RIEMANN SURFACES

Notes to a national master's course

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Introduction

As you may know, in complex function theory it is often convenient to work with the extension of the complex plane $\mathbb{C}$ obtained by adding a point at infinity, the socalled *Riemann sphere* $\hat{\mathbb{C}}$. Strictly speaking this is more than just an extension as a topological space, since we understand not only what it means for a complex valued function f on the open neighborhood $|z| > R$ of ∞ to be continuous, but also what it means to be there holomorphic (the answer is: the function on the disk $|z| < R^{-1}$ defined by $z \mapsto f(z^{-1})$ for $z \neq 0$, and $0 \mapsto f(\infty)$ is holomorphic). Apparently complex function theory makes sense on the Riemann sphere as much as it does on an open subset of $\mathbb{C}$. This observation suggests that it possible to do complex function theory on general surfaces. That turns out to be indeed the case. But the surface on which we want to do this must, in order to start at all, come with more structure than just that of a topological space. If that structure is present we call it a *Riemann surface*. Although this can be thought of as a natural extension of complex function theory, both the central issues and the techniques force upon us a way of thinking of a more geometric nature.

Riemann ran into the surfaces now named after him during his investigations of multivalued functions in a single complex variable. The naming of this class of surfaces is therefore appropriate, especially since he was the first to conceive the notion and to emphasize the geometric view on complex function theory on which it is based. Such surfaces are ubiquitous, although we sometimes encounter these in a guise that makes them hard to recognize. Thus a compact Riemann surface is synonymous to what in algebraic geometry is called a *nonsingular complex-projective curve*. And this in turn is essentially the same thing as finitely generated field extensions $K/\mathbb{C}$ of transcendence degree 1 (i.e., K is $\mathbb{C}$-isomorphic to a finite extension of $\mathbb{C}(z)$). A Riemann surface can also be gotten as an oriented surface with a conformal structure (i.e., a notion of angle in each tangent plane) and it is in this guise that they appear in string theory as the surface traced out in a higher dimensional space-time universe by an evolving closed string.

Finally a word about the way this course is set up. Although Riemann surfaces are introduced at an early stage, they do not play an important role in the first half of these notes. This is related to our wish to confine the prerequisites to a basic course of complex function theory and a bit of elementary point set topology. Thus chapter 2 is mostly of a topological nature: we develop here the theory of covering spaces (that provides the topological foundation for the notion of a multivalued function) and explain its relationship to the fundamental group. The latter is computed for a closed surface. Algebraic topology is not a prerequisite and that is why we define the first homology group of an arcwise connected space in a somewhat unusual manner, namely as the abelianized fundamental group. This group comes up naturally when one is dealing with countour integrals. In the same spirit we deal in chapters 3 and 4 with the theorems of Stokes and De Rham for compact surfaces (they hold and make sense in a much more general context). (So if you are already familiar with these theorems, you may just skim through these chapters.) From that point onward, Riemann surfaces take the stage. We state in chapter 5 the Hodge-Weyl decomposition of differentials, a result we make plausible, but do not prove (the proof rests on the theory of elliptic differential equations and falls beyond the scope of this course). Incidentally, this too is a special case of a much more general result that was first stated by De Rham (harmonic representation of De Rham cohomology for compact Riemannian spaces). After all this preparation, it is relatively straight sailing to the principal results of this field: the Riemann-Roch theorem, Serre-duality, eventually culminating in the Abel-Jacobi isomorphism.

CHAPTER 1

Manifolds and Riemann surfaces

1. Manifolds

We begin with a basic definition.

DEFINITION 1.1. A *topological* m*-manifold* is a Hausdorff space that can be covered by open subsets each of which is homeomorphic to some open subset of $\mathbb{R}^m$. A topological 2-manifold is often called a topological *surface*.

Clearly, an open subset U of a topological m-manifold M is also a topological m-manifold. It is also easily verified that the topological product of a topological m-manifold with a topological n-manifold is a een topological $(m+n)$-manifold.

Given a topological m-manifold M, then a *chart* for M is a pair (U, κ) where $U \subset M$ is a (possibly empty) open subset of M and κ is a homeomorphism from U onto an open subset of $\mathbb{R}^m$. A collection of charts whose domains cover M is called an *atlas* for M. Notice that an ordered pair of charts (U, κ), (U', κ') determines a homeomorphism between two subsets of $\mathbb{R}^m$:

$$\kappa'\kappa^{-1} : \kappa(U \cap U') \to \kappa'(U \cap U').$$

Such a homeomorphism is often referred to as a *coordinate change* or a *chart transition*.

EXAMPLES 1.2. (i) The unit sphere $S^m := \{x \in \mathbb{R}^{m+1} | \; \|x\| = 1\}$ is a topological m-manifold. An atlas is for instance the collection charts 'by half spheres' $(U_i^\varepsilon, \kappa_i^\varepsilon)$, $\varepsilon = \pm$, $i = 0, \dots, m$, where $U_i^\pm := \{x \in S^m | \; \pm x_i > 0\}$ and $\kappa_i^\pm(x) = (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_m)$.

(ii) The projective space P^m. A point of P^m is by definition represented by a one-dimensional linear subspace (briefly, a line) in $\mathbb{R}^{m+1}$ (this is equivalent to giving an antipodal pair on the unit sphere $S^m \subset \mathbb{R}^{m+1}$). This is a topological m-manifold: every pair of charts $(U_i^\pm, \kappa_i^\pm)$ in (i) determines a single chart (U_i, κ_i) for P^m and the collection of charts thus obtained is an atlas for P^m.

(iii) The n-dimensional complex projective space $\mathbb{P}^n$. By definition a point of $\mathbb{P}^n$ is represented by a complex line (= a one-dimensional complex linear subspace) in $\mathbb{C}^{n+1}$. To be precise, $\mathbb{P}^n$ is as a topological space the quotient of $\mathbb{C}^{n+1} - \{0\}$ by scalar multiplication: two vectors $\zeta, \zeta' \in \mathbb{C}^{n+1} - \{0\}$ represent the same point of $\mathbb{P}^n$ if and only if $\zeta' = \lambda\zeta$ for some $\lambda \in \mathbb{C} - \{0\}$. We therefore write (quite appropriately) $[\zeta_0 : \cdots : \zeta_n]$ for the point represented by $(\zeta_0, \dots, \zeta_n) \in \mathbb{C}^{n+1} - \{0\}$. The quotient topology on $\mathbb{P}^n$ is compact and Hausdorff. We here use charts of a different nature than in the previous example: for $i = 0, \dots, n$, we have an open subset $U_i \subset \mathbb{P}^n$ defined by $\zeta_i \neq 0$ and we have a continuous bijection $\mathbb{C}^n \to U_i$ defined by $(z_1, \dots, z_n) \mapsto [z_1 : \cdots : z_{i-1} : 1 : z_i : \cdots : z_n]$ whose inverse is $\kappa_i : U_i \to \mathbb{C}^n$, $[\zeta_0 : \cdots : \zeta_n] \mapsto (\zeta_0/\zeta_i, \dots, \zeta_{i-1}/\zeta_i, \zeta_{i+1}/\zeta_i, \dots, \zeta_n/\zeta_i)$ is also continuous. So $\mathbb{P}^n$ is a topological $2n$-manifold and the collection $\{(U_i, \kappa_i)\}_{i=1}^n$ is an atlas for $\mathbb{P}^n$.

(iv) Let $L \subset \mathbb{R}^m$ be a lattice, that is, the additive subgroup generated of a basis of $\mathbb{R}^m$, and let $T := \mathbb{R}^m/L$. Notice that T is an abelian group with a topology. As such T is isomorphic with an m-torus, that is, a product of m circles: if $(\nu_1, \dots, \nu_m)$ is a basis for L, then

$$(\theta_1, \dots, \theta_m) \in (\mathbb{R}/\mathbb{Z})^m \mapsto \sum_i \theta_i \nu_i \in \mathbb{R}^m/L$$

is well-defined, is a homeomorphism and an isomorphism of groups at the same time. In particular T is a topological m-manifold. Let $\varepsilon := \min\{\|\nu\| \ : \ \nu \in L - \{0\}\}$. Then the projection $\pi : \mathbb{R}^m \to T$ is injective on every open disk in $\mathbb{R}^m$ of radius $< \frac{1}{2}\varepsilon$. It is easy to verify that for such a disk B, $\pi(B)$ is open in T, and that π maps B homeomorphically onto $\pi(B)$. If we denote the inverse of this homeomorphism by $\kappa_B : \pi(B) \to B \subset \mathbb{R}^m$, then $\{(\pi(B), \kappa_B)\}_B$ is an atlas for T. It has the property that every chart transition whose domain is nonempty is a translation over some element of L.

(v) A compact topological surface *of genus* g, $g = 0, 1, \dots$, is een surface homeomorphic to the boundary surface of a body in $\mathbb{R}^3$ that 'has g holes' (for $g = 0$ this surface is homeomorphic to S^2).

We shall see that one can put additional structure on a topological manifold M by giving an atlas for which the chart transitions are of a restricted nature (e.g., differentiable or complex-analytic). As a first example, say that an atlas for a topological manifold M is *smooth* if all the chart transitions are differentiable[1]. Such chart transitions are then necessarily diffeomorphisms, because the inverse of a chart transition is one as well.

LEMMA 1.3. *Let $\mathcal{A} := \{(U_i, \kappa_i)\}_i$ be a smooth atlas for M. Then the collection $\hat{\mathcal{A}}$ of charts (U, κ) with the property that $\kappa\kappa_i^{-1} : \kappa_i(U_i \cap U) \to \kappa(U_i \cap U)$ is a diffeomorphism for every i makes up a differentiable atlas for M which contains $\mathcal{A}$. This atlas is maximal for this property: every smooth atlas for M that contains $\mathcal{A}$ is contained in $\hat{\mathcal{A}}$.*

PROOF. We first show that a chart transition attached to a pair $(U, \kappa), (U', \kappa') \in \hat{\mathcal{A}}$ is of class C^∞. Let $x \in \kappa(U \cap U')$ be arbitrary. If (U_i, κ_i) is a chart in $\mathcal{A}$ with $\kappa^{-1}(x) \in U_i$, then $\kappa(U \cap U' \cap U_i)$ is a neighborhood of x in $\mathbb{R}^m$. The restriction of $\kappa'\kappa^{-1}$ to this neighborhood is the composition of two C^∞-maps:

$$\kappa'\kappa^{-1} : \kappa(U \cap U' \cap U_i) \xrightarrow{(\kappa\kappa_i^{-1})^{-1}} \kappa_i(U \cap U' \cap U_i) \xrightarrow{\kappa'\kappa_i^{-1}} \kappa'(U \cap U' \cap U_i)$$

and hence is itself C^∞. So $\hat{\mathcal{A}}$ is a smooth atlas. Its maximality property is clear. □

EXERCISE 1.1. Given a smooth atlas $\mathcal{A}$ for M, then we say that a function $f : M \to \mathbb{R}$ defined on an open $U \subset M$ is C^∞ *relative to* $\mathcal{A}$ if for every chart (U_i, κ_i) in $\mathcal{A}$, $f\kappa_i^{-1} : \kappa(U_i \cap U) \to \mathbb{R}$ is C^∞. Show that this property is equivalent to f being C^∞ relative to $\hat{\mathcal{A}}$.

DEFINITION 1.4. A *smooth structure* on M is a maximal smooth atlas $\mathcal{A}$ for M (and hence one for which $\mathcal{A} = \hat{\mathcal{A}}$) and the pair $(M, \mathcal{A})$ is called a *smooth* m-manifold.

[1]In this course *differentiable* means indefinitely differentiable, that is, of class C^∞.

Usually we refer to a smooth manifold $(M, \mathcal{A})$ by the single symbol M and unless we explicitly state the contrary, we tacitly assume that a chart for M is taken from $\mathcal{A}$. It is clear from the preceding lemma that every smooth atlas for a topological manifold makes the latter a smooth m-manifold.

EXERCISE 1.2. Verify that the atlasses described in 1.2 are all smooth and thus define smooth manifolds.

EXERCISE 1.3. Define a different atlas for P^m by imitating the construction for $\mathbb{P}^n$. Show that this atlas defines the same smooth structure on P^m as the given one.

DEFINITION 1.5. We say that a map $f : M \to N$ of smooth manifolds is *differentiable* if for every pair of charts (U, κ) of M and (V, λ) of N with $f(U) \subset V$ (taken from their respective smooth structure), the composite map $\lambda f \kappa^{-1} : \kappa(U) \to \lambda(V)$ is of class C^∞. A differentiable map $f : M \to N$ that is bijective and whose inverse is also differentiable is called a *diffeomorphism*; we then say that M and N are *diffeomorphic*.

Clearly the identity map of a manifold M, $\mathbf{1}_M : M \to M$ is differentiable. Also, the composition of two composable differentiable maps $M \to N$ and $N \to P$ is differentiable. In other words, we have a category with smooth manifolds as objects and differentiable maps as morphisms. This category provides the natural setting for many results in analysis in several variables (such as the implicit function theorem). This category is the subject of investigation of the field called *differential topology*.

One can show that any topological manifold M of dimension ≤ 3 possesses a differentiable structure. Moreover, that structure is essentially unique in the sense that for every pair of differentiable structures $\mathcal{A}$, $\mathcal{A}'$ on M there exists a homeomorphism $h : M \to M$ that takes $\mathcal{A}'$ to $\mathcal{A}$ (in other words, $(M, \mathcal{A})$ and $(M, \mathcal{A}')$ are diffeomorphic). It is quite remarkable that in dimension ≥ 4 neither assertion holds. But this will hardly concern us as we shall mainly deal with smooth surfaces.

2. Complex manifolds and Riemann surfaces

We recall that a $\mathbb{C}^k$-valued function defined on an open subset $U \subset \mathbb{C}^n$ is *holomorphic* if each component of that function can at any $p \in U$ be represented by an absolute convergent power series in the complex variables $z_1 - p_1, z_n - p_n$.

In what follows we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via $z = x + \sqrt{-1}y \mapsto (x, y)$.

DEFINITION 1.6. If M is a topological $2n$-manifold, then we say that an atlas for M is *holomorphic* if every chart transition is (we here make the above identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$). A *holomorphic structure* on M is the given of a maximal holomorphic atlas; M is then called a *complex manifold* of complex dimension n.

Both statement and proof of Lemma 1.3 can be adapted in straightforward manner to such atlasses. So every holomorphic atlas is contained in a (unique) maximal holomorphic atlas and thus turns the manifold in question into a complex manifold. Clearly, a complex structure is very special kind of smooth structure: a complex manifold is a smooth manifold. In analogy to the notion of a differentiable map we have the notion of a *holomorphic map* (between complex manifolds). So a complex-valued function f on a complex manifold is holomorphic if for every chart (U, κ) taken from the holomorphic atlas, $f\kappa^{-1}$ is holomorphic. A holomorphic bijection $f : M \to N$ of complex manifolds whose inverse is also holomorphic is called an *isomorphism* (and we then say that M and N are *biholomorphic*). Just as for smooth manifolds we usually omit mention of the holomorphic atlas, and it is tacitly understood that any chart is taken from the holomorphic atlas. We have a category of complex manifolds and holomorphic maps and the part of mathematics that takes care of it is called *complex-analytic geometry*.

EXAMPLES 1.7. (i) The complex projective space $\mathbb{P}^n$ is a complex manifold of complex dimension n, for the given atlas is in fact holomorphic.

(ii) If $L \subset \mathbb{C}^n$ is a lattice, then the any chart transition of the given atlas for $T := \mathbb{C}^n/L$ is a (restriction of) a translation in L. A translation is evidently holomorphic, and hence so is the atlas. This atlas therefore turns T into a complex manifold. We call it an *complex n-torus*.

DEFINITION 1.8. A *Riemann surface* is a complex manifold of (complex) dimension 1.

EXAMPLES 1.9. We have of course the previous examples of complex manifolds in complex dimension one:

(i) The complex projective line $\mathbb{P}^1$ is also known as the Riemann sphere. We have $\mathbb{P}^1 = U_0 \cup \{[0:1]\}$. Since $\mathbb{P}^1$ is compact, we may interpret $\mathbb{P}^1$ as the one point compactification of $U_0 \cong \mathbb{C}$ and this helps us to recognize that $\mathbb{P}^1$ is indeed the *Riemann sphere*.

(ii) A complex 1-torus $T = \mathbb{C}/L$. In algebraic geometry such a Riemann surface is called an *elliptic curve* (curve, because algebraic geometers emphasize the complex dimension (which is 1); elliptic, because of a connection with measuring the arc length along an ellipse). We shall see that two lattices L and L' yield isomorphic Riemann surfaces if and only if $L' = \lambda L$ for some nonzero complex scalar λ (see Proposition 2.13). Yet all these Riemann surfaces are pairwise diffeomorphic.

(iii) A slew of examples is obtained by means of the holomorphic analogue of the implicit function theorem. Let $U \subset \mathbb{C}^2$ be open and let $f : U \to \mathbb{C}$ be a holomorphic function. Let $N \subset U$ be the zero set of f and suppose that in no point of N both partial derivatives simultaneously vanish. We show that N is then in a natural manner a Riemann surface. If $p \in N$, then by definition $\frac{\partial f}{\partial z_1}(p) \neq 0$ or $\frac{\partial f}{\partial z_2}(p) \neq 0$. Suppose the latter occurs. The complex analogue of the implicit function theorem (which we will not prove, but which may be derived from the familiar one for C^∞ maps) then says that N is at p the graph of a holomorphic function: there is an open product neighborhood $U_1 \times U_2 \subset U$ of $p = (p_1, p_2)$ and a holomorphic $\phi : U_1 \to U_2$ such that $(U_1 \times U_2) \cap N = \{(z, \phi(z)) \mid z \in U_1\}$. So the projection of $(U_1 \times U_2) \cap N$ on U_1 is a homeomorphism with inverse $z \mapsto (z, \phi(z))$. We thus find that N is a surface, for a chart transition is the identity map or is the restriction of a function ϕ for which N is locally the graph and hence is holomorphic. (This construction can be generalized to higher dimensions and we thus obtain more examples of complex manifolds.)

(iv) A variation on the previous example yields compact Riemann surfaces. For this we take a polynomial $F \in \mathbb{C}[\zeta_0, \zeta_1, \zeta_2]$ that is homogeneous of degree $d > 0$. If $\zeta \in \mathbb{C}^3 - \{0\}$ lies on the zero set of F, then so does the complex line spanned by ζ. Hence the zero set of F determines a subset N of the complex projective plane $\mathbb{P}^2$. This subset is closed in $\mathbb{P}^2$ and hence compact (since $\mathbb{P}^2$ is compact).

Now assume that F has the property that the partial derivatives of F nowhere vanish simultaneously except perhaps in the origin. We claim that N has then the structure of a Riemann surface. As before, let U_i be the open part of $\mathbb{P}^2$ defined by door $\zeta_i \neq 0$, $(i = 0, 1, 2)$. We first show that $N \cap U_0$ is a Riemann surface.

Recall that we may identify U_0 with $\mathbb{C}^2$ by means of the coordinates $z_i := \zeta_i/\zeta_0$, $(i = 1, 2)$. Now $F(\zeta_0, \zeta_1, \zeta_2)$ can be written $\zeta_0^d f(\zeta_1/\zeta_0, \zeta_2/\zeta_0)$ with $f(z_1, z_2) := F(1, z_1, z_2)$. So $N \cap U_0$ is the zero set of f. We now check that in no zero of f, both partial derivatives of f vanish. For this, we note that by the chain rule

$$\frac{\partial F}{\partial \zeta_i}(\zeta) = \zeta_0^{d-1} \frac{\partial f}{\partial z_i}\Big(\frac{\zeta_1}{\zeta_0}, \frac{\zeta_2}{\zeta_0}\Big), \quad (i = 1, 2).$$

Since F is homogeneous of degree d, Euler's formula (which is easily verified per monomial) says that:

$$d.F(\zeta) = \sum_{i=0}^{2} \zeta_i \frac{\partial F}{\partial \zeta_i}(\zeta).$$

Suppose now $(\zeta_1/\zeta_0, \zeta_2/\zeta_0) \in U_0$ is a common zero of f and its partial derivatives. Then $F(\zeta) = 0$ and from the first equation we see that $\frac{\partial F}{\partial \zeta_i}(\zeta) = 0$ for $i = 1, 2$. Euler's formula then implies that $\frac{\partial F}{\partial \zeta_0}(\zeta) = 0$ also. Since $\zeta \neq 0$ this contradicts our assumption. We conclude that $N \cap U_0$ is Riemann surface.

Similarly we find that $N \cap U_1$ and $N \cap U_2$ are Riemann surfaces and as these three form a covering of N by open subsets we only need to verify that the two Riemann surface structures on $N \cap U_i \cap U_j$ coincide. This is left to you.

Such Riemann surfaces are studied in algebraic geometry from a more algebraic point of view and go there under the name *nonsingular complex-algebraic plane curves of degree* d. It can be shown that for $d = 3$ such a Riemann surface is isomorphic to an elliptic curve, and that every elliptic curve thus arises up to isomorphism.

EXERCISE 1.4. Two lattices $L, L' \subset \mathbb{C}$ are called *similar* if there is a $\lambda \in \mathbb{C}$ such that $L' = \lambda L$.

(a) Prove that similar lattices define isomorphic elliptic curves. (The converse also holds: two lattices in $\mathbb{C}$ that determine isomorphic elliptic curves, are similar.)

(b) Prove that every lattice is similar to one of the form $L(\tau) := \mathbb{Z} + \mathbb{Z}\tau$, with $\mathrm{Im}(\tau) > 0$.

(c) Let τ, τ' both lie in the upper half plane $\mathrm{Im}(z) > 0$. Prove that $L(\tau)$ and $L(\tau')$ are similar if and only if there exists a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with integral entries and of determinant 1 such that $\tau' = (a\tau + b)/(c\tau + d)$.

We close this chapter with two elementary properties of Riemann surfaces.

PROPOSITION 1.10. *Let* $f : S \to \mathbb{C}$ *be a holomorphic function on a connected Riemann surface* S *which is not constant zero. Then* $f^{-1}(0)$ *is discrete.*

PROOF. Let $A \subset S$ be the set of accumulation points of $f^{-1}(0)$. It is clear that A is closed. We prove that A is also open in S. If $p \in A$, then choose a chart (U, κ) from the holomorphic atlas with $p \in U$. Then $\kappa(p)$ is an accumulation point of the zero set of $f\kappa^{-1} : \kappa(U) \to \mathbb{C}$. Complex function theory tells us that such a function must vanish on a neighborhood of $\kappa(p)$. So f vanishes on a neighborhood of p. This shows that A is open. Since S is connected we have $A = \emptyset$ (in other words, $f^{-1}(0)$ is discrete) or $A = S$. But in the last case, $f \equiv 0$, which we excluded. □

COROLLARY 1.11 (Unicity of analytic continuation). *Two holomorphic functions on a connected Riemann surface which coincide on a non-discrete subset are equal.*

PROOF. Apply the previous proposition to their difference. □

On a smooth m-manifold M with $m > 0$ there are plenty of differentiable $\mathbb{R}$-valued functions. For instance, for every pair distinct points $p, q \in M$ there is a differentiable function that takes in p the value 0 en in q the value 1. In contrast, there are no interesting holomorphic functions on a compact Riemann surface:

PROPOSITION 1.12. *Every holomorphic function f on a compact connected Riemann surface S is constant.*

PROOF. Since S is compact and $|f|$ is continuous, the latter has a maximum. Let that maximum be taken in $p \in S$ and choose a chart (U, κ) from the holomorphic atlas of S with $p \in U$. Then $f\kappa^{-1} : \kappa(U) \to \mathbb{C}$ is a holomorphic function whose absolute value takes its maximum in $\kappa(p)$. The maximum principle then says that $f\kappa^{-1}$ is constant on a neighborhood of $\kappa(p)$. Hence f is constant on a neighborhood of p. The previous corollary then implies that f is constant on all of S. □

In particular, there are no nonconstant holomorphic functions on elliptic curves. One can show that a nonsingular complex-algebraic plane curve is connected and so admits no nonconstant holomorphic function either.

EXERCISE 1.5. Is there a compact Riemann surface of the form 1.9-iii?

EXERCISE 1.6. Prove that any holomorphic function on a connected compact complex manifold is constant.

CHAPTER 2

Fundamental group and covering spaces

1. Covering spaces

We begin with the definition.

DEFINITION 2.1. Let d be positive number, or more generally a nonzero cardinal number (in practice always countable). A map $f : \tilde{X} \to X$ between topological spaces is called a *covering map of degree* d (and $\tilde{X}$ a *covering space* of X) if every point of X is contained in an open subset V such that $f^{-1}V$ is nonempty and can be decomposed into d open subsets each of which is mapped by f homeomorphically onto V.

If V is as in the lemma, then we say that f is *trivial* over V. This terminology is explained as follows: if $\{V_i\}_{i\in I}$ is the decomposition of $f^{-1}V$, with $|I| = d$, then we have an obvious surjective map $r : f^{-1}V \to I$ such that the fiber over i is V_i and (hence) $(f, r) : f^{-1}V \to V \times I$ is a bijection. But if we endow I with the discrete topology, then this is even a homeomorphism and therefore $f : f^{-1}V \to V$ is really like the projection $V \times I \to V$. It is useful to note that if V is connected, then the partition of $f^{-1}V$ is necessarily the partition in connected components (and is hence unique).

We also observe that for every subspace $A \subset X$, $f : f^{-1}A \to A$ is also a covering map of degree d.

EXAMPLES 2.2. Here are some examples.

(i) The dullest example is the following: Let I be a nonempty set endowed with the discrete topology. Then the projection $X \times I \to X$ is covering map. We call such a covering map *trivial*. (So any covering map is locally trivial.)

(ii) Somewhat more interesting are: $z \in \mathbb{C} - \{0\} \mapsto z^k \in \mathbb{C} - \{0\}$ is for $k \neq 0$ a covering map of degree $|k|$ and $z \in \mathbb{C} \mapsto e^z \in \mathbb{C} - \{0\}$ is a covering map whose degree is countably infinity.

(iii) The map $(x_0, \dots, x_m) \in S^m \mapsto [x_0 : \cdots : x_m] \in P^m$ is a covering map of degree 2.

