SMOOTH DELIGNE-MUMFORD COMPACTIFICATIONS BY MEANS OF PRYM LEVEL STRUCTURES

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Abstract. We show that the Deligne–Mumford compactification of the moduli space of smooth complex curves of genus g admits a smooth Galois covering whose general point classifies curves with a level structure on their universal Prym cover. We also correct Brylinski's proof that the Teichmüller group ramifies universally with respect to the Deligne–Mumford compactification.

Introduction

Given a compact oriented connected two-manifold S (a *surface*, for short) of genus g, then a *universal Prym cover* of S is a connected unramified abelian Galois covering $\tilde{S} \to S$ corresponding to the normal subgroup of $\pi_1(S,x)$ generated by the squares in $\pi_1(S,x)$ (where $x \in S$ is some base point). Its Galois group G is canonically isomorphic to $H_1(S;\mathbb{Z}/2)$, but we shall write the group law of G multiplicatively. Of course all such covers are mutually isomorphic, though not canonically.

Let us fix S and a universal Prym cover $\tilde{S} \to S$. If $f : S \to S$ is an orientation preserving homeorphism, then f will lift to an orientation preserving homeomorphism $\tilde{f} : \tilde{S} \to \tilde{S}$. This lift commutes with the covering transformations, and the possible lifts form a G-coset. Fix an integer $n > 0$. The condition that $\tilde{f}$ acts as an element of G on $H_1(\tilde{S};\mathbb{Z}/n)$ is clearly independent of the lift. Homeomorphisms f with this property define a normal subgroup $\Gamma_{S,\binom{n}{2}}$ of the group Γ_S of mapping classes of S. Clearly, the quotient group $\Gamma[{n \atop 2}] := \Gamma_S/\Gamma_{S,\binom{n}{2}}$ is a subquotient of $\operatorname{Aut} H_1(\tilde{S};\mathbb{Z}/n)$, and hence finite.

If X is a complete smooth algebraic curve of genus g, then a *Prym level n-structure* on X is defined by an orientation preserving homeomorphism $f : S \to X$, with the understanding that two such homeomorphisms f, f' define the same structure iff $f^{-1}f'$ belongs to $\Gamma_{S,\binom{n}{2}}$. The curves with such a structure define a moduli variety $\mathcal{M}_g[{n \atop 2}]$ which lies as a finite Galois cover over $\mathcal{M}_g$ with Galois group $\Gamma[{n \atop 2}]$. It has a Deligne–Mumford compactification $\overline{\mathcal{M}}_g[{n \atop 2}]$ which by definition is the normalization of $\mathcal{M}_g[{n \atop 2}]$ in the Deligne–Mumford compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$. We prove that $\overline{\mathcal{M}}_g[{n \atop 2}]$ is smooth if n is even and ≥ 6. This implies that $\overline{\mathcal{M}}_g$ is the quotient of a smooth variety by a finite group action, a fact which greatly simplifies the foundational aspects of its Chow theory, see Mumford [5], p. 281. It should be noted here that Mostafa [6], and independently, Van Geemen & Oort [3], have shown that for $g \geq 3$ the Deligne–Mumford compactification is never smooth had we used ordinary (abelian) level structures.

At the same time we show that the collection Galois covers $\overline{\mathcal{M}}_g[{n \atop 2}] \to \overline{\mathcal{M}}_g$, $n = 1, 2, \ldots$ is *universally ramified* along the Deligne–Mumford boundary. This means that for every $p \in \overline{\mathcal{M}}_g$ and every finite connected Galois covering of the analytic germ of $\overline{\mathcal{M}}_g$ at p with ramification locus contained in the Deligne–Mumford boundary, there exists an $n > 0$ and a $\tilde{p} \in \overline{\mathcal{M}}_g[{n \atop 2}]$ over p such that the map-germ $(\overline{\mathcal{M}}_g[{n \atop 2}], \tilde{p}) \to (\overline{\mathcal{M}}_g, p)$ factors through this covering. This is in fact the principal result of Brylinski [1], but Van Geemen discovered that the proof given there has a gap. (It occurs on page 327, fifth line from below, where it is falsely claimed that after an isotopy, d' and $\tilde{a}$ will not meet.) As is explained by Mumford [4], the property of universal ramification implies that $\overline{\mathcal{M}}_g[{n \atop 2}]$ is of general type when n is sufficiently divisible. Our approach differs from Brylinski's in that we make fuller use of a result of K. Baclawski.

