proper simplex of A is of the form $a(G, f)$ and that over such a simplex we have defined in section 6 the family $\mathcal{C}(G) \to a(G, f)$. As each inclusion of proper simplices is canonically covered by an inclusion of the corresponding families, this gives us a global family $\pi : \mathcal{C} \to A$. This family comes with sections labeled by P.

Summing up:

(7.4) Proposition. *The set of points of A° is naturally interpreted as the set isomorphism classes of marked ribbon graphs endowed with a unital metric. It is obtained from the spaces $\mathrm{ncf}(G, f)$ by identifying the space $\mathrm{ncf}(G/G_Z, f/Z)$ with its image in $\mathrm{ncf}(G, f)$ for every negligible $Z \subset X_1(G)$.*

Moreover, A supports a family $\pi : \mathcal{C} \to A$ of weakly normal curves with sections indexed by P. Over A° these sections are disjoint, the family is locally trivial with fiber S and each fiber comes with a complex structure that varies continuously with the base point.

In the next section we shall discuss the fibers over A_∞.

The family π restricted to A° defines a classifying map $\Phi : A^\circ \to \mathfrak{T}$. This map is continuous and clearly Γ-equivariant. The following theorem is a rather direct consequence of the work of Strebel.

(7.5) Theorem. *The map*

$$\Psi^\circ := (\Phi, \lambda) : A^\circ \to \mathfrak{T} \times \Delta_P$$

is a homeomorphism.

The observation that Strebel's work leads to theorems of this type is attributed by Harer to Mumford [3]. (We did not come across this version, though.)

For the proof we must discuss Jenkins-Strebel differentials first. Let R be a Riemann surface. If q is a meromorphic quadratic differential on R, then at each point p of R where q has neither a zero nor a pole the tangent vectors at p on which q takes a real value ≥ 0 form a real line in $T_z C$. This defines a foliation on R minus the singular set of q. If the union of the closed leaves of this foliation is dense in R, then q is called a *Jenkins-Strebel differential*. Suppose q is such a differential. Then a local consideration shows that q has no poles of order > 2 and that the double residue at a pole of order 2 is a negative real number. The form q determines a Riemann metric $|q|$ on the complement of the singular set of q. This metric is locally like $|dz|^2$ and hence flat. The union K of the non-closed leaves and the singular points of q of order ≥ -1 is closed in R. It is an embedded graph with a singularity of order k being a vertex of valency $k+2$; it is called the *critical graph* of q. Each connected component of the complement of K is either a flat annulus (metrically a flat cylinder) or a disk containing a unique pole of order two (metrically outside this pole a flat semi-infinite cylinder) or a copy of $\mathbb{C} - \{0\}$.

Suppose that R is the complement of a finite subset of a compact Riemann surface C. Then q is also a Jenkins-Strebel differential on C and the closure $\overline{K}$ of K in C is an embedded graph. (When C has genus zero it may happen that this closure becomes a closed orbit on C, so $\overline{K}$ may depend on R. It can be shown however, that this is the only such case.) Clearly, $\overline{K}$ has the structure of a ribbon graph. Notice that q defines a metric on it.

(7.6) Theorem. (Strebel) *Let (C, x) be a compact connected P-pointed Riemann surface that is not the two-pointed Riemann sphere and let $\lambda \in \Delta_P$. Then there exists a Jenkins–Strebel differential q on C with the property that the union of the closed leaves of q form semi-infinite cylinders* *around the points of $x(p)$ with $\lambda(p) \neq 0$ (of circumference $\lambda(p)$) and the points $x(p)$ with $\lambda(p) = 0$ lie on the critical graph of q. Moreover, such a q is unique.*

Proof. Denote by $Q \subset P$ denote the zero set of λ and put $Q' := P - Q$. If $|Q'| \geq 2$, then the asserted properties follow from Theorem 23.5 of [9] applied to the Riemann surface $C - x(Q)$ with circumferences given by $p \in Q' \mapsto \lambda(p)$. (The fact that q will have order ≥ -1 at the points of Q follows from the discussion above.) In case Q' is a singleton $\{p\}$, then Theorem 23.2 of [9] implies that there is a Jenkins–Strebel differential on $C - x(Q)$ all of whose closed leaves belong to the cylinder about p. This differential is unique up to a positive real scalar factor and hence the theorem follows in this case, too.

We shall refer to λ as a *circumference function* of (C, x), the name being suggested by the above theorem. So such a function determines a metrized ribbon graph (G_λ, l_λ) in C (denoted by $\overline{K}$ in the discussion above). Notice that if $\lambda(p) = 0$, then $x(p)$ is a univalent vertex or an interior point of an edge of G_λ; if $\lambda(p) \neq 0$, then $x(p)$ defines a boundary cycle of G_λ. Moreover, all univalent vertices and boundary cycles of G_λ are thus obtained. In other words, G_λ is in a natural manner a P-pointed ribbon graph. The associated P-pointed curve $C(G_\lambda, l_\lambda)$ is canonically isomorphic to (C, x): this is clear on the complement of the union of $x(P)$ and the vertex set of G_λ. Hence it is true everywhere.