(iv) If G is a group of homeomorphisms of a space X that acts *freely discontinuously*, in the sense that every point of X has a neighborhood U such that $gU \cap U = \emptyset$ for all $g\Gamma - \{1\}$, then the quotient map $f : X \to G\backslash X$ which forms the orbit space (the latter has the quotient topology) is a covering map of degree $|G|$. For $f^{-1}f(U) = \cup_{g\in G}\, gU$ is then open in X and hence $f(U)$ is open in $G\backslash X$. The same argument shows that $U \to f(U)$ is a homeomorphism. So $gU \to U$ is a homeomorphism for every $g \in G$ and the assertion follows.

EXERCISE 2.1. Show that the cylinder $S^1 \times (-1, 1)$ covers the Möbius band twice.

EXERCISE 2.2. Let S be a connected Riemann surface. We denote by $\mathcal{H}(S)$ the $\mathbb{C}$-algebra of holomorphic functions on S. Let $F \in \mathcal{H}(S)[t]$ be a monic polynomial of degree d, considered as a function on $\mathbb{C} \times S$: $F(t,p) := t^d + c_1(p)t^{d-1} + \cdots + c_d(p)$, with c_i holomorphic on S. Suppose that for no $p \in S$, the polynomial $F(t,p) \in \mathbb{C}[t]$ has a multiple root.

(a) Prove that the zero set $S_F \subset \mathbb{C} \times S$ of F with its projection onto S is a covering map of degree d. (Hint: use the implicit function theorem.)

(b) We say that F is *irreducible* over $\mathcal{H}(S)$ if it is not a product of two monic polynomials in $\mathcal{H}(S)[t]$ of positive degree. Prove that S_F is connected if and only F is irreducible.

(c) Suppose given a covering map $\pi : \tilde{S} \to S$ with $\tilde{S}$ connected, such that $\pi^* F \in \mathcal{H}(\tilde{S})[t]$ has a continous root in the sense that there is a continuous function $\tau : \tilde{S} \to \mathbb{C}$ such that $F(\tau(\tilde{p}), \pi(\tilde{p})) = 0$ for all $\tilde{p} \in \tilde{S}$. Prove that this root is holomorphic.

The proof of the following lemma is left to you.

LEMMA 2.3. *Let $\tilde{X} \to X$ be a covering map. If X is a topological m-manifold, the so is $\tilde{X}$. If X has also a differentiable resp. holomorphic structure, then there is one on $\tilde{X}$ that makes f een differentiable resp. holomorphic (and this structure is unique). In particular, a covering space of a Riemann surface is a Riemann surface.*

EXERCISE 2.3. Let $f : \tilde{S} \to S$ be a covering of Riemann surfaces of finite degree d and let $\phi : \tilde{S} \to \mathbb{C}$ be holomorphic. Define $F_\phi : \mathbb{C} \times S \to \mathbb{C}$ by

$$F_\phi(t,p) := \prod_{\tilde{p} \in f^{-1}(p)} (t - \phi(\tilde{p})).$$

Prove that F_ϕ is in $\mathcal{H}(S)[t]$, i.e., of the form $t^d + c_1 t^{d-1} + \cdots + c_d$, with c_k holomorphic on S. Prove also that ϕ is a 'root' of F_ϕ in the sense that $F_\phi(\phi(\tilde{p}), f(\tilde{p}))$ is identially zero. (This says that the ring homomorphism $f^* : \mathcal{H}(S) \to \mathcal{H}(\tilde{S})$ is what one calls in commutative algebra, an integral extension.)

2. The fundamental groupoid

We recall a few topological notions. Given a topological space X, then a *path* in X from $p \in X$ (its *starting point* or *point of departure*) to $q \in X$ (its *end point* or *point of arrival*) is a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$. A path γ can be composed with a path γ' if the latter departs where the former arrives:

$$\gamma'\gamma(t) = \begin{cases} \gamma(2t) & \text{when } t \in [0, \frac{1}{2}]; \\ \gamma'(2t-1), & \text{when } t \in [\frac{1}{2}, 1]. \end{cases}$$

(N.B.: some authors use the opposite order.) This composition is in general not associative.

We say that two paths $\gamma_0, \gamma_1 : [0,1] \to X$ from p to q are *homotopic* if there is an interval of paths $\{\gamma_s\}_{0 \le s \le 1}$ from p to q connecting them which is continuous in the sense that $(s,t) \in [0,1] \times [0,1] \mapsto \gamma_s(t) \in X$ is so. This is equivalence relation. An equivalence class is usually referred to as a *homotopy class* of paths from p to q and the collection of these homotopy classes is denoted $\Pi_X(p,q)$.

Composition respects homotopy classes (check this!), so that we get a map

$$\Pi_X(q,r) \times \Pi_X(p,q) \to \Pi_X(p,r), \quad ([\gamma'], [\gamma]) \mapsto [\gamma'][\gamma] := [\gamma'\gamma].$$

Denote for $p \in X$ the path that is constant equal to p by $\mathbf{1}_p$. Then $[\gamma][\mathbf{1}_p] = [\gamma\mathbf{1}_p] = [\gamma]$ and similarly $[\mathbf{1}_q][\gamma] = [\gamma]$. We may sum this up by saying that we have a category Π_X whose objects are the points of X and a morphism from p to q is a homotopy class of paths from p to q. If we define the *inverse* of the path γ by $\bar{\gamma}(t) = \gamma(1-t)$, then you easily verify that this name is deserved, at least for homotopy classes: we have $[\bar{\gamma}][\gamma] = [e_p]$ and $[\gamma][\bar{\gamma}] = [e_q]$. In other words, this categorie has the property that every of its morphisms is an isomorphism: the inverse of $[\gamma]$ is $[\bar{\gamma}]$. A category with this property is called a *groupoid* and that is why Π_X is referred to as the *fundamental groupoid* of X. We list some consequences.

PROPERTIES 2.4. We have

(i) The homotopy classes of *loops* at p, $\Pi_X(p,p)$, form a group under composition; this group is called *fundamental group* of (X,p) and is denoted $\pi(X,p)$.

(ii) The group $\pi(X,p)$ resp. $\pi(X,q)$ acts on $\Pi_X(p,q)$ from the right resp. from the left by composition:

$$\Pi_X(p,q)\times\pi(X,p)\to\Pi_X(p,q),\quad ([\gamma],[\alpha])\mapsto[\gamma][\alpha],$$
$$\pi(X,q)\times\Pi_X(p,q)\to\Pi_X(p,q),\quad ([\beta],[\gamma])\mapsto[\beta][\gamma].$$

Both actions are free and transitive. Furthermore, $[\gamma]\in\Pi_X(p,q)$ determines a group isomorphism

$$[\gamma]_\# : \pi(X,p)\to\pi(X,q),\quad [\alpha]\mapsto[\gamma][\alpha][\gamma]^{-1}.$$

(So in case X is path connected, then the isomorphism type of $\pi(X,p)$ is independent of p, reason why we then sometimes speak of 'the' fundamental group of X without specifying a base point.)

(iii) A continuous map $f : X\to Y$ determines a functor $\Pi_f : \Pi_X\to\Pi_Y$, in particular we have for every $p\in X$ a group homomorphism $\pi(X,p)\to\pi(Y,f(p))$.

(iv) Homotopic maps $f_0, f_1 : X\to Y$ induce the same map on fundamental groups, precisely, if $F:[0,1]\times X\to Y$ is a homotopy from f_0 to f_1, and is δ the path in Y from $f_0(p)$ to $f_1(p)$ defined by $\delta(t)=F(t,p)$, then

$$\Pi_{f_1}([\alpha]) = [\delta]\Pi_{f_0}([\alpha])[\delta]^{-1}.$$

This can be seen as follows: for $s\in[0,1]$, let $f_s := F(s,\ )$ and let δ_s be the path in Y from $f_s(p)$ to $f_1(p)$ defined by $\delta_s(t) = F((1-t)s+ts,p)$. Then

$$(s,t)\mapsto\delta_s((f_s\alpha)(\bar{\delta}_s))$$

is a homotopy of loops at $f_1(p)$ that shows that $f_1\alpha$ is homotopic to $\delta((f_0\alpha)(\bar{\delta}))$.

So the fundamental groups of homotopy-equivalent path connected spaces are isomorphic. In particular, every *contractible* space (that is, a space which is homotopy-equivalent to a singleton) has trivial fundamental group.

DEFINITION 2.5. We say that a space X is *simply connected* if it is nonempty and for all $p,q\in X$, $\Pi_X(p,q)$ consists of precisely one element, or equivalently, if X is path connected and has trivial fundamental group.

Without proof we mention that a simply connected (smooth) surface that has a countable basis for its topology is homeomorphic (diffeomorphic) to $\mathbb{R}^2$ or S^2. Also without proof we quote

PROPOSITION 2.6 (Uniformization theorem). *A simply connected Riemann surface that possesses a countable basis for its topology is biholomorphic to the Riemann sphere $\mathbb{P}^1$, the complex plane $\mathbb{C}$ or to the upper half plane $\mathbb{H}$.*

We shall not use these results.

EXERCISE 2.4. Prove that no two of these three Riemann surfaces are biholomorphic. (Hint: look at the properties of the algebra of holomorphic functions on each of these. For instance, only one of them admits a nonconstant bounded holomorphic function.)

If $f : \tilde{X} \to X$ and $g : A \to X$ are continuous maps between topological spaces, then a *lift of* g *over* f is a continuous map $\tilde{g} \colon A \to \tilde{X}$ such that $f\tilde{g} = g$. We have the following very useful

PROPOSITION 2.7. *Let $f : \tilde{X} \to X$ be a covering map and γ a path in X departing from $p \in X$. Then for every $\tilde{p} \in f^{-1}(p)$ there is precisely one lift $\tilde{\gamma}_{\tilde{p}}$ of γ over f that starts in $\tilde{p}$. Moreover the homotopy class of $\tilde{\gamma}_{\tilde{p}}$ only depends on the homotopy class of γ (in particular, the point of arrival of $\tilde{\gamma}_{\tilde{p}}$ only depends on $[\gamma]$ and $\tilde{p}$).*

Is γ' another path starting in p, then $\tilde{\gamma}'_{\tilde{p}}$ has the same end point as $\tilde{\gamma}_{\tilde{p}}$ if and only if γ' has the same end point as γ and $[\gamma]^{-1}[\gamma'] \in \pi(X, p)$ lies in the image of $\pi(\tilde{X}, \tilde{p}) \to \pi(X, p)$.

PROOF. If the image of γ lies in an open subset $V \subset X$ over which the covering is trivial, the the first assertion is clear: if $\tilde{V}$ is the copy of V in $f^{-1}V$ that contains $\tilde{p}$, then γ composed with the inverse of the homeomorphism $\tilde{V} \to V$ yields the $\tilde{\gamma} = \tilde{\gamma}_{\tilde{p}}$ we asked for. As $\tilde{V}$ contains the connected component of $\tilde{p}$ of $f^{-1}V$, every solution must stay in $\tilde{V}$, and hence this is also the only solution.

The general case is easily reduced to this: since $[0, 1]$ is compact, there is a division $0 = t_0 < t_1 < \cdots < t_N = 1$ of $[0, 1]$ such that $\gamma([t_{i-1}, t_i])$ is contained in an open $V_i \subset X$ over which f is trivial. Once we have constructed $\tilde{\gamma}|[0, t_{i-1}]$ (inductively), then we extend it over $[0, t_i]$ by applying the preceding to $\gamma|[t_{i-1}, t_i]$ en $\tilde{\gamma}(t_{i-1})$.

Let now $\Gamma(s, t) = \gamma_s(t)$ be a path homotopy from p to q. If the image of Γ is contained in an open $V \subset X$ over which f is trivial, then we argue as above and find that there is precisely one continuous $\tilde{\Gamma} : [0, 1] \times [0, 1] \to \tilde{X}$ with $\Gamma = f\tilde{\Gamma}$ and $\tilde{\Gamma}(0, 0) = \tilde{p}$. In general we choose a division of $[0, 1] \times [0, 1]$ in squares $\{B_{ij}\}_{i,j}$ such that $\Gamma(B_{ij})$ is contained in an open V_{ij} over which f is trivial and we apply the preceding (in lexicographic order) to the squares B_{ij}: we thus get a continuous $\tilde{\Gamma}$ with $\Gamma = f\tilde{\Gamma}$ and $\tilde{\Gamma}(0, 0) = \tilde{p}$. It remains to see that $\tilde{\Gamma}$ is a homotopy, i.e., that for $i = 0, 1$, the path $s \mapsto \tilde{\Gamma}(s, i)$ is constant. This is immediate from the fact that this is a lift of the constant path.

Now let γ' be another path departing from p. If $\tilde{\gamma}'$ arrives at the same point as $\tilde{\gamma}$, then $[\tilde{\gamma}]^{-1}[\tilde{\gamma}']$ is defined as an element of $\pi(\tilde{X}, \tilde{p})$ and has $[\gamma]^{-1}[\gamma'] \in \pi(X, p)$ as its image under f. Suppose conversely, that $[\gamma]^{-1}[\gamma'] \in \pi(X, p)$ is in the image of Π_f. This means that $\bar{\gamma}\gamma'$ is homotopic to the image of a loop at $\tilde{p}$. In view of the preceding $\widetilde{(\bar{\gamma}\gamma')}$ then closes up: the point of arrival is $\tilde{p}$. But that means that the last half is a reparameterisation of $\tilde{\gamma}$. So the point of arrival of $\tilde{\gamma}'$ is the point of departure of the inverse of $\tilde{\gamma}$, in other words, the point of arrival of $\tilde{\gamma}$. $\square$

COROLLARY 2.8. *Let be given a covering map* $f : \tilde{X} \to X$ *and* $p \in X$. *Then*

$$([\alpha], \tilde{p}) \in \pi(X, p) \times f^{-1}(p) \mapsto \tilde{\alpha}_{\tilde{p}}(1) \in f^{-1}(p)$$

defines an action of $\pi(X, p)$ *on* $f^{-1}(p)$. *It has the property that for any* $\tilde{p} \in f^{-1}(p)$, *the homomorphism* $\Pi_f : \pi(\tilde{X}, \tilde{p}) \to \pi(X, p)$ *is injective with image the stabilizer* $\pi(X, p)_{\tilde{p}}$ *of* $\tilde{p}$ *for this action. If* $\tilde{X}$ *is path connected, then this action is transitive.*

PROOF. That the map in question is well-defined follows from Proposition 2.7. So we obtain for every $[\alpha] \in \pi(X, p)$ a map $\sigma_{[\alpha]} : f^{-1}(p) \to f^{-1}(p)$. That this defines an action is easy to see: clearly $\sigma_{[\mathbf{1}_p]}$ is the identity on $f^{-1}(p)$ and if α and β are loops in X at p, and $\tilde{\alpha}$ and $\tilde{\beta}$ lifts of these such that $\tilde{\beta}$ departs where $\tilde{\alpha}$ arrives, then $\tilde{\beta}\tilde{\alpha}$ is a lift of $\beta\alpha$. This implies that $\sigma_{[\beta]}\sigma_{[\alpha]} = \sigma_{[\beta][\alpha]}$. Since the map $\pi(X, p)_{\tilde{p}} \to \pi(\tilde{X}, \tilde{p})$ composed with $\Pi_f : \pi(\tilde{X}, \tilde{p}) \to \pi(X, p)$ is the inclusion, the latter is injective with image $\pi(X, p)_{\tilde{p}}$. The last clause is also clear. □

PROPOSITION 2.9. *Let be given a covering map* $f : \tilde{X} \to X$, *a connected, locally path connected*[1]*space* A *and a continuous map* $g : A \to X$. *Let also be given* $a \in A$, $p \in X$ *and* $\tilde{p} \in \tilde{X}$ *such that* $g(a) = p = f(\tilde{p})$. *Then a lift* $\tilde{g} : A \to \tilde{X}$ *of* g *over* f *with* $\tilde{g}(a) = \tilde{p}$ *exists if and only if the image of* $\Pi_g : \pi(A, a) \to \pi(X, p)$ *is contained in* $\pi(X, p)_{\tilde{p}}$; *this lift is then unique.*

PROOF. If $\tilde{g}$ exists, then the image of $\Pi_g = \Pi_{f\tilde{g}} = \Pi_f\Pi_{\tilde{g}} : \pi(A, a) \to \pi(X, p)$ is contained in the image $\Pi_{\tilde{g}} : \pi(\tilde{X}, \tilde{p}) \to \pi(X, p)$, which is just $\pi(X, p)_{\tilde{p}}$.

Suppose conversely that the image of $\Pi_g : \pi(A, a) \to \pi(X, p)$ is contained in $\pi(X, p)_{\tilde{p}}$. Choose for every $b \in A$ a path α from a to b. According to Proposition 2.7 there is precisely one lift $\widetilde{g\alpha}$ of $g\alpha$ over f departing at $\tilde{p}$. We claim that the point of arrival of $\widetilde{g\alpha}$ only depends of b: for another path α' from a to b is $\overline{g\alpha}(g\alpha') = g(\bar{\alpha}\alpha')$, the g-image of a loop in A at a. By assumption its homotopy class is then contained in the image of $\Pi_f : \pi(\tilde{X}, \tilde{p}) \to \pi(X, p)$. Corollary 2.8 now implies that $\overline{g\alpha}$ and $\overline{g\alpha'}$ arrive at the same point. So if we denote that point $\tilde{g}(b)$, then we have constructed a map $\tilde{g} : A \to \tilde{X}$ with $f\tilde{g} = g$ and $\tilde{g}(a) = \tilde{p}$.

We verify that this map is continuous at b. For this we choose an open neighborhood V of $g(b)$ over which f is trivial and we let $\tilde{V} \subset \tilde{X}$ be a copy of V which contains $\widetilde{g\alpha}(1)$. Is U a path connected neighborhood of a in $f^{-1}V$, then it is easily verified that $\tilde{f}|V$ is the composite of $f|V$ and the inverse of the homeomorphism $f : \tilde{V} \to V$ and therefore continuous.

Finally we show that $\tilde{g}$ is unique. If $\tilde{g}'$ is another solution, then $\tilde{g}\alpha$ and $\tilde{g}'\alpha$ are lifts of $f\alpha$ over f with the same point of departure $\tilde{p}$. By Proposition 2.7 they must then be equal, in particular we have $\tilde{g}(b) = \tilde{g}'(b)$. □

Notice that if in the above proposition A and X are smooth (resp. complex) manifolds, en is f differentiable (resp. holomorphic), then so is $\tilde{f}$.

COROLLARY 2.10. *Let* X *be a connected, locally path connected space and* $p \in X$. *If* $(\tilde{X}, \tilde{p}) \to (X, p)$ *and* $(\tilde{X}', \tilde{p}') \to (X, p)$ *are covering maps with path connected covering spaces, then the former factors through the latter via a covering map* $(\tilde{X}', \tilde{p}') \to (\tilde{X}, \tilde{p})$ *if and only if the image of* $\pi(\tilde{X}', \tilde{p}') \to \pi(X, p)$ *is contained in the image of* $\pi(\tilde{X}, \tilde{p}) \to \pi(X, p)$ *(and the factorisation is then unique).*

[1]I.e., the space in question has a basis of open subsets that are path connected. The path components of such a space are open and coincide with the connected components.

3. Galois covers

DEFINITION 2.11. Given a covering map $f : \tilde{X} \to X$, then a *covering transformation* of f is a homeomorphism Φ of $\tilde{X}$ onto itself with $f\Phi = f$ (so Φ preserves every fiber of f). The covering transformations of f form evidently group under composition; this group is called the *Galois group* of the covering and will be denoted $G(\tilde{X}/X)$. We say that f is a *Galois covering* if the Galois group acts transitively on the fibers of f (so that X can be regarded as the G-orbit space of $\tilde{X}$)

As the terminology suggests, there is a connection with the theory of field extensions.

COROLLARY 2.12. *A covering map* $f : (\tilde{X}, \tilde{p}) \to (X, p)$ *between path connected, locally path connected spaces is a Galois covering if and only if the stabilizer* $\pi(X, p)_{\tilde{p}}$ *of* $\tilde{p}$ *(which is also the image of* $\Pi_f : \pi(\tilde{X}, \tilde{p}) \to \pi(X, p)$*) is a normal subgroup of* $\pi(X, p)$*. In that case the Galois group can be identified with the factor group* $\pi(X, p)/\pi(X, p)_{\tilde{p}}$.

PROOF. We first make the following general observation. If a group G acts transitively on a set F, then the stabilizers $\{G_y\}_{y \in F}$ make up a conjugacy class of subgroups of G. So if one of them is a normal subgroup, then they are all equal to each other and the resulting action of G/G_y on Y is free.

Now let $\tilde{q} \in \tilde{X}$ be arbitrary and put $q := f(\tilde{q})$. A path $\tilde{\gamma}$ from $\tilde{p}$ to $\tilde{q}$ determines according to 2.4 an isomorphism $[\tilde{\gamma}]_\# : \pi(\tilde{X}, \tilde{p}) \cong \pi(\tilde{X}, \tilde{q})$; similarly $\gamma := f\tilde{\gamma}$ determines $[\gamma]_\# : \pi(X, p) \cong \pi(X, q)$ and the diagram

$$\begin{array}{ccc} \pi(\tilde{X}, \tilde{p}) & \xrightarrow{[\tilde{\gamma}]_\#} & \pi(\tilde{X}, \tilde{q}) \\ {\scriptstyle \Pi_f}\downarrow & & {\scriptstyle \Pi_f}\downarrow \\ \pi(X, p) & \xrightarrow{[\gamma]_\#} & \pi(X, q) \end{array}$$

commutes. So $[\gamma]_\#$ maps the image of the left vertical map, $\pi(X, p)_{\tilde{p}}$ onto the image of the right vertical map, $\pi(X, p)_{\tilde{q}}$. Hence if one is a normal subgroup, then so is the other. Moreover, we see that if $q = p$, then $\pi(X, p)_{\tilde{q}}$ and $\pi(X, p)_{\tilde{p}})$ are conjugate subgroups of $\pi(X, p)$ and that we get a full conjugacy class of such subgroups if we let $\tilde{q}$ run over $f^{-1}(p)$.

If $\tilde{q} \in f^{-1}(p)$, then by Corollary 2.10 then there exists a covering map $\Phi : (\tilde{X}, \tilde{q}) \to \tilde{X}, \tilde{p})$ which commutes with f if and only if $\pi(X, p)_{\tilde{q}} \subset \pi(X, p)_{\tilde{p}}$. Since both these subgroups of $\pi(X, p)$ belong the same conjugacy class, the latter inclusion comes down to equality and so we have then a covering map $\Psi : (\tilde{X}, \tilde{p}) \to (\tilde{X}, \tilde{q})$ as well. The unicity clause of 2.9 applied to $\Psi\Phi$ and $\Phi\Psi$ shows that both compositions are the identity. In other words, Φ is then a Galois transformation. The action of Φ on $f^{-1}(p)$ equals that of an element of $\pi(X, p)$. We conclude that $f^{-1}(p)$ is an orbit of the Galois group $G(\tilde{X}/X)$ if and only if $\pi(X, p)_{\tilde{p}}$ is a normal subgroup of $\pi(X, p)$ and that $G(\tilde{X}/X)$ then may be identified with the factor group $\pi(X, p)/\pi(X, p)_{\tilde{p}}$. We noticed that the property: '$\pi(X, p)_{\tilde{p}}$ is a normal subgroup of $\pi(X, p)$' is independent of p. □

EXERCISE 2.5. Let $f : \tilde{X} \to X$ be a covering map between path connected spaces and let G be its Galois group. Prove that f is the composite of a Galois covering $\tilde{X} \to G\backslash\tilde{X}$ and a covering $G\backslash\tilde{X} \to X$. Show that if $\tilde{p} \in \tilde{X}$ has image $p \in X$, then

G can be indentified with the normalizer of $\pi(\tilde{X}, \tilde{p})$ in $\pi(X, p)$, that is the group of $g \in \pi(X, p)$ with $g\pi(\tilde{X}, \tilde{p})g^{-1} = \pi(\tilde{X}, \tilde{p})$.

Here is an amusing application of 2.9.

PROPOSITION 2.13. *Two lattices* $L, L' \subset \mathbb{C}$ *define isomorphic elliptic curves if and only if they are proportional, i.e.,* $L' = \lambda L$ *for some* $\lambda \in \mathbb{C} - \{0\}$.

PROOF. If we have a biholomorphic map $f : \mathbb{C}/L \to \mathbb{C}/L'$, then we can by composing f with a translation over $-f(0)$ assume that $f(0) = 0$. Denote the projections $\pi : \mathbb{C} \to \mathbb{C}/L$ and $\pi' : \mathbb{C} \to \mathbb{C}/L'$. Then 2.9 applied to $g := f\pi : \mathbb{C} \to \mathbb{C}/L'$ shows that there is a holomorphic $\tilde{g} : \mathbb{C} \to \mathbb{C}$ with $\pi'\tilde{g} = g$ and $\tilde{g}(0) = 0$. If $a \in L$, then $g(z + a) = g(z)$, and so $\tilde{g}(z + a) - \tilde{g}(z) \in L'$ for all $z \in \mathbb{C}$. Since $\tilde{g}$ is continuous, it follows that $\tilde{g}(z + a) - \tilde{g}(z)$ is constant in z. Hence for the derivative $\tilde{g}'$ we have $\tilde{g}'(z+a) = \tilde{g}'(z)$ for all $z \in \mathbb{C}$. Since a was taken arbitrary from L' it follows that $\tilde{g}'$ factors through a holomorphic function on $\mathbb{C}/L$. Such a function must be constant by 1.12, say equal to λ. Since $\tilde{g}(0) = 0$ it follows that $\tilde{g}(z) = \lambda z$. Since $g(L) \subset L'$, it follows that $\lambda L \subset L'$. But g is bijective and so $\lambda L = L'$. □

EXERCISE 2.6. Prove that every holomorphic map $f : \mathbb{C}/L \to \mathbb{C}/L'$ between elliptic curves is the composition of a homomorphism and a translation: $f(z + L) = \lambda z + \mu + L'$ for some $\mu \in \mathbb{C}/L'$ and $\lambda \in \mathbb{C}$ with $\lambda L \subset L'$.

EXERCISE 2.7. Give a bijection between the isomorphism classes of elliptic curves and the points of a Riemann surface $PSL(2, \mathbb{Z})\backslash\mathbb{H}$. (Use Exercise 1.4.)

Let X be a fixed connected, locally simply connected space with base point p. Suppose $f : \tilde{X} \to X$ is a covering map with $\tilde{X}$ connected and $\tilde{p} \in f^{-1}(p)$. We show that these covering data are entirely determined by the image $H \subset \pi(X, p)$ of $\Pi_f : \pi(\tilde{X}, \tilde{p}) \to \pi(X, p)$.

