A MINICOURSE ON MODULI OF CURVES

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These are notes that accompany a short course given at the School on Algebraic Geometry 1999 at the ICTP, Trieste. A major goal is to outline various approaches to moduli spaces of curves. In the last part I discuss the algebraic classes that naturally live on these spaces; these can be thought of as the characteristic classes for bundles of curves.

1. Structures on a surface

We start with two notions from linear algebra. Let T be a real vector space. A *conformal structure* on T is a positive definite inner product $(\ \cdot\ )$ on T given up to multiplication by a positive scalar. The notion of length is lost, but we retain the notion of angle, for if $v_1, v_2 \in T$ are independent, then

$$\angle(v_1, v_2) := \frac{(v_1 \cdot v_2)}{||v_1|| ||v_2||}$$

does not change if we multiply $(\ \cdot\ )$ with a positive scalar. A *complex structure* on T is a linear automorphism J of T such that $J^2 = -1_V$; this makes T a complex vector space by stipulating that multiplication by $\sqrt{-1}$ is given by J. If $\dim T = 2$, then these notions almost coincide: if we are given an orientation plus a conformal structure, then 'rotation over $\frac{\pi}{2}$' is a complex structure on T. Conversely, if we are given a complex structure J, then we have an orientation prescribed by the condition that (v, Jv) is oriented whenever $v \neq 0$ and a conformal structure by taking any nonzero inner product preserved by J.

Let S be an oriented C^∞ surface. A *conformal structure* on S is given by a smooth Riemann metric on S given up to multiplication by a positive C^∞ function on S. By the preceding remark this is equivalent to giving an *almost-complex structure* on S, i.e., an automorphism J of the tangent bundle TS with $J^2 = -1_{TS}$, that is compatible with the given orientation. Given such a structure, then we have a notion of holomorphic function: a C^∞ function $f : U \to \mathbb{C}$ on an open subset U os S open is said to be *holomorphic* if for all $p \in S$, $df_p \circ J_p = \sqrt{-1} df_p : T_pS \to \mathbb{C}$. This generalizes the familiar notion for if S happens to be $\mathbb{C}$ with its standard almost-complex structure, then we are just saying that f satisfies the Cauchy-Riemann equations. It is a special property of dimension two that S admits an atlas consisting of holomorphic charts. (This amounts to the property that for every Riemann metric on a neighborhood of $p \in S$ we can find local coordinates x, y at p such that the the metric takes the form $c(x, y)(dx^2 + dy^2)$.) Coordinate changes will be holomorphic also (but now in the conventional sense), and we thus find that S has actually a complex-analytic structure.

A surface equipped with such a structure is called a *Riemann surface*. We shall usually denote such a surface by C. I will assume you are familiar with some of the basic facts regarding compact Riemann surfaces such as the Riemann-Roch theorem and Serre-duality and the notions that enter here. These give us for instance

Theorem 1.1. *A compact connected Riemann surface C of genus g can be complex-analytically embedded in $\mathbb{P}^{g+1}$ such that the image is a nonsingular complex projective curve of degree $2g+1$. The algebraic structure that C thus receives is canonical.*

The proof may be sketched as follows. Choose $p \in C$. Then the complete linear system generated by $(2g+1)(p)$ is of dimension $g+2$ and defines an complex-analytic embedding of C in $\mathbb{P}^{g+1}$ of degree $2g+1$. A theorem of Chow asserts that a closed analytic subvariety of a complex projective space is algebraic. So the image of this embedding is algebraic. It also shows that the algebraic structure is unique: if C is complex analytically embedded into two projective spaces as $C_1 \subset \mathbb{P}^k$ and $C_2 \subset \mathbb{P}^l$, then consider the diagonal embedding of C in $\mathbb{P}^k \times \mathbb{P}^l$, composed with the Segre embedding of $\mathbb{P}^k \times \mathbb{P}^l$ in $\mathbb{P}^{kl+k+l}$; By Chow's theorem the image is a complex projective curve C_3. The curves C_1 and C_2 are now obtained as images of C_3 under linear projections. These are therefore (algebraic) morphisms that are complex-analytic isomorphisms. Such morphisms are always algebraic isomorphisms.

The above theorem shows in particular that a compact Riemann surface of genus zero resp. one is isomorphic to $\mathbb{P}^1$ resp. to a nonsingular curve in $\mathbb{P}^2$ of degree 3.

What can we say about the automorphism group of a compact Riemann surface C of genus g? If $g = 0$, then we can assume $C = \mathbb{P}^1$ and $\mathrm{Aut}(\mathbb{P}^1)$ is then just the group of fractional linear transformations $z \mapsto (az+b)(cz+d)^{-1}$ with $ad - bc \neq 0$. If $g = 1$, then the classical theory tells us that C is isomorphic to a complex torus, and so $\mathrm{Aut}(C)$ contains that torus as a 'translation' group. This subgroup is normal and the factor group is finite. In all other cases ($g \geq 2$), $\mathrm{Aut}(C)$ is finite. There are several ways to see this, one could be based on the uniformization theorem, another on the fact that $\mathrm{Aut}(C)$ acts faithfully on $H_1(C, \mathbb{Z})$ (any automorphism acting trivially has Lefschetz number $2 - 2g < 0$, so cannot have a finite fixed point set, hence must be the identity) and preserves a positive definite Hermitian form on $H_1(C, \mathbb{C})$.

We denote by $\mathcal{M}_g$ the *set* of isomorphism classes of nonsingular genus g curves. For the moment it is just that: a set and nothing more, but our aim is to put more structure on $\mathcal{M}_g$ when $g \geq 2$. We will discuss four approaches to this:

- Riemann's original (heuristic) approach, that we will discuss very briefly.
- The approach through Teichmüller theory. Actually there are several of this type, but we mention just one. More is said about this in Hain's lectures.
- The introduction of an orbifold structure on $\mathcal{M}_g$ in the spirit of Grothendieck's formalization and generalization of the Kodaira-Spencer theory.
- The introduction of a quasi-projective structure on $\mathcal{M}_g$ by means of geometric invariant theory.

The last two approaches lead us consider a compactification of $\mathcal{M}_g$ as well.