Proof of (7.5). The above discussion shows that Ψ° has a unique inverse, in other words, that it is bijective. Since Ψ° is continuous and has locally compact domain and range, it must be a homeomorphism.

(7.7) Corollary. (Harer [3]) *For nonempty P, the moduli space $\mathfrak{M}_g^P$ has the homotopy type of a finite semi-simplicial complex of dimension $\leq 4g - 4 + |P|$. In particular, $\mathfrak{M}_g^P$ has no homology or cohomology in dimension $> 4g - 4 + |P|$.*

Proof. Choose $p \in P$ and regard p as a vertex of Δ_P. Then $\mathfrak{T}$ is by (7.5) equivariantly homeomorphic to $\lambda^{-1}(p) \cap A^\circ$. Now apply (4.3).

8. Minimal models

In this section we introduce a combinatorial analogue of a Q-minimal P-pointed curve. Here (G, x) is a connected marked ribbon graph.

(8.1) We say that a set Z of edges of G is *semistable* if no component of G_Z is the set of edges of a negligible subset and every univalent vertex of G_Z is in the image of x. Then every component of G_Z that is contractible contains at least two vertices in $x(P)$. A component that is a homotopy circle without a vertex in $x(P)$ is necessarily a topological circle that is not a boundary cycle of G. It is clear that every subset $Z \subset X_1(G)$ has a maximal semistable subset Z^{sst}. Notice that $Z - Z^{\mathrm{sst}}$ is a negligible subset of $X_1(G)$ so that if we put $G' := G/G_{Z-Z^{\mathrm{sst}}}$, then $S(G')$ is quasiconformal homeomorphic relative to P to $S(G)$. We sometimes regard $G_{Z^{\mathrm{sst}}}$ as a graph on $S(G')$, so that with this convention $G/G_Z = G'/G'_{Z^{\mathrm{sst}}}$.

(8.2) Let there be given a proper subset Z of $X_1(G)$. We can associate to Z two ribbon graphs: one with edge set Z and another with edge set $X_1(G) - Z$. In the first case we give G_Z an induced structure of ribbon graph by telling how the corresponding operator σ_0 acts on $X(G_Z)$: it sends $e \in X(G_Z)$ to the first term of the sequence $(\sigma_0^k(e))_{k\geq 1}$ that is in $X(G_Z)$. The second case is in a sense dual to the first: we define a ribbon graph G/G_Z with $X_1(G) - Z$ as its set of edges and the corresponding operator σ_∞ sends $e \in X(G) - X(G_Z)$ to the first term of the sequence $(\sigma_\infty^k(e))_{k\geq 1}$ that is not in $X(G_Z)$. This ribbon graph naturally maps onto a subgraph of G, but this map need not be injective as it may identify distinct vertices of G/G_Z.

A vertex of G/G_Z that is in the image of an oriented edge in Z^{sst} will be called *exceptional*. Any such vertex corresponds to a boundary cycle of $G_{Z^{\mathrm{sst}}}$ that is not a boundary cycle of G (and vice versa) and we call such boundary cycles *exceptional* also.

(8.3) Lemma. *There is a natural identification mapping of $S(G/G_Z) \to S(G)_Z$. This map identifies two distinct points if and only if both are exceptional vertices of $S(G/G_Z)$ that come from a boundary cycle of the same component of $G_{Z^{\mathrm{sst}}}$. In particular, $S(G)_Z$ is a pseudosurface whose combinatorial normalization is $S(G/G_Z)$. Moreover, every distinguished point of G/G_Z comes from a distinguished point of G or is exceptional.*

Proof. Straightforward.

In this situation we have a genus defect function $\epsilon : S(G)_Z \to \mathbb{Z}_{\geq 0}$ that assigns to the image of an exceptional vertex the genus of the corresponding component of $S(G_{Z^{\mathrm{sst}}})$ and is zero otherwise.

(8.4) Choose an $l \in a(G)$. In (6.4) we constructed a map $\pi_G : \mathcal{C}(G) \to a(G)$ and we noticed that that the fiber over l, $\mathcal{C}(G)_l$, can be identified with $S(G)_Z$, where Z is the zero set of l. Since l determines a unital metric on G/G_Z, we have a Riemann surface $C(G/G_Z, l)$ with underlying space $S(G/G_Z)$. We use the previous lemma to give $\mathcal{C}(G)_l$ the unique complex-analytic structure for which $\mathcal{C}(G)_l$ is weakly normal and $C(G/G_Z, l) \to C(G)_l$ is its normalization.

(8.5) Proposition. *Let Q be the set of $p \in P$ that map to a boundary cycle of G of positive length. Then $(Q, \epsilon, P \to S(G) \to \mathcal{C}(G)_l)$ give $\mathcal{C}(G)_l$ the structure of a Q-minimal P-pointed curve.*

Proof. We verify the defining properties of (3.4). The property for $p \in P$ to belong to Q is equivalent to p mapping to a cusp of G/G_Z. The first property now follows. For the second we must show that $S(G/G_Z) - X_\infty(G/G_Z) - \{\text{exceptional vertices}\}$ has negative Euler characteristic. But this follows from the fact that this is (by (8.3)) just the complement of the set of distinguished points on G/G_Z. The verification of the third property is left to the reader.