For $\tilde{q} \in \tilde{X}$ arbitrary, there is always a path γ from p such that the lift $\tilde{\gamma}_{\tilde{p}}$ has the property that it arrives at $\tilde{q}$: simply take for γ the image of a path $\tilde{\gamma}$ from $\tilde{p}$ to $\tilde{q}$; then $\tilde{\gamma}_{\tilde{p}} = \tilde{\gamma}$. According to Proposition 2.7 two paths γ and γ' departing at p have this property if and only if they arrive at the same point and $[\gamma]^{-1}[\gamma'] \in H$. In other words, if $\Pi_X(p)$ denotes the set of homotopy classes of paths departing at p (i.e., the disjoint union $\cup_{q \in X} \Pi_X(p, q)$), the we have a bijection

$$\Pi_X(p)/H \cong \tilde{X}, \quad [\gamma]H \mapsto \tilde{\gamma}_{\tilde{p}}(1).$$

We show that under this bijection the covering map $\tilde{X} \to X$ and the topology of $\tilde{X}$ can be described solelyl in terms of the lefthand side. The former is clearly given by assigning to $[\gamma] \in \Pi_X(p)$ its end point $\gamma(1)$ and the topology is recovered as follows: if is a $U \subset X$ simply connected open subset, then choose $q_0 \in U$. For every $q \in U$, $\Pi_U(q_0, q)$ is a singleton. If we call this element $[\delta_q]$, then we have a bijection

$$U \times (\Pi_X(p, q_0)/H) \cong \cup_{q \in U} \Pi_X(p, q)/H, \quad (q, [\gamma]H) \mapsto [\delta_q][\gamma]H$$

whose composition with the bijection above is a bijection onto $f^{-1}U$. If we give $\Pi_X(p, q_0)/H$ the discrete topology and $U \times (\Pi_X(p, q_0)/H)$ the product topology, then this bijection is a homeomorphism.

In view of this we can also proceed in the opposite direction by starting with a subgroup H of $\pi(X, p)$.

PROPOSITION 2.14. *Let X be a connected, locally simply connected space and $p \in X$. Then for every subgroup $H \subset \pi_1(X,p)$ there exists a covering map $f : \tilde{X} \to X$ with $\tilde{X}$ connected and a $\tilde{p} \in f^{-1}(p)$ such that $\pi(X,p)_{\tilde{p}} = H$. The pair $(f : \tilde{X} \to X, \tilde{p})$ is unique in the sense that if $f' : (\tilde{X}', \tilde{p}') \to (X,p)$ is also a solution, then there is precisely one homeomorphism $h : (\tilde{X}', \tilde{p}') \cong (\tilde{X}, \tilde{p})$ for which $f' = fh$.*

If we apply this to the trivial subgroup $\{1\} \subset \pi(X,p)$, then we find that in this situation there exists a covering map $\tilde{X} \to X$ with $\tilde{X}$ simply connected.

DEFINITION 2.15. We say that a covering map $\pi : \hat{X} \to X$ is *universal* if $\hat{X}$ is simpy connected and X is locally simply connected.

So this is a Galois covering whose Galois group can be identified (after a choice of a base point $\hat{p}\hat{X}$) with the fundamental group of X with base point $\pi(\hat{p})$.

EXAMPLE 2.16. The covering map

$$\pi : \mathbb{R}^n \to T^n, \quad (x_1, \dots, x_n) \mapsto (e^{2\pi i x_1}, \dots, e^{2\pi i x_n})$$

is universal. Its Galois group is the translation group $\mathbb{Z}^n$, hence the fundamental group of T^n is isomorphic to $\mathbb{Z}^n$.

In the exercises below $\pi : \tilde{X} \to X$ is a covering map with $\tilde{X}$ connected, X locally simply connected, $\tilde{p} \in \tilde{X}$ and $p = \pi(\tilde{p})$.

EXERCISE 2.8. Prove that every point of $\tilde{X}$ has a neighborhood $\tilde{U}$ such that $g\tilde{U} \cap \tilde{U} = \emptyset$ for all Galois transformations $g \neq \mathbf{1}$.

EXERCISE 2.9. Two subgroups $H_1 \subset \pi_1(X,p_1)$, $H_2 \subset \pi_1(X,p_2)$ with $p_1, p_2 \in X$ are said to be *conjugate* if there is a path α from p_1 to p_2 with $H_2 = [\alpha]H_1[\alpha]^{-1}$. Let $f : X \to X$ be a homeomorphism of X onto itself.

(a) Prove that f lifts to a homeomorphism from $\tilde{X}$ onto $\tilde{X}$ if and only if $\pi(\tilde{X}, \tilde{p}) \subset \pi(X,p)$ and $\Pi_f \pi(\tilde{X}, \tilde{p}) \subset \pi(X, f(p))$ are conjugate.

(b) Prove that for every Galois transformation h, $h\tilde{f}$ is also a lift of f and that every solution of (a) is thus obtained.

EXERCISE 2.10. Determine the automorphism group of the two elliptic curves $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and $\mathbb{C}/(\mathbb{Z} + \rho\mathbb{Z})$, where $\rho := e^{2\pi i/3}$. Prove that an elliptic curve E admits an automorphism that is *not* of the type $x \mapsto \pm x + a$ (for some $a \in E$) must be isomorphic to one of these.

EXERCISE 2.11. Compute the fundamental group of P^m (you may use the fact that for $m \geq 2$, S^m is simply connected).

EXERCISE 2.12. Let $f : S \to S'$ be a non-constant holomorphic map between connected compact Riemann surfaces.

(a) Prove that the set Σ of $p \in S$ where f fails to be local isomorphism is finite.

(b) Prove that $f : S - f^{-1}f(\Sigma) \to S' - f(\Sigma)$ is a covering map.

EXERCISE 2.13. We write Δ for the open complex unit disk and Δ^* for $\Delta - \{0\}$.

(a) Let $\pi : W \to \Delta^*$ be a covering map of finite degree d with W connected. Prove that there exists a homeomorphism $h : \Delta^* \cong W$ such that $\pi h(z) = z^d$.

(b) Let $\hat{W}$ be the disjoint union of W and a singleton $*$. Prove that $\hat{W}$ can be given the structure of a Riemann surface that makes the map which extends π by mapping $*$ to 0 is holomorphic. Show also that this structure must be unique.

(c) Let S be a Riemann surface , $\Sigma \subset S$ een discrete subset, $S' := S - \Sigma$, and $\pi' : \tilde{S}' \to S'$ a covering map of finite degree d. Prove that there exists a Riemann surface $\tilde{S} \supset \tilde{S}'$ and a *proper* holomorphic extension $\pi : \tilde{S} \to S$ of π such that $\pi^{-1}S' = \tilde{S}'$. (A map between topological spaces is said to be *proper* if the preimage of every compact set is compact.) (Hint: choose at every $s \in \Sigma$ a chart $(U, z : U \cong \Delta)$ with $U \cap \Sigma = \{s\}$ and s mapping to 0, and introduce for every connected component $\pi'^{-1}(U - \{s\})$ a point for $\tilde{S}'$.)

(d) Prove that in (c) every Galois transformation of π' extends to an automorphism of $\tilde{S}$ and conclude that the Galois group of $\tilde{S}' \to S'$ can be identified with the group of automorphisms g of S with $\pi g = \pi$.

EXERCISE 2.14 (Supplement to Exercise 2.2). Let S be a connected Riemann surface and let $F = t^d + c_1 t^{d-1} + \cdots + c_d \in \mathcal{H}(S)[t]$ be an irreducible monic polynomial of degree $d > 0$. We shall also consider F as a holomorphic function on $\mathbb{C} \times S$.

(a) Let $S' \subset S$ be the set of $p \in S$ for which $F(t,p)$ has no double root in t. Prove that $S - S'$ is discrete.

(b) Let $S'_F := \{(t,p) \in \mathbb{C} \times S' : F(t,p) = 0\}$. According to Exercise 2.2, the projection $\pi' : S'_F \to S'$ is a covering map of degree d with S'_F connected. Following Exercise 2.13 π' extends to a proper holomorphic map between Riemann surfaces $\pi : S_F \to S$. Prove that the first projection $t : S'_F \to \mathbb{C}$ extends holomorphically to S_F and that the map $S_F \to \mathbb{C} \times S$ thus obtained maps to the zero set of F.

(c) Conclude that F has a root in $\mathcal{H}(S_F)[t]$.

4. Computation of some fundamental groups

An interesting class of examples is provided by group theory. A *directed graph* may be given by a set $\mathcal{V}$ (of *vertices*) and a subset $\mathcal{K} \subset \mathcal{V} \times \mathcal{V}$ (of *directed edges*) that is disjoint from the diagonal: its geometric realization is then obtained by connecting two vertices p and q by an interval $[p,q]$ precisely when $(p,q) \in \mathcal{K}$ (the direction is then from p to q). Such a graph arises from a group G and a subset $S \subset G - \{e\}$: the vertex set is G and the set of directed edges is the collection $\{(g, gs) \mid g \in G, s \in S\}$. So the geometric realization $\tilde{X}(G,S)$ of this graph has its vertices labeled by G and its edges labeled by $G \times S$. The group G acts on $\tilde{X}(G,S)$ by $h \in G : g \mapsto hg$ (this maps the edge $[g, gs]$ to the edge $[hg, hgs]$). It is easily checked that this action is freely discontinuous, and so the orbit map $\tilde{X}(G,S) \to X(G,S) := G \backslash \tilde{X}(G,S)$ is a covering map. We shall denote the imag In view of the fact that $[g, gs] = g[e,s]$, $X(G,S)$ is already a quotient space of $\cup_{s \in S}[e,s]$ and is obtained by identifying each end point $s \in S$ with e: so $X(G,S)$ is homeomorphic to a collection of circles—one every $s \in S$—which have one point in common.

For instance, if $G = \mathbb{Z}$ and $S = \{1\}$, then $\tilde{X} = \cup_{n \in \mathbb{Z}}[n, n+1] \cong \mathbb{R}$ and $n \in \mathbb{Z}$ acts as $t \mapsto t + n$. The orbit space is $\mathbb{R}/\mathbb{Z}$ which is homeomorphic to a circle.

PROPOSITION 2.17. *The graph $\tilde{X}(G,S)$ is connected if and only if S generates G as a group; it is a tree (or what amounts to the same, simply connected) if and only if S generates G freely.*

PROOF. To say that $\tilde{X}(G,S)$ is connected means that we can connect the vertex 0_e with any other vertex 0_g by means of a edge path, that is, a sequence of edges $0_e, 0_{g_1}, \dots, 0_{g_n} = 0_g$, where $g_i = g_{i-1}s_i$ or $g_{i-1} = g_i s_i$ for some $s_i \in S$ (and we stipulate that $g_0 = e$). We may write this as $g_i = g_{i-1}s_i^{\varepsilon_i}$ with $\varepsilon_i \in \{\pm 1\}$. This shows that

$$g = g_n = g_{n-1}s_n^{\varepsilon_n} = \cdots = g_0 s_1^{\varepsilon_1}\cdots s_n^{\varepsilon_n} = s_1^{\varepsilon_1}\cdots s_n^{\varepsilon_n}$$

and hence an element of the subgroup generated by S. Conversely, any signed word in the elements of S defines an edge path as above.

Saying that $\tilde{X}(G,S)$ is a tree amounts to saying that the edge path is unique if we do not allow backtracking in the sense that we prohibit that $g_{i+1} = g_{i-1}$. This means that any $g \in G$ is uniquely written as a signed word $s_1^{\varepsilon_1}\cdots s_n^{\varepsilon_n}$ in the elements of S, provided we prohibit that $s_{i+1}^{\varepsilon_{i+1}} \neq s_i^{-\varepsilon_i}$. This last property says that S generated G freely. □

COROLLARY 2.18. *Let n be a positive integer and X_n the space obtained from n circles S^1_k, $k = 1,\dots,n$ by identifying the points $1_k \in S^1_k$. If the common image of these points in X_n is denoted 1 and $a_k \in \pi(X_n, 1)$ is the class of $t \mapsto (e^{2\pi i t})_k$, then $a_1,\dots,a_n$ generate $\pi(X_n,1)$ freely.*

PROOF. Let F_n be the group freely generated by the symbols $S_n := \{s_1,\dots,s_n\}$. Then $\tilde{X}(F_n,S_n) \to X(F_n,S_n)$ is a covering with F_n as Galois group and $X(F_n,S_n)$ is homeomorphic to X_n. Following Proposition 2.17 is $\tilde{X}(F_n,S_n)$ simply connected. So F_n can be indentified with $\pi_1(X_n,1)$. This identification sends s_k to a_k. □

We now compute the fundamental group of a compact orientable surface. This will be based on:

PROPOSITION 2.19. *Let A be a connected locally simply connected space and $g : S^1 \to A$ a continuous map. Let X be obtained from the disjoint union of A and the 2-disk D^2 by identifying $y \in S^1$ with $g(y)$ (endowed with the quotient topology). Then the inclusion $i : A \subset X$ yields a surjection*

$$\Pi_i : \pi(A, g(1)) \to \pi(X, g(1)),$$

whose kernel is the normal subgroup of $\pi(A, g(1))$ generated by the class of g.

PROOF. We do this in four steps with the union of the last two steps giving the proposition. Write p for $g(1)$, $j : D^2 \to X$ for the obvious map and U for $X - j(\{0\})$.

Step 1: The inclusion $A \subset U$ induces an isomorphism on fundamental groups.

Define a map $F : [0,1] \times U \to U$ by

$$F(t,x) = \begin{cases} x & \text{if } x \in A;\\ j(t\frac{u}{|u|} + (1-t)u), & \text{if } x = j(u). \end{cases}$$

Observe that F is well-defined and continuous. It has the property that F_0 is the identity map of U, F_1 maps U to A and F_t is the identity on A for all t. So if $\alpha : [0,1] \to U$ is a loop at p, then $F_t\alpha$ is a homotopy from α to a loop at A. So $A \subset U$ induces a surjection of fundamental groups. This is also an injection, because $F_1 : U \to A$ is a left inverse for this inclusion.

Step 2: Construction of a Galois covering of X with Galois group $G := \pi(A,p)/N$.

According to 2.14 there exists a connected covering map $f : (\tilde{U}, \tilde{p}) \to (U,p)$ such that N is the image of $\pi(\tilde{U},\tilde{p}) \to \pi(U,p) \cong \pi(A,p)$. Since N is a normal subgroup, this will be a Galois covering with Galois group $G = \pi(A,p)/N$. We show that the covering admits a continuous section σ over $U - A$. This suffices, for such a section defines a trivialisation of the G-covering $f^{-1}(U-A) \to U - A$

by $(U - A) \times G \cong f^{-1}(U - A)$, $(x, g) \mapsto g\sigma(x)$ and this trivilisation enables us to extend f to a Galois covering $\tilde{X} \to X$ that is trivial over U (so that G keeps acting) by glueing on $U \times G \to U$.

To this end, we consider the restriction $(D^2 - \{0\}, 1) \to (U, p)$ of j. The inclusion $(S^1, 1) \subset (D^2 - \{0\}, 1)$ induces an isomorphism on fundamental groups. The former is generated by the counterclockwise loop, so is the latter. But the image of this loop under the map $\pi(D^2 - \{0\}, 1) \to \pi(U, p)$ lies in N by definition. Hence the image of $\pi(D^2 - \{0\}, 1) \to \pi(U, p)$ lies in N and so by Proposition 2.7 there is lift of $D^2 - \{0\} \to U$ over f to a map $\tilde{\sigma} : D^2 - \{0\} \to \tilde{U}$ that maps p to $\tilde{p}$. Since $\mathrm{Int}(D^2) - \{0\}$ maps homeomorphically onto $U - A$ the restriction of $\tilde{\sigma}$ to $\mathrm{Int}(D^2) - \{0\}$ yields the desired section.

Step 3: The kernel of $\pi(A, p) \to \pi(X, p)$ equals N.

Since we have constructed a connected Galois covering $\tilde{X} \to X$ with Galois group G, the latter must arise as a quotient of $\pi(X, p)$. But we also have $G \cong \pi(A, p)/N$ and so this means that N must contain the kernel of the homomorphism $\Pi_i : \pi(A, p) \to \pi(X, p)$. On the other hand, the loop defined by g is evidently homotopic to the constant loop ($(s, u) \in [0, 1] \times S^1 \mapsto j((1 - s)u + s)$ does the job) and hence $[g] \in \mathrm{Ker}\, \Pi_i$. Since N is the normal subgroup generated by $[g]$, N is also contained in the kernel of $\pi(A, p) \to \pi(X, p)$.

Step 4: The homomorphism $\pi(A, p) \to \pi(X, p)$ is onto.

If $\alpha : [0, 1] :\to X$ is a loop at p, then there is a subdivision $0 = t_0 < t_1 < \cdots < t_N = 1$ of $[0, 1]$ such that $\omega([t_{i-1}, t_i])$ lies in U or in $j(\mathrm{Int}\, D^2)$. In the last case there is a homotopy in $j(\mathrm{Int}\, D^2)$ from the path $\alpha|[t_{i-1}, t_i]$ to one which avoids $j(0)$. This makes α is homotopic to a path which avoids $j(0)$. This shows that $[\alpha] \in \pi(X, p)$ is in the image of $\pi(A, p)$. □

REMARKS 2.20. The construction that led from A and $g : S^1 \to A$ to X is usually referred to the *attaching a 2-cell to A (with attaching map g)*.

The covering space $\tilde{X}$ constructed in Step 3 is simply connected because the Galoisgroep G of $\tilde{X} \to X$ is also the fundamental group of X.

Let M be a closed surface of genus g with base point $o \in M$. We draw on M $2g$ topological circles $A_1, A_{-1}, \dots, A_g, A_{-g}$ as in *fig. ?*. So they have in common the point o, but are otherwise disjoint. We orient them as indicated. If $A \subset M$ denotes their union, then according to 2.18 $\pi_1(A, o)$ is freely generated by the classes $\{\alpha_i = [A_i]\}_{i=\pm 1}^{\pm g}$.

If we cut M open along these circles, then we find that M can be obtained as a quotient space of D^2. If we denote the restriction of the projection $D^2 \to M$ to $S^1 \to A$ by g, then we see that M is gotten by attaching a 2-cell to A as in Proposition 2.19. The class of g in $\pi(A, o)$ is the inverse of the product of commutators $(\alpha_1, \alpha_{-1}) \cdots (\alpha_g, \alpha_{-g})$. So from Proposition 2.19 we see that:

PROPOSITION 2.21. *The fundamental group $\pi(M, o)$ has the following presentation: it is generated by the classes $\{\alpha_i = [A_i]\}_{i=\pm 1}^{\pm g}$, subject to the single relation $(\alpha_1, \alpha_{-1}) \cdots (\alpha_g, \alpha_{-g}) = 1$.*

We can see that the fundamental group determines the genus simply by abelianizing the former. It will then be convenient to dispose over he following definition.

DEFINITION 2.22. The *first homology group* of a path connected space X, denoted, $H_1(X)$, is the abalianized (and additively written) fundamental group of X, denoted $H_1(X)$.

REMARKS 2.23. Property 2.4-ii implies that $H_1(X)$ is independent of a choice of base point.

An element of $H_1(X)$ can be represented by a homotopy class of maps $\alpha : S^1 \to X$ (no fixed base point here). Two such are then added by picking representatives $\alpha, \beta : S^1 \to X$ with $\alpha(1) = \beta(1)$ and then the sum is the represented by the composite $\alpha\beta$. In an algebraic topology course this group is introduced differently, but it is not hard to show that the two definitions agree for path connected spaces.

COROLLARY 2.24. *The first homology group* $H_1(M)$ *of* M *is the free abelian group of rank* $2g$ *generated by the classes* $a_1, a_{-1}, \ldots, a_g, a_{-g}$ *of* $\alpha_1, \alpha_{-1}, \ldots, \alpha_g, \alpha_{-g}$*; in particular,* g *is a homotopy invariant of* M.

EXERCISE 2.15. Let M be a closed surface of genus g and let $p \in M$. Prove that the fundamental group of $M - \{p\}$ is a free group on $2g$ generators.

EXERCISE 2.16. (a) Prove that the Klein bottle admits a degree 2 covering by the torus.

(b) Prove that the fundamental group of the Klein bottle has a subgroup of index two that is free abelian of rank two.

(c) Show that the Klein bottle can be obtained by attaching a 2-cell to a bouquet of two circles.

(d) Find a presentation of the fundamental group of the Klein bottle that exhibits the property found in (b).

CHAPTER 3

Forms and integrals

1. Differentials

A *differential* on an open subset U of $\mathbb{R}^m$ is an expression of the form

$$\omega = \sum_{i=1}^{m} p_i dx_i,$$

where $p_1, \ldots, p_n$ are differentiable functions. Such a differential can be integrated over a differentiable path $\gamma : [a, b] \to U$: by definition

$$\int_\gamma \omega := \int_a^b \sum_{i=1}^{m} p_i(\gamma(t))\dot{\gamma}_i(t)dt.$$

It is easy to check that this does not change after a differentiable reparameterization of γ.

It is clear that the set of differentials on U (which we shall denote by $\mathcal{E}^1(M)$) not only form a vector space (infinite dimensional if $U \neq \emptyset$), but even admit multiplication by differentiable functions on U: $\mathcal{E}^1(U)$ is a module over the ring of differentiable functions on U (this ring is sometimes denoted $\mathcal{E}^0(U)$ instead of $C^\infty(U)$). As such it is freely generated by $dx_1, \ldots, dx_m$.

An example of a differential is the total differential of a differentiable function $\phi : U \to \mathbb{R}$:

$$d\phi := \sum_i \frac{\partial \phi}{\partial x_i} dx_i.$$

It is clear that $d\phi$ is constant zero precisely when ϕ is locally constant. A differential of the form $d\phi$ is called *exact*. Notice that

$$\int_\gamma d\phi = \phi\gamma(b) - \phi\gamma(a) \tag{1}$$

We say that the differential $\omega = \sum_i p_i dx_i$ is *closed* if we have for all i, j that

$$\frac{\partial p_i}{\partial x_j} = \frac{\partial p_j}{\partial x_i}.$$

For $m = 1$ this condition is empty and so every differential is then closed. It is clear that an exact differential is closed. Locally the converse is also true:

LEMMA 3.1. *A closed differential on an open block in $\mathbb{R}^m$ is exact on that block*

PROOF. Let $\omega = \sum_i p_i dx_i$ be a closed differential on the open block $B \subset \mathbb{R}^m$. Choose $u \in B$ fixed and let for $x \in B$,

$$\phi(x) := \sum_{i=1}^{m} \int_{u_i}^{x_i} p_i(x_1, \ldots, x_{i-1}, t, u_{i+1}, \ldots, u_m)dt.$$

Now check that $d\phi = \omega$. □

In order to transfer this notion to smooth manifolds we must first understand how it transforms under a differentiable map. Let be given an open $V \subset \mathbb{R}^n$ and a differentiable map f from V to an open $U \subset \mathbb{R}^m$. If $\omega = \sum_i p_i dx_i$ is a differential on U, then the *pull-back of* ω *along* f is by definition the differential

$$f^*\omega(y) := \sum_{i,j} p_i f(y).\frac{\partial f_i}{\partial y_j}(y)dy_j.$$

If $\gamma : [a, b] \to U$ is a differentiable path, then $\gamma^*\omega$ is the integrand of $\int_\gamma \omega$, and so

$$\int_\gamma \omega = \int_a^b \gamma^*\omega$$

This indicates that our notion of pull-back is a natural one.

PROPERTIES 3.2. We list a few.

(i) If $g : W \to V$ is a differentiable map from an open $W \subset \mathbb{R}^p$ to V, then $(fg)^*\omega = g^*f^*\omega$ (this follows from the chain rule).
(ii) If $\phi : U \to \mathbb{R}$ is differentiable, then $f^*d\phi = d(\phi f)$ (and so f^* takes exact differentials to exact differentials).
(iii) If ω is closed, then so is $f^*\omega$.
(iv) For every differentiable $\gamma : [a, b] \to V$ we have:

$$\int_\gamma f^*\omega = \int_{f\gamma} \omega$$

(v) If γ' is a path in U departing where γ arrives, then

$$\int_{\gamma'\gamma} \omega = \int_\gamma \omega + \int_{\gamma'} \omega.$$

Property (iv) follows from (i) and formula (1):

$$\int_{f\gamma} \omega = \int_a^b (f\gamma)^*\omega = \int_a^b \gamma^*f^*\omega = \int_\gamma f^*\omega,$$

but can also be checked directly.

As to Property (v), if $\delta : [0, 2] \to U$ is the path that we get by first traversing γ and then γ', then it is clear that the integral of ω over δ is the sum of those over γ and γ'. From the fact that $\delta(t) = \gamma'\gamma(2t)$ (a reparameterization) we easily deduce that $\int_\delta \omega = \int_{\gamma'\gamma} \omega$.

The preceding suggests how we can transfer this notion to a manifold.

DEFINITION 3.3. A *differential* ω on a manifold M assigns to every differentiable chart (U, κ) of M a differential ω_κ on $\kappa(U)$ such that for every coordinate change $\kappa'\kappa^{-1} : \kappa(U \cap U') \to \kappa'(U \cap U')$ we have:

$$(\kappa'\kappa^{-1})^*\omega_{\kappa'} = \omega_\kappa \tag{2}$$

REMARKS 3.4. (i) A differential is already given if we have a collection $\{\omega_{\kappa_i}\}_i$ for a subatlas $(U_i, \kappa_i)_i$ that satisfies (2), for it easily follows from (i) that this collection uniquely extends to a collection $\{\omega_\kappa\}_\kappa$ that still satisfies (2).

(ii) We expect that a differentiable function $\phi : M \to \mathbb{R}$ defines a differential $d\phi$ on M and this is indeed the case: take for $(d\phi)_\kappa$ the (ordinary) total differential of $\phi\kappa^{-1} : \kappa(U) \to \mathbb{R}$. A differential thus obtained is called *exact*.

(iii) A differential ω on M can be multiplied by a differentiable function $\phi : M \to \mathbb{R}$ to produce another differential $\phi\omega$: simply let $(\phi\omega)_\kappa$ be $\phi\kappa^{-1}.\omega_\kappa$. Thus the set $\mathcal{E}^1(M)$ of differentials on M becomes a module over the ring $\mathcal{E}^0(M)$ of differentiable functions on M.