2. Riemann's moduli count

Fix integers $g \geq 2$ and $d \geq 2g - 1$. Let C be a smooth genus g curve. Choose a point $p \in C$. By Riemann-Roch the linear system $|d(p)|$ has dimension $g - d$. Choose a generic line L in this linear system that passes through $d(p)$, in other words, L is a pencil through $d(p)$. The genericity assumption ensures that this pencil has no fixed points. Choose an affine coordinate w on L such that $w = \infty$ defines $d(p)$. We now have a finite morphism $C \to \mathbb{P}^1$ of degree d that restricts to a finite morphism $f : C - \{p\} \to \mathbb{C}$. We invoke the Riemann-Hurwitz formula (which is basically an euler characteristic computation):

$$\sum_{x\in C-\{p\}} \nu_x(f) = 2g-1+d,$$

where $\nu_x(f)$ is the *ramification index* of f at x (= the order of vanishing of df at x). The *discriminant divisor* D_f is $\sum_{x\in C-\{p\}} \nu_x(f)(f(x))$ (so the coefficient of $w\in\mathbb{C}$ is the sum of the ramification indices of the points of $f^{-1}(w)$). Its degree is clearly $2g-1+d$. The passage to the discriminant divisor loses only a finite amount of information: from that divisor we can reconstruct C and the covering $C\to\mathbb{P}^1$ (up to isomorphism) with finite ambiguity. Furthermore, it is easy to convince yourself that in the $(2g-1+d)$-dimensional projective space of effective degree $2g-1+d$ divisors on $\mathbb{P}^1$ the discriminant divisors make up a Zariski open subset. We now count moduli as follows: in order to arrive at f we needed for a given C, the choice of $p\in C$ (one parameter), the choice of a line L in $|d(p)|$ through $d(p)$ ($d-g-1$ parameters) and the affine coordinate w (2 parameters). Hence the number of parameters remaining for C is

$$(2g-1+d)-\big(1+(d-g-1)+2\big)=3g-3.$$

This suggests that $\mathcal{M}_g$ is like a variety of dimension $3g-3$.

3. Orbifolds and the Teichmüller approach

We begin with a modest discussion of orbifolds. Let G be a Lie group acting smoothly and properly on a manifold M. *Proper* means that the map $(g,p)\in G\times M\to(g(p),p)\in M\times M$ is proper; this guarantees that the orbit space $G\backslash M$ is Hausdorff.

If G acts freely on M, then the orbit space $G\backslash M$ is in a natural way a smooth manifold: for every $p\in M$, choose a submanifold S of M through p such that T_pS supplements the tangent space of the G-orbit of p at p. After shrinking S if necessary, S will meet every orbit transversally and at most once. Hence the map $S\to G\backslash M$ is injective. It is not hard to see that the collection of these maps defines a smooth atlas for $G\backslash M$, making it a manifold.

If G acts only with finite stabilizers, then we can choose S in such a way that it is invariant under the finite group G_p. After shrinking S in a suitable way we can ensure that every G-orbit that meets S, meets it in a G_p-orbit and that the intersection is transversal. So we then have an injection $G_p\backslash S\to G\backslash M$. This is in fact an open embedding and hence $G\backslash M$ is locally like a manifold modulo a finite group. It is often very useful to remember the local genesis of such a space, because this information cannot be recovered from the space itself (example: the obvious action of the nth roots of unity on a one dimensional complex vector space has orbit space isomorphic to $\mathbb{R}^2$, so that we cannot read off n from just the orbit space). This leads to Thurston's notion of *orbifold*: this is a Hausdorff space X for which we are given an 'atlas of charts' of the form $(U_\alpha,G_\alpha,h_\alpha)_\alpha$, where U_α is a smooth manifold on which a finite group G_α acts, and h_α is an open embedding of the orbits space $U_\alpha\backslash G_\alpha$ in X. The images of these open embeddings must cover X and there should be compatibility relations on overlaps. It is understood that two such atlasses whose union is also atlas define the same orbifold structure. So if F is a discrete space on which a finite group H acts simply transitively, then we may add the chart given by $U_\alpha\times F$ with its obvious action of $G_\alpha\times H$ (its orbit space is $U_\alpha\backslash G_\alpha$) and h_α. This allows us to express the compatibility relation simply by saying that the atlas is closed under the formation of fibered products ('intersections'): $U_\alpha \times_X U_\beta$ with its $G_\alpha \times G_\beta$-action and the identification of the orbit space with a subset of X should also be in it. It also implies that we have an atlas of charts for which the group actions are *effective*. I leave it to you to verify that $G\backslash M$ has that structure. There exist parallel notions in various settings, e.g., complex-analytic and algebraic. In the last two cases, it is often useful to work with a more refined notion of orbifold (a ' stack'), but we will not go into this now.

The case that concerns us is in infinite dimensional analogue of the above situation: we fix a closed oriented surface S_g of genus g and we let the space of conformal structures on S take the role of M and the group of orientation preserving diffeomorphisms take the role of G. It is understood here that these carry certain structures that allow us to think of an action of an infinite dimensional Lie group on an infinite dimensional space. This action turns out to be proper with finite stabilizers. It turns out that all orbits have codimension $6g-6$ and so it is at least plausible that $\mathcal{M}_g$ has the structure of an orbifold of real dimension $6g-6$. This heuristic reasoning has been justified by Earle and Eells.

4. Grothendieck's view point

It is worthwhile to discuss things in a more general setting that is strictly necessary for the present purpose, for the methods and notions that we need come up in virtually all deformation problems.

4.1. **Kodaira-Spencer maps.** Suppose $\pi : \mathcal{C} \to B$ is a proper (holomorphic) submersion between complex manifolds. According to Ehresmann's fibration theorem, π is then locally trivial in the C^∞-category, that is, for every $b \in B$ we can find an open $U \ni b$ and a smooth retraction $h : \mathcal{C}_U = \pi^{-1}U \to C_b = \pi^{-1}(b)$ such that $\tilde{h} = (h, \pi) : \mathcal{C}_U \to C_b \times U$ is a diffeomorphism. In particular, when B is connected, then all fibers of π are mutually diffeomorphic. If both B and the fibers are connected, we will call π a *family of compex manifolds with smooth base*. We assume that this is the case and we wish to address the question whether the fibers C_b are mutually isomorphic as complex manifolds. Suppose that the family is trivial over U, in other words, that the retraction h can be chosen holomorphically so that $\tilde{h}$ is an analytic isomorphism. Then each holomorphic vector field on U lifts to $C_b \times U$ in an obvious way and hence also lifts to $\mathcal{C}_U$ (via $\tilde{h}$). Suppose now the converse, namely, that every holomorphic vector field at b lifts holomorphically. Then π is locally trivial at b. To see this, assume for simplicity that $\dim B = 1$. Choose a nowhere zero vector field on an open open $U \ni b$ (always obtainable by means of a coordiante chart) that lifts holomorphically to a vector field on $\mathcal{C}_U$. The properness of π ensures that this lift is integrable to a holomorphic flow on $\mathcal{C}_U$. The flow surfaces (= complex flow lines) produces the desired retraction $h : \mathcal{C}_U \to C_b$. The case of a higher dimensional base goes by induction and the induction step is a parametrized version of the one dimensional case just discussed.

