

UNITARITY OF $SL(2)$ -CONFORMAL BLOCKS IN GENUS ZERO

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ABSTRACT. It has been conjectured for quite some time that a bundle of conformal blocks carries a unitary structure that is (projectively) flat for the Hitchin connection. This was recently established by T.R. Ramadas in the simplest nontrivial case, namely where the genus is zero and the group is $SL(2)$. In this paper we present a shorter and more direct version of his proof. We also identify the conformal block space with the bidegree $(N, 0)$ -part of an eigenspace of a finite group acting on a Hodge structure of weight N .

INTRODUCTION

Physicists have told us that a bundle of (what they call) conformal blocks or (what mathematicians often refer to as) generalized theta functions ought to come with a unitary structure that is flat for the Friedan-Shenker (or Hitchin) connection. This challenge from one community to another has been left unanswered for quite a while. We confess that the current state of affairs is a bit opaque to us, but it is our understanding that this has now been settled via topological field theory and the representation theory of quantum groups. What does seem to be clear however is that this approach does not produce a concrete inner product on a given space of conformal blocks. This, we believe, is certainly desirable, for such an explicit description would not only be more satisfying, it is also bound to be accompanied by a better structural understanding of the space in question. Gawedzki made a proposal in this direction for the genus zero case (see [5] and its references), but as it involves functional integrals and is fraught with convergence issues, mathematicians have not yet managed to implement his ideas. There is however one exception: it has inspired T.R. Ramadas to find this unitary structure in case the group is $SL(2)$ (and genus zero). In a remarkable paper [8] he converted the Gawedzki proposal into a purely algebro-geometric approach that starts out from the GIT-moduli space of parabolic rank two bundles on a punctured Riemann sphere. Ramadas invents and develops a GIT-concept, called (by him) the Harder-Narasimhan trace, and it is with the help of this notion that he is able to embed the space of conformal blocks in a space of logarithmic forms of top degree on a product of Riemann spheres (or rather on a covering thereof). A careful analysis

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leads him to conclude that these forms are in fact regular on this covering and thus receive a Hermitian inner product via classical Hodge theory. In passing he shows that the embedding is flat relative to the Hitchin connection (which in the genus zero case equals the KZ connection) on the domain and the Gauß-Manin connection on the range.

This evidently ties in with earlier work of Schechtman and Varschenko [9], who gave an explicit map from the larger KZ-domain to a space of logarithmic differentials and which is defined in a generality that includes the full genus zero case (they allow any simple complex-algebraic Lie group G ; the KZ-domain is then the space of G -invariants of a tensor product of irreducible finite dimensional representations). It therefore raises the question of whether Ramadas' proof could be adapted in such a way that one might start out immediately with the map introduced by Schechtman-Varchenko. Admittedly, this would be at the expense of the conceptual link between GIT and the Gawedzki problem found by Ramadas, but might on the other hand stand a better chance of generalizing to groups other than $SL(2)$.

The present paper is our (affirmative) answer to this question. We believe that our arguments are quite elementary and as the length of this paper shows, the proof is relatively short. Our main result is also slightly sharper: we find that the image of a KZ-vector under the Schechtman-Varchenko map is square integrable precisely when the conformal block condition is fulfilled. By combining this with a refinement of ours of a theorem of Esnault-Schechtman-Viehweg that takes care of the resonance issues, we are able to identify the space in question purely in terms of Hodge theory, namely as the bidegree $(N, 0)$ -part of an eigenspace of a finite group acting on a Hodge structure of weight N . We expect however even a topological characterization, just as there is one for the solution space of the KZ equation (and which is recalled in the appendix).

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Conventions. For a sequence $\mathbf{k} := (k_1, \dots, k_n)$ of nonnegative integers, we abbreviate $|\mathbf{k}| := \sum_{\nu} k_{\nu}$ and $\mathbf{k}! := k_1! \cdots k_n!$.