We note here that a dyhedral level structure as defined by Brylinski determines a Prym level structure, but that the converse need not hold. We explain the relation between the two notions in section 3.

This paper has its origin in work with Dick Hain on the related problem of determining the representation of the mapping class group of a surface on the homology of its universal Prym cover. It was then that Bert van Geemen drew our attention to Brylinski's paper. It is a pleasure to acknowledge the lively and stimulating discussions I had with both of them. I also thank the referee for the careful reading of the manuscript and the useful comments.

1. Topological discussion

Suppose a smooth curve X degenerates into a stable curve. Each singular point of the stable curve determines an embedded circle on X which is unique up to isotopy: this is a *vanishing circle* of the degeneration. These vanishing circles may be taken to be disjoint, and if that is the case, then topologically the stable curve is reconstructed from X by collapsing each circle to a point. The stability of the degenerate curve implies that every connected component of the complement of the union of these circles in X has negative Euler characteristic, or equivalently, that none of these circles bounds a disk and no two of them bound a cylinder. In this section we will investigate the topological aspects that these data give rise to.

So let now S be a surface of genus $g \geq 2$ equipped with a collection of disjoint circles as above: none of these circles bounds a disk, and no two of them bound a cylinder. To this there is associated a graph Γ: a vertex corresponds to a connected component of the complement of the union of circles, and every circle of our collection determines an edge joining the vertices that are labeled by the components to which the circle is adjacent. We shall index the circles by the edges of Γ: an edge e of Γ determines a circle C_e on S. Clearly, Γ is connected. It is obtained as a quotient space of S as follows: choose a tubular neighborhood N_e of C_e whose closure is homeomorphic to a closed cylinder and such that the closures of N_e's are pairwise disjoint. Choose a trivialisation $N_e \cong C_e \times e$, and let $p : S \to \Gamma$ be the continuous map which on N_e is the projection onto e and which collapses every connected component of $S - \cup_e N_e$ onto the corresponding vertex.

Brylinski shows ([1], Prop. 1) that $p_* : H_1(S) \to H_1(\Gamma)$ is surjective. (Unless otherwise indicated coefficients are integral.) This implies that $p^* : H^1(\Gamma) \to H^1(S)$

is an isomorphism onto a *primitive* isotropic sublattice of $H^1(S)$. We shall identify $H^1(\Gamma)$ with a sublattice of $H^1(S)$ via this map.

An orientation $\vec{e}$ of an edge e makes it a 1-cocycle on Γ and therefore yields a class $[\vec{e}] \in H^1(\Gamma)$. In fact, $\vec{e}$ determines a generator of $H^1(\Gamma, \Gamma - e)$, and $[\vec{e}]$ is the image of that generator in $H^1(\Gamma)$. From the cohomology exact sequence of the pair $(\Gamma, \Gamma - e)$ it immediately follows that e disconnects Γ if and only if $[\vec{e}] = 0$. A similar argument shows that two distinct edges which do not disconnect Γ by themselves, together disconnect Γ if and only if after suitable orientations they define the same class in $H^1(\Gamma)$. The set of the classes in $H^1(\Gamma)$ of the form $[\vec{e}]$ is *unimodular*, meaning that every linearly independent subset of it generates a primitive sublattice ([1], Lemme 8).