Liftability issues lead inevitably to cohomology. Let us begin with the noting that the fact that π is a submersion implies that for every $x \in \mathcal{C}$ we have an exact sequence

$$0 \to T_x C_b \to T_x \mathcal{C} \to T_b B \to 0, \quad b := \pi(x).$$

This sheafifies as an exact sequence of $\mathcal{O}_\mathcal{C}$-modules

$$0 \to \theta_{\mathcal{C}/B} \to \theta_\mathcal{C} \to \pi^*\theta_B \to 0$$

($\theta_{\mathcal{C}}$ stands for the sheaf of holomorphic vector fields on $\mathcal{C}$, $\theta_{\mathcal{C}/B}$ for the subsheaf of $\theta_{\mathcal{C}}$ of vector fields that are tangent to the fibers of π). Now take the direct image under π; since the fibers are connected, we get:

$$0 \to \pi_*\theta_{\mathcal{C}/B} \to \pi_*\theta_{\mathcal{C}} \to \theta_B \to 0.$$

This sequence diplays our lifting problem: an element of θ_B is holomorphically liftable iff it is in the image of $\pi_*\theta_{\mathcal{C}/B}$. But the sequence may fail to be exact at θ_B since π_* is only left exact. We need the right derived functors of π_* in order to continue the sequence in an exact manner:

$$0 \to \pi_*\theta_{\mathcal{C}/B} \to \pi_*\theta_{\mathcal{C}} \to \theta_B \xrightarrow{\delta} R^1\pi_*\theta_{\mathcal{C}/B} \to \cdots.$$

The sheaf $R^1\pi_*\theta_{\mathcal{C}/B}$ is a coherent $\mathcal{O}_B$-module, whose value at b is equal to the cohomology group $H^1(C_b, \theta_{C_b})$. So an element of θ_B is holomorphically liftable iff it image under δ vanishes. Hence δ gives us a good idea of how nontrivial the family at a point b is: the family is locally trivial at b iff δ is zero in b. Both δ and its value at a point b, $\delta(b) : T_bB \to H^1(C_b, \theta_{C_b})$, are called the *Kodaira-Spencer map.*

4.2. **The deformation category.** Let us fix a connected compact complex manifold C. A *deformation of C with smooth base* (B, b_0) is given by a proper holomorphic submersion $\pi : \mathcal{C} \to B$ of complex manifolds, a distinguished point $b_0 \in B$ and an isomorphism $\iota : C \cong C_{b_0}$, with the understanding that replacing π by its restriction to a neighborhood of b_0 in B defines the same deformation (in particular, B may be assumed to connected). From the preceding discussion it is clear that we may think of this as a variation of complex structure on C parametrized by the manifold germ (B, b_0).

The deformations of C are objects of a category: a morphism from $(\pi,' \iota')$ to (π, ι) is given by a pair of holomorphic map germs $(\tilde{\phi}, \phi)$ in the diagram

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{\tilde{\phi}} & \mathcal{C} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ (B', b_0') & \xrightarrow{\phi} & (B, b_0) \end{array}$$

such that the square is cartesian (this basically says that $\tilde{\phi}$ sends the fiber $C'_{b'}$ isomorphically to the fibre $C_{\phi(b')}$) and $\tilde{\phi}\iota' = \iota$. So (π, ι) is the more interesting then $(\pi,' \iota')$ as every fiber of the latter is present in the former. In this sense, the most interesting object would be a final object, if it exists (which is often not the case). A deformation (π, ι) is said to be *universal* if is a final object for this category. So this means that that for every deformation (π', ι') of C there is a unique morphism $(\pi', \iota') \to (\pi, \iota)$. A universal deformation is unique up to unique isomorphism (a general property of final objects). We also observe that the automorphism group $\mathrm{Aut}(C)$ acts on (π, ι): if $g \in \mathrm{Aut}(C)$, then $(\pi, \iota g^{-1})$ is another defromation of C and so there is a unique morphism $(\tilde{\phi}_g, \phi_g)) : (\pi, \iota g^{-1}) \to (\pi, \iota)$. The uniqueness implies that $\tilde{\phi}_{gh} = \tilde{\phi}_g\tilde{\phi}_h$ and that $\tilde{\phi}_1$ is the identity. So the action of $\mathrm{Aut}(C)$ extends to $(\mathcal{C}, C_{b_0})$. Similarly, $\mathrm{Aut}(C)$ acts on (B, b_0) such that π is equivariant.

Remark 4.1. The restriction to deformations over a smooth base turns out to be inconvenient. The custom is to allow B to be singular. The submersivity requirement for π is then replaced be the condition that π be locally trivial on $\mathcal{C}$: for every $x \in C_{b_o}$, there is a local holomorphic retraction $h : (\mathcal{C}, x) \to (C_{b_o}, x)$ such that $\tilde{h} = (h, \pi)$ is an isomorphism of analytic germs. This enlarges the deformation category and consequently the notion of universal deformation changes. Our restriction to deformations with smooth base was only for didactical purposes: a universal deformation is always understood to be the final object of this bigger category (and therefore need not have a smooth base).

We can go a step further and allow C to be singular as well (in fact, we shall have to deal with that case). Then the right condition to impose on π is that it be *flat*, which is an algebraic way of saying that the map must be open. In contrast to the situation considered above, the topological type of C can now change (simple example: take the family of conics in $\mathbb{P}^2$ with affine equation $y^2 = x^2 + t$). The Kodaira-Spencer theory has to be modified as well. For example, $H^1(C, \theta_C)$ must be replaced by $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$.

Remark 4.2. As said earlier, a universal deformation need not exist. But what always exists is a deformation (π, ι) with the property that for any deformation (π, ι') there exists a morphism $(\tilde{\phi}, \phi) : (\pi', \iota') \to (\pi, \iota)$ with ϕ *unique up to first order only.* This is called a *semi-universal deformation* of C (others call it a *Kuranishi family* for C). Such a deformation is still unique, but may have automorphisms (inducing the identity on C and the Zariski tangent space of the base).

4.3. Orbifold structure on $\mathcal{M}_g$.

We can now state the basic

Theorem 4.3. *For a smooth curve C of genus $g \geq 2$ we have*

(i) *C has a universal deformation with smooth base.*
(ii) *A deformation (π, ι) of C is universal iff its Kodaira-Spencer map $T_{b_0}B \to H^1(C_{b_0}, \theta_{C_{b_0}}) \cong H^1(C, \theta_C)$ is an isomorphism.*
(iii) *A universal deformation of C can be represented by a family $\mathcal{C} \to B$ such $\mathrm{Aut}(C)$ acts on this family (and not just on the germ) so that every isomorphism between fibers $C_{b_1} \to C_{b_2}$ is the restriction of the action of an automorphism of C.*

So the universal deformation of C is smooth of dimension $h^1(\theta_C)$. A simple application of Riemann-Roch shows that this number is $3g - 3$.

A universal deformation as in (iii) defines a map $B \to \mathcal{M}_g$ that factorizes over an injection $\mathrm{Aut}(C)\backslash B \to \mathcal{M}_g$. We give $\mathcal{M}_g$ the finest topology that makes all those maps continuous. It is not difficult to derive from the above theorem that with this topology the maps $\mathrm{Aut}(C)\backslash B \to \mathcal{M}_g$ become open embeddings. It is harder to prove that the topology is Hausdorff. Thus $\mathcal{M}_g$ acquires the structure of a (complex-analytic) orbifold of complex dimension $3g - 3$.