(iv) Our definition of a differential looks a bit artificial and there are indeed more intrinsic ways of introducing this notion. One way is to think of a differential ω on M as assigning to any differentiable path in M 'given up to first order' a real number subject to certain differentiability conditions. (For $\omega = \sum_i p_i dx_i$ on $U \subset \mathbb{R}^m$ this assigns to the curve $\gamma : (a, b) \to U$, and $c \in (a, b)$, the value $\sum_i p_i(\gamma(c))\dot{\gamma}_i(c)$. This depends clearly only on the first order behaviour of γ at c.)

EXERCISE 3.1. Let ϕ and ψ be differentiable functions on a manifold M. Prove that $d(\phi.\psi) = \phi d\psi + \psi d\phi$.

REMARK 3.5. We mention without proof that in case M has a countable basis for its topology (a property always verified in practice) then $\mathcal{E}^1(M)$ is as a $\mathcal{E}^0(M)$-module generated by finitely many total differentials. Moreover $\mathcal{E}^1(M)$ is the quotient of the free $\mathcal{E}^0(M)$-module generated by the total differentials modulo the submodule generate by the expressions $d(\phi.\psi) - \phi d\psi - \psi d\phi$. So we might forget our earlier definition and say that a typical element of $\mathcal{E}^1(M)$ is formal expression $\sum_{i=1}^N \phi_i d\psi_i$ with $\phi_i, \psi_i \in \mathcal{E}^0(M)$ subject to the relations $d(\phi.\psi) = \phi d\psi + \psi d\phi$. One says that $d : \mathcal{E}^0(M) \to \mathcal{E}^1(M)$ is the universal derivation of $\mathcal{E}^0(M)$. This is another characterization of $\mathcal{E}^1(M)$.

We may 'pull-back' a differential ω on M along a differentiable map of manifolds $f : N \to M$ as follows: choose an atlas $\{(V_i, \lambda_i)\}$ for N with the property that for every i, $f(V_i)$ is contained in the domain U_i of a chart κ_i of M. Then a differential $f^*\omega$ on N is defined by $(f^*\omega)_{\lambda_i} = (\kappa_i f \lambda_i^{-1})^* \omega_{\kappa_i}$. Check that this is independent of the choices made.

If we are given a differential ω on M, then Formula (1) suggests a definition for integrating this differential along a differentiable path $\gamma : [a, b] \to M$:

$$\int_\gamma \omega := \int_a^b \gamma^*\omega. \tag{3}$$

In more concrete terms: if $a = t_0 < t_1 < \cdots < t_N = b$ is a division of $[a, b]$ such that $\gamma([t_{i-1}, t_i])$ is the domain U_i of a chart κ_i, then

$$\int_\gamma \omega := \sum_i \int_{\kappa_i \gamma|[t_{i-1}, t_i]} \omega_{\kappa_i}. \tag{3'}$$

We say that ω is *closed* if every ω_κ is so. Property (iii) shows that this only needs to be verified for κ running through an atlas and is we choose these charts as to map onto an open block, then according to Lemma 3.1 we can even arrange that each is ω_{κ_i} is exact, say equal to the total differential of some $\phi_\kappa : \kappa(U) \to \mathbb{R}$. This means that $\omega|U$ is equal to $d(\phi_\kappa \kappa)$. In other words, to say that ω is closed is equivalent to saying it is locally exact. An exact differential is clearly closed.

We list the natural generalization of the properties 3.2 to the manifold setting.

PROPERTIES 3.6. Let $g : P \to N$ and $f : N \to M$ be differentiable maps between manifolds and let ω be a differential on M.

(i) $(fg)^*\omega = g^*f^*\omega$.
(ii) If $\phi : M \to \mathbb{R}$ is a differentiable function , then $f^*d\phi = d(\phi f)$.
(iii) If ω is closed, then so is $f^*\omega$.
(iv) If $\gamma : [a, b] \to N$ is a differentiable path, then $\int_\gamma f^*\omega = \int_{f\gamma} \omega$.
(v) If γ' is a path in M departing where γ arrives, then

$$\int_{\gamma'\gamma} \omega = \int_\gamma \omega + \int_{\gamma'} \omega.$$

Property (iv) follows from (i) and the defining formula (3). It follows from (i) and (ii) that for a differentiable path $\gamma : [a, b] \to M$,

$$\int_\gamma d\phi = \int_a^b \gamma^* d\phi = \int_a^b d(\phi\gamma) = \phi\gamma(b) - \phi\gamma(a). \tag{4}$$

In particular, $d\phi = 0$ implies that ϕ is locally constant .

Notice that (iv) implies that an integral over a path is not affected by differentiable reparameterization.

Suppose that ω is a closed differential on M, and let $\gamma : [a, b] \to M$ be a differentiable path that starts in $p \in M$ and ends in $q \in M$. Then $\int_\gamma \omega$ may be computed as follows. Cover the image γ by finitely many open subsets on which ω is exact: so we obtain a division $a = t_0 < \cdots < t_N = b$, open subsets $U_i \subset M$ and and differentiable functions $\{\phi_i : U_i \to \mathbb{R}\}_{i=1}^N$ such that $\gamma_i([t_{i-1}, t_i]) \subset U_i$ and $d\phi_i = \omega|U_i$. Denote by C_i the path component of $\gamma(t_i)$ in $U_i \cap U_{i+1}$. Since $d(\phi_{i+1} - \phi_i)$ is zero $U_i \cap U_{i+1}$ it is locally constant there and hence constant, say c_i, on C_i (for C_i is connected). Formulae (3′) and (4) show that

$$\int_\gamma \omega = \sum_{i=1}^N \big(\phi_i\gamma(t_i) - \phi_i\gamma(t_{i-1})\big) = -\phi_1(p) - c_1 - \cdots - c_N + \phi_N(q) \tag{5}$$

Let us refer to such a sequence $(U_1, C_1, U_2, C_2, \ldots, C_{N-1}, U_N)$ as a *Cech sequence* for γ. Formula (5) implies that if two paths from p to q have a common Cech sequence, then they yield the same integral. The notion Cech sequence allows us to abandon the requirement that γ be differentiable: continuity suffices. It is easy to check that two Cech sequences for a continuous γ define the same integral, so that $\int_\gamma \omega$ still makes sense. More is true:

PROPOSITION 3.7. *If ω is a closed differential on the manifold M, then the integral $\int_\gamma \omega$ only depends on the homotopy class of γ.*

PROOF. Let $(\gamma_s)_{0\le s\le 1}$ be a path homotopy of paths from p to q. Then a Cech sequence for γ_s ($s \in [0, 1]$) will still be one for $\gamma_{s'}$ for s' in a neighborhood I_s of s in $[0, 1]$. So the integral of ω over $\gamma_{s'}$ is constant on I_s. Since $s \in [0, 1]$ was arbitrary, the connectedness of $[0, 1]$ implies that this is constant on all of $[0, 1]$. In particular,

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

□

Assume that M is connected and let ω be a closed differential on M. If $p \in M$, then the previous proposition and Property 3.6-v imply that integration of ω defines a homomorphism

$$[\alpha] \in \pi_1(M, p) \mapsto \int_\alpha \omega \in \mathbb{R}$$

Since $\mathbb{R}$ is an abelian group, this factors through a homomorphism $H_1(M) \to \mathbb{R}$.

LEMMA 3.8. *This homomorphism is constant zero* $\iff$ ω *is exact.*

PROOF. $\Leftarrow$) If $\omega = d\phi$, then we have that for any loop α at p, $\int_\alpha \omega = \phi(p) - \phi(p) = 0$.

$\Rightarrow$) For any point $q \in M$ we choose a path γ_q from p to q. Then $\phi(q) := \int_{\gamma_q} \omega$ only depends on q: another choice for γ_q yields an integral whose difference with the first integral is the integral over a loop and hence zero by assumption. We claim that ϕ is differentiable and that $\omega = d\phi$. For an arbitrary $q_0 \in M$ there is a neighborhood $U \ni q_0$ and a differentiable function $\psi : U \to \mathbb{R}$ such that $\omega|U = d\psi$. If $q \in U$, then for every path δ_q in U from q_0 to q we have

$$\phi(q) - \phi(q_0) = \int_\delta \omega = \int_\delta d\psi = \psi(q) - \psi(q_0).$$

So $\phi|U$ and ψ differ by a constant and hence $\phi|U$ is differentiable en $d\phi|U = d\psi = \omega|U$. □

EXERCISE 3.2. Let M be a connected manifold and let ω be a closed differential on M. Prove that there exists a Galois covering $f : \tilde{M} \to M$ with abelian Galois group (this is often abbreviated as an *abelian covering*) such that $f^*\omega$ is exact. (Remark: This makes precise the following phenomenon that one for instance encounters in complex function theory: if we try to integrate ω, that is, try to write it as the total differential of a differentiable function, then locally there is no problem, but globally we may get into trouble: the function might become multi-valued. A simple example is the differential $d\theta$ on S^1, where θ is the angular parameter, which indeed cannot exist as a differentiable function on all of S^1. But passage to a covering, in this case $\mathbb{R} \to S^1$, eliminates the multivaluedness.)

We found in 2.24 that for a compact surface M of genus g, $H_1(M) \cong \mathbb{Z}^{2g}$; a basis may be given by the maps $A_1, \ldots, A_{-g}$ in fig. ?.

DEFINITION 3.9. The *first De Rham cohomology group* of M, denoted $H^1(M, \mathbb{R})$, is the space of closed differentials on M modulo exact differentials.

So by 3.8 we have an injective homomorphism

$$\int : H^1(M, \mathbb{R}) \to \mathrm{Hom}(H_1(M), \mathbb{R}).$$

It is in fact an isomorphism. We will only check this for the circle and for a surface of genus g. We note that for a circle the group $H_1(S^1)$ is freely generated by the parameterization of S^1 by arc length $\gamma : [0, 1] \to S^1$, $\alpha(t) = (\cos(2\pi t), \sin(2\pi t))$. Then $\eta := d\gamma$ is a closed differential on S^1 and the associated homomorphism $H_1(S^1) \to \mathbb{R}$ sends $[\gamma]$ to $\int_\gamma \eta = 1$. So $\int$ is nonzero and hence surjective.

PROPOSITION 3.10. *For a compact surface* M *of genus* g *the homomorphism*

$$\int : H^1(M, \mathbb{R}) \to \mathrm{Hom}(H_1(M), \mathbb{R})$$

is an isomorphism, in particular $H^1(M,\mathbb{R})$ *is of dimension* $2g$.

PROOF. It suffices to prove that $\int$ is surjective. For $g = 0$ there is not much to do, as the fundamental group is then trivial. Therefore, suppose that $g > 0$. So we need to find $2g$ closed differentials $\eta_{\pm 1},\dots,\eta_{\pm g}$ on M for which $\int_{A_i}\eta_j = \delta_{i,j}$.

First choose a differential η on S^1 which vanishes on a neighborhood of $-1 \in S^1$ and for which $\int_{S^1}\eta = 1$. We subsequently choose for $i = 1,\dots,g$ an open neighborhood U_i of the union of (the images of) A_i and A_{-i} in M which is diffeomorphic to the punctured torus $S^1 \times S^1 - \{(-1,-1)\}$ under a diffeomorphism H_i for which $H_iA_i(\theta) = (\theta, 0)$ and $H_iA_{-i}(\theta) = (0,\theta)$. Moreover we take $U_1,\dots,U_g$ disjoint. Then H_i can be extended continuously to a map $M \to S^1 \times S^1$ by sending the complement of U_i to $(-1,-1)$ (zie fig. ?). We denote the components of this extension of H_i by (h_i, h_{-i}), so that we have defined $h_j : M \to S^1$ for $j \in \{\pm 1,\dots,\pm g\}$. We claim that the $\eta_j := h_j^*\eta$ are as desired: since for $j \neq k$, h_jA_k is constant, we have $\int_{A_k}\eta_j = 0$. On the other hand, $h_jA_j : S^1 \to S^1$ is the identity map so that $\int_{A_j}\eta_j = \int_{S^1}\eta = 1$. □

2. 2-forms and the exterior derivative

Let V be a real vector space of dimension m. A bilinear function $A : V \times V \to \mathbb{R}$ is called *alternating* if $A(v,w) = -A(w,v)$. The alternating bilinear functions on V make up a vector space that we shall denote by $\wedge^2 V^*$. For every pair linear functions $\xi, \xi' \in V^*$ on V we define their *exterior product* $\xi \wedge \xi' \in \wedge^2 V^*$ by

$$\xi \wedge \xi'(v,v') := \xi(v)\xi'(v') - \xi(v')\xi'(v).$$

Notice that $\xi \wedge \xi' = -\xi' \wedge \xi$ and that $\xi \wedge \xi = 0$. If $e_1,\dots,e_m$ is a basis of V and $\xi_1,\dots,\xi_m$ the basis of V^* dual to this (characterized by $\xi_i(e_j) = \delta_{ij}$), then $\{\xi_i \wedge \xi_j\}_{i<j}$ is a basis of $\wedge^2 V^*$: any $A \in \wedge^2 V^*$ is determined by its values on the pairs (e_k, e_l) with $k < l$ and we have for $i < j$ and $k < l$:

$$\xi_i \wedge \xi_j(e_k, e_l) = \delta_{ik}\delta_{jl}.$$

We may regard V as the dual of V^*: $V^{**} = V$, and so the preceding applied to V^* yields the definition of the vector space $\wedge^2 V$ and the assertion that the $\{e_i \wedge e_j\}_{i<j}$ form a basis for that space. If $f : W \to V$ is a linear map and $A \in \wedge^2 V^*$, then $(w,w') \mapsto A(f(w), f(w'))$ is alternating bilinear and so belongs to $\wedge^2 W^*$. The resulting map $\wedge^2 V^* \to \wedge^2 W^*$ is linear; we denote it by f^*.

We do this with differentials. Recall that the differentials defined on an open neighborhood of a fixed $x \in \mathbb{R}^m$ have values in a vector space with basis $dx_1,\dots,dx_m$. Pointwise construction of the preceding yields the concept of a *2-form* on an open $U \subset \mathbb{R}^m$. So this will look like

$$\zeta = \sum_{i<j} r_{ij}\, dx_i \wedge dx_j$$

where r_{ij} is differentiable on U. This terminology suggest that we might have a definition of a k-form for every k. This is indeed the case: a 0-form resp. 1-form on U is a differentiable function resp. a differential on U. So two 1-forms $\omega = \sum_i p_i\, dx_i$, $\eta = \sum_i q_i\, dx_i$ have a pointwise defined exterior product:

$$\omega \wedge \eta = \sum_{i,j} p_i q_j\, dx_i \wedge dx_j = \sum_{i<j} (p_i q_j - p_j q_i)\, dx_i \wedge dx_j.$$

There is also an analogue of the formation of a differential, called the *exterior derivative*, which assigns to a 1-form ω a 2-form $d\omega$:

$$d\Big(\sum_i p_i\,dx_i\Big) := \sum_i dp_i \wedge dx_i = \sum_{i<j}\Big(\frac{\partial p_j}{\partial x_i} - \frac{\partial p_i}{\partial x_j}\Big)dx_i \wedge dx_j.$$

PROPERTIES 3.11. It is easy to check the following properties:

(i) ω is closed $\iff$ $d\omega = 0$,
(ii) d is $\mathbb{R}$-linear: $d(\omega+\eta) = d\omega + d\eta$ and if $c \in \mathbb{R}$, then $d(c\omega) = c d\omega$,
(iii) the *Leibniz rule*: if $\phi : U \to \mathbb{R}$ is differentiable, then

$$d(\phi\omega) = \phi d\omega + d\phi \wedge \omega.$$

If $V \subset \mathbb{R}^n$ is open and $f : V \to U$ is a differentiable map, then we define the pull-back of the 2-form $\zeta = \sum_{i<j} r_{ij}\,dx_i \wedge dx_j$ on U in a pointwise manner:

$$f^*\zeta(y) := \sum_{i<j} r_{ij} f(y).df_i \wedge df_j = \sum_{i<j,k<l} r_{ij} f(y).\Big(\frac{\partial f_i}{\partial y_k}\frac{\partial f_j}{\partial y_l} - \frac{\partial f_i}{\partial y_l}\frac{\partial f_j}{\partial y_k}\Big)dy_k \wedge dy_l.$$

(iv) $f^*(\zeta+\zeta') = f^*\zeta + f^*\zeta'$,
(v) $f^*(\phi\zeta) = \phi f.f^*\zeta$ and $f^*(\omega\wedge\eta) = f^*\omega \wedge f^*\eta$,
(vi) $f^*d\omega = df^*\omega$.

This generalizes in an evident manner to smooth manifolds: a 2-form ζ on a smooth manifold M consists of giving a 2-form ζ_κ on $\kappa(U)$ for every chart (U,κ) such that for every coordinate change $\kappa'\kappa^{-1} : \kappa(U\cap U') \to \kappa'(U\cap U')$, we have $\zeta_\kappa|U\cap U' = (\kappa'\kappa^{-1})^*\zeta_{\kappa'}|U\cap U'$.

EXERCISE 3.3. Show that it actually suffices to have given the ζ_κ for (U,κ) running over *some* atlas $(U_i,\kappa_i)_i$ with the above compatibility property (you must show that this system extends uniquely to the whole atlas)

Given differentials ω and η on M, then the collection $\{\omega_\kappa \wedge \eta_\kappa\}_\kappa$ defines a 2-form $\omega\wedge\eta$ on M, called their *exterior product*. Likewise the collection $\{d\omega_\kappa\}_\kappa$ defines a 2-form $d\omega$ on M, its exterior derivative. The properties (i) – (vi) continue to hold in this setting.

If we denote the space of differentiable 2-forms on M by $\mathcal{E}^2(M)$, then the exterior derivative defines a map

$$d : \mathcal{E}^1(M) \to \mathcal{E}^2(M).$$

We now suppose that $m = 2$: if $\zeta = r(x_1,x_2)\,dx_1 \wedge dx_2$ is a 2-form on an open $U \subset \mathbb{R}^2$ and $D \subset U$ measurable, then we define the *surface integral* of ζ over K as

$$\int_D \zeta := \iint_D r(x_1,x_2)\,dx_1\,dx_2.$$

Stokes's theorem may now be stated as follows:

THEOREM 3.12. *Let $U \subset \mathbb{R}^2$ be open and D a closed subset of U with a smooth boundary* [1]. *Orient the boundary such that if we traverse ∂D in that direction D is on th left. If ω is a differential on $U \supset D$, with $\omega|D$ zero outside a compact subsetof D, then*

$$\int_{\partial D} \omega = \int_D d\omega.$$

[1] this means that D is at every point of ∂D, given by an inequality $\pm x_2 \le \phi(x_1)$ or $\pm x_1 \le \phi(x_2)$ for some differentiable function ϕ. We allow $\partial D = \emptyset$ or $D = U$.

We assume this proposition known. The behaviour of the surface integral with respect to diffeomorphisms like the integral over a path. This follows from the transformation formula for surface integrals, which we can now state as follows:

PROPOSITION 3.13. *Let* f *be a diffeomorphism from an open* $U' \subset \mathbb{R}^2$ *onto an open deel* $U \subset \mathbb{R}^2$ *with the property that the determinant of the derivative matrix,* $\det(df)$ *is* > 0 *everywhere. If* $D \subset U$ *is a closed subset with differentiable boundary* ∂D *and* ζ *a is 2-form on* U' *which vanishes outside a compact subset of* U'*, then*

$$\int_{f(D)} \zeta = \int_D f^* \zeta.$$

This fact is the point of departure for the definition of a surface integral. Since we required that $\det(df) > 0$ we first introduce an associated concept. We say that a diffeomorphism f from an open part of $\mathbb{R}^m$ to an open part of $\mathbb{R}^m$ is *orientation preserving* if $\det(df) > 0$ everywhere.

DEFINITION 3.14. We say that a (smooth) atlas of a smooth manifold$M$ is *oriented* if every change of charts of their atlas is orientation preserving. A maximal such atlas is called an *orientation* of M.

Since an oriented atlas can always be extended to a maximal one, an oriented atlas determines an orientation of M. An oriented atlas need not exist (examples are the Möbius band, the real projective plane and the Klein bottle). The geometric meaning of an orientation of a surface is that if we traverse a differentiable curve, we have a sense of left and right.

EXERCISE 3.4. (a) Prove that a holomorphic diffeomorphism from an open part of $\mathbb{C}$ to an open part of $\mathbb{C}$ preserves orientation.
(b) Show that a Riemann surface comes with a given orientation.
(c) Conclude that the Möbius band, the real projective plane and the Klein bottle cannot be given the structure of a Riemann surface.

EXERCISE 3.5. Prove that a connected smooth manifold M of dimension $m > 0$ has either zero or two orientations. More specifically, prove that if $\{(U_i, \kappa_i)_i\}$ is a maximal oriented atlas, and $h : \mathbb{R}^m \to \mathbb{R}^m$ a linear transformation with determinant < 0, then $\{(U_i, h\kappa_i)_i\}$ is another maximal oriented atlas and that there are no others.

3. The theorems of Stokes and De Rham for surfaces

Let M be an oriented surface and $D \subset M$ a closed subset with differentiable (one-sided) boundary ∂D: there is an oriented atlas of charts (U, κ) such that D is given by $\kappa_1 \leq 0$.

We show how to integrate a 2-form ζ on M that vanishes outside a compact $K \subset M$ over D. Denote by $B \supset B' \supset B''$ the open disks in $\mathbb{R}^2$ centered at 0 of radius $1, \frac{2}{3}, \frac{1}{3}$ respectively. Then there is a differentiable function $\phi : B \to [0, 1]$ with $\phi|B''$ constant 1 and $\phi|B{-}B'$ constant 0. Choose finitely many oriented charts $(U_i, \kappa_i)_{i=1}^N$ for which $\kappa_i(U_i) = B$ and the $\kappa_i^{-1}B''$ cover K. We extend $\phi\kappa_i : U_i \to [0, 1]$ to a function $\tilde{\phi}_i : M \to [0, 1]$ by letting it be zero on $M - U_i$. Since this function is then zero on the open part $M - \kappa_i^{-1}\bar{B}'$ (whose union with the open U_i covers M), it is differentiable. Then $\sum_i \tilde{\phi}_i|K \geq 1$ as every term is ≥ 0 and at least one equals 1. So $\phi_i := (\sum_j \tilde{\phi}_j)^{-1}\tilde{\phi}_i$ satisfies:

(i) ϕ_i is a differentiable function defined on an open neighborhood of K; it is ≥ 0 on its domain and $\phi_i|D$ is zero outside a compact subset of $D \cap U_i$,

(ii) $\sum_i \phi_i|K = 1$.

We call such a collection functions a *partition of 1* on K with respect to the covering $\{U_i\}_i$. Notice that then $(\phi_i.\zeta)_{\kappa_i} = \phi_i\kappa_i^{-1}.\zeta_{\kappa_i}$ is zero outside B'. The image of $D \cap U_i$ under κ_i satisfies the hypotheses of Theorem 3.12, and hence the integral of $(\phi_i.\zeta)_{\kappa_i}$ over $D \cap U_i$ is defined.

PROPOSITION-DEFINITION 3.15. *The sum of integrals*

$$\sum_i \int_{\kappa_i(D\cap U_i)} (\phi_i.\zeta)_{\kappa_i}$$

only depends on D *and the 2-form* ζ*. It is called the integral of* ζ *over* D *and denoted* $\int_D \zeta$*. This integral is* $\mathbb{R}$*-linear in* ζ*.*

PROOF. If $\{(V_j, \lambda_j, \psi_j)\}_j$ is another choice for $\{(U_i, \kappa_i, \phi_i)\}_i$, then

$$\sum_j \int_{\lambda_j(D\cap V_j)} (\psi_j\zeta)_{\lambda_j} = \sum_{i,j} \int_{\lambda_j(D\cap V_j)} (\phi_i\psi_j\zeta)_{\lambda_j} \quad (\text{for } \sum_i \phi_i = 1)$$
$$= \sum_{i,j} \int_{\lambda_j(D\cap U_i\cap V_j)} (\phi_i\psi_j\zeta)_{\lambda_j}$$

(for the integrand restricted to D is zero outside a compact subset of $D \cap U_i \cap V_j$)

$$= \sum_{i,j} \int_{\kappa_i(D\cap U_i\cap V_j)} (\phi_i\psi_j\zeta)_{\kappa_i} \quad (\text{transformation formula})$$
$$= \sum_i \int_{\kappa_i(D\cap U_i)} (\phi_i\zeta)_{\kappa_i} \quad (\text{for } \sum_j \psi_j = 1).$$

This proves independence of the choices made. The $\mathbb{R}$-linearity is obvious. □

COROLLARY 3.16 (Stokes' theorem for surfaces). *Let* M *be an oriented surface and* $D \subset M$ *a closed subset with differentiable one-sided boundary* ∂D*. Orient the latter by the convention that if we traverse* ∂D*,* D *is on the left. Then for every differential* ω *on* M *that is zero outside a compact subset* $\int_D d\omega = \int_{\partial D} \omega$*.*

PROOF. For $\{(U_i, \kappa_i, \phi_i)\}_i$ as above, we have

$$\int_D d\omega = \int_D d(\sum_i \phi_i\omega) = \sum_i \int_{\kappa_i(D\cap U_i)} d(\phi_i\omega)_{\kappa_i}$$
$$= \sum_i \int_{\partial(\kappa_i(D\cap U_i))} (\phi_i\omega)_{\kappa_i} \quad (\text{by Stokes})$$
$$= \sum_i \int_{\kappa_i(\partial D\cap U_i)} (\phi_i\omega)_{\kappa_i} = \int_{\partial D} \omega.$$

□

In particular, if M is compact, then any exact 2-form on M has zero integral.

Now let M be a compact oriented surface f genus g. If ω and η run over the differentials on M, then $\int_M \omega \wedge \eta$ is bilinear and antisymmetric in (ω, η): $\int_M \omega \wedge \eta = -\int_M \eta \wedge \omega$.

COROLLARY 3.17. *A differential* ω *on* M *is closed if and only if* $\int_M \omega \wedge d\phi = 0$ *for every differentiable* $\phi : M \to \mathbb{R}$.

By the Leibniz rule we have $\omega \wedge d\phi = -d(\phi\omega) + \phi d\omega$. So by 3.16, integration over M yields $\int_M \omega \wedge d\phi = \int_M \phi d\omega$. Hence the corollary frollows from:

LEMMA 3.18. *If* ζ *is 2-form on* M *with* $\int_M f\zeta = 0$ *for all differentiable* $f : M \to \mathbb{R}$, *then* $\zeta = 0$.

PROOF. We must show that if $\zeta(p) = 0$ for some $p \in M$ then $\int_M f\zeta \neq 0$ for some differentiable $f : M \to \mathbb{R}$.