Throughout this paper we fix a tuple $\mathbf{m} = (m_1, \dots, m_n)$ of nonnegative integers whose sum $\sum_{\nu} m_{\nu} = 2N$ is even. We also fix a positive integer ℓ (to which we shall refer as the *level*).

1. THE SPACE OF CONFORMAL BLOCKS

We denote by $L(\mathfrak{m})$ an irreducible representation of $SL(2)$ with highest weight \mathfrak{m} and identify it with the \mathfrak{m} th symmetric power of $L(1)$, precisely, with the degree \mathfrak{m} part $\mathbb{C}[x, y]_{\mathfrak{m}}$ of $\mathbb{C}[x, y]$, where $SL(2)$ acts by substitution. The infinitesimal action of the Lie algebra $\mathfrak{sl}(2)$ on $\mathbb{C}[x, y]_{\mathfrak{m}}$ is in terms of the standard basis (e, f, h) of $\mathfrak{sl}(2)$ the map which sends e resp. f to $x\partial_y$ resp. $y\partial_x$ and hence h to $x\partial_x - y\partial_y$. A highest weight vector is $x^{\mathfrak{m}}$ and $f^{\mathfrak{p}}x^{\mathfrak{m}-\mathfrak{p}} = \mathfrak{m}(\mathfrak{m}-1)\cdots(\mathfrak{m}+1-\mathfrak{p})x^{\mathfrak{m}-\mathfrak{p}}y^{\mathfrak{p}}$. By taking $x = 1$, $L(\mathfrak{m})$ gets identified with the space of polynomials in y of degree $\leq \mathfrak{m}$, $\mathbb{C}[y]_{\leq \mathfrak{m}}$, with e acting as ∂_y .

Consider now the $SL(2)$ -representation $L(\mathbf{m}) := L(\mathfrak{m}_1) \otimes \cdots \otimes L(\mathfrak{m}_n)$. It is graded by the action $h \in \mathfrak{sl}(2)$: $L(\mathbf{m}) = \bigoplus_{|k| \leq N} L(\mathbf{m})_{2k}$. For each $\nu = 1, \dots, n$ we have another representation $\sigma \in SL(2) \mapsto \sigma_{\nu}$ of $SL(2)$ on this space by letting σ_{ν} act in the standard way on the ν th tensor factor and as the identity on the others. It is clear that for the corresponding infinitesimal actions, $X \in \mathfrak{sl}(2)$ acts on $L(\mathbf{m})$ as $X_1 + \cdots + X_n$. The preceding identification yields an isomorphism P of $L(\mathbf{m})$ onto the space $\mathbb{C}[\mathbf{y}]_{\leq \mathbf{m}}$ of polynomials in $\mathbb{C}[y_1, \dots, y_n]$ that are of degree $\leq \mathfrak{m}_{\nu}$ in y_{ν} with 1 corresponding to highest weight vector $\nu \in L(\mathbf{m})_{2N}$. This transforms the operator e_{ν} into partial derivation with respect to y_{ν} . In view of the identity

$$P(\mathbf{f}^{\mathfrak{p}}_{\nu}) = \frac{\mathbf{m}!}{(\mathbf{m}-\mathfrak{p})!} \mathbf{y}^{\mathfrak{p}},$$

we shall use the generators

$$\Phi^{\mathfrak{p}} := \frac{(\mathbf{m}-\mathfrak{p})!}{\mathbf{m}!} \mathbf{f}^{\mathfrak{p}}_{\nu},$$

so that $P(\Phi^{\mathfrak{p}}) = \mathbf{y}^{\mathfrak{p}}$ and $e_{\nu}\Phi^{\mathfrak{p}} = \mathfrak{p}_{\nu}\Phi^{\mathfrak{p}-1_{\nu}}$. Notice that P maps $L(\mathbf{m})_{2N-2k}$ to polynomials homogeneous of degree k .

A vector in $L(\mathbf{m})$ is invariant under $SL(2)$ if and only if it lies in $L(\mathbf{m})_0$ and is primitive, that is, is killed by e . This also applies to its image under P of course: P maps the primitive elements of $L(\mathbf{m})_0$ onto the space of $SL(2)$ -invariant polynomials.