Remark 4.4. The orbifold structure on $\mathcal{M}_g$ has the property that every orbifold chart $(U, G, h : G\backslash U \to \mathcal{M}_g$ is induced by a family of genus g curves over U, provided that $g \geq 3$. For $g = 2$ we run into trouble since every genus 2 curve has a nontrivial involution (it is hyperelliptic). For this reason, the orbifold structure as defined here is not quite adequate and we have to resort to a more sophisticated version: ultimately we want only charts that support honest families of curves, with a change of charts covered by an isomorphism of families.

The space $\mathcal{M}_g$ is not compact. The reason is that one easily defines families of smooth genus g curves over the punctured unit disk $\mathcal{C} \to \Delta - \{0\}$ with monodromy of infinite order. A classic example is that of degerating family of genus one curves given by the affine equation $y^2 = x^3 + x^2 + t$, with t the parameter of the unit disk.

I admit that we dismissed genus one, but it is not difficult to generalize to higher genus: for example, replace x^3 by x^{2g+1}). Such a family defines an analytic map $\Delta - \{0\} \to \mathcal{M}_g$ such that the image of its intersection with a closed disk of radius < 1 is closed. To see this, suppose the opposite. One then shows that the map $\Delta - \{0\} \to \mathcal{M}_g$ must extend holomorphically over Δ. If the image of the origin is represented by the curve C, then a finite ramified cover of $(\Delta, 0)$ will map to the universal deformation of C so that the family on this finite cover will have trivial monodromy. This contradicts our assumption. The remedy is simple in principle: compactify $\mathcal{M}_g$ by allowing the curves to degenerate (as mildly as possible). This leads us to the next topic.

4.4. **Stable curves.** A *stable curve* is by definition a *nodal* curve C (that is, a connected complex projective whose singularities are normal crossings (nodes), analytically locally isomorphic to the union of the two coordinate axes in $\mathbb{C}^2$ at the origin) such that

- the euler characteristic of every connected component of the smooth part of C is negative.

The *genus* of such a curve can be defined algebro-geometrically as $h^1(\mathcal{O}_C)$ or topologically by the formula $2 - 2g = e(C_{\mathrm{reg}})$. So it has to be ≥ 2. The itemized condition is equivalent to each of the following ones:

- $g \geq 2$ and C_{reg} has no connected component isomorphic to $\mathbb{P}^1 - \{\infty\}$ or $\mathbb{P}^1 - \{0, \infty\}$,
- $\mathrm{Aut}(C)$ is finite,
- C has no infinitesimal automorphisms.

Topologically a stable genus g curve is gotten as follows. Let S_g be a closed oriented genus g surface. Choose on S a finite collection of embedded circles in distinct isotopy classes and such that none of these is trivial in the sense that it bounds a disk. Then the space obtained by contracting each of the circles underlies a stable curve and all these topological types are thus obtained. (Note that removal of an embedded circle from a surface does not alter its euler characteristic.) There is a deformation theory for stable curves which is almost as good as if the curve were smooth:

Theorem 4.5. *A stable curve of genus $g \geq 2$ has a universal deformation with smooth base of dimension $3g - 3$. This deformation can be represented by a family $\mathcal{C} \to B$ such that $\mathrm{Aut}(C)$ acts on this family in such a way that*

(i) *every isomorphism between fibers $C_{b_1} \to C_{b_2}$ is the restriction of the action of an automorphism of C,*
(ii) *all fibers are stable genus g curves,*
(iii) *for any singular point x of C, the locus $\Delta_x \subset B$ that parametrizes the curves for which x persists as a singular point is a smooth hypersurface and the $(\Delta_x)_{x \in C_{\mathrm{sing}}}$ cross normally.*

So the complement of the normal crossing hypersurface $\Delta := \cup_{x \in C_{\mathrm{sing}}} \Delta_x$ in B parametrizes smooth genus g curves. If there is just one singular point, then you may picture the degeneration from a smooth curve C_b to the singular curve $C_{b_0} \cong C$ in metric terms as by letting the circumference of the embedded circle on on S_g that defines the topological type of C go to zero. The monodromy around B_x is in this picture the Dehn twist along that circle (see the lectures by Hain).

Let $\overline{\mathcal{M}}_g$ be the set of isomorphism classes of stable genus g curves. The above theorem leads to a compact orbifold structure on this set:

Theorem 4.6 (Deligne-Mumford). *The universal deformations of stable genus g curves put a complex-analytic orbifold structure on $\overline{\mathcal{M}}_g$ of dimension $3g-3$. The space $\overline{\mathcal{M}}_g$ is compact and the locus $\partial\mathcal{M}_g = \overline{\mathcal{M}}_g - \mathcal{M}_g$ parametrizing singular curves is a normal crossing divisor in the orbifold sense.*

This is why $\overline{\mathcal{M}}_g$ is called the *Deligne-Mumford* compactification of $\mathcal{M}_g$. A generic point of the boundary divisor corresponds to a stable curve C with just one singular point. The underlying topological type of such a curve is determined by a nontrivial isotopy class of an embedded circle δ on S_g. The following cases occur:

- Δ_0: C is irreducible ($S_g - \delta$ is connected) or
- $\Delta_{\{g',g''\}}$: C is the one point union of two smooth curves of positive genera g', g'' with sum $g' + g'' = g$ ($S_g - \delta$ disconnected with components punctured surfaces of genera g' and g'').

These cases correspond to irreducible components of $\partial\mathcal{M}_g$. We denote them by Δ_0 and $\Delta_{\{g',g''\}}$.

5. The approach through geometric invariant theory

Perhaps the most appealing way to arrive at $\mathcal{M}_g$ and its Deligne-Mumford compactification is by means of geometric invariant theory. Conceptually this approach is more direct than the one discussed in the previous section. Best of all, we stay in the projective category. The disadvantage is that we do not know a priori what objects we are parametrizing.

Let us begin with a minimalist discussion of the general theory. Let be given a semisimple algebraic subgroup G of $\mathrm{SL}(r+1,\mathbb{C})$. That group acts on $\mathbb{P}^r$. Let $X \subset \mathbb{P}^r$ be a closed G-invariant subvariety. A G-orbit in $\mathbb{P}^r$ is called *semistable* if it is the projection of a G-orbit in $\mathbb{C}^{r+1} - \{0\}$ that does not have the origin in its closure. The union X^{ss} of the semistable orbits contained in X is a subvariety of X. This subvariety need not be closed and may be empty (in which case there is little reason to proceed). The basic results of Hilbert and Mumford are as follows:

1. Every semistable orbit in X^{ss} has in its closure a unique semistable orbit that is closed in X^{ss}.
2. There exists a positive integer N such that the semistable orbits that are closed in X^{ss} can be separated by the G-invariant homogeneous polynomials of degree N.
3. If R_X stands for the homogeneous coordinate ring of X, then the subring of its G-invariants R_X^G is noetherian.