Considered as an element of $H^1(S)$, $\vec{e}$ determines an orientation of C_e, such that its class in $H_1(S)$ is Poincaré dual to $[\vec{e}] \in H^1(S)$. We therefore denote that class by $[\vec{e}]^\vee \in H_1(S)$. Let τ_e be a Dehn twist about C_e with support in N_e. The τ_e's commute with each other. So if $e(\Gamma)$ denotes the set of edges of Γ, then these Dehn twists determine a homomorphism τ from $\mathbb{Z}^{e(\Gamma)}$ to the mapping class group of S. We shall write the group law of $\mathbb{Z}^{e(\Gamma)}$ multiplicatively, so that a typical element is given by a Laurent monomial in the e's. The action of τ_e on $H_1(S)$ is given by the Picard–Lefschetz transformation

$$T_e : x \mapsto x + (x, [\vec{e}]^\vee)[\vec{e}]^\vee.$$

Now consider the 'integral Lie algebra' of $Sp(H_1(S))$, i.e., the set of endomorphisms σ of $H_1(S)$ satisfying $(\sigma x, y) + (x, \sigma y) = 0$. The symplectic form defines an isomorphism of lattices $H^1(S) \otimes H^1(S) \cong \mathrm{End}(H_1(S))$, $a \otimes b \mapsto (-, a^\vee)b^\vee$ and under this isomorphism the Lie algebra gets identified with the lattice of symmetric tensors $\mathrm{Sym}^2 H^1(S)$ in $H^1(S) \otimes H^1(S)$. It is clear that $\mathrm{Sym}^2 H^1(\Gamma)$ is a primitive sublattice of $\mathrm{Sym}^2 H^1(S)$, and is abelian as a subLie algebra.

In this way $\log T_e : x \mapsto (x, [\vec{e}]^\vee)[\vec{e}]^\vee$ corresponds to $[\vec{e}] \otimes [\vec{e}] \in \mathrm{Sym}^2 H^1(\Gamma)$. To emphasize its quadratic character and the independence of the orientation, we write $[e^2]$ for this element. So in this description S no longer enters.

Proposition 5 of Brylinski's paper amounts to the statement that the non-zero elements of $\mathrm{Sym}^2 H^1(\Gamma)$ of the form $[e^2]$ are linearly independent. This follows from the unimodularity of the set of edge classes in $H^1(\Gamma)$ and the following

Lemma. *(Baclawski) Let L be a free $\mathbb{Z}$-module of finite rank and E a unimodular subset of L with the property that $E \cap (-E) = \emptyset$. Then the tensors $\{e \otimes e\}_{e \in E}$ are mutually distinct, make up a linearly independent subset of $\mathrm{Sym}^2(L)$, and generate a primitive sublattice of $\mathrm{Sym}^2(L)$.*

This lemma is equivalent to the theorem stated by Baclawski in the appendix of Brylinski's paper, except for the primitivity assertion. But inspection of his proof reveals that he proves this as well. So we have a corresponding strengthening of Brylinski's proposition 5:

Proposition 1. *The set of non-zero elements of $\mathrm{Sym}^2 H^1(\Gamma)$ of the form $[e^2]$ is linearly independent and spans a primitive sublattice of $\mathrm{Sym}^2 H^1(\Gamma)$.*

Let $\tilde{S} \to S$ be a universal Prym covering with Galois group G. The circles on $\tilde{S}$ lying over the circles C_e determine a graph $\tilde{\Gamma}$. The abelian group G acts on $\tilde{\Gamma}$, and the orbit space $G\backslash\tilde\Gamma$ may be identified with Γ. The inertia group G_e of the edges over the edge e may be identified with the image of $H_1(C_e;\mathbb{Z}/2)\to H_1(S;\mathbb{Z}/2)\cong G$. This image is of course generated by the mod 2 reduction of $[\vec e]^\vee$.