Suppose we are given a marking f of G that extends the pointing by x. This determines a marking of $\mathcal{C}(G)_l$ by (S,P). In view of the moduli interpretation (3.5), the structure present on $\mathcal{C}(G)_l$ determines a point of $K_Q\mathfrak{T}$. By letting l vary over the elements of $a(G,f)$, we thus obtain a map $a(G,f) \to |K_\bullet\mathfrak{T}|$ commuting with the given maps of domain and range to Δ_P. For a negligible edge s of G the restriction of this map to $a(G/s, f/s)$ coincides with the one defined for that simplex. This results in a Γ-equivariant map $\Psi : A \to |K_\bullet\mathfrak{T}|$. We can now state our first main result. It gives an analytic interpretation of A:

(8.6) Theorem. *The map $\Psi : A \to |K_\bullet\mathfrak{T}|$ is a Γ-equivariant continuous bijection that commutes with the given maps to Δ_P.*

The main difficulty is to show that Ψ is continuous. We postpone the proof to a point where we have treated the combinatorial version of the Deligne–Mumford compactification. The reader may wonder whether Ψ is a homeomorphism. The answer is that it is not, as is illustated by the case $g = 1$, P a singleton: then $|K_\bullet\mathfrak{T}|$ is the union of the upper half plane and $P^1(\mathbb{Q})$. Near ∞ it has the horocyclic topology but the topology it receives from its triangulation is much finer: a subset of the upper half plane is the complement of a neighborhood of ∞ if and only if its intersection with any vertical strip of bounded width is bounded.

9. Stable ribbon graphs

Here we introduce the ribbon graph analogue of a stable P-pointed curve. That our definition is the natural one may not be immediately obvious, but that this is indeed the case will become apparent in the discussion following the definition and in section 10.

(9.1) Suppose we are given a ribbon graph G and an injection $x : P \to X_0(G) \sqcup X_\infty(G)$. We no longer assume that $x(P)$ contains the set of distinguished points of $S(G)$, but instead we suppose given a subset $\Sigma \subset X_0(G) \sqcup X_\infty(G)$ that contains both $x(P)$ and the distinguished points of G and an involution ι on the complement $\Sigma - x(P)$. We define inductively the *order* of a connected component of G as follows: a connected component is of order zero if it contains a cusp hit by the image of x; a connected component has order $\leq k+1$ if it contains a distinguished point p such that $\iota(p)$ lies on a component of order $\leq k$.

We say that (G, x, ι) is a *stable P-pointed ribbon graph* if

(1) every component has an order and

(2) for every cusp p of G on a component of order $k > 0$, $\iota(p)$ is on a component of order $k-1$.

(So in the situation (2) $\iota(p)$ must be a vertex of G.)

(9.2) A stable P-pointed ribbon graph (G, x, ι) determines a stable P-pointed pseudosurface $(S(G,\iota), x)$: it is obtained from the surface $S(G)$ by identifying the points (of $\Sigma - x(P)$) according to the involution ι. If this surface is connected, then it has a *genus* g characterized by the condition that $2-2g$ is the Euler characteristic of the smooth part of $S(G,\iota)$.

We have seen that a conformal structure l on G determines a conformal structure on $S(G)$ so that we have a compact Riemann surface $C(G,l)$. This in turn, determines a weakly normal complex-analytic structure on $S(G,\iota)$. With that structure, $(S(G,\iota), x)$ becomes a stable P-pointed curve $(C(G,\iota,l), x)$. This curve has additional structure: to every point $p \in x(P) \cup S(G,\iota)_{\mathrm{sing}}$ is assigned a nonnegative number $\lambda(p)$, namely half the length of the corresponding boundary cycle (with respect to the componentwise unital metric defining the conformal structure) in case the point is a cusp and zero otherwise. Notice that $\lambda(p) = 0$ if $x(p)$ lies on a single irreducible component of $S(G,\iota)$ or if $p \in P$ and $x(p) \in X_0(G)$, and that the sum of the values of λ on each irreducible component is 1.

This suggests to extend the notion of a *circumference function* to the case of a stable connected P-pointed pseudosurface (S', x) as a function $\lambda : x(P) \cup S'_{sing} \to \mathbb{R}_{>0}$ that possesses these properties. So the space of circumference functions on (S', x) is a product of simplices (with a factor for each irreducible component).

(9.3) Just as for smooth P-pointed curves, the datum of a circumference function λ on a stable P-pointed curve (C, x) permits us to go in the opposite direction: apply Strebel's theorem (7.6) componentwise to the normalisation $(\hat{C}, \lambda)$. This determines a Jenkins-Strebel differential q on $\hat{C}$ with the properties mentioned there. In particular, we have a critical graph (G, l) in $\hat{C}$ that contains the zeroes of λ. Moreover, each $p \in \operatorname{supp}(\lambda)$ determines (and is determined by) a boundary cycle of G and the length of that boundary cycle is $\lambda(p)$. The associated Riemann surface $C(G,l)$ is naturally isomorphic to $\hat{C}$.