Choose a chart (U, κ) with $p \in U$ with $\kappa(p) = 0$ and $\kappa(U)$ the open unit disk $B \subset \mathbb{R}^2$. If we write $\zeta_\kappa = r(x)dx_1 \wedge dx_2$, then $r(0) \neq 0$. Let us assume that $r(0) > 0$ (otherwise proceed with $-\zeta$). Then there is a $\varepsilon \in (0, 1)$ such that $r(x) \geq 0$ if $|x| \leq \varepsilon$. Choose a differentiable function $\phi : B \to [0, 1]$ with $\phi(0) = 1$ and $\phi(x) = 0$ for $|x| \geq \varepsilon$. Then $\phi.r \geq 0$ and $\phi(0)r(0) > 0$. Now $\phi\kappa$ can be extended to a differentiable function f on M by letting $f|M - U$ be zero. Then

$$\int_M f\zeta = \int_B \phi r \, dx_1 \, dx_2 > 0.$$

□

The (easy)⇒ part of Corollary 3.17 implies that for closed differentials ω, η on M the integral $\int_M \omega \wedge \eta$ only depends on their images in $H^1(M, \mathbb{R})$. We thus find an antisymmetric bilinear map

$$H^1(M, \mathbb{R}) \times H^1(M, \mathbb{R}) \to \mathbb{R}, \quad ([\omega], [\eta]) \mapsto [\omega] \cdot [\eta] := \int_M \omega \wedge \eta.$$

Let $A_1, \ldots, A_{-g} : S^1 \to M$ embed circles in M as in fig. ?. We suppose here compatibility with the orientation in the sense that if p_i is the unique point of intersection of A_i and A_{-i} (where $i > 0$), then the direction of A_{-i} at p_i is to the left of the direction of A_i, to be precise, there is a chart κ at p_i taken from the orientation with $\kappa A_i(\theta) = (\theta, 0)$ and $\kappa A_{-i}(\theta) = (0, \theta)$. We know that the classes $a_i := [A_i] \in H_1(M)$ form a basis of $H_1(M)$.

PROPOSITION 3.19. *If* $(\alpha_i := [h_i^*\eta] \in H^1(M, \mathbb{R}))_{i=\pm 1}^{\pm g}$ *is the basis of* $H^1(M, \mathbb{R})$ *dual to* $(a_i)_{i=\pm 1}^{\pm g}$, *then* $a_i \cdot a_j = \delta_{ij}$ *(and so the* $\cdot$*-product reflects the intersection behavior of the* A_i*'s).*

PROOF. Let for $i = \pm 1, \cdots \pm g$, $h_i : M \to S^1$ be the map and η the differential on S^1 that we constructed in the proof of 3.10. In terms of the notation used there, the differential $h_i^*\eta$ is zero outside $U_{|k|}$ and $(\alpha_i := [h_i^*\eta] \in H^1(M, \mathbb{R}))_{i=\pm 1}^{\pm g}$ is the basis dual to $(a_i)_{i=\pm 1}^{\pm g}$. For $i \neq \pm j$, we have $U_{|i|} \cap U_{|j} = \emptyset$. Hence $h_i^*\eta \wedge h_j^*\eta = 0$ and so $\alpha_i \cdot \alpha_j = 0$. It is clear that $\alpha_i \cdot \alpha_i = 0$. It remains to see that for $i > 0$, $\alpha_i \cdot \alpha_{-i} = 1$. For this we observe that $h_i^*\eta \wedge h_{-i}^*\eta$ is the pull-back to M of the differential $p_1^*\eta \wedge p_2^*\eta$ on $S^1 \times S^1$, where $p_r : S^1 \times S^1 \to S^1$ is the projection onto the rth factor. So

$$\int_M \eta_i \wedge \eta_{-i} = \int_{S^1 \times S^1} p_1^*\eta \wedge p_2^*\eta = \int_{S^1} \eta . \int_{S^1} \eta = 1.$$

(The last identity used the compatibility with the orientation.) □

COROLLARY 3.20. *A differential ω on M is exact if and only if for every closed differential η we have $\int_M \omega \wedge \eta = 0$. In particular, the alternating bilinear form $\cdot$ on $H^1(M, \mathbb{R})$ is nondegenerate, that is, if $u \in H^1(M, \mathbb{R})$ has the property that $u \cdot v = 0$ for all $v \in H^1(M, \mathbb{R})$, then $u = 0$.*

PROOF. The implication $\Rightarrow$ follows from 3.17. If conversely, $\int_M \omega \wedge \eta = 0$ for every closed differential η, then $\Leftarrow$ of 3.17 implies that ω is closed. If we let η run over the $\eta_{\pm 1}, \dots, \eta_{\pm g}$, then the above discussion shows that the integral of ω over any closed loop is zero. So ω is exact. □

Here is a sort of converse to Stokes' theorem (which at the same time is a special case of the De Rham theorem). Let us say that a 2-form μ is *exact* if it is the exterior derivative of a differential.

THEOREM 3.21. *Let M be a compact oriented connected surface and μ is a 2-form on M. If $\int_M \mu = 0$, then μ is exact.*

REMARK 3.22. We may express is in a way similar to Proposition 3.10: if we define the *second De Rham cohomology group of the surface* M, denoted $H^2(M), \mathbb{R})$, as the space 2-forms modulo then exact 2-forms. Stokes' theorem shows that for M compact and oriented, integration over M defines a map $\int_M : H^2(M), \mathbb{R}) \to \mathbb{R}$. The above theorem says that for connected M this is an isomorphism.

We prove this using a variant of the Poincaré lemma:

LEMMA 3.23. *Let μ be a 2-form on $\mathbb{R}^2$ and $C > 0$.*
- (i) *There is a differential η with compact support such that $\mu - d\eta$ is zero on $[-C, C]^2$.*
- (ii) *If the support μ is contained in the band $[-C, C] \times \mathbb{R}$, then we can take η in (i) such that its support is also contained in $[-C, C] \times \mathbb{R}$.*
- (iii) *If the support of μ is contained in $[-C, C]^2$ and $\int_{\mathbb{R}^2} \mu = 0$, then $\mu = d\eta$ for some 1-form η whose support is also contained in $[-C, C]^2$.*

PROOF. Write $\mu = f(x, y)dx \wedge dy$. Let $F(x, y) := \int_{-C}^{y} f(x, t)dt$. Then the differential $-F(x, y)dx$ satisfies $d(-F(x, y)dx) = \mu$, but $-F(x, y)dx$ has in general no compact support. It is true that if $f(x, y)$ is zero for $|x| > C$, then the same holds for $F(x, y)$. In the cases (i) en(ii) we therefore choose a differentiable function $g : \mathbb{R}^2 \to \mathbb{R}$ with compact support that is constant 1 on $[-C, C]^2$ and we take $\eta := -gFdx$.

In case (iii) we observe that $F(x, y)$ is zero for $|x| \geq C$ or $y < -C$, but for $y \geq C$ we can only say that $F(x, y) = F(x, C)$. The assumption $\int_{\mathbb{R}^2} \mu = 0$ amounts to: $\int_{-C}^{C} F(x, C)dx = 0$. Let $G(x) := \int_{-C}^{x} F(s, C)ds$. So $G(x) = 0$ for $|x| \geq C$ and $dG(x) = F(x, C)dx$. Choose a differentiable function $h : \mathbb{R} \to \mathbb{R}$ which is zero for $t \leq -C$ and 1 for $t \geq C$ and consider $\tilde{G}(x, y) := G(x)h(y)$. We have that $d\tilde{G}$ is zero if $|x| \geq C$ or if $y \leq -C$, and equals dG if $y \geq C$. So $\eta := -F + d\tilde{G}$ is as required. □

PROOF OF THEOREM 3.21. Let $o \in M$ and choose an open neighborhood U_o of o differomorphic to $\mathbb{R}^2$. We show how to write μ as the sum of three exact 2-forms.

Choose oriented embedded circles $A_{\pm 1}, \dots, A_{\pm g}$ through o which are pairwise disjoint elsewhere and are such that the complement U of their union in M is diffeomorphic to $\mathbb{R}^2$. Let D_o be a closed disk D_o about o inside U_o and choose for every A_i a neighborhood pair $U_i \supset V_i$ of $A_i \setminus D_o$ that is diffeomorphic to the pair $\mathbb{R}^2 \supset (-1,1)\times\mathbb{R}^2$. Then there is a $C>0$ such under the diffeomorphism $\mathbb{R}^2 \cong U$, the image of $[-C,C]^2$ in U, D_o and the V_i's cover all of M. We apply case (i) of Lemma 3.23 to U: subtracting from μ an exact 2-form produces a μ' whose support is contained in the union of D_o and the V_i's. Nest we apply case (ii) of this lemma to every pair (U_i, V_i): subtracting from μ' an exact 2-form produces a μ'' with support in D_o. From case (iii) it then follows that μ'' is exact also. □

4. Holomorphic differentials

A complex valued function on a smooth manifold, $f = u + iv : M \to \mathbb{C}$, is differentiable simply when its real part u and its imaginary part v are. The complex conjugate of f, $\bar{f} = u - iv : M \to \mathbb{C}$ is then also differentiable. More generally, a *complex valued* k*-form* ω $(k = 0,1,2)$ is simply an expression $\alpha + i\beta$, where α and β are ordinary k-forms. In other words, it is an element of the complexification of the real vector space $\mathcal{E}^k(M)$ of the space of k-forms on M. We shall denote that complexification $\mathcal{E}^k(M,\mathbb{C})$, so that $\mathcal{E}^k(M,\mathbb{C}) = \mathcal{E}^k(M,\mathbb{C}) + i\mathcal{E}^k(M,\mathbb{C})$. We can integrate a complex valued differential along a differentiable path or a complex valued 2-form over a compact oriented surface and get a complex number. The complex conjugate of ω is $\bar{\omega} = \alpha - i\beta$. We extend the d-operator $\mathbb{C}$-linearly as a map $\mathcal{E}^k(M,\mathbb{C}) \to \mathcal{E}^{k+1}(M,\mathbb{C})$ for $k = 0,1$: $df := du + idv$ and, if ω is a 1-form, $d\omega := d\alpha + id\beta$. Likewise for the wedge map $\wedge : \mathcal{E}^1(M,\mathbb{C})\times\mathcal{E}^1(M,\mathbb{C}) \to \mathcal{E}^2(M,\mathbb{C})$: $\omega\wedge\omega' = (\alpha + i\beta)\wedge(\alpha' + i\beta') = (\alpha\wedge\alpha' - \beta\wedge\beta') + i(\alpha\wedge\beta' + \beta\wedge\alpha')$.

In this section all k*-forms are in principle complex valued.*

If $U \subset \mathbb{C}$ is open, then think of the inclusion $U \subset \mathbb{C}$ as a complex valued function $z = x + iy$. So we have the total differential $dz := dx + idy$. Its complex conjugate $d\bar{z}$ is $dx - idy$. Notice that then $dz\wedge d\bar{z} = -2idx\wedge dy$. Since we have $dx = \frac{1}{2}(dz + d\bar{z})$ and $dy = \frac{1}{2i}(dz - d\bar{z})$, every differential $\omega = pdx + qdy$ on an open $U \subset \mathbb{C}$ is uniquely written as $fdz + gd\bar{z}$, where f and g are differentiable functions on U. We write ω' resp. ω'' for fdz resp. $gd\bar{z}$, and we call this the $(1,0)$-*part* resp. $(0,1)$-*part* of ω. If $f : U \to \mathbb{C}$ is differentiable, then we shall write ∂f for $(1,0)$-part of df and $\bar{\partial} f$ for its $(0,1)$-part: $df = \partial f + \bar{\partial} f$. We denote the corresponding coefficients $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ so that $\partial f = \frac{\partial f}{\partial z}dz$ and $\bar{\partial} f = \frac{\partial f}{\partial \bar{z}}d\bar{z}$. Since

$$\begin{aligned} df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial x}.\frac{1}{2}(dz + d\bar{z}) + \frac{\partial f}{\partial y}.\frac{1}{2i}(dz - d\bar{z}) \\ = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})dz + \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})d\bar{z}, \end{aligned}$$

we see that

$$\frac{\partial f}{\partial z} := \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}), \quad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}).$$

Notice that the vanishing of $\frac{\partial f}{\partial \bar{z}}$ (or equivalently, of $\bar{\partial} f$) amounts to the Cauchy-Riemann equation being satisfied. So $\bar{\partial} f = 0$ precisely when f is holomorphic. We say that a differential is *holomorphic* if it is of the form fdz with f holomorphic. A differential is called *anti-holomorphic* if its complex conjugate is holomorphic (so it is of the form $\bar{f}d\bar{z}$, with f holomorphic).

LEMMA 3.24. *A differential of type* $(1,0)$ *(resp.* $(0,1)$*) is closed if and only if it is holomorphic (resp. anti-holomorphic).*

PROOF. Let $\omega = f dz$. Then $d\omega = df \wedge dz = \frac{\partial f}{\partial \bar z} d\bar z \wedge dz$ and hence ω is closed if and only if $\frac{\partial f}{\partial \bar z}$ vanishes. The last condition is equivalent to f being holomorphic. The second assertion follows form the first by complex conjugation. □

If we combine this with 3.8, then we recover the well-known fact from complex function theory that says that the integral of a holomorphic differential $f dz$ on a simply connected domain over a closed path is zero.

We define the *Hodge star operator* on differentials by:

$$\star(p dx + q dy) := -\bar q dx + \bar p dy.$$

So $\star dz = -\bar i dx + dy = i(dx - i dy) = i d\bar z$ and similarly $\star d\bar z = \bar i dx + dy = -i(dx + i dy) = -i dz$.

PROPERTIES 3.25. The following properties are clear:

(i) $\star(\omega + \eta) = \star\omega + \star\eta$,
(ii) $\star(f\omega) = \bar f . \star \omega$,
(iii) $\star \star \omega = -\omega$,
(iv) $(\star\omega)' = -i\overline{\omega''}$ and $(\star\omega)'' = i\overline{\omega'}$, in particular $\star\partial f = i\bar\partial \bar f$ and $\star\bar\partial f = -i\partial \bar f$.
(v) $\overline{\omega \wedge \star\eta} = \eta \wedge \star\omega$
(vi) $\omega \wedge \star\omega = (|p|^2 + |q|^2) dx \wedge dy$,

So $\star$ exchanges holomorphic and anti-holomorphic differentials.

LEMMA 3.26. *If f is a differentiable $\mathbb{C}$-valued function on een open part of $\mathbb{C}$, then $d \star \partial f = d \star \bar\partial f$ and hence $d \star df = 2d \star \bar\partial f$.*

PROOF. This is a simple exercise:

$$d(\star\partial f) = d\Big(\star \frac{\partial f}{\partial z} dz\Big) = d\Big(\overline{\frac{\partial f}{\partial z}} i d\bar z\Big) = d\Big(\frac{\partial \bar f}{\partial \bar z} i d\bar z\Big) = i \frac{\partial^2 \bar f}{\partial z \partial \bar z} dz \wedge d\bar z$$

and similarly

$$d(\star\bar\partial f) = d\Big(\star \frac{\partial f}{\partial \bar z} d\bar z\Big) = d\Big(\frac{\partial \bar f}{\partial z} . - i dz\Big) = -i \frac{\partial^2 \bar f}{\partial \bar z \partial z} d\bar z \wedge dz . = i \frac{\partial^2 \bar f}{\partial z \partial \bar z} dz \wedge d\bar z.$$

□

PROPOSITION-DEFINITION 3.27. *For a differential ω on an open part of $\mathbb{C}$ the following are equivalent:*

(i) $d\omega = d \star \omega = 0$,
(ii) $d\omega' = d\omega'' = 0$,
(iii) *ω' is holomorphic en ω'' is anti-holomorphic.*

If (any of) these properties are (is) fulfilled we say that ω is harmonic.

PROOF. (i) ⇒ (ii) We have $0 = d\omega = d\omega' + d\omega''$ and $0 = d \star \omega = d(-i\overline{\omega''} + i\overline{\omega'}) = -i(\overline{d\omega'' - d\omega'})$ by assumption. This implies that ω' and ω'' are both closed.

(ii) ⇒ (iii) has already been proved in Lemma 3.24.

(iii) ⇒ (i) follows from Lemma 3.24 and property (iv) follows from 3.25. □

EXERCISE 3.6. Prove that if $h : V \to U$ is a holomorphic map between open parts of $\mathbb{C}$, then for every differential ω on U, we have $(h^*\omega)' = h^*(\omega')$, $(h^*\omega)'' = h^*(\omega'')$ and $h^*(\star\omega) = \star(h^*\omega)$. Conclude that for a differentiable function f on U, $h^*\partial f = \partial(fh)$ and $h^*\bar\partial f = \bar\partial(fh)$.

This exercise shows that the preceding can be transfered to Riemann surfaces:

(i) On a Riemann surface S we have the notion of differential of type $(1,0)$ resp. $(0,1)$; every differential ω on S is uniquely written as $\omega' + \omega''$ with ω' resp. ω'' of type $(1,0)$ resp. $(0,1)$.

(i) The closed differentials of type $(1,0)$ are locally the differential of a holomorphic function; such differentials are said to be *holomorphic*. We have a similarly the notion of a *anti-holomorphic differential*: this is the complex conjugate of a holomorphic differential or equivalently a closed differential of type $(0,1)$.

(ii) For a differentiable function f on S, we write $\partial f := (df)'$ en $\bar{\partial} f := (df)''$. We have: f is holomorphic $\iff \bar{\partial} f = 0$.

(iii) We have defined a Hodge star operator $\star$ acting on the differentials on S. It satisfies the properties (i)–(v) of 3.25. Differentials ω for which $d\omega = d \star \omega = 0$ are called *harmonic*.

(iv) Lemma 3.26 and Proposition 3.27 hold on S.

In the remainder of this section S denotes a compact connected Riemann surface of genus g. We orient S by its holomorphic structure as in Exercise 3.4. For $\omega, \eta \in \mathcal{E}^1(S, \mathbb{C})$, we define

$$\langle \omega, \eta \rangle := \int_S \omega \wedge \star \eta.$$

This expression is complex linear in the first variable and complex anti-linear in the second.

PROPOSITION 3.28. *The map $\langle\, , \,\rangle : \mathcal{E}^1(S,\mathbb{C}) \times \mathcal{E}^1(S,\mathbb{C}) \to \mathbb{C}$ is an inner product: it gives $\mathcal{E}^1(S,\mathbb{C})$ the structure of a preHilbert space (i.e., satisfies all the properties that define a Hilbert space, except completeness):*

(i) *$\langle \omega, \eta \rangle$ is complex-linear as a function in ω,*

(ii) *$\overline{\langle \omega, \eta \rangle} = \langle \eta, \omega \rangle$,*

(iii) *$\langle \omega, \omega \rangle \geq 0$ with equality holding only if $\omega = 0$.*

PROOF. Property (i) is trivial, property (ii) follows from 3.25-v, and property (iii) is a consequence of 3.25-vi. □

EXERCISE 3.7. Denoting by $\mathcal{E}^1(S,\mathbb{C})_{\mathrm{cl}}$ the space of closed differentials on S, prove that for $\omega \in \mathcal{E}^1(S,\mathbb{C})$ the following equivalences hold:
(a) $\omega \perp d\mathcal{E}^0(S,\mathbb{C}) \iff d \star \omega = 0$,
(b) $\omega \perp \mathcal{E}^1(S,\mathbb{C})_{\mathrm{cl}} \iff \star\omega$ is exact,
(c) $\omega \perp \star d\mathcal{E}^0(S,\mathbb{C}) \iff d\omega = 0$,
(d) $\omega \perp \star\mathcal{E}^1(S,\mathbb{C})_{\mathrm{cl}} \iff \omega$ is exact.
(Hint: use 3.17 and 3.20.)

We denote the space of holomorphic (resp. harmonic) differentials on S by $\Omega(S)$ (resp. $\mathcal{H}^1(S)$). So $\mathcal{H}^1(S) = \Omega(S) \oplus \overline{\Omega(S)}$. We denote by $H^1(S,\mathbb{C})$ the complexification of $H^1(S,\mathbb{R})$, i.e., $H^1(S,\mathbb{R}) \oplus iH^1(S,\mathbb{R})$. This is a complex vector space of complex dimension $2g$ that is equal to $\mathcal{E}^1(S,\mathbb{C})_{\mathrm{cl}}/d\mathcal{E}^0(S,\mathbb{C})$. Since $\mathcal{H}^1(S)$ consist of closed differentials there is an evident map $\mathcal{H}^1(S) \to H^1(S,\mathbb{C})$.

The preceding proposition yields:

COROLLARY 3.29. *The subspaces $d\mathcal{E}^0(S,\mathbb{C})$ and $\star d\mathcal{E}^0(S,\mathbb{C})$ of $\mathcal{E}^1(S,\mathbb{C})$ are perpendicular to each other, and the orthogonal complement of their direct sum is $\mathcal{H}^1(S)$.*

Moreover, $\mathcal{H}^1(S)$ *is also the orthogonal complement of* $d\mathcal{E}^0(S,\mathbb{C})$ *in* $\mathcal{E}^1(S,\mathbb{C})_{cl}$ *so that the evident map* $\mathcal{H}^1(S)\to H^1(S,\mathbb{C})$ *is injective.*

If A is a Hilbert space, then it is easy to see that a *closed* subspace $B\subset A$ is a Hilbert subspace and that A is the direct som of B and its orthogonal complement $B^\perp$. (N.B. The orthogonal complement of an arbitrary subspace of a Hilbert space is always closed. Why?) Conversely, if $A=B\oplus B^\perp$, then B is closed, for $B^\perp=\overline{B}^\perp$, and hence $B\oplus B^\perp=A=\overline{B}\oplus B^\perp$ implies that $B=\overline{B}$. It is not surprising that we must be even more careful with preHilbert spaces. In particular, we my not conclude from the above corollary that $\mathcal{E}^1(S,\mathbb{C})$ is the direct sum of $d\mathcal{E}^0(S,\mathbb{C})$, $\star d\mathcal{E}^0(S,\mathbb{C})$ and $\mathcal{H}^1(S)$.

We recall that the inner product of a preHilbert space A extends continuously to the metric completerion $\hat{A}$ of A, and that $\hat{A}$ is a Hilbert space.

LEMMA 3.30. *Let* A *be a preHilbert space and* $B\subset A$ *a subspace such that*

(i) B *is closed in* A *and*
(ii) *the orthogonal complement of* B *in* $\hat{A}$ *is contained in* A.

Then $A=B\oplus B^\perp$.

PROOF. If $\hat{B}$ is the closure of B in $\hat{A}$, then $\hat{A}=\hat{B}\oplus(\hat{B})^\perp$. So every $a\in A$ is uniquely written as $b+c$ with $b\in\hat{B}$ and $c\in(\hat{B})^\perp$. By assumption (ii) we have $(\hat{B})^\perp\subset A$, and hence $c\in A$. This implies that $b=a-c\in A$. By assumption (i) we have $A\cap\hat{B}=B$, and so $b\in B$. □

THEOREM 3.31 (Hodge, Weyl). *On the compact Riemann surface* S *we have the orthogonal decomposition* $\mathcal{E}^1(S,\mathbb{C})=\mathcal{H}^1(S)\oplus d\mathcal{E}^0(S,\mathbb{C})\oplus\star d\mathcal{E}^0(S)$.

We will not prove this theorem, but only comment on its nature. According to Exercise 3.7, $d\mathcal{E}^0(S,\mathbb{C})$ is the orthogonal complement of $\star\mathcal{E}^1(S,\mathbb{C})_{cl}$ in $\mathcal{E}^1(S,\mathbb{C})$ and hence closed in $\mathcal{E}^1(S,\mathbb{C})$. Since $\star$ is a homeomorphism, $\star d\mathcal{E}^0(S,\mathbb{C})$ is also closed in $\mathcal{E}^1(S,\mathbb{C})$. Therefore their (orthogonal) direct sum is closed. The theorem then follows from Lemma 3.30 if we can see that:

Assertion. The orthogonal complement of $d\mathcal{E}^0(S,\mathbb{C})\oplus\star d\mathcal{E}^0(S,\mathbb{C})$ *in* $\hat{\mathcal{E}}^1(S,\mathbb{C})$ *is contained in* $\mathcal{E}^1(S,\mathbb{C})$ *(and hence equal to* $\mathcal{H}^1(S)$*).*

This orthogonal complement has the concrete interpretation as the space of L^2-solutions of an *elliptic partial differential equation*. The general theory of such equations indeed asserts that an L^2-solution is in fact a genuine one, that is, is differentiable.

Theorem 3.31 has as immediate consequence:

COROLLARY 3.32. *The evident map* $\mathcal{H}^1(S)\to H^1(S,\mathbb{C})$ *is an isomorphism and* $\dim\Omega(S)=g$.

PROOF. The orthogonal complement of $\star d\mathcal{E}^0(S)$ is $\mathcal{E}^1(S,\mathbb{C})_{cl}$ and so it follows from Theorem 3.31 that $\mathcal{E}^1(S,\mathbb{C})_{cl}$ equals the orthogonal direct sum of $\mathcal{H}^1(S)\oplus d\mathcal{E}^0(S,\mathbb{C})$. Hence $\mathcal{H}^1(S)\to H^1(S,\mathbb{C})$ is an isomorphism. The last assertion follows from the fact that $\dim\mathcal{H}^1(S)=2\dim\Omega(S)$ and $\dim H^1(S,\mathbb{C})=2g$. □

CHAPTER 4

Divisors

In this chapter S stands for a connected *Riemann surface.*

1. Principal and canonical divisors

DEFINITION 4.1. A *divisor* on S is a $\mathbb{Z}$-valued function $D : S \to \mathbb{Z}$ whose support $\mathrm{supp}(D)$ is discrete: every point of S has a punctured neighborhood on which this function is constant zero. We say that D is *effective* if it only takes values ≥ 0.

The value of the divisor D in p is rarely denoted $D(p)$; rather we may write D as $\sum_{p \in S} n_p(p)$, with $n_p \in \mathbb{Z}$ the value of D in $p \in S$. The divisors on S form an abelian group (under pointwise addition) which we denote by $\mathrm{Div}(S)$. The divisors are partially ordered by $D \geq D'$ (as functions on S).