If e disconnects Γ, then $[\vec e]^\vee=0$. Hence G_e is trivial and G acts freely on its pre-image in $\tilde\Gamma$. Moreover, the pre-image of C_e in $\tilde S$ consists of 2^{2g} circles, each of which maps homeomorphically onto C_e and the Dehn twist τ_e lifts canonically to $\tilde\tau_e=\prod_{\tilde e/e}\tau_{\tilde e}$. (Here $\tilde e/e$ means that $\tilde e$ is an edge of $\tilde\Gamma$ over e.) Notice that this lift commutes with the covering transformations. If $\tilde T_e$ denotes its action on $H_1(\tilde S)$, then

$$\log\tilde T_e=\sum_{\tilde e/e}[\tilde e^2]\in \operatorname{Sym}^2 H^1(\tilde\Gamma).$$

If e does not disconnect Γ, then the mod 2 reduction of $[\vec e]^\vee$ in nonzero. So G_e has order two and we find 2^{2g-1} circles in $\tilde S$ lying over C_e. If $\tilde C_e$ is one of them, then $\tilde C_e\to C_e$ has degree two and the Dehn twist τ_e lifts near $\tilde C_e$ to a 'half Dehn twist', but we have a choice which component of a tubular neighborhood boundary of $\tilde C_e$ we wish to leave pointwise fixed. The two choices are interchanged by a covering transformation. Likewise, the global lifts all commute with the covering transformations and are in the same G-coset. We choose one such lift $\tilde\tau_e$ and denote by $\tilde T_e$ the corresponding automorphism of $H_1(\tilde S)$. The square of $\tilde\tau_e$ is equal to $\prod_{\tilde e/e}\tau_{\tilde e}$, and is therefore independent of the choice. In particular,

$$\log(\tilde T_e^2)=\sum_{\tilde e/e}[\tilde e^2]\in \operatorname{Sym}^2 H^1(\tilde\Gamma).$$

Notice that the automorphisms $\tilde T_e$, $e\in e(\Gamma)$, mutually commute.

Proposition 2. *No pair of edges of $\tilde\Gamma$ disconnects $\tilde\Gamma$.*

We postpone the proof for a moment.

Corollary 1. *The homomorphism $\mathbb{Z}^{e(\Gamma)}\to \mathrm{Sym}^2 H^1(\tilde\Gamma)$ defined by $e\mapsto\sum_{\tilde e/e}[\tilde e^2]$ is injective and its image is a primitive sublattice of $\mathrm{Sym}^2H^1(\tilde\Gamma)$.*

Proof. It follows from prop. 2 that two different edges of $\tilde\Gamma$ cannot give (after orienting them) opposite elements in $H^1(\tilde\Gamma)$. Now apply prop. 1.

Let $e_0(\Gamma)$ be the set of edges of Γ that disconnect Γ, and $e_1(\Gamma)$ those that do not.

Corollary 2. *The representation of $\mathbb{Z}^{e(\Gamma)}$ on $H_1(\tilde S)$ defined by $e\in e(\Gamma)\mapsto\tilde T_e$ is faithful. If n is an even positive integer, then the kernel of the corresponding action on $H_1(\tilde S;\mathbb{Z}/n)$ is equal to $(n\mathbb{Z})^{e_0(G)}\times(2n\mathbb{Z})^{e_1(G)}$.*

Proof. Let $u:=\prod_e e^{u_e}\in\mathbb{Z}^{e(\Gamma)}$. The action of u^2 on $H_1(\tilde S)$ is unipotent and its logarithm is given by

$$\sum_{e\in e_0(\Gamma)}2u_e\sum_{\tilde e/e}[\tilde e^2]+\sum_{e\in e_1(\Gamma)}u_e\sum_{\tilde e/e}[\tilde e^2].$$

It follows from cor. 1 that this is zero only if all u_e are zero. This proves the first assertion. Now suppose that u acts trivially on $H_1(\tilde{S};\mathbb{Z}/n)$, where n is even. Then we see that n divides u_e if e is in $e_1(\Gamma)$. In particular, for such e the exponent u_e is even. Hence the action of u on $H_1(\tilde{S})$ is unipotent and its logarithm is just half the displayed expression. Again we apply cor. 1 and find that the second assertion also holds.