Let $G\backslash\backslash X$ denote the set of semistable G-orbits in X that are closed in X^{ss}. Property 1 implies that there is a natural quotient map $X^{ss} \to G\backslash\backslash X$. If G happens to act properly on X^{ss}, then every orbit in X^{ss} is closed in X^{ss} and so $G\backslash\backslash X$ will be just the orbit set $G\backslash X^{ss}$. From property 2 it follows that if $f_0, \dots, f_m$ is a basis of the degree N part of R_X^G, then the map $[f_0 : \cdots : f_m] : X^{ss} \to \mathbb{P}^m$ is well defined and factorizes over an injection $G\backslash\backslash X \to \mathbb{P}^m$. The image of this injection is a closed subvariety of $\mathbb{P}^m$ and thus $G\backslash\backslash X$ acquires the structure of a projective variety. (A more intrinsic way to give it that structure is to identify it with $\mathrm{Proj}(R_X^G)$.)

So much for the general theory. For the case that interests us, you need to know what the *dualizing sheaf* ω_C of a nodal curve C is: it is coherent subsheaf of the sheaf of meromorphic differentals characterized by the property that on C_{reg} it is the sheaf of regular differentials, whereas at a node p we allow a local section to have on each of the two branches a pole of order one, provided that the residues sum up to zero. It is easy to see that ω_C is always a line bundle (as opposed to Ω_C) and that its degree is $2g(C)-2$. It is ample precisely when C is stable and in that case $\omega_C^{\otimes k}$ is very ample for $k \geq 3$. (The name dualizing sheaf has to do with the fact that it governs Serre duality. But that property is of no concern to us; what matters here is that every stable curve comes naturally with an ample line bundle.)

If C is stable of genus g, then a small computation shows that for $k \geq 2$, $h^0(\omega_C^{\otimes k}) = (2k-1)(g-1)$. Let us fix for each $k \geq 2$ a complex vector space V_k of dimension $d_k := (2k-1)(g-1)$ $(k \geq 2)$. Choose $k \geq 3$. Since $\omega_C^{\otimes k}$ is very ample, we have an embedding of C in $\mathbb{P}(H^0(C, \omega_C^{\otimes k})^*)$. The choice of an isomorphism $\sigma : H^0(C, \omega_C^{\otimes k})^* \cong V_k$ allows us identify C with a curve $C_\sigma \subset \mathbb{P}(V_k)$. It is a standard result of projective geometry that for m large enough,

1. the degree m hypersurfaces in $\mathbb{P}(V_k)$ induce on C_σ a complete linear system,
2. C_σ is an intersection of degree m hypersurfaces.

In other words, the natural map

$$\mathrm{Sym}^m(V_k^*) \cong \mathrm{Sym}^m H^0(C, \omega_C^{\otimes k}) \to H^0(C, \omega_C^{\otimes mk})$$

is surjective (property 1) and its kernel defines C_σ (property 2). So the image $W_\sigma \subset \mathrm{Sym}^m V_k$ of the dual of this map is of dimension d_{mk} and determines C_σ. Nothing is lost if we take the d_{mk}th exterior power of W_σ and regard it as a point of $\mathbb{P}(\wedge^{d_{mk}} \mathrm{Sym}^m V_k)$. Now let $X_{k,m}$ be the set of points in $\mathbb{P}(\wedge^{d_{mk}} \mathrm{Sym}^m V_k)$ that we obtain by letting C run over all the stable genus g curves and σ over all choices of isomorphism. This is a (not necessarily closed) subvariety that is invariant under the obvious $SL(V_k)$-action on $\mathbb{P}(\wedge^{d_{mk}} \mathrm{Sym}^m V_k)$. One can show that $SL(V_k)$ acts properly on $X_{k,m}$. It is clear from the construction that as a set, $SL(V_k)\backslash X_{k,m}$ may be identified with $\overline{\mathcal{M}}_g$. The fundamental result is

Theorem 5.1 (Gieseker). *For k and m sufficiently large, the semistable locus of the closure of $X_{k,m}$ in $\mathbb{P}(\wedge^{d_{mk}} \mathrm{Sym}^m V_k)$ is $X_{k,m}$ itself.*

Corollary 5.2. *The set $\overline{\mathcal{M}}_g$ is in a natural way a projective variety containing $\mathcal{M}_g$ as an open dense subvariety*

In particular, $\mathcal{M}_g$ acquires a quasi-projective structure. As one may expect, the structure of projective variety $\overline{\mathcal{M}}_g$ is compatible with the analytic structure defined before. Incidentally, geometric invariant theory allows us also put the orbifold structure on $\overline{\mathcal{M}}_g$, but we shall not discuss that here.

6. Pointed stable curves

It is quite natural (and very worthwhile) to extend the preceding to the case of pointed curves. If n is a nonnegative integer, then an *n-pointed curve* is a curve C together with n numbered points $x_1, \dots, x_n$ on its smooth part C_{reg}. If $(C; x_1, \dots, x_n)$ is an n-pointed smooth projective genus g curve, then its automorphism group is finite unless $2g-2+n \leq 0$ (so the exceptions are $(g,n) = (0,0), (0,1), (0,2), (1,0)$). We therefore always assume that $2g-2+n > 0$. In much the same way as for $\mathcal{M}_g$ one shows that the set of isomorphism classes $\mathcal{M}_{g,n}$ of n-pointed smooth projective genus g curves has the structure of a smooth orbifold of dimension $3g-3+n$.

Just as we did for $\mathcal{M}_g$, we compactify $\mathcal{M}_{g,n}$ by allowing mild degenerations. The relevant definition is as follows:

An n-pointed curve $(C; x_1, \dots, x_n)$ is said to be *stable* if C is a nodal curve C such that

- the euler characteristic of every connected component of $C_{\mathrm{reg}} - \{x_1, \dots, x_n\}$ is negative,

a condition that is equivalent to each of the following ones:

- $2g - 2 + n > 0$ (where g is the genus defined as before) and and C_{reg} has no connected component isomorphic to $\mathbb{P}^1 - \{\infty\}$ or $\mathbb{P}^1 - \{0, \infty\}$,
- $\mathrm{Aut}(C; x_1, \dots, x_n)$ is finite,
- $(C; x_1, \dots, x_n)$ has no infinitesimal automorphisms.

We will also refer to a stable n-pointed genus g curve as a *stable curve of type* (g, n). The underlying topology is obtained as follows: fix $p_1, \dots, p_n$ distinct points of our surface S_g and choose on $S_g - \{p_1, \dots, p_n\}$ a finite collection of embedded circles $(\delta_e)_{e \in E}$ in distinct isotopy classes relative to $S_g - \{p_1, \dots, p_n\}$ such that none of these is bounds a disk on S_g containing at most one p_i, and contract each of these circles. There is a more combinatorial way of describing the topological type that we will use later. It is given by a finite connected graph Γ that may have multiple bonds and for which we allow *loose ends* (that is, edges of which only one end is attached to a vertex, some call them legs). We need the additional data consisting of

- for each vertex a nonnegative integer g_v and
- a numbering of the loose ends by $\{1, 2, \dots, n\}$.

We say that these date define a *stable graph* if

- for every vertex v we have $2g_v - 2 + \deg(v) > 0$.