Let $(z_{\nu} \in \mathbb{C} \subset \mathbb{P}^1)_{\nu=1}^n$ be pairwise distinct and put $\mathbf{z} = (z_1, \dots, z_n)$.

Definition 1.1. The space of conformal blocks of level ℓ relative to \mathbf{z} in $L(\mathbf{m})$ is the space of $SL(2)$ -invariant vectors in $L(\mathbf{m})$ that in addition are killed by the operator $(\sum_{\nu=1}^n z_{\nu}e_{\nu})^{\ell+1}$. We shall denote that space by $W(\mathbf{m})_{\ell}$.

Remark 1.2. This definition is nonstandard and has been adapted to the situation that we consider here. The usual notion of a space of conformal blocks of level ℓ is defined in the much more general setting where we are given a compact Riemann surface S , an n -tuple $(z_1, \dots, z_n) \in S^n$ of distinct points of S , a finite dimensional complex simple Lie algebra \mathfrak{g} and n finite dimensional irreducible representations L_1, \dots, L_n of \mathfrak{g} . If S is the Riemann

sphere and an affine coordinate on it is chosen such that all the z_ν 's lie in its domain, then we recover the above notion. This is due to Feigin and Beilinson; an elementary proof appears for example in [7]. In particular, this space can be defined independent of an affine coordinate, a property, that is perhaps not so obvious a priori. We also note that some authors prefer to call the space of conformal blocks, the dual of the one defined here.

The following lemma is equivalent to Theorem 4.3 of [8].

Lemma 1.3. *An element of $\Phi \in L(\mathbf{m})^{SL(2)}$ satisfies the conformal block condition of level ℓ relative to \mathbf{z} if and only if the polynomial $P(\Phi)$ has order $\geq N - \ell$ along \mathbf{z} (and hence also along the orbit of \mathbf{z} under $SL(2)$ acting diagonally on $(\mathbb{P}^1)^n$).*

Proof. We first observe that since $P(\Phi)$ is $SL(2)$ -invariant, the vanishing property along the $SL(2)$ -orbit of \mathbf{z} is indeed implied by the vanishing property at \mathbf{z} . In order to verify that the latter is equivalent to the conformal block condition, we write $\Phi = \sum_{\mathbf{p}} a_{\mathbf{p}} \Phi^{\mathbf{p}}$. Since $P e_\nu = \partial_\nu P$, we have

$$\begin{aligned} P\left(\sum_{\nu=1}^n z_\nu e_\nu\right)^k \Phi &= \left(\sum_{\nu} z_\nu \partial_\nu\right)^k \sum_{\mathbf{p}} a_{\mathbf{p}} \mathbf{y}^{\mathbf{p}} = \sum_{\mathbf{p}} a_{\mathbf{p}} \sum_{|\mathbf{q}|=k} \frac{k! \mathbf{z}^{\mathbf{q}}}{\mathbf{q}!} \cdot \frac{\mathbf{p}! \mathbf{y}^{\mathbf{p}-\mathbf{q}}}{(\mathbf{p}-\mathbf{q})!} = \\ &= k! \sum_{\mathbf{p}} a_{\mathbf{p}} \sum_{|\mathbf{r}|=N-k} \frac{\mathbf{p}! \mathbf{z}^{\mathbf{p}-\mathbf{r}}}{(\mathbf{p}-\mathbf{r})!} \cdot \frac{\mathbf{y}^{\mathbf{r}}}{\mathbf{r}!} = k! \sum_{|\mathbf{r}|=N-k} (\partial_{\mathbf{r}} P(\Phi))(\mathbf{z}) \cdot \frac{\mathbf{y}^{\mathbf{r}}}{\mathbf{r}!}. \end{aligned}$$