(9.4) Let (G, x) be a P-pointed ribbon graph. We describe how a proper subset of $X_1(G)$ (or rather, strictly decreasing sequences of such) define stable P-pointed ribbon graphs. First two definitions.

Let Z be a semistable set of edges of G. Recall that then every component of G_Z that is a homotopy circle without a vertex in $x(P)$ is necessarily a topological circle (and is not a boundary cycle of G). If this does not happen, i.e., if every component of G_Z that is a topological circle contains a vertex in the image of x, then we say that Z is *stable*. It is clear that every subset $Z \subset X_1(G)$ has a maximal semistable subset Z^{st}; it is a union of components of Z^{sst}.

Forgetting the bivalent vertices of $G_{Z^{\mathrm{st}}}$ that are in $x(P)$ yields a ribbon graph with the same underlying topological space as $G_{Z^{\mathrm{st}}}$; we denote this ribbon graph by $\bar{G}_{Z^{\mathrm{st}}}$ and its set of edges by $\bar{Z}^{\mathrm{st}}$. It is clear that the set of distinguished points of $S(G_{Z^{\mathrm{st}}})$ coincides with set of cusps of $G_{\bar{Z}^{\mathrm{st}}}$.

A metric on $G_{Z^{\mathrm{st}}}$ determines one on $\bar{G}_{Z^{\mathrm{st}}}$.

(9.5) Let Z be a proper subset of $X_1(G)$ and put $G(Z) := G/G_Z \sqcup G_{Z^{\mathrm{st}}}$. It is clear that the pointing x determines an injection $\tilde{x}$ of P in the set of 0-simplices of $G(Z)$. The proof of the following lemma is easy and left to the reader

(9.6) Lemma. *The set of distinguished points of $G(Z)$ that are not in the image of $\tilde{x}$ comes with a natural involution ι so that $G(Z)$, $\tilde{x}$ and ι define a stable P-pointed ribbon graph. The associated P-pointed stable pseudosurface $S(G;Z)$ is obtained from $S(G/G_Z)$ and $G_{Z^{\mathrm{st}}}$ by identifying each exceptional vertex of $S(G/G_Z)$ with the corresponding exceptional element of $X_\infty(G_{Z^{\mathrm{st}}})$ and then contracting every irreducible component that corresponds to a component of $G_{Z^{\mathrm{sst}}} - G_{Z^{\mathrm{st}}}$. A conformal structure on $\tilde{G}$ determines one on $S(\tilde{G},\iota)$ and turns the latter into a stable P-pointed curve.*

(9.7) We may of course repeat this construction for a set of edges of $G_{\bar{Z}^{\mathrm{st}}}$. In order to be able to state this we introduce the following notions.

A *permissible sequence* for (G,x) is a sequence

$$Z_\bullet = (X_1(G) = Z_0, Z_1, Z_2, \dots, Z_k)$$

such that $Z_\kappa \subset \bar{Z}^{\mathrm{st}}_{\kappa-1}$ and G_{Z_κ} does not contain a connected component of $\bar{G}_{Z^{\mathrm{st}}_{\kappa-1}}$.

A *stable metric* with respect to such a sequence is given by a conformal structure on every difference $\bar{G}_{Z^{\mathrm{st}}_\kappa} - G_{Z_{\kappa+1}}$. So this may be given by a sequence of functions $l_\kappa : Z^{\mathrm{st}}_\kappa \to \mathbb{R}{\geq 0}$ such that l_κ has zero set $Z_{\kappa+1}$ ($\kappa = 0,1,\dots$). (So $l_\bullet$ determines $Z_\bullet$.)

The previous discussion generalizes in a straightforward way to:

(9.8) Proposition. *Let $Z_\bullet$ be a permissible sequence for (G,x). Then the disjoint union of the ribbon graphs $G_{\bar{Z}^{\mathrm{st}}_\kappa}/G_{Z_{\kappa+1}}$ ($\kappa = 0,1,\dots$) is in a natural way a stable P-pointed ribbon graph $(G(Z_\bullet),\tilde{x},\iota)$. A stable metric $l_\bullet$ with respect to $Z_\bullet$ defines a conformal structure on $S(G,Z_\bullet)$ and turns it into a stable P-pointed curve $C(G,l_\bullet)$.*

10. Stable limits

In this section we fix a connected P-pointed ribbon graph (G,x). We explain how the stable pseudosurface associated to a permissible sequence for G arises as a limit of Riemann surfaces $C(G,l(t))$.

(10.1) We shall use a blowing up construction in the piecewise differentiable category. The basic construction starts out from a collection β of oriented edges of G that defines an oriented circular subgraph G_β of G. Let U_β be the union of the tiles less cusps that meet G_β and are on the same side of G_β as the tiles indexed by the elements of β; this is "half a regular neighborhood of G_β" in $S(G)$ and is homeomorphic to a half-open cylinder. Let introduce a the piecewise linear function $\phi_\beta : U_\beta \to \mathbb{R}_{\geq 0}$ that has G_β as zero set and so in a sense measures the distance to G_β. that on K_e is the euclidean distance to $G_\beta \cap K_e$ (where $|e|$ has been given unit length, say). So for $e \in \beta$, $\phi_\beta | K_e$ is just the height.