We define the *genus* by $g(\Gamma) := \sum_v g_v + b_1(\Gamma)$, and we call the pair $(g(\Gamma), n)$ the *type* of the stable graph. The receipe for assigning a stable graph of type (g, n) to $(S_g; p_1, \dots, p_n; (\delta_e)_{e \in E})$ is as follows: the vertex set is the set of connected components of $S_g - \{p_1, \dots, p_n\} - \cup_e \delta_e$, the set of bonds is indexed by E: the two sides of δ_e define one or two connected components and we insert a bond between the corresponding vertices (so this might be a loop), and we attach the ith loose end to the vertex v if the corresponding connected component contains p_i. You should check that the topological type is faithfully represented in this way. Note that the stable graph of a smooth curve of type (g, n) is like a star: n loose ends attached to a single vertex of weight g.

Let $\overline{\mathcal{M}}_{g,n}$ denote the set of isomorphism classes of stable curves of type (g, n). We have the expected theorem:

Theorem 6.1 (Knudsen-Mumford). *The universal deformations of stable curves of type (g, n) put a complex-analytic orbifold structure on $\overline{\mathcal{M}}_{g,n}$ of dimension $3g - 3 + n$. The space $\overline{\mathcal{M}}_{g,n}$ is compact and the locus $\partial\mathcal{M}_{g,n}$ parametrizing singular curves is a normal crossing divisor in the orbifold sense.*

The construction of $\overline{\mathcal{M}}_{g,n}$ can also obtained by means of geometric invariant theory, which implies that it is a projective orbifold. (The role of the dualizing sheaf is taken by $\omega_C(x_1 + \cdots + x_n)$; details can be found in a forthcoming sequel to [1].) The irreducible components of the boundary $\partial\mathcal{M}_{g,n}$ are in bijective correspondence with the embedded circles in $S - \{p_1, \dots, p_n\}$ given up to orientation preserving diffeomorphism. These are:

Δ_0: C is irreducible or

$\Delta_{\{(g',I'),(g'',I'')\}}$: C is a one point union of smooth curves of genera g' and g'' with the former containing the points x_i with $i \in I'$ and the latter the points indexed by I'' (so $\{I', I''\}$ is a partition of $\{1, \dots, n\}$). We allow g' to be zero. provided that $|I'| \geq 2$ and similarly for g''.

6.1. **The universal stable curve.** Let $(C; x_1, \dots, x_n)$ be a stable curve of type (g, n). Let us show that any $x \in C$ determines a stable curve $(\tilde{C}; \tilde{x}_1, \dots, \tilde{x}_{n+1})$ of type $(g, n+1)$.

- If $x \in C_{\text{reg}} - \{x_1, \dots, x_n\}$, then take $(\tilde{C}; \tilde{x}_1, \dots, \tilde{x}_{n+1}) = (C; x_1, \dots, x_n, x)$.
- If $x = x_i$ for some i, we let $\tilde{C}$ be the disjoint union of C and $\mathbb{P}^1$ with the points x_i and ∞ indentified. We let $\tilde{x}_i = 1 \in \mathbb{P}^1$ and $\tilde{x}_{n+1} = 0 \in \mathbb{P}^1$; whereas for $j \neq i, n+1$, $\tilde{x}_j = x_j$, viewed as a point of $\tilde{C}$. We denote this $(n+1)$-pointed curve by $\sigma_i(C; x_1, \dots, x_n)$.
- If $x \in C_{\text{sing}}$, then $\tilde{C}$ is gotten by separating the branches of C in x (i.e., we normalize C in this point only) and by putting back a copy of $\mathbb{P}^1$ with $\{0, \infty\}$ identified with the preimage of x. Then $\tilde{x}_{n+1} = 1 \in \mathbb{P}^1$ and for $i \leq n$, $\tilde{x}_i = x_i$, viewed as a point of $\tilde{C}$.

We thus have defined a map $C \to \overline{\mathcal{M}}_{g,n+1}$ that maps x_i to $\sigma_i((C; x_1, \dots, x_n)$.

There is also a converse construction: given stable curve $(\tilde{C}; \tilde{x}_1, \dots, \tilde{x}_{n+1})$ of type $(g, n+1)$, then we can associate to it a a stable curve $(C; x_1, \dots, x_n)$ of type (g, n) basically by forgetting $\tilde{x}_{n+1}$; this yields a stable pointed curve unless $\tilde{x}_{n+1}$ lies on a smooth rational component which has only two other special points. Let $\tilde{C}$ be obtained by contracting this component and let x_i be the image of $\tilde{x}_i$ $(i \leq n)$. This defines a map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$. Notice that the map $C \to \overline{\mathcal{M}}_{g,n+1}$ defined above parametrizes the fiber of π over the point defined by $(C; x_1, \dots, x_n)$.

Proposition 6.2. *The map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is morphism and so are its sections $\sigma_1, \dots, \sigma_n$. The fiber of π over the point defined by $(C; x_1, \dots, x_n)$ can be identified with the quotient of C by* $\mathrm{Aut}(C; x_1, \dots, x_n)$.

This Proposition says that in a sense the projection π with its n sections defines the universal stable curve of type (g, n). For this reason we often refer to $\mathcal{M}_{g,1}$ as the *universal smooth genus g curve* $(g \geq 2)$ and denote it by $\mathcal{C}_g$. Likewise $\overline{\mathcal{C}}_g := \overline{\mathcal{M}}_{g,1}$ is the *universal stable genus g curve.*

6.2. **Stratification of $\overline{\mathcal{M}}_{g,n}$.** The normal crossing boundary $\partial \mathcal{M}_{g,n}$ determines a stratification of $\overline{\mathcal{M}}_{g,n}$ in an obvious way: a *stratum* is by definition a connected component of the locus of points of $\overline{\mathcal{M}}_{g,n}$ where the number of local branches of $\partial \mathcal{M}_{g,n}$ at that point is equal to fixed number. That number may be zero, so that $\mathcal{M}_{g,n}$ is a stratum. It is clear that the strata decompose $\overline{\mathcal{M}}_{g,n}$ into subvarieties. A stratum of codimension k parametrizes stable curves of type (g, n) with fixed topological type (and k singular points). You may check that distinct strata correspond to distinct topological types.