Proof of prop. 2. We first note that removing a circle C_e from our collection has on Γ (resp. $\tilde{\Gamma}$) the effect of collapsing e (resp. each edge lying over e). So if e and f are edges of Γ, then to prove that no two edges lying over e resp. f disconnect $\tilde{\Gamma}$, we may—and will—assume that Γ has no edges besides e and f. Then there are six cases to consider, but before we do that we observe the following: if v is a vertex of Γ corresponding to a connected component S_v of $S-C_e-C_f$, then the inertia group G_v of the pre-image of v in $\tilde{\Gamma}$ is just the image of $H_1(S_v;\mathbb{Z}/2)\to H_1(S;\mathbb{Z}/2)\cong G$.

Case 1: Γ has a single vertex v and $e=f$ is a loop. Then G_v has index two in G, and $\tilde{\Gamma}$ has two vertices that are joined by 2^{2g-1} edges. Since $g\geq 2$, removal of two such edges will not disconnect $\tilde{\Gamma}$.

Case 2: Γ has a single vertex v and e, f are distinct loops. Then G_v has index four in G, and $\tilde{\Gamma}$ has four vertices. In fact, the orbit space of $\tilde{\Gamma}$ with respect to G_v is a square. Over every edge of this square lie 2^{2g-2} edges joining the same pair of vertices. Again it follows that removal of two edges will not disconnect $\tilde{\Gamma}$.

Case 3: Γ has two vertices v_1, v_2 connected by $e=f$. If the corresponding connected components have genus g_1 resp. g_2, then $g=g_1+g_2$, G_{v_i} has order 2^{2g_i}, and $G\to G/G_{v_1}\times G/G_{v_2}$ is an isomorphism. The graph $\tilde{\Gamma}$ with its G-action is isomorphic to the join of G/G_{v_1} and G/G_{v_2}. Since $g_i\geq 1$, G/G_{v_i} has order ≥ 4. It follows that two edges of $\tilde{\Gamma}$ cannot disconnect it.

Case 4: Γ has two vertices v_1, v_2 both joined by e and f, and $e\neq f$. In this case G_{v_i} has odd order $2g_i+1\geq 3$, and these groups intersect in a group of order two and span a subgroup of G of index two. One verifies that $\tilde{G}$ is again G-equivariantly isomorphic to the join of G/G_{v_1} and G/G_{v_2}.

There remain

Case 5: Γ has two vertices connected by e and f is a loop, and

Case 6: Γ has three vertices connected by e and f.

These can be dealt with in a similar fashion, where one should keep in mind that a disconnecting edge gives rise to a join type of situation.

We shall need a slight sharpening of cor. 2.

Proposition 3. *If $n=2m$ is an even integer ≥ 6, then the kernel of the composite homomorphism $\mathbb{Z}^{e(\Gamma)}\to\Gamma_S\to\Gamma_S[\frac{n}{2}]$ is equal to $(n\mathbb{Z})^{e_0(G)}\times(2n\mathbb{Z})^{e_1(G)}$.*

Proof. Since the $\tilde{\tau}_e$'s commute with the covering transformations, we have an action of $G\times\mathbb{Z}^{e(\Gamma)}$ on $H_1(\tilde{S};\mathbb{Z}/n)$. The proposition comes then down to the assertion that for n as given, $(n\mathbb{Z})^{e_0(G)}\times(2n\mathbb{Z})^{e_1(G)}$ is the kernel of this action.

To prove this, we first note that given any $g\in G-\{1\}$, g acts on $H_1(\tilde{S},\mathbb{R})$ as an involution whose (-1)-eigenspace is nonzero (for the universal Prym covering factors through a degree two covering whose sheets are interchanged by g). It follows that $H_1(\tilde{S};\mathbb{Z}/n)$ contains a cyclic subgroup of order n on which g acts by inversion. Since $n > 4$, $(g-1)^2$ acts non-trivially on this cyclic subgroup and hence also non-trivially on $H_1(\tilde{S};\mathbb{Z}/n)$.