Divisors can occur as follows. Let ϕ be a meromorphic function defined on a connected open subset $U \subset \mathbb{C}$ that is not identically zero. Then for any $p \in U$, ϕ is on a neighborhood of p uniquely written as $(z-p)^n g(z)$ with $n \in \mathbb{Z}$ and g holomorphic and nonzero at p. We then say that n is the order of ϕ at p and we denote it by $\mathrm{ord}_p(\phi)$. (So if $n \geq 0$, then ϕ is holomorphic at p (of order n) and if $n < 0$, then ϕ has there a pole of order $|n| = -n$.) So $\mathrm{div}(\phi) = \sum_{p \in S} \mathrm{ord}_p(\phi)(p)$ is a divisor on U. The notions meromorphic function and order are invariant under holomorphic diffeomorphisms and hence carry over to the Riemann surface S. A meromorphic function ϕ on S that is not constant zero has in any $p \in S$ an order $\mathrm{ord}_p(\phi)$ and we can form the divisor

$$\mathrm{div}(\phi) = \sum_{p \in S} \mathrm{ord}_p(\phi)(p).$$

To say that ϕ is not constant zero means that ϕ is invertible as a meromorphic function, for $1/\phi$ is also meromorphic on S. So the meromorphic functions on S form a field; we denote that field by $\mathcal{M}(S)$. It contains $\mathbb{C}$ as a subfield of constant functions. Observe that ϕ is holomorphic precisely when its divisor is effective. For two invertertible meromorphic functions ϕ en ψ we have

$$\mathrm{div}(\psi/\phi) := \mathrm{div}(\psi) - \mathrm{div}(\phi).$$

DEFINITION 4.2. A divisor on a Riemann surface S is called a *principal divisor* if it is the divisor of a nonzero meromorphic function on S. Such divisors make up a subgroup of $\mathrm{Div}(S)$ that we shall denote by $\mathrm{PDiv}(S)$. Two divisors D en D' are said to be *linearly equivalent* (we then write $D \equiv D'$) if their difference $D - D'$ is a principal divisor. The quotient group $\mathrm{Div}(S)/\mathrm{PDiv}(S)$ is called the *Picard group* of S, and is denoted $\mathrm{Pic}(S)$.

EXERCISE 4.1. Suppose S compact (and connected, as always in this chapter). Prove that two nonzero meromorphic functions define the same divisor if and only if one is a scalar multiple of the other.

A meromorphic differential ω on an open $U \subset \mathbb{C}$ is a differential with meromorphic coefficient: ϕdz, with ϕ meromorphic on U. The *order* $\mathrm{ord}_p(\omega)$ of ω at $p \in U$ is then defined as the order of ϕ at p. Clearly ω is holomorphic at p if and only if $\mathrm{ord}_p(\omega) \geq 0$. Suppose f is a holomorphic map from an open part V of $\mathbb{C}$ to U which maps $q \in V$ to $p \in U$ and is not constant near q. If we put $e := \mathrm{ord}_q(f')$, then $f(w) - p = g(w)(w-q)^{1+e}$ with $e \geq 0$ and g holomorphic and nonzero at q. We call $1+e$ the *multiplicity* of f at q (also denoted $\mathrm{mult}_q(f)$). By the inverse function theorem $e = 0$ ($\mathrm{mult}_q(f) = 1$) is equivalent to: f is a local isomorphism at q. If ϕ is a nonzero meromorphic function at p, then $f^*\phi = \phi f$ is a nonzero meromorphic function at q and we have $\mathrm{ord}_q(f^*\phi) = \mathrm{mult}_q(f)\,\mathrm{ord}_p(\phi)$. Similarly, the meromorphic differential $\omega = \phi dz$ pulls back as a meromorphic differential: we have $f^*\omega = \phi f.f' dz$ on a neighborhood of q and

$$(\dagger) \qquad \mathrm{ord}_q(f^*\omega) = \mathrm{mult}_q(f)\,\mathrm{ord}_p(\omega) + (\mathrm{mult}_q(f) - 1)$$

In particular, $\mathrm{ord}_q(f^*\omega) = \mathrm{ord}_{f(q)}(\omega)$ if f is a local isomorphism at q. So the notions *meromorphic differential* and order of such a differential can be transfered to Riemann surfaces.

If ω is a meromorphic differential on S that is not identically zero, then ω is not identically zero on any nonempty open subset so that we have defined its divisor

$$\mathrm{div}(\omega) := \sum_{p \in S} \mathrm{ord}_p(\omega)(p).$$

If ϕ is an arbitrary meromorphic function on S, then $\phi\omega$ is a meromorphic differential and any meromorphic differential on S is thus obtained. In other words, the meromorphic differentials form a vector space of dimension one over the field $\mathcal{M}(S)$. Is $\phi \neq 0$, then:

$$\mathrm{div}(\phi\omega) = \mathrm{div}(\phi) + \mathrm{div}(\omega).$$

But beware that a meromorphic differential on S is not necessary of the form $\phi\zeta$ with ζ a differential without zeroes and poles (like dz on $\mathbb{C}$) and ϕ meromorphic on S, for such a ζ need not exist.

DEFINITION 4.3. A divisor on a Riemann surface S is called *canonical* if it is the divisor of a meromorphic differential on S.

In view of the preceding, the canonical divisors on S form a linear equivalence class of PDiv(S).

If S is compact, then the support of any divisor is finite so that we have defined a homomorphism

$$\deg : \mathrm{Div}(S) \to \mathbb{Z}, \quad \sum_p n_p(p) \mapsto \sum_p n_p,$$

called the *degree*. Its kernel, the group of degree zero divisors, is denoted $\mathrm{Div}^0(S)$. If we pick a $o \in S$, then $\mathrm{Div}^0(S)$ is freely generated by the divisors $\{(p) - (o)\}_{p \in S}$, for if $\sum_p n_p = 0$, then $\sum_p n_p(p) = \sum_p n_p((p) - (o))$.

EXERCISE 4.2. Prove that two divisors on $\mathbb{P}^1$ are linearly equivalent if and only if their degrees are the same and conclude that the degree homomorphism $\deg : \mathrm{Pic}(\mathbb{P}^1) \to \mathbb{Z}$ is an isomorphism.

2. The analytic Riemann-Hurwitz formula

The notion of multiplicity makes sense for any nonconstant holomorphic map $f : S \to S'$ between connected Riemann surfaces. If $p \in S$ and w is a local coordinate at $f(p) \in S'$, then the order of wf at p is independent of the choice of w; this is the *multiplicity* $\mathrm{mult}_p(f)$ of f at p. So $\mathrm{mult}_p(f) \geq 1$ and equality holds if and only if f is a local isomorphism at p. More relevant will be $e_p(f) := \mathrm{mult}_p(f) - 1$ (which is the order of the differential $d(wf)$ at p). The preceding shows that we have thus defined a divisor

$$R_f := \sum_{p \in S} e_p(f)(p),$$

called the *ramification divisor*.

We define a homomorphism $f^* : \mathrm{Div}(S') \to \mathrm{Div}(S)$ by

$$f^*(\sum_{p' \in S'} n_{p'}(p')) := \sum_{p \in S} \mathrm{mult}_p(f) n_{f(p)}(p)$$

THEOREM 4.4. *Let ϕ resp. ω be a nonzero meromorphic function resp. differential on S'. Then $f^*\phi$ resp. $f^*\omega$ is a nonzero meromorphic function resp. differential on S and we have*

$$\mathrm{div}(f^*\phi) = f^* \mathrm{div}(\phi) \quad \textit{and} \quad \mathrm{div}(f^*\omega) = f^* \mathrm{div}(\omega) + R_f.$$

In particular, f^* takes principal divisors to principal divisors and hence induces a homomorphism $f^* : \mathrm{Pic}(S') \to \mathrm{Pic}(S)$.

PROOF. This is clear (use formula (†)). □

A continuous map between two topological spaces $F : X \to Y$ is called *proper* if the preimage of a compact subset Y is compact in X. This is automatic if X is compact and Y is Hausdorff: a compact subset of Y is then automatically closed, its inverse image in X is the also closed (because F is continuous) and hence that preimage is compact (since X is). It is clear that if F is proper, then for every subspace $B \subset Y$, the restriction $F^{-1}B \to B$ is also proper.

PROPOSITION-DEFINITION 4.5. *For a proper, nonconstant holomorphic map between connected Riemann surfaces, $f : S \to S'$, the degree of the divisor $f^*(p')$ (i.e., $\sum_{p \in f^{-1}(p')} \mathrm{mult}_p(f)$) is independent of p'; we call this common value the* degree *of f,* $\deg(f)$.

PROOF. Let $p' \in S'$. The preimage $f^{-1}(p')$ is discrete, but also compact and hence finite. Let $p_1, \dots, p_N$ be the distinct points of $f^{-1}(p')$ and put $m_i := \mathrm{mult}_{p_i}(f)$. So the degree of $f^*(p')$ equals $\sigma_i m_i$. Since f is at p_i locally equivalent to $z \mapsto z^{m_i}$, there exist open neighborhoods $U_i \ni p_i$ and $U_i' \ni p'$ such that f has multiplicity 1 in every point of $U_i - \{p_i\}$ and for every $q' \in U_i' - \{p'\}$, $f^{-1}(q') \cap U_i$ consists of exactly m_i points. We choose U_i so small that the $U_1, \dots, U_N$ are pairwise disjoint. We show that there exists an open $U' \ni p'$ such that $f^{-1}U' \subset \cup_i U_i$. This suffices because the U_i's determine for every $q' \in U' - \{p'\}$ a partition of $f^{-1}(q')$ and since $f^{-1}(q') \cap U_i$ has exactly m_i points, each of multiplicitity 1, $f^*(q')$ has degree $\sum_i m_i$.

The desired U' can be obtained as follows: choose a compact susbet K' of S' that has p' in its interior. Then $f^{-}K'$ is compact and contains $\{p_1, \dots, p_N\}$ in its interior. Now $K := f^{-1}K' - \cup_i(U_i \cap f^{-1}K')$ is closed in $F^{-1}K'$ and hence compact. But then $f(K)$ is compact and hence closed (for S' is Hausdorff). Since $p_i \notin K$ for all i, we have $p' \notin f(K)$. Now let U' be the interior of K' minus its intersection with $f(K)$. This is an open subset containing p'. If $q \in f^{-1}(p)$, then $q \in f^{-1}K'$. But $q \notin K$ and hence $q \in U_i$ for some i. □

COROLLARY 4.6. *Let $f : S \to S'$ be a nonconstant holomorphic map between compact (and as ever, connected) Riemann surfaces. Then this map has a degree and if D' is a divisor on S', then* $\deg(f^*D') = \deg(f)\deg(D)$.

A nonconstant meromorphic function ϕ on S can be geometrically understood as a holomorphic map to the Riemann sphere $\hat\phi : S \to \mathbb{P}^1$: if $\mathrm{ord}_p(\phi) = n$, then in terms of a local coordinate z with $z(p) = 0$ we can write $\phi = z^n\psi$ as before (so with ψ holomorphic and $\psi(p) \neq 0$) and $\hat\phi$ is then at p given by $z \mapsto [1 : \phi] \in \mathbb{P}^1$, which for $n < 0$ can also be written as $z \mapsto [z^{-n} : \psi]$. The preceding corollary shows that every nonconstant meromorphic function on a compact Riemann surface has a degree. This implies that ϕ has as many poles as zeroes if we count them with appropriate multiplicities:

COROLLARY 4.7. *If S is compact, then every principal divisor on S has degree zero: the degree homomorphism* $\deg : \mathrm{Div}(S) \to \mathbb{Z}$ *factors through a homomorphism* $\deg : \mathrm{Pic}(S) \to \mathbb{Z}$.

PROOF. Let ϕ be a nonconstant meromorphic function on S. This determines a nonconstant holomorphic map $\hat\phi : S \to \mathbb{P}^1$. It is clear that then $\mathrm{div}(\phi) = \hat\phi^*(0) - \hat\phi^*(\infty)$. It follows that $\mathrm{div}(\phi)$ has degree zero. □

EXERCISE 4.3 (Supplements Exercise 2.3). Let $\pi : \tilde S \to S$ be a proper, nonconstant holomorphic map between connected Riemann surfaces and let $\phi : \tilde S \to \mathbb{C}$ be a holomorphic function. Denote by d the degree of π.
(a) Prove that the expression

$$F_\phi(t, p) := \prod_{\tilde p \in \pi^{-1}(p)} (t - \phi(\tilde p))^{\mathrm{mult}_{\tilde p}(\pi)}$$

is a polynomial of degree d in t whose coefficients are holomorphic and which has ϕ as a root. (So $\pi^* : \mathcal{H}(S) \to \mathcal{H}(\tilde S)$ is an integral extension.)
(b) Let $F = t^k + c_1t^{k-1} + \cdots + c_k$ be a minimum polynomial of ϕ (i.e., the monic irreducible factor of F_ϕ that has ϕ as a root) and let $\pi_F : S_F \to S$ be defined as in Exercise 2.2. Prove that $\pi : \tilde S \to S$ factors through a proper holomorphic map $\tilde S \to S_F$.

EXERCISE 4.4. (Parts (d) and (e) require you to know some basic field theory.) Let $\pi : \tilde S \to S$ be a nonconstant holomorphic map between compact, connected Riemann surfaces, of degree d say, and let $\phi \in \mathcal{M}(\tilde S)$.
(a) Let P be a finite subset of S with the property that π is a covering map over $S - P$ and ϕ is holomorphic on $\pi^{-1}(S - P)$. Prove that the expression

$$F_\phi(t, p) := \prod_{\tilde p \in \pi^{-1}(p)} (t - \phi(\tilde p)), \quad p \in S - P,$$

is a polynomial of degree d in t whose coefficients are meromorphic on S.
(b) We now view F_ϕ as an element of $\mathcal{M}(S)[t]$. Prove that ϕ is a root of F_ϕ. (So $\pi^* : \mathcal{M}(S) \to \mathcal{M}(\tilde{S})$ is an algebraic extension.)
(c) Let $F \in \mathcal{M}(S)[t]$ be an irreducible polynomial of degree $k > 0$. Construct a compact connected Riemann surface S_F and a nonconstant holomorphic map $\pi_F : S_F \to S$ such that $\pi_F^* F$ has a root in $\mathcal{M}(S_F)$.
(d) Prove that $\pi^* : \mathcal{M}(S) \to \mathcal{M}(\tilde{S})$ is a finite field extension.
(e) Prove that every finite field extension of $\mathcal{M}(S)$ of degree d is isomorphic to one that is given by a holomorphic map of degree d from a compact connected Riemann surface to S.

3. The topological Riemann-Hurwitz formula

A polygon has as many edges as it has vertices. Euler discovered a beautiful two dimensional generalization of this simple fact: for a regular Platonic solid the number of its vertices minus the number of its edges plus the number of its faces is always equal to 2. This holds in fact for every decomposition of the 2-sphere into polygonal faces: the proof proceeds by induction on the number of edges: removing an edge such we are still left with a decomposition into polygonal faces makes not only the number of edges drop by one, but also the number of faces, whereas the number of vertices stays the same. We can continue until no edges are left: we are the in the situation that we have just one vertex and one face so that the expression in question is 2 indeed. A similar argument applies to a compact surface M; the number thus defined is called the *Euler characteristic* of M, denoted by $\chi(M)$. If M is connected of genus g, then we already found a decomposition with one vertex, $2g$ edges and one face (cf. the discussion preceding 2.21). So then $\chi(M) = 2 - 2g$.

THEOREM 4.8 (Riemann-Hurwitz). *Let $f : S \to S'$ be a nonconstant holomorphic map of degree d between compact connected Riemann surfaces of genus resp. g and g'. Then*

$$2g - 2 = d(2g' - 2) + \deg(R_f)$$

(in particular the degree of R_f is even).

PROOF. Let $p' \in S'$ be arbitrary. Then $\sum_{p \in f^{-1}(p')} \mathrm{mult}_p(f) = d$, by definition and so the number of points of $f^{-1}(p')$ is d minus $\sum_{p \in f^{-1}(p')} (\mathrm{mult}_p(f) - 1) = \sum_{p \in f^{-1}(p')} e_p(f)$.

Now choose a decomposition of S' into polygonal faces so that every point over which f has a point of multiplicity > 1 is a vertex. The we get a decomposition of S into polygonal faces such that every face of of S maps onto one of S'. Over every face or edge of S' lie exactly d copies of it in S. For a vertex p' this number may be less, the difference being $\sum_{p \in f^{-1}(p')} e_p(f)$. The total deficit is therefore $\sum_{p \in S} e_p(f) = \deg(R_f)$ and hence we find that $\chi(S) = d\chi(S') - \deg(R_f)$. □

We shall later find that a canonical divisor has degree $2g - 2$. So the topological Riemann-Hurwitzformula 4.8 can be obtained from the last formula of Theorem 4.4 by taking the degree of both sides.

CHAPTER 5

Linear systems

In this chapter S *denotes a* compact connected *Riemann surface.*

1. The map defined by a linear system

For a divisor D on S we put

$$L(D) := \{\phi \in \mathcal{M}(S) : \phi \neq 0 \Rightarrow \mathrm{div}(\phi) + D \geq 0\}$$

So if $D = \sum_{i=1}^{N} n_{p_i}(p_i)$, then $L(D)$ is the set of meromorphic functions on S that are holomorphic on $S - \{p_1, \dots, p_N\}$ and whose order at p_i is $\geq -n_{p_i}$. It is then clear that $L(D)$ is a complex vector space.

LEMMA 5.1. *Let* D' *be a divisor on* S *which is linearly equivalent to* D *by means of the meromorphic function* ψ*:* $D' = \mathrm{div}(\psi) + D$*. Then multiplication by* ψ *defines a linear isomorphism of vector spaces* $L(D') \to L(D)$.

PROOF. Given $\phi \in \mathcal{M}(S) - \{0\}$, then $\phi \in L(D')$ if and only if $\mathrm{div}(\phi) + \mathrm{div}(\psi) + D \geq 0$. This is equivalent to $\mathrm{div}(\phi\psi) + D \geq 0$. □

In order to better understand $L(D)$ it is convenient to have the following notation at our disposal. Given $p \in S$, then denote by $\mathcal{O}_{S,p}$ the ring of holomorphic functions on an unspecified neighborhood of p. Via a local coordinate we may identify this with the ring $\mathbb{C}\{z\}$ of complex power series with a positive radius of convergence. If $n \in \mathbb{Z}$, then denote by $\mathcal{O}_{S,p}(n)$ the set of meromorphic functions on an unspecified neighborhood of p in S of order $\geq -n$ at p. Our local coordinate identified this with $z^{-n}\mathbb{C}\{z\}$. So $\mathcal{O}_{S,p}(0) = \mathcal{O}_{S,p}$. Notice that $\mathcal{O}_{S,p}(n-1) \subset \mathcal{O}_{S,p}(n)$ and that the quotient space has dimension 1. In particular, we have that $n \geq 0$, $\dim(\mathcal{O}_{S,p}(n)/\mathcal{O}_{S,p}) = n$ (it has basis $z^{-1}, \dots, z^{-n}$).

LEMMA 5.2. *We have* $\dim L(D) \leq \max\{0, 1 + \deg(D)\}$*, so that in particular* $L(D)$ *is finite dimensional.*

PROOF. Let $D = \sum_{p \in S} n_p(p)$. We may of course assume that $\dim L(D) > 0$. Because of Lemma 5.1 we can replace D by an element of $L(D)$ and thus assume without loss of generality that $D \geq 0$, so that $n_p \geq 0$ for all $p \in S$. If $P \subset S$ is the finite set of $p \in S$ for which $n_p > 0$, then we have an evident linear map

$$L(D) \to \oplus_{p \in P} \mathcal{O}_{S,p}(n_p)/\mathcal{O}_{S,p}.$$

The kernel consists of the holomorphic functions on S. These are constant and so the kernel is of dimension 1. Hence $\dim L(D) \leq 1 + \sum_{p \in P} \dim(\mathcal{O}_{S,p}(n_p)/\mathcal{O}_{S,p}) = 1 + \sum_{p \in P} n_p = 1 + \deg(D)$. □

We recall some terminology regarding complex-projective spaces. Let V be a complex vector space of finite dimension d. The associated projective space $\mathbb{P}(V)$

is the set of one dimensional linear subspaces of V (so an element of $\mathbb{P}(V)$ is representable by a nonzero vector in V). A choice of basis in V identifies V with $\mathbb{C}^d$ and hence $\mathbb{P}(V)$ with $\mathbb{P}^{d-1}$. Via this identification we may regard $\mathbb{P}(V)$ as a complex manifold. This structure is independent of the choice of basis, for another basis yields another identification with $\mathbb{P}^{d-1}$ that differs from the first by a map $\mathbb{P}^{d-1} \to \mathbb{P}^{d-1}$ induced by a complex-linear transformation $\mathbb{C}^d \to \mathbb{C}^d$ and such a map is biholomorphic.

If $W \subset V$ is a linear subspace, then clearly $\mathbb{P}(W) \subset \mathbb{P}(V)$. Such a subset of $\mathbb{P}(V)$ is called *(projective-)linear subspace*. A linear injection $f : V \to V'$ determines an injection $\mathbb{P}(f) : \mathbb{P}(V) \to \mathbb{P}(V')$. An injection of projective spaces that so arises is said to be *projective*. Its image is clearly a linear subspace. If f is a linear isomorphism, $\mathbb{P}(f)$ is called a *linear isomorphism*[1]. In case $V' = V$, the map $\mathbb{P}(f) : \mathbb{P}(V) \to \mathbb{P}(V)$ is called a linear transformation in $\mathbb{P}(V)$.

PROPOSITION 5.3. *The set of all effective divisors in a linear equivalence class (that may be empty) has in a natural manner the structure of a (finite dimensional) complex-projective space. Precisely, given a divisor D, then let $|D|$ be the set of effective divisors that are linearly equivalent to D. Then the map which assigns to $\phi \in L(D) - \{0\}$ the divisor $\operatorname{div}(\phi) + D \in |D|$ is a bijection*

$$\mathbb{P}(L(D)) \to |D|, \quad [\phi] \mapsto \operatorname{div}(\phi) + D$$

and the resulting structure of a complex-projective space on $|D|$ only depends on $|D|$ (and not on the particular D).

PROOF. It is clear that for $\phi \in L(D) - \{0\}$, $\operatorname{div}(\phi) + D$ is an effective divisor linearly equivalent to D. Conversely, an effective divisor linearly equivalent to D is of this form (by definition). Now is $\operatorname{div}(\phi) + D = \operatorname{div}(\psi) + D$ if and only if $\operatorname{div}(\phi) = \operatorname{div}(\psi)$, in other words if $\operatorname{div}(\phi/\psi) = 0$. This means that ϕ/ψ is constant, or what amounts to the same, that ϕ and ψ span the same line in $L(D)$. We thus obtain a bijection $J_D : \mathbb{P}(L(D)) \to |D|$. If $|D| = |D'|$, then D' is of the form $\operatorname{div}(\psi)+D$ for some $\psi \in \mathcal{M}(S)$. Multiplication by ψ then gives (according to Lemma 5.1) a linear isomorphism $\mu : L(D') \to L(D)$ and hence a projective-linear isomorphism $\mathbb{P}(\mu) : \mathbb{P}(L(D')) \to \mathbb{P}(L(D))$. It is clear that $J_{D'} = J_D \mathbb{P}(\mu)$. So D' and D define the same structure of a complex-projective space on $|D| = |D'|$. □

DEFINITION 5.4. A set of effective divisors is called a *linear system* if it makes up a linear subspace in the projective space of all effective divisors within a fixed linear equivalence class. The set of all effective divisors within a fixed linear equivalence class is called a *complete linear system*.

EXERCISE 5.1. Prove that two divisors on $\mathbb{P}^1$ of the same degree d are linearly equivalent. So if $d \geq 0$, then the effective divisors of degree d make up a complete linear system. Determine the dimension of that system.

EXERCISE 5.2. Let D_0 be an effective divisor and let $\mathcal{D}$ be a linear system. Prove that $D + \mathcal{D}$ is also a linear system and that $D \in \mathcal{D} \mapsto D + D_0 \in D + \mathcal{D}$ is a linear isomorphism.

EXERCISE 5.3. Let $\mathcal{D}$ be a linear system. We say that $p \in S$ is a *fixed point* of $\mathcal{D}$ if $\mathcal{D} - (p)$ is a linear system. Prove that $\mathcal{D}$ can be uniquely written as $D_0 + \mathcal{D}'$,

[1] One can show that every biholomorphic map between complex projective spaces is of this form.

where D_0 is effective and $\mathcal{D}'$ is a linear system without fixed points. (We call D_0 the *fixed part* and $\mathcal{D}'$ the *moving part* of $\mathcal{D}$; an empty linear system is without fixed points by convention.)

We run into a linear system when we are given a holomorphic map from S to a complex-projective space. In order to explain this, we have another look at complex projective spaces. So let P be a complex-projective space: $P = \mathbb{P}(V)$ for some finite dimensional complex vector space V. A linear hyperplane $H \subset P$ is by definition the projectivization of a linear hyperplane in V. The latter is the kernel of some linear form $l \in V^* - \{0\}$. The linear form is unique up to multiplication by a scalar and so H is complete detrmined by the line l spans in in V^*. Conversely, a one dimensional linear subspace of V^* is a hyperplane in V and hence a hyperplane in $\mathbb{P}(V^*)$. We conclude that $\mathbb{P}(V^*)$ is bijectively indexes the hyperplanes of P. In other words, the collection van hyperplanes in P has the structure of complex projective space. It is denoted $\check{P}$ and we call it the *dual of* $\mathbb{P}(V)$.

We observe that for $H \in \check{P}$, $P - H$ is in a natural manner a complex-affine space: if $l \in V^*$ defines H, then $l^{-1}(1)$ is an affine hyperplane with $l^{-1}(0)$ as translation space. The map

$$j_l : l^{-1}(1) \to P - H, \quad v \in l^{-1}(1) \mapsto [v]$$

is a bijection, for every point of $P - H$ can be represented by a one dimensional linear subspace $L \subset V - l^{-1}(0)$ and this subspace will meet $l^{-1}(1)$ in precisely one point. Any other $l' \in V^*$ that defines H is of the form λl for some $\lambda \in \mathbb{C} - \{0\}$ and it is clear that $j_{l'}(v) = j_l(\lambda v)$. Since multiplication by λ maps the affine space $l^{-1}(1)$ isomorphically and affine-linearly onto the affine space ${l'}^{-1}(1)$, $P - H$ has indeed in a natural manner the structure of a complex-affine space. (We may regard P as a compactification this affine space that has H as its hyperplane at infinity.)