We next show that the closure of every stratum is naturally covered by a product of moduli spaces of stable curves. Let Y be a stratum. If $(C; x_1, \dots, x_n)$ represents a point of Y, then consider the normalization $n : \hat{C} \to C$ and the preimage of the set of special points $\hat{X} := n^{-1}(C_{\text{sing}} \cup \{x_1, \dots, x_n\})$. The connected components of the pair $(\hat{C}, \hat{X})$ are stable curves, at least after suitably (re)numbering the points of $\hat{X}$, for every component of $\hat{C} - \hat{X}$ maps homeomorphically to a component of $C_{\text{reg}} - \{x_1, \dots, x_n\}$, hence has negative euler characteristic. So if the types of the stable curves are $(g_i, n_i)_{i \in I}$, then we find an element of $\prod_{i \in I} \mathcal{M}_{g_i, n_i}$. Conversely if we are given a finite collection of smooth curves of type $(g_i, n_i)_{i \in I}$, then by identifying some pairs of points we find a stable curve of type (g, n). In terms of stable graphs: the first procedure amounts to cutting all the bonds in the middle of the stable graph associated to Y so that we end up with a finite set of stars, whereas the second builds out of stars a stable graph by identifying certain pairs of edges. This receipe defines a map $\prod_i \mathcal{M}_{g_i, n_i} \to Y$ of which it is not difficult to see that it is a finite surjective morphism. The same receipe can be used to glue stable (not necessarily smooth) curves of type (g_i, n_i), so that the map extends to

$$f : \prod_i \overline{\mathcal{M}}_{g_i, n_i} \to \overline{\mathcal{M}}_{g,n}$$

It is easy to verify that this is a finite morphism with image the closure of Y. (It is in fact an orbifold cover of that closure.) The section σ_i of the universal curve is a special case of this construction.

7. Tautological classes

Throughout the discussion that follows we take rational coefficients for our cohomology groups. One reason is that the space underlying an orbifold is a rational homology manifold, so that it satisfies Poincaré duality when oriented (which is always the case in the complex-analytic setting). We shall be dealing with complex quasi-projective orbifolds and the coholomology classes that we consider happen to have Poincaré duals that are $\mathbb{Q}$-linear combinations of closed subvarieties. Such classes are called *algebraic classes*.

7.1. **The Witten classes.** Given a stable curve $(C; x_1, \dots, x_n)$, then for a fixed $i \in \{1, \dots, n\}$, we may associate to it the one dimensional complex vector space $T^*_{x_i} C$. This generalizes to families: if $(\mathcal{C} \to B : (\xi_i : B \to \mathcal{C})_{i=1}^n)$ is a family of stable curves, then for a fixed i, the conormal bundle of ξ_i defines a line bundle over B. In the universal example $(\overline{\mathcal{M}}_{g,n+1} \to \mathcal{M}_{g,n}; \sigma_1, \dots, \sigma_n)$ this produces an orbifold line bundle over $\mathcal{M}_{g,n}$. Its first Chern class, denoted $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n})$, is called the *ith Witten class*. Since the notation wants to travel lightly, it is a little ambiguous. For example, it is not true that the image of ψ_i in $H^2(\overline{\mathcal{M}}_{g,n+1}$ is the ith Witten class of $\overline{\mathcal{M}}_{g,n+1}$. The euler class of the normal bundle of a parametrization $f : \prod_i \overline{\mathcal{M}}_{g_i, n_i} \to \overline{\mathcal{M}}_{g,n}$ of a closed stratum is a product of Witten classes of the factors. This illustrates a point we are going to make, namely, that all classes of interest appear to be obtainable from the Witten classes.

7.2. **The Mumford classes.** Mumford defined these classes for $\overline{\mathcal{M}}_g$ before the Witten classes were considered; Arbarello-Cornalba used the Witten classes to extend Mumford's definition to the pointed case. It goes as follows: if $\tilde{\psi}_{n+1}$ denotes the last Witten class on $H^2(\overline{\mathcal{M}}_{g,n+1})$, and r is a nonnegative integer, then the rth Mumford class is

$$\kappa_r := \pi_*(\tilde{\psi}_{n+1}^{r+1}) \in H^{2r}(\overline{\mathcal{M}}_{g,n}).$$

(It is an interesting exercise to check that $\kappa_0 = 2g - 2 + n$.) Arbarello and Cornalba showed that κ_1 is the first Chern class of an ample line bundle. They also proved that $H^2(\overline{\mathcal{M}}_{g,n})$ is generated by $\psi_1, \dots, \psi_n, \kappa_1$ and the Poincaré duals of the irreducible components of the boundary $\partial\overline{\mathcal{M}}_{g,n}$ and that these classes form a basis when $g \geq 3$.

7.3. **The tautological algebra.** It is convenient to make an auxiliary definition first: define the *basic algebra* $\mathcal{B}(\overline{\mathcal{M}}_{g,n})$ of $\overline{\mathcal{M}}_{g,n}$ as the subalgebra of $H^\bullet(\overline{\mathcal{M}}_{g,n})$ generated by the Witten classes $(\psi_j)_j$ and the Mumford classes $(\kappa_r)_{r\geq 0}$. Then the *tautological algebra* of $\overline{\mathcal{M}}_{g,n}$, $\mathcal{R}(\overline{\mathcal{M}}_{g,n})$, is by definition the subalgebra of $H^\bullet(\overline{\mathcal{M}}_{g,n})$ generated by the direct images $f_*(\otimes_i \mathcal{B}(\overline{\mathcal{M}}_{g_i,n_i}) \subset H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n})$, where the maps $f : \prod_i \overline{\mathcal{M}}_{g_i,n_i} \to \overline{\mathcal{M}}_{g,n}$ run over the parametrizations of the strata. Since the tautological algebra is made of algebraic classes we grade it by half the cohomolocical degree: $\mathcal{R}^k(\overline{\mathcal{M}}_{g,n}) \subset H^{2k}(\overline{\mathcal{M}}_{g,n})$. It is clear that the tautological algebra and the basic algebra have the same restriction to $\mathcal{M}_{g,n}$; we denote that restriction by $\mathcal{R}(\mathcal{M}_{g,n})$ and call the *tautological algebra* of $\mathcal{M}_{g,n}$.

It is remarkable that all the known algebraic classes on $\overline{\mathcal{M}}_{g,n}$ are in the tautological subalgebra. We illustrate this with two examples.

Example 7.1 (The Hodge bundle). If C is a stable curve of genus $g \geq 2$, then $H^0(C, \omega_C)$ is a g-dimensional vector space. On the universal example this gives a rank g vector bundle $\mathbb{E}$ over $\overline{\mathcal{M}}_g$ called the *Hodge bundle*. (Since $H^0(C, \omega_C)$ only depends on the (generalized) Jacobian of C, $\mathbb{E}$ is the pull-back of a bundle that is naturally defined on a certain compactification of the moduli space of principally polarized abelian varieties of dimension g.) Mumford [3] expressed the Chern class $\lambda_i := c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_g)$ as an element of $\mathcal{R}^i(\overline{\mathcal{M}}_g)$.

Example 7.2 (The Weierstraß loci). Suppose C is a smooth, connected projective curve C of genus $g \geq 2$, $p \in C$, and an l a positive integer. It is easy to see that the following conditions are equivalent:

- The linear system $|l(p)|$ is of dimension ≥ 1.
- There exists a nonconstant regular function on $C - \{p\}$ that has in p with a pole of order $\leq l$.
- There exists a finite morphism $C \to \mathbb{P}^1$ of degree $\leq l$ that is totally ramified in p.

By Riemann-Roch these conditions are always fulfilled if $l \geq g+1$. If $l = 2$, then the morphism f appearing in the last item must have degree 2 so that C is hyperelliptic and p is a Weierstraß point.