Denote by $\mathbb{R}^*_{\geq 0}$ the one-point compactification of $\mathbb{R}_{\geq 0}$ and let $(U_\beta \times \mathbb{R}_{\geq 0})^{\sim}$ be the closure of the graph of the function

$$(u,t) \in U_\beta \times \mathbb{R}_{>0} \mapsto t^{-1}\phi_\beta(u) \in \mathbb{R}_{\geq 0}$$

in $U_\beta \times \mathbb{R}_{\geq 0} \times \mathbb{R}^*_{\geq 0}$. The projection $(U_\beta \times \mathbb{R}_{\geq 0})^{\sim} \to U_\beta \times \mathbb{R}_{\geq 0}$ is clearly a homeomorphism over the complement of $G_\beta \times \{0\}$ whereas the pre-image of $G_\beta \times \{0\}$ (the *exceptional set*) is $G_\beta \times \{0\} \times \mathbb{R}^*_{\geq 0}$. The closure of the pre-image of $(U_\beta - G_\beta) \times \{0\}$ (the *strict transform* of $U_\beta \times \{0\}$) meets the exceptional set in $G_\beta \times \{0\} \times \{\infty\}$. So the pre-image of $U_\beta \times \{0\}$ (its *total transform*) is a kind of thickening of U_β. In particular, this total transform is homeomorphic to U_β; indeed, the projection $(U_\beta \times \mathbb{R}_{\geq 0})^{\sim} \to \mathbb{R}_{\geq 0}$ is a trivial fibration (piecewise differentiably).

We glue $(U_\beta \times \mathbb{R}_{\geq 0})^{\sim} \to \mathbb{R}_{\geq 0}$ to $(S(G) \times \mathbb{R}_{\geq 0}) - (\phi_\beta^{-1}(0,1)) \times \{0\})$ in obvious way and obtain a modification

$$(S(G) \times \mathbb{R}_{\geq 0})^{\sim}_\beta \to S(G) \times \mathbb{R}_{\geq 0}.$$

This is what we call the *blowing up of the cycle* defined by β.

Fig. 5 Blowing up of a cycle

For $e \in \beta$, the strict transform of the tile $K_e^* \times \{0\}$ in $(S(G) \times \mathbb{R}_{\geq 0})\tilde{}_\beta$ is in fact a lift of $K_e^* \times \{0\}$ and hence homeomorphic to it. We apply an edge collapse to each of the strict transforms of the tiles indexed by β and we denote the result $(S(G) \times \mathbb{R}_{\geq 0})\hat{}_\beta$. The pre-image of $S(G) \times \{0\}$ is denoted by $S(G;\beta)$. It is a pseudosurface that is homeomorphic to the space obtained from $S(G)$ by contracting G_β. It comes with an injection of P into its regular part.

(10.2) We now fix a proper subset Z of $X_1(G)$ and show how $S(G;Z)$ is obtained as a one-parameter degeneration of $S(G)$. First we assume that Z is stable. We carry out the previous construction for each boundary cycle of Z. It is easily seen that these can be performed independently so that we have defined a modification

$$(S(G) \times \mathbb{R}_{\geq 0})\hat{}_Z \to S(G) \times \mathbb{R}_{\geq 0}.$$

Notice that this projection $(S(G) \times \mathbb{R}_{\geq 0})\hat{}_Z \to \mathbb{R}_{\geq 0}$ is trivial over $\mathbb{R}_{>0}$ with fiber $S(G)$, whereas the fiber over 0 is canonically isomorphic to $S(G;Z)$.

In case Z is not stable, we first apply the preceding procedure to Z^{st} and next we collapse the strict transforms of the tiles indexed by the oriented members of $Z - Z^{\mathrm{st}}$ (a total collapse or an edge collapse, depending on whether the boundary cycle generated by the corresponding oriented edge is in G_Z or not). The order of these operations can be reversed; in particular, we can first pass to $G' := G/G_{Z-Z^{\mathrm{sst}}}$ and the image Z' in $X_1(G')$ (so that Z' is now semistable), then perform edge collapses on the tiles indexed by the oriented members of $Z' - Z'^{\mathrm{st}}$ (these make up a union of circular components of $G_{Z'}$) and finally apply the preceding construction with Z'^{st}. Then the fiber over 0 can be identified with $S(G;Z)$ as before.

We already observed that conformal structures l_0 on G/G_Z and l_1 on G_Z determine a conformal structure on $S(G;Z)$, turning it into a stable P-pointed curve $C(G,(l_0,l_1))$ whose normalization is the disjoint union of the Riemann surfaces $C(G/G_Z,l_0)$ and $C(G_{Z^{\mathrm{st}}},l_1)$. We may obtain such conformal structures by means of a degeneration of a family of metrics on $S(G)$. To be concrete, let l be a metric on G and let for $t>0$, $l(t)$ be the metric on G that takes on an edge s the value $tl(s)$ if $s \in Z$ and remains $l(s)$ if not. We give the fiber of $(S(G)\times\mathbb{R}_{\geq 0})\hat{}_Z \to \mathbb{R}_{\geq 0}$ over $t \in \mathbb{R}_{>0}$ (which is just $S(G)$) the corresponding metric structure (denoted m_t). The regular part of the fiber over 0 is given the metric structure m_0 defined by the restrictions l_0 resp. l_1 of l to $X_1(G) - Z$ resp. Z. This is in general not a continuous family of metrics, but for the underlying conformal structures the situation is different. To see this, we first look at what happens on K_e, where e is an oriented edge of Z^{sst}. If β is a boundary cycle of Z^{sst} that contains e, then it is clear from our definition of m_t that the set $K_e \cap \phi_\beta^{-1}(0,a)$ with metric m_t is conformally equivalent to subset $K_e \cap \phi_\beta^{-1}(0,t^{-1}a)$ with metric m_1. In fact, we have