Thus we have in fact a defined a holomorphic atlas for P: if $H, H' \in \check{P}$ are defined by $l, l' \in V^*$, then we have bijections

$$l^{-1}(1) \setminus {l'}^{-1}(0) \xrightarrow[\cong]{j_l} P - H - H' \xleftarrow[\cong]{j_{l'}} {l'}^{-1}(1) \setminus l^{-1}(0).$$

The composite from left to right is simply $v \mapsto l'(v)^{-1}v$ and its inverse is $v \mapsto l(v)^{-1}v$. So this map is biholomorphic. Let us also notice here that l/l' is homogeneous of degree zero and can be regarded as a function $P - H \to \mathbb{C}$. This function is holomorphic and has zero set $H' \setminus H$. Similarly its reciproque l'/l is holomorphic on $P - H'$ with zero set $H \setminus H'$.

Let $f : S \to P$ be a holomorphic map from the (compact, connected) Riemann surface S to a projective space P. We suppose that f is *nondegenerate* in the sense that its image doe not lie in a hyperplane of P (this is no serious restriction, for we can arrive at that situation simply by replacing P by the intersection of all hyperplanes that contain the image of f). So if H is any hyperplane of P, then $f^{-1}H \neq S$. We associate to H an effective divisor f^*H with support $f^{-1}H$ as follows. Let H be defined by $l \in V^*$. Choose a hyperplane $H' \subset \mathbb{P}$ that does not contain p and let $l' \in V^*$ be a defining linear form for H'. Then $f^*(l/l')$ is a holomorphic function on $S - f^{-1}H'$ that has $f^{-1}H \setminus f^{-1}H'$ as zero set. So this function is not constant zero in p. Hence $\mathrm{ord}_p\, f^*(l/l')$ is finite and ≥ 0. If H'', l'' is another choice for H', l', then

$$\mathrm{ord}_p\, f^*(l/l'') = \mathrm{ord}_p\, f^*(l/l') + \mathrm{ord}_p\, f^*(l''/l') = \mathrm{ord}\, f^*(l/l'),$$

as $f^*(l''/l')$ has order zero at p. The value of the divisor f^*H in $p \in S$ is by definition the common value of these orders. It is clear that the support of f^*H equals $f^{-1}H$.

LEMMA 5.5. *The map* $H \in \check{P} \mapsto f^*H$ *is a projective linear isomorphism of* $\check{P}$ *onto a linear system* $f^*\check{P}$ *on* S. *This linear system is without fixed points.*

PROOF. Let $H_o \in \check{P}$ be defined by $l_o \in V^* - \{0\}$. If $H \in \check{P}$ is defined by $l \in V^*$, then $f^*H - f^*H_o = \mathrm{div}(f^*(l/l_o))$, and so $f^*H \equiv f^*H_o$ and $f^*(l/l_o) \in L(f^*H_o)$. The map

$$l \in V^* \mapsto f^*(l/l_o) \in L(f^*H_o)$$

is evidently linear en injective. The projectivization of that map yields a linear map $\check{P} = \mathbb{P}(V^*) \to \mathbb{P}(L(f^*H_o))$.

If $\dim P > 0$, then for every $p \in S$ there is a $H \in \check{P}$ with $f(p) \notin H$ and so p is not in the support of f^*H and so $f^*\check{P}$ has no fixed points. If $\dim P = 0$, then the linear system in question is empty and hence without fixed points by convention. □

We also have a converse:

PROPOSITION 5.6. *Let* $\mathcal{D}$ *be a nonempty linear system on* S *without fixed points. Then for any* $p \in S$, $\mathcal{D}_p := \{D \in \mathcal{D};\, D \geq (p)\}$ *is a hyperplane in* $\mathcal{D}$ *and the resulting map* $S \to \check{\mathcal{D}}$, $p \mapsto \mathcal{D}_p$ *is holomorphic and nondegenerate.*

PROOF. Since $\mathcal{D}$ has no fixed points, there exists for every $p \in S$ a $D \in \mathcal{D}$ with $p \notin \mathrm{supp}(D)$. So it suffices to show that given any $D \in \mathcal{D}$, then for every $p \in S - \mathrm{supp}(D)$, $\mathcal{D}_p$ is a hyperplane in $\mathcal{D}$ and that this defines a holomorphic map $S - \mathrm{supp}(D) \to \check{\mathcal{D}}$. To this end, we consider the vector space $L(D) = \{\phi \in \mathcal{M}(S) : \phi \neq 0 \Rightarrow \mathrm{div}(\phi) + D \geq 0\}$ of which we observe that it contains the constant functions.

For $\phi \in L(D)$ we have $\mathrm{div}(\phi) + D \geq 0$, and so ϕ is holomorphic on $S - \mathrm{supp}(D)$. Letting ϕ run over $L(D)$, we get a map $F_D : S - \mathrm{supp}(D) \to L(D)^*$: $F_D(p)$ is the linear form on $L(D)$ that assigns to $\phi \in L(D)$ its value $\phi(p)$ in p. This map is holomorphic: if $(\phi_0, \ldots, \phi_N)$ is a basis for $L(D)$, then F_D is given by $p \in S - \mathrm{supp}(D) \mapsto (\phi_0(p), \ldots, \phi_N(p))$. The constant function $1 \in L(D)$ defines a linear form on $L(D)$ that we denote by l. It is clear that the image of F_D is contained in the affine hyperplane $l^{-1}(1)$ (in coordinates: if we let $\phi_0 = 1$, then the image of F_D lies in the affine hyperplane in $\mathbb{C}^N$ met first coordinate 1). The composite

$$f_D : S - \mathrm{supp}(D) \xrightarrow{F_D} l^{-1}(1) \longrightarrow \mathbb{P}(L(D)^*) = \check{\mathbb{P}}(L(D))$$

is of course also holomorphic. This map lets $p \in S - \mathrm{supp}(D)$ correspond to the hyperplane in $\mathbb{P}(L(D))$ defined by the $\phi \in L(D)$ with $\phi(p) = 0$. Under the identification $\mathbb{P}(L(D)) \cong \mathcal{D}$, $[\phi] \leftrightarrow D + \mathrm{div}(\phi)$, this corresponds in turn to the set of divisors $D' = D + \mathrm{div}(\phi) \in \mathcal{D}$, for which $D' \geq (p)$, i.e., to $\mathcal{D}_p$. In particular, $\mathcal{D}_p$ is a hyperplane in $\mathcal{D}$ and f_D is the resulting map $p \in S - \mathrm{supp}(D) \mapsto [\mathcal{D}_p] \in \check{\mathcal{D}}$. □

A theorem of Chow asserts that the image of a holomorphic map $f : M \to \mathbb{P}^N$ from a compact complex manifold to to a complex projective space is always a complex projective variety, that is, the common zero set of a collection homogeneous polynomials in $N+1$ variables). If M is our Riemann surface S, then $f(S)$ is a nonsingular algebraic curve (in the sense of algebraic geometry) if f is injective and the derivative of f is nowhere zero. We then say that f is a *holomorphic embedding* (it is perhaps a bit confusing that in the language of algebraic geometry this is called a *closed immersion*). In the situation of the previous proposition we have:

PROPOSITION 5.7 (Embedding criterion). *In the situation of Proposition 5.6, when* $p, q \in S$*, denote by* $\mathcal{D}_{p,q}$ *the collection of* $D \in \mathcal{D}$ *with* $D - (p) - (q) \geq 0$*. Then the map* $p \in S \to [\mathcal{D}_p] \in \check{\mathcal{D}}$ *is a holomorphic embedding precisely when* $\mathcal{D}_{p,q} \subset \check{\mathcal{D}}$ *is a linear subspace of codimension 2 for all* $p, q \in$ *(including the case* $p = q$*).*

PROOF. Denote the map in question by $f : S \to \check{\mathcal{D}}$. Suppose $p \neq q$. Then $\mathcal{D}_{p,q} = \mathcal{D}_p \cap \mathcal{D}_q$. It is the clear that $\mathcal{D}_{p,q}$ is of codimension 2 in $\mathcal{D}$ if and only if $\mathcal{D}_p \neq \mathcal{D}_q$, which amounts to $f(p) \neq f(q)$.

We finish the proof by showing that the derivative of f in p is nonzero precisely when $\mathcal{D}_{p,p}$ has codimension 2 in $\mathcal{D}$. Let $D \in \mathcal{D}$ be such that $p \notin \mathrm{supp}(D)$. If we recall that $\mathcal{D}$ is identified with the projectivization of $L(D) = \{\phi \in \mathcal{M}(S) : \phi \neq 0 \Rightarrow \mathrm{div}(\phi) \geq -D\}$, then the subspaces $\mathcal{D}_{p,p} \subset \mathcal{D}_p \subset \mathcal{D}$ are defined by the linear subspaces $L(D - 2(p)) \subset L(D - (p)) \subset L(D)$. Now $L(D - 2(p))$ is as a subspace of $L(D - (p))$ the kernel of the linear map $\phi \in L(D - (p)) \mapsto d\phi(p) \in T^*_p S$. Hence $L(D - 2(p))$ is of codimension 2 in $L(D)$ precisely when there is a $\phi \in L(D)$ with $\mathrm{ord}_p(\phi) = 1$. If $(1 = \phi_0, \phi_1, \dots, \phi_N)$ is a basis for V^*, then this is equivalent to: the map $(\phi_1, \dots, \phi_N) : S - \mathrm{supp}(D) \to \mathbb{C}^N$ has at p a derivative $\neq 0$, or equivalently, the derivative of f at p is nonzero. □

REMARK 5.8. If in the previous proposition S is identified with its image in $\check{\mathcal{D}}$, then every $D \in \mathcal{D}$ may we understood as a *hyperplane section*, that is, the intersection of S with a hyperplane, where we take into account intersection multiplicities (the value of D at p is the intresection multiplicity of S with the corresponding hyperplane). So the total number of intersection points, each counted with its multiplicitity, is just the degree of D. This is why we call this number also the *degree of the embedding*.

EXERCISE 5.4. Show that in the situation of 5.7, $\mathcal{D}_{p,p}$ consists of the hyperplanes through $f(p)$ that are tangent to $f(S)$ at $f(p)$.

EXAMPLE 5.9 (Rational curves in $\mathbb{P}^n$). If $z_0 \in \mathbb{C}$, then $(z_0) - (\infty)$ is the divisor of the meromorphic function $z - z_0$ on $\mathbb{P}^1$, and hence is a principal divisor. It follows that the effective divisors on $\mathbb{P}^1$ of degree $d \geq 0$ make up a complete linear system $\mathcal{D}_d$, equal to $|d(\infty)|$. This linear system has dimension d and is without fixed points. The associated map $\mathbb{P}^1 \to \check{\mathcal{D}}_d$ can be made explicit: $L(d(\infty))$ is simply the space of polynomials in z of degree $\leq d$. and has therefore $(1, z, z^2, \dots, z^d)$ as basis. This identifies $\check{\mathcal{D}}_d$ with $\mathbb{P}^d$ and the map in question is then $[1 : z] \mapsto [1 : z : z^2 : \cdots : z^d] \in \mathbb{P}^d$, or rather,

$$f_d : [Z_0 : Z_1] \in \mathbb{P}^1 \mapsto [Z_0^d : Z_0^{d-1} Z_1 : Z_0^{d-2} Z_1^2 : \cdots : Z_1^d] \in \mathbb{P}^d.$$

This is for $d > 0$ a holomorphic embedding. The image of this map is called the *rational curve of degree* d. (For $d = 1$ we get an isomorphism.)

EXERCISE 5.5. Let $f : \mathbb{P}^1 \to \mathbb{P}^d$ be a nondegenerate holomorphic map of degree d. Prove that there is a linear transformation T of $\mathbb{P}^d$ such that $f = Tf_d$.

EXERCISE 5.6. Prove that the image f_d is defined by the set of $[W_0 : W_1 : \cdots : W_d] \in \mathbb{P}^d$ that satisfy the quadratic equations $W_k W_l = W_{k'} W_{l'}$ for all (k, l, k', l') in $\{0, \dots, d\}$ with $k + l = k' + l'$.

2. Hyperelliptic Riemann surfaces

Suppose that S is admits a one dimensional linear system $\mathcal{D}$ of degree $d \leq 2$ (we then say that S is *hyperelliptic*). By passing to its moving part we may assume that $\mathcal{D}$ has no fixed points. We then get a morphism $f : S \to \check{\mathcal{D}}$ of degree d, where $\check{\mathcal{D}}$ is a one-dimensional projective space (the Riemann sphere).

The situation is rather dull when $d = 1$, for then f is an isomorphism and so $S \cong \mathbb{P}^1$. Let us therefore assume that $d = 2$. Then all the points of ramification of f will have multiplicity 2. Following Riemann-Hurwitz (4.8) their number must be $2g - 2 - 2(0 - 2) = 2g + 2$. Denote by $B \subset \check{\mathcal{D}}$ the set of images of these ramification points. If we identify $\check{\mathcal{D}}$ with $\mathbb{P}^1$ in such a manner that $\infty \notin B$, then we may identify $S - f^{-1}(\infty)$ with the Riemann surface in $\mathbb{C}^2$ defined $w^2 = \prod_{b \in B}(z - b)$.

Conversely, a $2g+2$-element subset $B \subset \mathbb{P}^1$ determines a hyperelliptic Riemann surface of genus g: after a possible linear transformation in $\mathbb{P}^1$ we can assume that $B \subset \mathbb{C}$ and if S is the surface $\mathbb{C}^2$ defined above plus the two points over infinity that correspond to $w \sim \pm z^{g+1}$, $|z| \ggg 0$, then S has genus g (by the Riemann-Hurwitz formula) and comes with a projection $f : S \to \mathbb{P}^1$. The corresponding 1-dimensional linear system has degree 2 (and is without fixed points).

CHAPTER 6

The Riemann-Roch theorem

In order to apply results such as the Embedding criterion 5.7 it is desirable to find the dimension of $|D|$ for a divisor D. The Riemann-Roch formula, the subject of this chapter, does almost that.

In this chapter S is a compact, connected *Riemann surface of genus g.*

1. An exact sequence

For $p \in S$ we denote by

$$\mathcal{M}_{S,p} = \cup_{n\in\mathbb{Z}} \mathcal{O}_{S,p}(n),$$

the ring of meromorphic functions on an unspecified neighborhood of p. This is in fact a field (isomorphic to $\mathbb{C}\{z\}[z^{-1}]$). By 'restricting' a meromorphic function on S to a neighborhood of p, we find a natural ring homomorphism $\mathcal{M}(S) \to \mathcal{M}_{S,p}$. Such a homomorphism is (as any ring homomorphism with domain a field), injective. Let now $D = \sum_{p\in S} n_p(p)$ be a divisor on S and consider the map

$$e_D : \mathcal{M}(S) \to \oplus_{p\in S} \mathcal{M}_{S,p}/\mathcal{O}_{S,p}(n_p),$$

whose p-component is the restriction map above followed by the evident quotient map $\mathcal{M}_{S,p} \to \mathcal{M}_{S,p}/\mathcal{O}_{S,p}(n_p)$. We should observe that this is well-defined, for the set P of $p \in S$ for which the image of $\phi \in \mathcal{M}(S)$ in $\mathcal{M}_{S,p}/\mathcal{O}_{S,p}(n_p)$ is nonzero is finite indeed: for all but finitely many p we have $n_p = 0$ and for all but finitely many p we have $\mathrm{ord}_p(\phi) < 0$. Hence for all but but finitely many p the image of p in $\mathcal{M}_{S,p}/\mathcal{O}_{S,p}(n_p)$ is nonzero. The map e_D is $\mathbb{C}$-linear and we have $e_D(\phi) = 0$, precisely when $\mathrm{ord}_p(\phi) \geq -n_p$ for all $p \in S$, in other words, when $\phi \in L(D)$. In particular e_D has finite dimensional kernel. We shall find that the cokernel of e_D (which we will denote by $H^1(D)$) is also finite dimensional.

PROPOSITION 6.1. *If $D' \leq D$, then we have a natural exact sequence*

$$0 \to L(D') \to L(D) \to L(D/D') \to H^1(D') \to H^1(D) \to 0,$$

where $\dim L(D/D')$ *is the vector space* $\oplus_{p\in S}\mathcal{O}_{S,p}(n_p)/\mathcal{O}_{S,p}(n'_p)$ *whose dimension is* $\deg(D) - \deg(D')$.

PROOF. Write $D' = \sum_{p\in S} n'_p(p)$, so that $n'_p \leq n_p$ for all $p \in S$. Since $\mathcal{O}_{S,p}(n'_p) \subset \mathcal{O}_{S,p}(n_p)$ we have a surjection $\mathcal{M}_{S,p}/\mathcal{O}_{S,p}(n'_p) \to \mathcal{M}_{S,p}/\mathcal{O}_{S,p}(n_p)$. The map e_D is therefore $e_{D'}$ composed with the surjection

$$\oplus_{p\in S}\mathcal{M}_{S,p}/\mathcal{O}_{S,p}(n'_p) \twoheadrightarrow \oplus_{p\in S}\mathcal{M}_{S,p}/\mathcal{O}_{S,p}(n_p)$$

The kernel of the latter is $L(D/D')$, which has indeed dimension $\sum_{p\in S}(n_p - n'_p) = \deg(D) - \deg(D')$. Dividing both sides by the image of $\mathcal{M}(S)$ therein, yields a surjection

$$H^1(D') \twoheadrightarrow H^1(D).$$

The kernel of this map can be identified with $L(D/D')$ modulo the image of $\mathcal{M}(S)$ in $tL(D/D')$. But that subspace is just $L(D)/L(D')$. □

2. Residues

The residue theorem from complex function theory says among other things that if f is a holomorphic function on a punctured disk at $0 \in \mathbb{C}$ of radius $> r$, the coefficient of z^{-1} in the Laurent expansion of f equals $(2\pi i)^{-1} \int_{|z|=r} f(z)dz$. This is an indication that the intrinsic definition of the residue concerns differentials rather than functions.

For $p \in S$ we denote by $\mathcal{M}\Omega_{S,p}$ the collection of meromorphic differentials on an unspecified neighborhood of p. This is a $\mathcal{M}_{S,p}$-vector space of dimension 1. A local coordonate identifies this with the $\mathbb{C}\{z\}[z^{-1}]$-vector space $\mathbb{C}\{z\}[z^{-1}]dz$.

DEFINITION 6.2. The *residu map at* p is the map

$$\mathrm{Res}_p : \mathcal{M}\Omega_{S,p} \to \mathbb{C}, \quad \omega \mapsto \frac{1}{2\pi i}\int_{\partial B} \omega,$$

where B is a closed disk in S that contains p in its interior and has the property that $B - \{p\}$ is contained in the domain where ω is defined as a holomorphic differential (so this B will depend on ω).

This is well-defined in the sense that the residue does not depend on the choice of B (this follows from Stokes' theorem for instance). It is clear that $\mathrm{Res}_p(\omega) = 0$ if ω is holomorphic at p and so a meromorphic differential on S has nonzero residues for only finitely many points of S (these must all be poles of the differental).

PROPOSITION 6.3 (Residue theorem). *The sum of the residues of a meromorphic differential on* S *is zero.*

PROOF. Let ω be a nonzero meromorphic differential on S and denote by $P \subset S$ the (finite) set of poles of ω. We choose for every $p \in P$ a closed disk B_p in S that has p in its interior and is such that the $\{B_p\}_{p \in P}$ are pairwise disjoint. We apply Stokes' theorem for surfaces on the domain with boundary $D := S - \cup_{p\in P}\mathrm{Int}(B_p)$: we find

$$\begin{aligned}\sum_{p\in S}\mathrm{Res}_p(\omega) = \sum_{p\in P}\mathrm{Res}_p(\omega) &= \sum_{p\in P}\frac{1}{2\pi i}\int_{\partial D_p}\omega \quad \text{(by definition)}\\ &= -\frac{1}{2\pi i}\int_{\partial D}\omega \quad \text{(the orientation reverses)}\\ &= -\frac{1}{2\pi i}\int_{D} d\omega = 0 \quad \text{(Corollary 3.16)}.\end{aligned}$$

□

EXERCISE 6.1. Let ϕ be a nonconstant meromorphic function on S. Prove that $\omega := \phi^{-1}d\phi$ is a meromorphic differential all of whose poles are simple (i.e., of order one) and whose residue at $p \in S$ equals $\mathrm{ord}_p(\phi)$. Use this to give an alternate proof that a principal divisor has degree zero.

The residue theorem tells us that the polar parts of a meromorphic differential on S cannot be arbitrarily prescribed, as a necessary condition is that the sum of its residues is zero. It is quite remarkable that this is the only restriction:

THEOREM 6.4 (The converse residue theorem). *An element in the kernel of the map*

$$\mathrm{Res} : \oplus_{p\in S}\mathcal{M}\Omega_{S,p}/\Omega_{S,p} \to \mathbb{C}, \quad (\omega_p)_{p\in S} \mapsto \sum_{P\in S} \mathrm{Res}_p\, \omega_p,$$

is the image of a meromorphic differential o S.

PROOF. Let $(\eta_p \in \mathcal{M}\Omega_{S,p}/\Omega_{S,p})_{p\in S}$ be an element of the kernel of the residue map Res. Denote by $P \subset S$ the finite set of $p \in S$ for which $\eta_p \neq 0$. For every $p \in P$ we represent η_p by a meromorphic differential $\tilde{\eta}_p$ defined on a neighborhood U_p of p, where we assume the U_p's to be mutually disjoint. It is easy to construct a differentiable $\mathbb{R}$-valued function g on S whose support is contained in $\cup_{p\in P} U_p$ and that is for every $p \in P$ constant 1 on a disklike neighborhood B_p of p.

Let η be the form $S - P$ that is zero on $S - \cup_{p\in P} U_p$ and $g\tilde{\eta}_p$ on $U_p - \{p\}$. So η is of type $(1,0)$. Moreover $d\eta$ is a 2-form on $S - P$ that is constant zero on $\cup_{p\in P} D_p - \{p\}$ and hence is defined on all of S. Stokes' theorem then shows:

$$\int_S d\eta = \int_{S-\cup_{p\in P} B_p} d\eta = -\sum_{p\in P}\int_{\partial B_p} \tilde{\eta}_p = -\sum_{p\in P} 2\pi i\, \mathrm{Res}_p(\tilde{\eta}_p) = 0.$$

Following Theorem 3.21 $d\eta$ may then be written as $d\zeta$, where (unlike η) ζ is a 1-form defined on all of S. We can always alter ζ by a closed 1-form. So if we write ζ according to the Hodge-Weyl decomposition 3.31: $\zeta = \zeta_0 + dh' + \star dh$, where ζ_0 is a harmonic 1-form and h, h' are $\mathbb{C}$-valued functions on S, then we may assume that $\zeta_0 = 0$ and $h' = 0$, so that $\zeta = \star dh$. Consider $\omega := \eta - 2\star\bar{\partial}h$. This is a form on $S - P$ of type $(1,0)$ and we have

$$d\omega = d\eta - 2d\star\bar{\partial}h = d\zeta - d\star dh \quad \text{(because of Lemma 3.26)} \quad = 0$$

But this means that ω is holomorphic on $S - P$. Moreover, ω and $\tilde{\eta}_p$ differ on a neighborhood of p by a differentiable form defined on that neighborhood; this difference is of type $(1,0)$ and closed, hence holomorphic. This implies that ω is a meromorphic differential on S that represents $(\eta_p)_{p\in S}$. □

3. Riemann-Roch

We begin with reformulating the converse residue theorem:

COROLLARY 6.5. *If* ω *is a nonzero meromorphic differential on* S *and* K *its divisor, then the linear map*

$$(\phi_p)_{p\in S} \in \oplus_{p\in S}\mathcal{M}_{S,p} \mapsto \sum_{p\in S} \mathrm{Res}_p\, \phi_p\omega \in \mathbb{C}$$

factors through an isomorphism $H^1(K) \cong \mathbb{C}$.

PROOF. Multiplication by ω identifies $H^1(K)$ with the cokernel of

$$\mathcal{M}\Omega(S) \to \oplus_{p\in S}\mathcal{M}\Omega_{S,p}/\Omega_{S,p}.$$

Theorem 6.4 tells us that the residue map identifies this cokernel with $\mathbb{C}$. □

COROLLARY 6.6 (Riemann-Roch). *For any divisor* D, $H^1(D)$ *is finite dimensional and*

$$\dim L(D) - \dim H^1(D) = 1 - g + \deg(D).$$

Moreover the degree of a canonical divisor equals $2g - 2$.

PROOF. Even if we do not know yet whether $\dim H^1(D) < \infty$, we can define

$$c(D) := \dim L(D) - \dim H^1(D) - \deg(D) \in \mathbb{Z} \cup \{-\infty\},$$

(so $c(D) = -\infty$ precisely when $\dim H^1(D) = \infty$). Proposition 6.1 shows that when $D \geq D'$, we have $c(D) = c(D')$. This implies that $c(D)$ is independent of D, for if D, D' are arbitrary, then we may choose D'' n such a manner that $D'' \leq D$ and $D'' \leq D$ and conclude that $c(D) = c(D'') = c(D')$. We denote this common value by c. If K is a canonical divisor, then Corollary 6.5 says that $\dim H^1(K) = 1 < \infty$, so that $c = g-1-\deg(K)$. In particular, $\dim H^1(D) < \infty$ and $\dim L(D) - \dim H^1(D) = c + \deg(D)$. So if we show that the degree of a canonical divisor equals $2g-2$, then we know the value of c: $c = \dim \Omega(S) - H^1(K) - \deg(K) = g-1-(2g-2) = 1-g$ and the remainder also follows.

To this end we choose $p \in S$. The for $k \in \mathbb{Z}$, $\dim L(k(p)) = c+k+\dim H^1(k(p)) \geq c+k$. So if we take $k = -c+2$, then we see that there exists a nonconstant meromorpic function on S. We regard this as a nonconstante holomorphic map $f : S \to \mathbb{P}^1$. An example of a canonical divisor on $\mathbb{P}^1$ is the divisor of dz: $-2(\infty)$. Following Theorem 4.4 then the (canonical) divisor K of f^*dz is such that $K = -2f^*(\infty) + R_f$. Taking the degree of both sides gives $\deg(K) = -2\deg(f) + \deg(R_f)$. But the topological Riemann-Hurwitz formula 4.8 shows that the last expression equals $2g-2$. □

EXERCISE 6.2. (The parts (b) en (c)require some basic knowledge of field theory.) Let S be a compact connected Riemann surface of genus g.

(a) Prove that there exists a nonconstant homolomorphic map $f : S \to \mathbb{P}^1$ of degree $\leq g+1$.