In view of cor. 2 it is enough to show that no involution $\neq 1$ in the image of $\mathbb{Z}^{e(\Gamma)}$ is in the image of G. The proof of cor. 2 shows that any $u = \prod_e e^{u_e} \in \mathbb{Z}^{e(\Gamma)}$ which induces an involution in $H_1(\tilde{S};\mathbb{Z}/n)$ must have u_e even if $e \in e_1(\Gamma)$. So u acts on $H_1(\tilde{S})$ as

$$x \mapsto x + \sum_{e\in e_0(\Gamma)} u_e \sum_{\tilde{e}/e} (x, [\vec{\tilde{e}}]^\vee)[\vec{\tilde{e}}]^\vee + \sum_{e\in e_1(\Gamma)} \tfrac{1}{2}u_e \sum_{\tilde{e}/e} (x, [\vec{\tilde{e}}]^\vee)[\vec{\tilde{e}}]^\vee.$$

Since the classes $[\vec{\tilde{e}}]^\vee$ span a totally isotropic sublattice of $H_1(\tilde{S})$, it follows that $(u-1)^2$ induces the zero map in $H_1(\tilde{S})$. Hence u cannot act as an element g of G unless $g = 1$. The proposition follows from cor. 2.

2. The main result

Let X_0 be a stable curve of genus g and $P_1, \dots, P_k$ its singular points. The curve X_0 admits a universal deformation $(\mathcal{X}, X_0) \to (B, 0)$, where B is a convex neighborhood of $0 \in \mathbb{C}^{3g-3}$, such that the divisor $z_i = 0$ parametrizes the curves for which the singular point P_i subsists ($i = 1, \dots, k$). We further assume B is such that there are no other singularities in the fibres, that $\mathrm{Aut}(X_0)$ acts on the family, and that no two fibres are isomorphic unless they are in the same $\mathrm{Aut}(X_0)$-orbit. Then $\mathrm{Aut}(X_0)\backslash B$ may be identified with an open subset of $\overline{\mathcal{M}}_g$.

Fix a positive integer $m \geq 3$ and let us recall from Mostafa [6] how one gets a corresponding open subset of the moduli space $\overline{\mathcal{M}}_g[m]$ of stable genus g-curves with a level m-structure. Let $\mathcal{X}' \to B'$ be the part of the deformation that parametrizes smooth curves. Choose a base point $z \in B'$, and let $f : S \to X_z$ be an orientation preserving homeomorphism. Every singular point P_i determines a non-trivial isotopy class of circles on S. These isotopy classes are distinct, and we can choose pairwise disjoint representative circles $C_1, \dots, C_k$. There is a natural isotopy class of homeomorphisms between $S - \cup_i C_i$ and the smooth part X_0' of X_0. In particular, the graph Γ attached to S and the circles C_i can be identified with the graph of the stable curve X_0. The monodromy transformation of $\mathcal{X}' \to B'$ about $z_i = 0$ can be represented by a Dehn twist about C_i. Let $\hat{B}' \to B'$ be the connected unramified covering defined by the monodromy representation $\pi_1(B', z) \to \mathrm{Aut} H_1(S;\mathbb{Z}/m)$. Then the pull-back $\hat{\mathcal{X}}' \to \hat{B}'$ of $\mathcal{X}' \to B'$ comes with a level m-structure. There is a natural homotopy class of maps $S \to X_0$ which induces a surjection $H_1(S;\mathbb{Z}/m) \to H_1(X_0;\mathbb{Z}/m)$. It is known that $\mathrm{Aut}(X_0)$ acts faithfully on $H_1(X_0;\mathbb{Z}/m)$, and hence no two fibres of $\hat{\mathcal{X}}' \to \hat{B}'$ are isomorphic as curves with a level m-structure. So the normalization $\hat{B} \to B$ of $\hat{B}' \to B'$ in B maps isomorphically to an open subset of $\overline{\mathcal{M}}_g[m]$. (The covering $\hat{B} \to B$ is dominated by the abelian Galois covering of B obtained by extracting the m-th root of z_i, whenever P_i does not disconnect X_0. We have equality only if no two C_i's define the same non-zero homology class up to sign; otherwise $\hat{B}$ is a singular quotient of this. If $g \geq 3$, this will definitely happen for certain X_0 and so $\overline{\mathcal{M}}_g[m]$ will not be smooth. A similar phenomenon occurs for $m = 2$.)