(10.3) Lemma. *Suppose that the pointing x of G has been extended to a marking by (S,P). Then the map $\mathbb{R}_{\geq 0} \to \overline{\mathfrak{T}}$ that assigns to $t>0$ resp. $t=0$ the isomorphism class of $C(G,l(t))$ resp. $C(G,(l_0,l_1))$ is continuous.*

Proof. There is no loss of generality in assuming that G_Z has no negligible components.

The continuity on $\mathbb{R}_{>0}$ is clear. To prove continuity at 0 we wish to invoke (2.1). This requires that we trivialize our family locally. At the points of $S(G;Z)$ outside the exceptional set this is no problem and it is clear that with respect to a suitable trivialisation the complex structures converges uniformly on compact subsets. At the points of $S(G;Z)$ outside the strict transform we trivialize as follows. Choose a piecewise linear retraction $r_\beta : U_\beta \to G_\beta$ so that (r_β, ϕ_β) defines a homeomorphism h of $U_\beta - G_\beta$ onto $G_\beta \times \mathbb{R}_{>0}$. Let k denote its inverse and for $t>0$, let $k_t(p,s) = k(p,st)$. Then

$$(p,s,t) \in G_\beta \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \mapsto (k_t(p,s),t)$$

extends to a homeomorphism of $\tilde{G}_Z \times \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}$ onto an open subset of $(S(G)\times\mathbb{R}_{\geq 0})\widehat{}_Z$ so that for $t=0$ we get a homeomorphism k_0 of $G_\beta \times \mathbb{R}_{>0}$ onto the complement of the union of the strict transform of $S(G)$ and G_Z in $S(G;Z)$. We must show that the conformal structure J_t, $t \geq 0$ on $G_\beta \times (0,1)$ defined by pull-back of the given conformal structure on $C(G,l(t))$ under k_t depends continuously on t. This is a somewhat tedious calculation that uses explicit coordinates. We omit this.

The preceding can be iterated in an obvious way and yields:

(10.4) Proposition. *If $Z_\bullet$ is a permissible sequence, then there is defined an iterated modification:*

$$(S(G)\times\mathbb{R}_{\geq 0})\widehat{}_{Z_\bullet} \to \mathbb{R}_{\geq 0}.$$

This fibration is canonically trivialized (relative to x) over $\mathbb{R}_{>0}$ with fiber $S(G)$, whereas the fiber over 0 is canonically homeomorphic to $S(G;Z_\bullet)$.

Suppose that the pointing x of G has been extended to a marking by (S,P). Given a metric l on G, let $l(t)$ be the metric on G that on $G_{Z_\kappa} - G_{Z_{\kappa+1}}$ is equal to $t^\kappa l$ $(t>0)$ and let $l_\bullet$ be the stable metric with respect to $Z_\bullet$ that is defined by the restrictions of l. Then the map $\mathbb{R}_{\geq 0} \to \overline{\mathfrak{T}}$ that assigns to $t \in \mathbb{R}_{>0}$ resp. $t=0$ the isomorphism type of $C(G,l(t))$ resp. $S(G) \cong C(G,l_\bullet)$ is continuous.

11. Deligne–Mumford modification of the arc complex

Let (G,x) be a connected P-pointed ribbon graph. Recall that we have defined the family $\pi : \mathcal{C}(G) \to a(G)$ which over the interior $\mathrm{cf}(G)$ of $a(G)$ is trivialized with fiber $S(G)$. We are going to modify this family over the locus where this family is not locally trivial. This will also modify the base and the result will be a family parametrizing stable pointed pseudosurfaces with stable conformal structures.

Let $\mathcal{Z}(G)$ denote the collection of stable subsets $Z \subset X_1(G)$ with G_Z connected. For $l \in \mathrm{cf}(G)$ and $Z \in \mathcal{Z}(G)$, we let $\pi_Z(l)$ denote the unital metric on $G_{\bar{Z}}$ that is proportional to $l|G_Z$. Let $\hat{a}(G)$ be the closure of the graph of the map $l \in \mathrm{cf}(G) \mapsto (\pi_Z(l) \in \mathrm{cf}(G_Z))_Z$ in $a(G) \times \prod_{Z \in \mathcal{Z}(G)} a(G_{\bar{Z}})$.