(b) Conclude with the help of Exercise 4.4 that as field extension of $\mathbb{C}$, $\mathcal{M}(S)$ is finitely generated and has transcendence degree 1.

(c) Prove that every finitely generated field extension of $\mathbb{C}$ of transcendenc degree 1 is isomorphic to the field of meromorphic functions on a compact connected Riemann surface.

4. Serre duality

The full power of the Riemann-Roch theorem is unleashed only in tandem with a duality property that relates the rather elusive space $H^1(D)$ via the residue theorem with the much more concrete space of meromorphic differentials.

Let be given a divisor $D := \sum_{p \in S} n_p(p)$. We denote the space of meromorphic differentials ω on S with $\mathrm{div}(\omega) \geq D$ by $L^1(-D)$. If we choose a nonzero meromorphic differential η, then we may identify this with a space of meromorphic functions, for if K is the (canonical) divisor $\mathrm{div}(\eta)$, then we have an isomorphism

$$L(K-D) \cong L^1(-D), \quad \phi \mapsto \phi\eta.$$

For $\phi \in L(K-D)$ is equivalent to $\mathrm{ord}_p(\phi) \geq -\mathrm{ord}_p(\eta) + n_p$ for all $p \in S$ and $\phi\eta \in L^1(-D)$ is equivalent to $\mathrm{ord}_p(\phi) + \mathrm{ord}_p(\eta) \geq n_p$ for all $p \in S$.

Now let $\omega \in L^1(-D)$ and consider the map

$$\mathrm{Res}(\omega) : \oplus_{p\in S}\mathcal{M}_{S,p} \to \mathbb{C}, \quad (\phi_p)_{p\in S} \mapsto \sum_{p\in S} \mathrm{Res}_p(\phi_p\omega).$$

If $\phi_p \in \mathcal{O}_{S,p}(n_p)$, then $\mathrm{ord}_p(\phi_p\omega) \geq \mathrm{ord}_p(\omega) - n_p \geq 0$ and hence the residue of $\phi_p\omega$ at p is zero. This means that $\oplus_{p\in S}\mathcal{O}_{S,p}(n_p)$ is in the kernel of $\mathrm{Res}(\omega)$ so that the latter factors through a map

$$\oplus_{p\in S}\mathcal{M}_{S,p}/\mathcal{O}_{S,p}(n_p)\to\mathbb{C}.$$

The residue theorem tells us that this map is zero on the image of $\mathcal{M}(S)$ in this direct sum, so that $\mathrm{Res}(\omega)$ determines a linear function $H^1(D)\to\mathbb{C}$. We thus have associated to ω an element of $H^1(D)^*$. This construction defines a map

$$\mathrm{Res}_D : L^1(-D)\to H^1(D)^*$$

which is clearly linear.

THEOREM 6.7 (Serre-duality). *The map* $\mathrm{Res}_D : L^1(-D)\to H^1(D)^*$ *is an isomorphism.*

PROOF. We first prove that Res_D is injective. Write $D=\sum_{p\in S} n_p(p)$ and let $0\neq\omega\in L^1(-D)$. Choose $o\in S$ such that both n_o and $\mathrm{ord}_o(\omega)$ are zero and let $\phi_o\in\mathcal{M}_{S,o}$ be such that $\mathrm{ord}_o\,\phi_o=-1$. Then $\phi_o\omega$ has nonzero residu at o. Then $\mathrm{Res}(\omega)$ takes a nonzero value on the element of $\oplus_{p\in S}\mathcal{M}_{S,p}/\mathcal{O}_{S,p}(n_p)$ that in the o-summand is the image of ϕ_o and in the remaining summands is zero. So the linear form on $H^1(D)$ defined by ω is nonzero.

In particular, $\dim L(K-D)\le\dim H^1(D)$. Replacing D by $K-D$ also yields $\dim L(D)\le\dim H^1(K-D)$. But the Riemann-Roch theorem implies that

$$\begin{aligned}\dim L(D)+\dim L(K-D)=&\\&=\dim H^1(D)+1-g+\deg(D)+\dim H^1(K-D)+1-g+\deg(K-D)\\&=\dim H^1(D)+\dim H^1(K-D)+2-2g+\deg(K)\\&=\dim H^1(D)+\dim H^1(K-D),\end{aligned}$$

and so necessarily $\dim H^1(D)=\dim L(K-D)$ (and likewise $\dim H^1(K-D)=\dim L(D)$). Hence Res_D is also surjective. □

COROLLARY 6.8. *Let* D *be a divisor of degree* $d\ge 0$ *on* S.

(i) *We have* $d-g\le\dim|D|\le d$,
(ii) *if* $d\ge 2g-1$, *then* $\dim|D|=d-g$,
(iii) *if* $d\ge 2g$, *then* $|D|$ *has no fixed points and hence defines a nondegenerate map of degree* d *to a projective space of dimension* $d-g$,
(iv) *if* $d\ge 2g+1$, *then* $|D|$ *defines an embedding in a* $(d-g)$*-dimensional projective space.*

PROOF. We know that $|D|$ is a projective space of dimension one less than that of $L(D)$. So the upper bound for $\dim|D|$ follows from Lemma 5.2 and the lower bound follows from Riemann-Roch.

Let K be a canonical divisor. Then according to Corollary 6.6 the degree of $K-D$ equals $2g-2-d$. So if $d\ge 2g-1$, then this degree is negative, so that $L(K-D)=0$ and hence $\dim|D|=d-g$. Propositions 5.6 and 5.7 imply that $|D|$ has no fixed points for $d\ge 2g$ and defines an embedding if $d\ge 2g+1$. □

If we combine this with Chow's theorem (which we stated without proof), then it follows that every compact Riemann surface S of genus g can be realized as a nonsingular projective algebraic curve in $\mathbb{P}^{g+1}$ of degree $2g+1$. Of special interest are the cases $g=0$ and $g=1$:

COROLLARY 6.9. *A compact connected Riemann surface of genus 0 is isomorphic to the Riemann sphere $\mathbb{P}^1$. A compact connected Riemann surface of genus 1 can be realized as a cubic curve in $\mathbb{P}^2$.*

COROLLARY 6.10. *If D is a divisor of degree $d > 0$, then the space of meromorphic differentials on S with divisor $\geq -D$, $\dim L^1(D)$, has dimension $g - 1 + d$.*

PROOF. We have $L^1(K + D) \cong H^1(-D)$ and by Riemann-Roch $\dim L(-D) - \dim H^1(-D) = 1 - g - d$. But $L(-D) = \{0\}$ (the degree $-d$ of $-D$ is negative) and so $\dim L^1(D) = g - 1 + d$. □

PROPOSITION 6.11. *If $g \leq 2$, then S is hyperelliptic*

PROOF. If $g = 0$ resp. $g = 1$, then choose $p \in S$ and consider the complete linear system $|2(p)|$. By part (i) resp. (iii) of Corollary 6.8 this has dimension at least 1. In case $g = 2$ the linear system of effective canonical divisors (so the divisors of holomorphic differentials) is of dimension 1 and has degree 2. □

EXERCISE 6.3. If we apply Corollary 6.10 to $D = (p)$, then we find that $L^1((p))$ has dimension g, which we know is also the dimension of $\Omega(S)$. In other words, the inclusion $\Omega(S) \subset L^1((p))$ is an isomorphism: there is no holomorphic differential on $S - \{p\}$ with a simple pole at p. How can we see this directly?

EXERCISE 6.4. For $p \in S$ we denote by L_p the space of meromorphic functions on S that are holomorphic on $S - \{p\}$. Observe that this is a ring and equal to $\cup_{k \geq 0} L(k(p))$.

(a) Prove that the image of $-\operatorname{ord}_p : L_p \to \mathbb{Z}_+$ is an additive semigroup (i.e., closed under addition and containing 0).

(b) Let k be a nonnegative integer. Prove that there is no $\phi \in L_p$ with $-\operatorname{ord}_p \phi = k$ if and only if there is a $\omega \in \Omega(S)$ with $\operatorname{ord}_p \omega = k - 1$.

(c) Conclude that there are precisely g positive integers not in $-\operatorname{ord}(L_p)$ and that these are all $\leq 2g - 1$; we may therefore order them $1 \leq n_1 < n_2 < \cdots < n_g \leq 2g - 1$. (These are the socalled *Weierstrass gaps* of p.)

CHAPTER 7

The Jacobian and the Abel-Jacobi map

In this chapter S is still a *compact, connected* Riemann surface of genus g.

1. The Jacobian

In wat follows we should remember that $H_1(S) \cong \mathbb{Z}^{2g}$ and that $\Omega(S) \cong \mathbb{C}^g$. Recall that a lattice in a real vector space of dimension d is a subgroup generated by a basis of that vector space (so such a subgroup is isomorphic to $\mathbb{Z}^d$). The quotient group is then isomorphic to the d-torus $(S^1)^d$ (also as a manifold). If the vector space happens to be a complex vector space (with the preceding applied to the underlying real vector space), then according to Example 1.7-ii, this torus is a complex manifold.

We define a homomorphism

$$e : H_1(S) \to \Omega(S)^*, \quad e(\gamma)(\omega) := \int_\gamma \omega.$$

PROPOSITION-DEFINITION 7.1 (Jacobi). *The homomorphism e maps $H_1(S)$ isomorphically onto a lattice in $\Omega(S)^*$, so that $\Omega(S)^*/e(H_1(S))$ is a complex torus. The image of e is called the* period lattice *of S and the complex torus is called the* jacobian *of S; we denote the latter by* $\mathrm{Jac}(S)$.

PROOF. Let $\gamma_1, \dots, \gamma_{2g}$ be a basis of $H_1(S)$. We must show that its image under e, $(e(\gamma_1), \dots, e(\gamma_{2g}))$, is a $\mathbb{R}$-basis for $\Omega(S)^*$. Since $\Omega(S)^*$ has real dimension $2g$, it suffices to show that $e(\gamma_1), \dots, e(\gamma_{2g})$ are $\mathbb{R}$-independent. So suppose that $c_1, \dots, c_{2g} \in \mathbb{R}$ are such that $\sum_i c_i e(\gamma_i) = 0$. This means that for all $\omega \in \Omega(S)$:

$$\sum_i c_i \int_{\gamma_i} \omega = 0.$$

Complex conjugation of this identity show that it also holds for all $\omega \in \overline{\Omega(S)}$. Theorem **??** implies that this then holds for all classes in $H^1(S, \mathbb{C})$. But then Proposition 3.10 tells us that we must have $c_i = 0$ $(i = 1, \dots, 2g)$. □

A choice of basis $\omega_1, \dots \omega_g$ of $\Omega(S)$ identifies $\Omega(S)^*$ with $\mathbb{C}^g$ (assign to $u \in \Omega(S)^*$ the g-tuple $(u(\omega_1), \dots, u(\omega_g))$). If we also choose a basis $\gamma_1, \dots, \gamma_{2g}$ of $H_1(S)$, then $\mathrm{Jac}(S)$ can be decribed more concretely (though less intrinsically) as follows: the element $e(\gamma_i)$ corresponds to the vector $(\int_{\gamma_i} \omega_1, \dots, \int_{\gamma_i} \omega_g) \in \mathbb{C}^g$. So these $2g$ are $\mathbb{R}$-independent and $\mathrm{Jac}(S)$ 'is' the quotient of $\mathbb{C}^g$ by the lattice spanned by these vectors.

The isomorphism type of the jacobian turns out be complete invariant of the isomorphism type of S (Torelli theorem). The more precise statement is that if S and S' are compact connected Riemann surfaces and $\phi : H_1(S) \to H_1(S')$ is an isomorphism of abelian groups such that the resulting isomorphism $\phi^*_{\mathbb{C}} : H^1(S', \mathbb{C}) \to$

$H^1(S,\mathbb{C})$ respects both the dot product and the decomposition into type (holomorphic and antiholomorphic), then and ϕ (or $-\phi$, for $-\phi$ will then have the same property) is induced by an isomorphism $f : S \to S'$.

2. The Abel-Jacobi map

Let $p, q \in S$. If γ is a path from p to q, then $\omega \in \Omega(S) \mapsto \int_\gamma \omega \in \mathbb{C}$ is linear in ω, and hence defines an element $\tilde{I}_\gamma \in \Omega(S)^*$. Since $\Omega(S)$ consists of closed forms, $\tilde{I}_\gamma$ only depends on the homotopy class of γ. If γ' is another path from p to q, then

$$\int_{\gamma'} \omega - \int_\gamma \omega = \int_{\gamma^*\gamma'} \omega = e([\gamma^*\gamma'])(\omega),$$

in other words, $\tilde{I}_{\gamma'} - \tilde{I}_\gamma = e([\gamma^*\gamma'])$. It follows that the image $\tilde{I}_\gamma$ in $\mathrm{Jac}(S) = \Omega(S)^*/e(H_1(S))$ only depends on p and q. If we denote that image I_p^q, then it is clear that:

$$I_p^p = 0, \quad I_p^q + I_q^r = I_p^r$$

LEMMA 7.2. *The map* $(p,q) \in S \times S \to I_p^q \in \mathrm{Jac}(S)$ *is holomorphic and additive in the sense that*

$$I_p^p = 0, \quad I_p^q + I_q^r = I_p^r.$$

PROOF. The additivity properties are clear. If we fix $o \in S$, then $I_p^q = I_o^q - I_o^p$. So it suffices to show that I is holomorphic in the upper variable. Choose a basis $\omega_1, \dots, \omega_g$ for $\Omega(S)$. Given a $q_0 \in S$, then we know that ω_i is at q_0 of the form df_i, with f_i holomorphic. We may of course also arrange that $f_i(q_0) = 0$. If q runs in a neighborhood of q_0 on which all f_ω are defined, then $I_o^q - I_o^{q_o}$ is represented by the element of $\Omega(S)^* \cong \mathbb{C}^g$ given by $(f_1(q), \dots, f_g(q))$. Since each f_i is holomorphic, it follows that I_o^q depends holomorphically on q. □

COROLLARY 7.3. *There is defined a homomorphism of abelian groups* $I : \mathrm{Div}^0(S) \to \mathrm{Jac}(S)$ *characterized by the property that* $I((q) - (p)) = I_p^q$.

PROOF. Choose $o \in S$. If $D = \sum_p n_p(p)$ is a degree zero divisor, then define $I(D) := \sum_{p\in S} n_p I_o^p$ (this sum is finite). We have $I((q) - (p)) = I_o^q - I_o^p = I_p^q$ by additivity. It is checked likewise that I is a homomorphism. □

LEMMA 7.4. *The homorphism* $I : \mathrm{Div}^0(S) \to \mathrm{Jac}(S)$ *is onto.*

PROOF. We first show that the image of I contains a nonempty open subset U. We prove this via the implicit function theorem, which states (among other things) that if the derivative of a holomorphic map in a given point is surjective, then the map is at that point like a projection, so that in particular, its image contains a nonempty open subset. Choose $o \in S$, let $n > 2g - 2$ and consider the map $F : S^n \to \mathrm{Jac}(S)$, $F(p_1, \dots, p_n) = \sum_{i=1}^n I_o^{p_i}$. This map is holomorphic. Choose a coordinate z_j at p_j. Let $(\omega_1, \dots, \omega_g)$ is a basis of $\Omega(S)$, then at p_j we can write $\omega_i = df_{ij}$ for a holomorphic f_{ij} and so near $(p_1, \dots, p_n)$, F is essentially given by $(z_1, \dots, z_n) \mapsto (\sum_j f_{1j}(z_j), \cdots \sum_j f_g(z_j)) \in \mathbb{C}^g$. Its derivative matrix at $(p_1, \dots, p_n)$ is the matrix $(\omega_i(p_j))_{i,j}$. It has indeed the maximal rank g: a nonzero element in the kernel is a nonzero differential (a linear combination of $\omega_1, \dots, \omega_g$) that has a zero at every p_i. But this cannot be as it will then have $n > 2g - 2$ zeroes and this contradicts Corollary 6.6.

Denote by J the image of I. This is a subgroup of $\mathrm{Jac}(S)$ that contains the image of F, and hence contains a nonempty open subset U of $\mathrm{Jac}(S)$. By taking differences in points of U we see that J contains a neighborhood of $0 \in \mathrm{Jac}(S)$. Such a neighborhood contains the image in $\mathrm{Jac}(S)$ of a an open neighborhood B van $0 \in \Omega(S)^*$. So every $\nu \in \Omega(S)^*$ there is a positive integer n such that $\frac{1}{n}\nu \in B$ and so the subgroup generated by B is all of $\Omega(S)^*$. This implies that $J = \mathrm{Jac}(S)$. $\square$

PROPOSITION 7.5 (Abel). *The kernel of the map* $I : \mathrm{Div}^0(S) \to \mathrm{Jac}(S)$ *contains the principal divisors.*

PROOF. A principal divisor $D \neq 0$ is by definition the divisor of a nonconstant meromorphic function ϕ on S. If $\hat{\phi}$ is the associated holomorphic map $S \to \mathbb{P}^1$, then $D = \hat{\phi}^*(0) - \hat{\phi}^*(\infty)$. We define a map $u : \mathbb{P}^1 \to \mathrm{Jac}(S)$ by $z \mapsto I(\hat{\phi}^*(z) - \hat{\phi}^*(\infty))$. With te help of local coordinates we see that u is holomorphic. Since $\mathbb{P}^1$ is simply connected, u can be lifted by Proposition 2.9 to a holomorphic map $\tilde{u} : \mathbb{P}^1 \to \Omega^*$. Every coordinate of $\tilde{u}$ is a holomorphic function on a connected compact Riemann surface, and hence constant. So $\tilde{u}$ is constant (necessarily zero). It follows that $I(D) = I(f^*(0) - f^*(\infty)) = u(0) = 0$. $\square$

So we have an induced map

$$I : \mathrm{Pic}^0(S) \to \mathrm{Jac}(S),$$

called the *Abel-Jacobi map* of S. It follows from Lemma 7.4 that it is surjective.

THEOREM 7.6 (Clebsch). *The Abel-Jacobi map* $I : \mathrm{Pic}^0(S) \to \mathrm{Jac}(S)$ *is also injective and is hence a group isomorphism.*

COROLLARY 7.7. *If the genus of* S *is* 1 *and* $o \in S$*, then the map* $I_o : S \to \mathrm{Jac}(S)$*,* $p \mapsto I_o^p$ *is an isomorphism, so that* S *is isomorphic to a complex torus.*

PROOF. The map I_o is holomorphic. We show that it is of degree one; this suffices, for it then follows that this map is an isomorphism.Suppose $I_o(p) = I_o(q)$. Then $I((q) - (p)) = 0$ and Theorem 7.6 tells us that then $(q) - (p)$ is a principal divisor: there is a meromorphic function ϕ on S with divisor $(q) - (p)$. If $p \neq q$, then this means that $\hat{\phi} : S \to \mathbb{P}^1$ is such that $\hat{\phi}^*(0) = q$ and $\hat{\phi}^*(\infty) = p$. But then $\hat{\phi}$ has degree 1 and is hence an isomorphism. This contradicts the fact that S has genus 1. $\square$

EXERCISE 7.1. Let S be a complex torus of dimension 1 with identity element $o \in S$ (we write the group law of S additively). Let $\mathcal{D}$ be a complete linear system on S of degree 2.

(a) Prove that $\mathcal{D}$ has no fixed points, is of dimension 1 and that the associated map $S \to \check{\mathcal{D}}$ has precisely 4 ramification points.

(b) Prove that $|2(o)|$ consists of the divisors $(p) + (-p)$, $p \in S$. Conclude that p is a ramification point of the associated map if and only if $p + p = o$ (there are indeed 4 such).

(c) Prove that there is a translation in S that takes $\mathcal{D}$ to $|2(o)|$.

The proof of Theorem 7.6 is beautiful, but requires a bit of preparation. In the discussion preceding Propositie 2.21 we saw that we can find on S a set of oriented piecewise differentiable circles $A_1, \dots, A_{-g}$ whose complement U in S is simply connected. We may therefore identify U with an open part $\tilde{U}$ in a universal covering $\pi : \tilde{S} \to S$ whose boundary is a $4g$-gon with respective sides

$$A^-_{-g}, A^-_g, \dots, A^+_2, A^-_{-1}, A^-_1, A^+_{-1}, A^+_1.$$

Here the order is defined by the orientation of U, that is, the one that we get if we traverse the boundary in such a manner that U stays on the left and A^+_i resp. A^-_i is a lift of A_i (with the superscript indicating what has happened to the orientation).

LEMMA 7.8. *If $D = \sum_{p\in S} n_p(p)$ is a degree 0 divisor on S with support in U, then there is precisely one meromorphic differential η_D on S with simple poles only such that $\mathrm{Res}_p(\eta) = n_p$ for all $p \in S$ and for which the integrals over each $A_{\pm i}$ is purely imaginary.*

PROOF. According to Theorem 6.4 there is a meromorphic differential η on S all of whose poles are simple and with $\mathrm{Res}_p(\eta) = n_p$ for all p. This η is unique up to a holomorphic differential. We use that ambiguity to make the integrals in question purely imaginary. For this we recall that $A_{\pm 1}, \dots, A_{\pm g}$ defines a basis $a_{\pm 1}, \dots, a_{\pm g}$ of $H_1(S)$. So integration of η over these arcs defines a homomorphism $H_1(S) \to \mathbb{C}$. If we regard this as an element of $H^1(M, \mathbb{C})$, then we can represent it harmonically by $\omega_1 + \bar{\omega}_2$ for certain $\omega_1, \omega_2 \in \Omega(S)$. So if we put $\eta_D := \eta - \omega_1 - \omega_2$, then the integrals of η_D over $A_1, \dots, A_{-g}$ are those of $-\omega_2 + \bar{\omega}_2$ and hence purely imaginary. The uniqueness of η_D follows from the fact that a holomorphic differential on S whose values on $H_1(S)$ are purely imaginary must be zero. □

Choose $o \in U$ and define for every $p \in U$, $\tilde{I}^p_o \in \Omega(S)^*$ as integration over a path U from o to p (this is independent of the chosen path because U is simply connected). For D as in the previous lemma we define $\tilde{I}(D) := \sum_{p\in U} n_p \tilde{I}^p_o \in \Omega(S)^*$. The next lemma tells us how this is written out on the $\mathbb{R}$-basis $e(a_{\pm 1}), \dots, e(a_{\pm g})$.

LEMMA 7.9. *In the situation of Lemma 7.8 we have*

$$\tilde{I}(D) := \sum_{i=1}^{g} \frac{1}{2\pi i} \int_{A_{-i}} \eta_D . e(a_i) - \sum_{i=1}^{g} \frac{1}{2\pi i} \int_{A_i} \eta_D . e(a_{-i}).$$

(Notice that the coefficients of $e(a_{\pm 1}), \dots, e(a_{\pm g})$ in the righthand side are all real.)

PROOF. Denote by $\tilde{o} \in \tilde{U}$ the point over $o \in U$. Let ω be a holomorphic differential on S and define $f : \tilde{S} \to \mathbb{C}$ by letting $f(\tilde{p})$ be the integral of $\pi^*\omega$ over a path in $\tilde{S}$ from $\tilde{o}$ to $\tilde{p}$. The f is holomorphic and $df = \pi^*\omega$. Since we may identify $\tilde{U}$ with U, we can consider $f|\tilde{U}$ also as a function U. It is clear that then

$$\tilde{I}(D)(\omega) = \sum_{p\in U} n_p f(p) = \sum_{p\in U} \mathrm{Res}_p(f\eta_D).$$

We can extend f to the boundary of $\tilde{U}$, but be aware that the identification of A^+_i and A^-_i with A_i makes that each of $f|A^+_i$ en $f|A^-_i$ yields function on A_i and these two functions are in general distinct.

Let $i > 0$. Then $f|A^+_i - f|A^-_i$ is constant equal to $\int_{A_{-i}} \omega$, for A^+_{-i} departs where A^-_i departs and arrives, where A^+_i arrives). Similarly, $f|A^+_{-i} - f|A^-_{-i}$ is constant equal to $-\int_{A_i} \omega$ (for A^-_i departs where A^+_{-i} departs and arrives where A^-_{-i}

departs). The residue formula applied to $f\pi^*\eta_D$ and the closure of $\tilde{U}$ yields:

$$\sum_{p\in U} 2\pi i\,\mathrm{Res}_p(f\eta_D) = \int_{\partial\tilde{U}} f\eta_D.$$

Since η_D has only simple poles the lefthand member equals $\sum_{p\in U} 2\pi i n_p f(p)$. The righthand side can be written as a sum of $4g$ integrals according to the sides of $\partial\tilde{U}$. The integrals over A_i^+ and A_i^- add up as $\int_{A_{-i}}\omega.\int_{A_i}\eta$ and likewise those of A_{-i}^+ and A_{-i}^- add up to $-\int_{A_i}\omega.\int_{A_{-i}}\eta$. Summing up:

$$\sum_{p\in U} 2\pi i n_p f(p) = \sum_{i=1}^{g}\Big(\int_{A_{-i}}\eta_D.\int_{A_i}\omega - \int_{A_i}\eta_D.\int_{A_{-i}}\omega\Big)$$

Since $f(p)$ is the value of $\tilde{I}_o^p$ on ω the lemma follows. □

PROOF OF THEOREM 7.6. Let $D = \sum_{p\in S} n_p(p)$ be an effective divisor on S with $I(D) = 0$. We choose the loops $A_1,\dots,A_{-g}$ in such a manner that they avoid $\mathrm{supp}(D)$. Then the Lemma's 7.8 and 7.9 can be applied. The lefthand side of Lemma 7.9 represents the origin of $\mathrm{Jac}(S)$ and hence lies in the image of e. Since the righthand side has real coefficients, these must all be integral. We conclude that the integrals $\int_{A_1}\eta,\dots,\int_{A_{-g}}\eta$ all lie in $2\pi i\mathbb{Z}$.

If we take $o\in S-\mathrm{supp}(D)$, then for $p\in S-\mathrm{supp}(D)$ the integral $\int_o^p\eta$ is multivalued (it depends on the chosen path in $S-\mathrm{supp}(D)$ from o to p), but the ambiguity lies in $2\pi i\mathbb{Z}$ and disappears if we consider $\phi(p) := \exp(\int_o^p\eta)$. This function is nowhere zero on $S-\mathrm{supp}(D)$ and satisfies $\phi^{-1}d\phi = \eta$. This shows that ϕ is meromorphic on S and that $\mathrm{ord}_p(\phi) = \mathrm{Res}_p(\eta)$ for all $p\in S$. This means that $\mathrm{div}(\phi) = D$: D is a principal divisor. □