Put $n := 2m$ and let $\tilde{B} \to B$ be the connected abelian Galois covering obtained by extracting an n-th (resp. $2n$-th) root of the coordinate z_i ($i = 1, \dots, k$) if P_i disconnects (resp. does not disconnect) X_0. Notice that $\tilde{B}$ is smooth. We denote the pre-image of B' in $\tilde{B}$ by $\tilde{B}'$.

Proposition 4. *Under the assumption that n is even and ≥ 6, the pull-back of $\mathcal{X}' \to B'$ over $\tilde{B}'$ admits a Prym level n-structure and distinct fibres of this pull-back are not isomorphic under an isomorphism that preserves their Prym level structure.*

Proof. It is clear from prop. 3 that $\tilde{B}' \to B'$ is just the covering defined by the kernel of the homomorphism $\pi_1(B', z) \to \Gamma_S[\frac{n}{2}]$. This implies that the Prym level n-structure on $X_z \cong S$ extends over the pull-back $\tilde{\mathcal{X}}' \to \tilde{B}'$.

We next observe that the image of the map $H_1(\tilde{S}; \mathbb{Z}/n) \to H_1(S; \mathbb{Z}/n)$ is $2H_1(S; \mathbb{Z}/n)$, and hence is naturally isomorphic to $H_1(S; \mathbb{Z}/m)$. In particular, a Prym level n-structure determines an ordinary level m-structure, so that we have a natural map $\mathcal{M}_g[\frac{n}{2}] \to \mathcal{M}_g[m]$. So if $X_{\tilde{w}}$, $X_{\tilde{w}'}$ are two fibres of $\tilde{\mathcal{X}}' \to \tilde{B}'$ which are isomorphic as curves with Prym level n-structure, then they are isomorphic as curves with level m-structure. It follows from our earlier discussion that they must then lie over the same point—call it w—of B' and that the isomorphism is the natural isomorphism $X_{\tilde{w}} \cong X_w \cong X_{\tilde{w}'}$. Since this isomorphism respects the Prym level structure, prop. 3 implies that $\tilde{w} = \tilde{w}'$.

Our main result is a corollary of this proposition:

Theorem. *The system of Galois coverings of $\overline{\mathcal{M}}_g$ defined by the Prym level n-structures ($n = 1, 2, \dots$) has universal ramification along the Deligne–Mumford boundary, and if n is even and ≥ 6, then $\overline{\mathcal{M}}_g[\frac{n}{2}]$ is smooth.*

Proof. If n is even and ≥ 6, then the analytic manifold $\tilde{B}$ of the previous proposition defines a chart of $\overline{\mathcal{M}}_g[\frac{n}{2}]$. Since $\overline{\mathcal{M}}_g[\frac{n}{2}]$ is covered by such charts, it is smooth. The covering $\tilde{B}' \to B'$ is defined by a subgroup of $\pi_1(B', z)$ which is contained in the subgroup of $\pi_1(B', z)$ generated by its n-th powers. The first statement now follows by letting n vary over all even integers ≥ 6, compare [1] Prop. 4.

3. Prym level versus dyhedral level

We compare our notion of Prym level structure with Brylinski's dyhedral level structure. We follow Brylinski in denoting for any group H the subgroup generated by the n-th powers h^n ($h \in H$) by $H^{(n)}$. Notice that this is a characteristic subgroup of H. We shall also write H/n for the 'mod n reduction' $H/H^{(n)}$.