We see that the left hand side vanishes if and only if $\partial_{\mathbf{r}} P(\Phi)(\mathbf{z}) = 0$ for all multi-indices $\mathbf{r} = (r_1, \dots, r_n)$ with $r_\nu \leq m_\nu$ and $\sum_{\nu} r_\nu = N - k$. The restriction $r_\nu \leq m_\nu$ is however empty as P has degree $\leq m_\nu$ in y_ν . The assertion now follows by taking $k = \ell + 1, \dots, N$. \square

2. THE PASSAGE TO LOGARITHMIC FORMS

Schechtman and Varchenko [9] found (in a much more general setting than discussed here) a linear map from the solution space of the KZ-equation to a space of logarithmic forms which has the virtue that if we pass to the corresponding bundles, it becomes flat if we endow the latter with a Gauß-Manin connection after a ‘twist’. For an appropriate choice of the parameter that enters in the KZ-equation, this solution space contains the space of conformal blocks of fixed level and the image of that subspace under this map was characterized by Feigin, Schechtman and Varchenko [4]. Let us recall how it is defined. It goes from $L(\mathbf{m})_{2k}$, $k = 0, \dots, N$, to the space of logarithmic forms of top degree on $(\mathbb{P}^1)^{N-k}$ and is given by the rule

$$\Omega_{\mathbf{z}}\left(\frac{\Phi^{\mathbf{p}}}{\mathbf{p}!}\right) := \sum_{\substack{\phi: [N-k] \rightarrow [n] \\ |\phi^{-1}(\nu)| = p_\nu}} \frac{dt_1 \wedge \dots \wedge dt_{N-k}}{(t_1 - z_{\phi(1)}) \cdots (t_{N-k} - z_{\phi(N-k)})}.$$

Here $[n]$ is short for $\{1, \dots, n\}$ and $[N-k]$ is similarly understood. Notice that this form is anti-invariant relative to the \mathcal{S}_{N-k} -action (it acts with the sign

character). (We take the occasion to point out that the defining formula for ω_{i_1, \dots, i_q} in [4] has a typo, for each coefficient c_{m_i, p_i} that appears in it must be divided by $p_i!$. Furthermore, our map differs by an innocent factor $(N-k)!$ from theirs.) It is shown in [9] that the map Ω_z is injective.

Lemma 2.1. *Given $\Phi \in L(\mathbf{m})_{2k}$, $k \geq 0$, then $\Omega_z(e_\nu \Phi) = \text{Res}_{(t_{N-k}=z_\nu)} \Omega_z(\Phi)$, where we identified the divisor $(t_{N-k}=z_\nu)$ with \mathbb{P}^{N-k-1} .*

Proof. A straightforward check shows that this is true for the basis elements $\Phi_p/p!$. \square

Corollary 2.2. *If $\Phi \in L(\mathbf{m})_{2k}$, $k \geq 0$, is primitive, then $\Omega_z(\Phi)$ is a logarithmic form on \mathbb{P}^{N-k} that is regular along the divisors $(t_i = \infty)$, $i = 1, \dots, N$.*

Proof. For $i = N-k$, this follows from

$$0 = \Omega_z\left(\sum_\nu e_\nu \Phi\right) = \sum_\nu \text{Res}_{(t_{N-k}=z_\nu)} \Omega(\Phi) = -\text{Res}_{(t_{N-k})=\infty} \Omega(\Phi).$$

This implies the general case, as the form in question is anti-invariant relative to the \mathcal{S}_{N-k} -action. \square

Proposition 2.3. *For an element $\Phi \in L(\mathbf{m})_0$ the following are equivalent:*

- (i) $\Phi \in W(\mathbf{m})_\ell$
- (ii) $\Omega_z(\Phi)$ vanishes along any codimension ℓ diagonal (i.e., the locus where $\ell+1$ t_i 's are equal to each other).
- (iii) $\Omega_z(\Phi)$ vanishes along the locus where $t_{N-\ell} = \dots = t_N = \infty$.