These equivalent conditions define a closed subvariety W_l^* of the universal smooth curve of genus g, $\mathcal{C}_g$. Arbarello, who introduced these varieties, noted that W_l^* is irreducible of codimension $g + 1 - l$ in $\mathcal{C}_g$. Mumford expressed the corresponding class in $H^{2(g+1-l)}(\mathcal{C}_g)$ as an element of $\mathcal{R}^{g+1-l}(\mathcal{C}_g)$.

The validity of Grothendieck's standard conjectures implies that the algebraic classes on $\overline{\mathcal{M}}_{g,n}$ make up a nondegenerate subspace of $H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n})$ with respect to the intersection product. Since we do not know any algebraic class that is not tautological, we ask:

Question 7.3. Does $\mathcal{R}(\overline{\mathcal{M}}_{g,n})$ satisfy Poincaré duality?

Since all tautological classes originate from Witten classes, this question could in principle be answered for a given pair (g_0, n_0) if we would know all the intersection numbers

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \in \mathbb{Q},$$

(where it is of course understood that this number is zero if the degree $k_1 + \cdots + k_n$ of the integrand fails to equal $3g - 3 + n$) for all (g, n) with $2g + n \leq 2g_0 + n_0$. A marvelous conjecture of Witten (which is a conjecture no longer) predicts the values of these numbers. We shall state it in a form that exhibits the algebro-geometric content best (this formulation is due to Dijkgraaf-E. Verlinde-H. Verlinde). For this purpose it is convenient to renormalize the intersection numbers as follows:

$$[\tau_{k_1}\tau_{k_2}\cdots\tau_{k_n}]_g := (2k_1+1)!!(2k_2+1)!!\cdots(2k_n+1)!! \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\cdots\psi_n^{k_n},$$

where $(2k+1)!! = 1.3.5.\cdots.(2k+1)$. We use these to form the series in the variables $t_0, t_1, t_2, \dots$:

$$F_g := \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k_1\geq 0,,k_2\geq 0,\dots,k_n\geq 0} [\tau_{k_1}\tau_{k_2}\cdots\tau_{k_n}]_g t_{k_1}t_{k_2}\cdots t_{k_n}.$$

It is invariant under permutation of variables. The Witten conjectures assert that these polynomials satisfy a series of differential equations indexed by the integers ≥ -1. For index -1 this is the *string equation*:

$$\frac{\partial F_g}{\partial t_0} = \sum_{m\geq 1} (2m+1)t_m \frac{\partial F_g}{\partial t_{m-1}} + \tfrac{1}{2}\delta_{0,g}t_0^2,$$

for index 0 the *dilaton equation*:

$$\frac{\partial F_g}{\partial t_1} = \sum_{m\geq 0} (2m+1)t_m \frac{\partial F_g}{\partial t_m} + \tfrac{1}{8}\delta_{1,g}t_0^2$$

and for $k \geq 1$ we get:

$$\begin{aligned}\frac{\partial F_g}{\partial t_{k+1}} &= \sum_{m\geq 1}(2m+1)t_m\frac{\partial F_g}{\partial t_{m+k}}\\ &\quad + \tfrac{1}{2}\sum_{m'+m''=k-1}\frac{\partial F_{g-1}}{\partial t_{m'}\partial t_{m''}}\\ &\quad + \tfrac{1}{2}\sum_{m'+m''=k-1}\sum_{g'+g''=g}\frac{\partial F_{g'}}{\partial t_{m'}}\frac{\partial F_{g''}}{\partial t_{m''}}.\end{aligned}$$

You may verify that these equations determine the functions F_g completely (note that F_g has no constant term). The string equation and the dilaton equation involve a single genus only and were verified by Witten using standard arguments from algebriac geometry. But the equations for $k \geq 1$ were proved in an entirely different manner: Kontsevich gave an amazing proof based on a triangulation of Teichmüller space on which the intersection numbers appear as integrals of explicitly given differential forms. Yet there are reasons to wish for a proof within the realm of algebraic geometry. The form of the equations is suggestive in this respect: the first line involves the $\overline{\mathcal{M}}_{g,n}$, but the second and third seem to be about intersection numbers formed on irreducible components of $\partial\mathcal{M}g, n$ (Δ_0 and the $\Delta_{\{(g',I'),(g'',I'')\}}$ respectively). Consider this a challenge.

7.4. **Faber's conjectures.** These concern the structure of $\mathcal{R}(\mathcal{M}_g)$. From the definition it is clear that these are generated by the restrictions of the Mumford classes. Let us for convenience denote these classes κ_r instead of $\kappa_r|\mathcal{M}_g$. Faber made the essential part of his conjectures around 1993. We shall not state them in their most precise form (we refer to [4] for that).

1. $\mathcal{R}_g$ is zero in degree $\geq g-1$ and is of dimension one in degree $g-2$ and the cup product
$$\mathcal{R}^i(\mathcal{M})g) \times \mathcal{R}^{g-2-i}(\mathcal{M}_g) \to \mathcal{R}^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$$
is nondegenerate.
2. The classes $\kappa_1, \dots, \kappa_{[g/3]}$ generate $\mathcal{R}(\mathcal{M}_g)$ and satisfy no polynomial relation in degree $\leq [g/3]$.

Let us review the status of these conjectures. First, there is the evidence provided Faber himself: he checked his conjectures up to genus 15.

As to conjecture 1: the vanishing assertion was proved in a paper of mine where is was also shown that $\dim \mathcal{R}^{g-2}(\mathcal{M}_g) \leq 1$. Subsequently Faber proved that the dimension is in fact equal to one. The rest of conjecture 1 remains open. Conjecture 2 now seems settled: Harer had shown around the time that Faber made his conjectures that the kappa classes have no polynomial relations in degree $\leq [g/3]$ and Morita has recently announced that $\kappa_1, \dots, \kappa_{[g/3]}$ generate $\mathcal{R}(\mathcal{M}_g)$.

Let me close with saying a bit more about Faber's nonvanishing proof. He observes that the class $\lambda_g\lambda_{g-1} \in \mathcal{R}^{2g-1}(\overline{\mathcal{M}}_g)$ restricts to zero on the boundary $\partial\mathcal{M}_g$. This implies that for every $u \in \mathcal{R}^{g-2}(\overline{\mathcal{M}}_g)$, the intersection product $\int_{\overline{\mathcal{M}}_g} \lambda_g\lambda_{g-1}u$ only depends on $u|\mathcal{M}_g$. So this defines a 'trace' $t : \mathcal{R}(\mathcal{M}_g) \to \mathbb{Q}$. Faber proves that this trace is nonzero on κ_{g-2}. The unproven part of the conjecture can be phrased as saying that the associated form $(u, v) \in \mathcal{R}(\mathcal{M}_g) \times \mathcal{R}(\mathcal{M}_g) \mapsto t(uv)$ is nondegerate. Faber has also an explicit proposal for the value of the trace on any monomial in the Mumford classes.

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