Let us write π for the fundamental group of S relative to some base point and put $\tilde{G} := \pi/[\pi^{(2)}, \pi^{(2)}]$. In the exact sequence

$$1 \to \pi^{(2)}/[\pi^{(2)}, \pi^{(2)}] \to \tilde{G} \to G \to 1$$

the term $\pi^{(2)}/[\pi^{(2)}, \pi^{(2)}]$ may be identified with $H_1(\tilde{S})$ and under this identification the actions of G on these groups correspond. We mention in passing that this sequence is not split.

If $n = 2m$ is an even positive integer, then the mod n reduction of the above sequence is still exact and yields the exact sequence

$$0 \to H_1(\tilde{S}; \mathbb{Z}/n) \to \tilde{G}/n \to G \to 1.$$

(For odd n, $H_1(\tilde{S}; \mathbb{Z}/n) \to \tilde{G}/n$ is an isomorphism.) The mapping class group Γ_S of S acts on $\tilde{G}/n$ via outer automorphisms:

$$\rho : \Gamma_S \to \mathrm{Out}(\tilde{G}/n).$$

This defines what Brylinski calls the *dyhedral level of order* m and is the main concern of his paper [1]. (A dyhedral level structure of order m on a smooth projective curve X of genus g can be given by an orientation preserving homeomorphism $f : S \to X$, where two such maps $f, f' : S \to X$ define the same structure if and only if the class of $f^{-1}f'$ is in the kernel of ρ.) Our Prym level structure is in a sense the residue of this notion on $H_1(\tilde{S}; \mathbb{Z}/n)$: The group $\mathrm{Aut}_G H_1(\tilde{S}; \mathbb{Z}/n)$ of G-equivariant automorphisms of $H_1(\tilde{S}; \mathbb{Z}/n)$ contains G as a normal subgroup, and we have a natural homomorphism

$$\sigma : \mathrm{Out}(\tilde{G}/n) \to \mathrm{Aut}_G H_1(\tilde{S}; \mathbb{Z}/n)/G.$$

Then the kernel of $\sigma\rho$ is equal to $\Gamma_{S,\binom{n}{2}}$, and so $\sigma\rho$ 'defines' the Prym level n-structure. We conclude that a dyhedral level of order m determines a Prym level of order $2m$. The converse need not hold, for it is conceivable that a mapping class maps under ρ to a nontrivial element of the kernel of σ. (The kernel of σ can be identified with $H_1(G, H_1(\tilde{S}; Z/n))$.)

4. Final remarks

We notice that for $n \geq 3$, $\overline{\mathcal{M}}_g[{n \atop 2}]$ supports a family of stable curves of genus $\tilde{g} := 2^{2g}(g-1)+1$ with G-action and level n-structure. This is clearly true over $\mathcal{M}_g[{n \atop 2}]$. In terms of the local coordinate patch $\tilde{B}$, the stable curve lying over X_0 is a Galois covering $\tilde{X}_0 \to X_0$ of degree 2^{2g} which ramifies (with degree 2) over the singular points of X_0 that do not disconnect X_0. The graph of $\tilde{X}_0$ is just the one attached to $\tilde{S}$ and the collection of circles lying over the circles $C_1, \dots, C_k$ and which was denoted by $\tilde{\Gamma}$ in section 1.

It seems likely that $\overline{\mathcal{M}}_g[{n \atop 2}]$ is smooth for all $n \geq 3$. M. Pikaart (Utrecht) has extended our results to higher order Prym structures that involve an abelian covering of S defined by $H_1(S; \mathbb{Z}/q)$ with $q \geq 2$. In fact, he proved that the moduli space $\overline{\mathcal{M}}_g[{mq \atop q}]$ (the definition of which should be obvious) is smooth and defined over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{mq}])$ whenever m and q are integers with $m \geq 3$ and $q \geq 2$.

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