Proof. (i) \Rightarrow (ii) The restriction of $\Omega_z(\Phi)$ to the diagonal $t_{N-k+1} = \dots = t_N =: t$ is written

$$k! \sum_{\mathbf{p}} \sum_{|q|=k} \alpha_{\mathbf{p}} \frac{\mathbf{p}!}{\mathbf{q}!} \left(\prod_\nu \frac{1}{(t-z_\nu)^{q_\nu}} \right) \Omega_z(\Phi_{\mathbf{p}-\mathbf{q}}) \wedge dt_{N-k+1} \wedge \dots \wedge dt_N.$$

If we pass to the summation over $r_\nu := p_\nu - q_\nu$ and divide by $k!$, this becomes

$$\begin{aligned} \sum_{\mathbf{p}} \sum_{|\mathbf{r}|=N-k} \alpha_{\mathbf{p}} \frac{\mathbf{p}!}{(\mathbf{p}-\mathbf{r})!} \left(\prod_\nu \frac{1}{(t-z_\nu)^{p_\nu-r_\nu}} \right) \Omega_z(\Phi_{\mathbf{r}}) \wedge dt_{N-k+1} \wedge \dots \wedge dt_N = \\ \sum_{|\mathbf{r}|=N-k} \partial_{\mathbf{r}}(P(\Phi)) \left(\frac{1}{t-z_1}, \dots, \frac{1}{t-z_n} \right) \Omega_z(\Phi_{\mathbf{r}}) \wedge dt_{N-k+1} \wedge \dots \wedge dt_N. \end{aligned}$$

The argument of $\partial_{\mathbf{r}}(P(\Phi))$ lies in the $SL(2)$ -orbit of \mathbf{z} and so by Lemma 1.3 the partial derivative vanishes there, provided that $k > \ell$. We conclude that $\Omega_z(\Phi)$ is identically zero on the diagonal $t_{N-\ell} = \dots = t_N$. By \mathcal{S}_N -anti-invariance, this then applies to any such diagonal of codimension ℓ .

(ii) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (i) Since $\Phi \in L(\mathbf{m})^{SL(2)}$, it follows from Corollary 2.2 that $\Omega_z(\Phi)$ is regular at infinity. Now note the following identity

$$\Omega_z\left(\sum_\nu z_\nu e_\nu \Phi\right) = \sum_\nu \text{Res}_{t_{N-k}=z_\nu} t_{N-k} \Omega(\Phi) = -\text{Res}_{t_{N-k}=\infty} t_{N-k} \Omega_z(\Phi).$$

If we iterate this we find

$$\Omega_{\mathbf{z}}\left(\sum_{\nu} z_{\nu} e_{\nu}\right)^{\ell+1} \Phi = \operatorname{Res}_{(t_{N-\ell}=\infty)} \cdots \operatorname{Res}_{(t_N=\infty)} t_{N-\ell} \cdots t_N \Omega_{\mathbf{z}}(\Phi).$$

Now replace the coordinates t_i , $i \geq N - \ell$, by $u_i := t_i^{-1}$, and write $\Omega_{\mathbf{z}}(\Phi)$ as

$$G(t_1, \dots, t_{N-\ell-1}, u_{N-\ell}, \dots, u_N) dt_1 \wedge \cdots \wedge dt_{N-\ell-1} \wedge du_{N-\ell} \wedge \cdots \wedge du_N.$$

By assumption G is in the ideal generated by $u_{N-\ell}, \dots, u_N$. But this just means that the iterated residue above vanishes. Since $\Omega_{\mathbf{z}}$ is injective, it follows that $(\sum_{\nu} z_{\nu} e_{\nu})^{\ell+1} \Phi = 0$, so that $\Phi \in \mathcal{W}(\mathbf{m})_{\ell}$. \square

3. A SQUARE INTEGRABILITY CRITERION

In this section M denotes a complex-analytic manifold of complex dimension d (we work here in the complex-analytic category). We first recall a bit of classical valuation theory. If f is a meromorphic function on M and S is an irreducible (locally closed) subvariety of M , then the *order of f along S* , $\operatorname{ord}_S(f)$, is the coefficient of the exceptional divisor of the blow up of S in the divisor of f , unless f is identically zero on an open subset meeting S : we then stipulate that $\operatorname{ord}_S(f) = \infty$. This notion only depends on the generic point of S and is insensitive to blowing up (and then taking the strict transform of S). It generalizes to the setting where f is of Nilsson class, but for us it is enough to restrict to the case where f has only finitely many determinations, by which we simply mean that f becomes univalued on a finite (possibly ramified) cover of M . Then f has at a generic point of the exceptional divisor a fractional order.