(11.1) Proposition. *There is a natural bijection between the points of $\hat{a}(G)$ and the set of stable conformal structures on G.*

Proof. Let $(l^{(n)})_{n=1}^{\infty}$ be a sequence in $\mathrm{cf}(G)$. By passing to a subsequence, we may assume that for every $Z \in \mathcal{Z}(G)$, the sequence $(\pi_Z(l^{(n)}))_n$ converges (to l_Z, say). Write l_0 for $l_{X_1(G)}$, let $Z(l_0)$ be the zero set of l_0 and put $Z_1 := Z(l_0)^{\mathrm{st}}$. Notice that $\mathcal{Z}(G_{Z_1})$ is just a subset of $\mathcal{Z}(G_Z)$. So for each $Z \in \mathcal{Z}(G_{Z_1})$ we have a function $l_Z : Z \to [0,1]$ whose sum is 1. Applying this to the connected components of G_{Z_1} yields a function $l_1 : Z_1 \to [0,1]$ that on each connected component of G_{Z_1} sums up to 1. We proceed with induction: if $l_\kappa : Z_\kappa \to [0,1]$ has been constructed, then let $Z(l_\kappa)$ be the zero set of l_κ. We put $Z_{\kappa+1} := Z(l_\kappa)^{\mathrm{st}}$ and define $l_{\kappa+1} : Z_{\kappa+1} \to [0,1]$ by letting it on each connected component G_Z of $G_{Z_{\kappa+1}}$ be equal to l_Z. Then $Z_\bullet$ is a permissible sequence for (G,x) by construction. It comes naturally with a unital stable metric $l_\bullet$ with respect tothis sequence. This stable metric determines every l_Z: for $Z \in \mathcal{Z}(G)$, let κ be such that $G_Z \subset G_{Z_\kappa}$ and $G_Z \not\subset G_{Z_{\kappa+1}}$. Then G_Z is contained in a connected component $G_{Z'}$ of G_{Z_κ}. Since $Z \not\subset Z_{\kappa+1}$, $l_{Z'}|Z$ (and hence $l_\kappa|Z$) is not identically zero. It then follows that l_Z is the unital metric proportional to $l_\kappa|Z$. On the other hand, (10.4) shows that every stable metric arises thus.

(11.2) If Z is a negligible set of edges of G, then $\hat{a}(G/G_Z)$ can be identified with the subset of $\hat{a}(G)$ parametrizing stable metrics $l_\bullet$ of which each term vanishes on Z. Hence if we endow the ribbon graphs with markings, then the closed cells $\hat{a}(G,f)$ can be glued together to yield a modification

$$\hat{A} \to A.$$

It is clear that $\hat{A}$ comes with a decomposition into cells. Such a cell admits a description in terms of arc complexes as follows: it is of the form $\sigma_0 \times \sigma_1 \times \dots$, where each σ_κ is a cell (a product of simplices) of the arc complex associated to a (not necessarily connected) pointed surface (S_κ, P_κ). These pointed surfaces (and hence these cells) are defined inductively: $(S_0, P_0) := (S, P)$ and σ_0 is an arbitrary simplex of A. For $\kappa \geq 1$, let $\bar{S}'_\kappa$ be the pseudosurface obtained from $S_{\kappa-1}$ by contracting the arcs that make up $\sigma_{\kappa-1}$, S'_κ its normalisation, and let $P'_\kappa \subset S'_\kappa$ the pre-image of the image of $P_{\kappa-1}$. Let (S_κ, P_κ) be obtained from (S'_κ, P'_κ) by discarding all components that are one- or two-pointed spheres. The connected components of S_κ label the factors of σ_κ so that each factor is made up of arcs in that component. We require that these arcs connect only points of P_κ that map to singular points of $\bar{S}'_{\kappa-1}$. Under the projection $\hat{A} \to A$ this cell maps to σ_0.

It is possible to give a complete description of the incidence relations between these cells, but we will not do that.

(11.3) We shall define a family of surfaces $\hat{\mathcal{C}}(G)$ over $\hat{a}(G)$. Let $Z_\bullet$ be a permissible sequence for G of *connected stable* subsets, which we here regard as a strictly decreasing sequence of connected stable subsets of $X_1(G)$, and consider the map

$$I_{Z_\bullet} : S(G) \times \mathrm{cf}(G) \to \prod_{\kappa \geq 1} (S(G) \times \mathbb{R}_{>0}), \quad (u, l) \mapsto (u, l(Z_\kappa)/l(Z_{\kappa-1}))_\kappa.$$

The closure of its graph in $S(G) \times a(G) \times \prod_{\kappa\geq 1}(S(G) \times \mathbb{R}_{\geq 0})\widehat{}_{Z_\kappa}$ is denoted by $(S(G) \times a(G))\widehat{}_{Z_\bullet}$.

Similarly, we denote the closure of the graph of

$$\mathrm{cf}(G) \to \prod_{\kappa\geq 1} \mathbb{R}_{>0}, \quad l \mapsto (l(Z_\kappa)/l(Z_{\kappa-1}))_\kappa$$

in $a(G) \times \prod_{\kappa\geq 1} \mathbb{R}_{\geq 0}$ by $a(G)\widehat{}_{Z_\bullet}$. Since the functions $l(Z_\kappa)/l(Z_{\kappa-1})$ extend continuously to $\hat{a}(G)$, this is a quotient of $\hat{a}(G)$. We have a projection

$$(S(G) \times a(G))\widehat{}_{Z_\bullet} \to a(G)\widehat{}_{Z_\bullet}.$$

Any fiber over a point of $a(G)\widehat{}_{Z_\bullet}$ that has all its coordinates in $\prod_{\kappa\geq 1} \mathbb{R}_{\geq 0}$ equal to zero is isomorphic to $S(G; Z_\bullet)$.

We do this for all such sequences simultaneously. To be precise, let $\mathcal{Z}_\bullet(G)$ be the collection of strictly decreasing sequences of connected stable subsets of $X_1(G)$, and consider the map

$$I = (I_{Z_\bullet}) : S(G) \times \mathrm{cf}(G) \to \prod_{Z_\bullet} \prod_{\kappa\geq 1} (S(G) \times \mathbb{R}_{>0}).$$

The closure of its graph in

$$\hat{a}(G) \times \prod_{Z_\bullet} \prod_{\kappa\geq 1} (S(G) \times \mathbb{R}_{\geq 0})\widehat{}_{Z_\kappa}$$

is denoted $\hat{\mathcal{C}}(G)$ and the projection of $\hat{\mathcal{C}}(G)$ onto $\hat{a}(G)$ by $\hat{\pi}_G$. The preceding discussion shows:

(11.4) Proposition. *If $l_\bullet$ is a stable metric with associated permissible sequence $Z_\bullet$, then the fibre $\hat\pi_G^{-1}(l_\bullet)$ is naturally homeomorphic to $S(G; Z_\bullet)$.*

We endow the fiber $\hat\pi_G^{-1}(l_\bullet)$ with the conformal structure prescribed by the stable metric $l_\bullet$ so that $\hat\pi_G$ defines a family of stable P-pointed stable curves.

For marked ribbon graphs this construction is compatible in the sense that if $Z \subset X_1(G)$ is negligible, then $\hat\pi_{G/G_Z} : \hat{\mathcal{C}}(G/G_Z) \to \hat{a}(G/G_Z)$ can be identified with the restriction of $\hat\pi_G$ over $\hat{a}(G/G_Z)$. We may therefore glue these maps to each other to get a family $\hat\pi : \hat{\mathcal{C}} \to \hat{A}$ of stable P-pointed curves. Each fiber of $\hat\pi$ maps to a fiber of π, so that we have a commutative diagram

$$\begin{array}{ccc} \hat{\mathcal{C}} & \longrightarrow & \mathcal{C} \\ {\scriptstyle\hat\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ \hat{A} & \longrightarrow & A \end{array}$$

of spaces with Γ-action. We have also have a classifying map that extends Φ:

$$\hat\Phi : \hat{A} \to \overline{\mathfrak{T}}.$$

It is clearly Γ-equivariant. Our second main result reads as follows:

(11.5) Theorem. *The map $\hat\Phi : \hat{A} \to \overline{\mathfrak{T}}$ is a Γ-equivariant continuous surjection. The pre-image of the class of a marked stable P-pointed curve $(C, [f])$ under $\hat\Phi$ can be identified with the space of circumference functions (9.2) of (C, x). In particular, $\hat\Phi$ drops to a continuous surjection of $\Gamma\backslash\hat{A}$ onto the Deligne–Mumford compactification $\overline{\mathfrak{M}}_g^P$.*

Proof. Let $(C, [f])$ be as in the theorem. The construction described in (9.3) produces for every circumference function of (C, x) a marked ribbon graph (G, f) plus a stable metric $l_\bullet$ on G that reconstructs $(C, [f])$ for us. This defines an element of $\hat{a}(G, f)$ and one verifies that its image in $\hat{A}$ is unique.

It remains to show that $\hat\Phi$ is continuous. It is enough to prove that its restriction to every closed cell $\hat{a}(G, f)$ is. Since $\hat{a}(G, f)$ is second countable and $\overline{\mathfrak{T}}$ is Hausdorff, we only need to verify that the image of a converging sequence $(l_\bullet^{(n)})_n$ in $\hat{a}(G, f)$ under $\hat\Phi$ has a limit point. Then after passing to a subsequence we may assume that $(l_\bullet^{(n)})_n$ is in the relative interior of a single cell, say of $\hat{a}(G, f)$. The desired property then follows from (2.1) as in the proof of (10.3).

We can now finish the proof of our first main result, too.

Proof of (8.6). The map $\hat\Phi$ and the projection $\hat{A} \to A \to \Delta_P$ together define a map from $\hat{A}$ to $\overline{\mathfrak{T}} \times \Delta_P$. If we compose the latter with the quotient map $\overline{\mathfrak{T}} \times \Delta_P \to |K_\bullet\mathfrak{T}|$ we get a map $\hat\Psi : \hat{A} \to |K_\bullet\mathfrak{T}|$. The theorem above implies that the fibers of $\hat{\Psi}$ and the fibers of $\hat{A} \to A$ coincide. The induced bijection $A \to |K_\bullet \mathfrak{T}|$ is just Ψ. Since A has the quotient topology for the projection $\hat{A} \to A$, it follows that Ψ is continuous.

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