The following notion has similar properties and is for that reason almost equally useful. It is closely related to the notion of log-discrepancy.

Definition 3.1. Let ω be a multivalued meromorphic d -form on M with only finitely many determinations and let S be an irreducible subvariety of M . We define the *logarithmic order* of ω along S as follows. Write ω at some point p of S as $f\omega_0$, where ω_0 is d -form on S that is nonzero in p and f is multivalued meromorphic at p . The logarithmic order of ω along S is then $\operatorname{codim}(S) + \operatorname{ord}_{S,p}(f)$. (It is easily seen that this only depends on ω and S .)

Suppose that $D \subset M$ a hypersurface which is *arrangementlike* in the sense that D can be covered by analytic coordinate charts of M on which D is given by a product of linear forms in the coordinates. It is clear that D then comes with a natural partition into connected, locally closed submanifolds, its *strata*. We say that a stratum S is *abnormal* (other authors call such a stratum *dense*) if at any $p \in S$ the germ (S_p, D_p) is not ‘normally decomposable’. In order to be more precise, note first that for every $p \in S$ the set of irreducible components of the germ D_p is in bijective correspondence with a subset of the projectivized conormal space $\mathbb{P}((T_p M / T_p S)^*)$ to S in M . Abnormality of S means that this set contains a projective basis: in other words, if c denotes the codimension of S in M (so that $\mathbb{P}((T_p M / T_p S)^*)$ is a

projective space of dimension $c - 1$), then either $c = 1$ or $c > 1$ and the set in question contains a $(c + 1)$ -element subset with the property that if we remove any one of its elements we have an independent set (once we order these points, this fixes a projective coordinate system for which the $c + 1$ elements in question are $[1 : 0 : \cdots : 0], \dots, [0 : 0 : \cdots : 1], [1 : 1 : \cdots : 1]$). The terminology is suggested by the fact that this is (in codimension > 1) in a sense the opposite of a normal crossing.

Proposition 3.2. *Suppose M compact and ω is a meromorphic multivalued d -form on M with only finitely many determinations and whose polar set is contained in an arrangementlike hypersurface D . Then equivalent are:*

- (i) ω is square integrable (in the sense that $\int_M \omega \wedge \bar{\omega}$ converges),
- (ii) ω has positive logarithmic order along any abnormal stratum of D ,
- (iii) if $\tilde{M} \rightarrow M$ is a proper surjective map with \tilde{M} a complex manifold of the same dimension as M and such that ω becomes a univalued d -form on \tilde{M} , then the latter form is regular.

Proof. We only prove (i) \Leftrightarrow (ii), the equivalence (i) \Leftrightarrow (iii) being left to the reader. For this we proceed with the induction on the number of abnormal strata of codimension > 1 in D . If there are none, then D is a normal crossing divisor. Then let $p \in D$ and choose local coordinates (u_1, \dots, u_d) so that D is given at p by $u_1 \cdots u_k = 0$ for some $0 < k \leq d$. We can then write

$$\omega = \phi u_1^{r_1} \cdots u_k^{r_k} \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_k}{u_k} \wedge du_{k+1} \wedge \cdots \wedge du_d,$$

with $r_i \in \mathbb{Q}$ and ϕ without poles, but not vanishing on $u_i = 0$ when $i \leq k$. A little exercise shows that ω is square integrable over a neighborhood of p precisely when each r_i is positive. As r_i is the logarithmic order of ω along the branch of D_p defined by $u_i = 0$, we see that the proposition follows in that case.

Now assume D has at least one abnormal stratum of codimension > 1 . If we choose such a stratum S of minimal dimension, then its closure \tilde{S} in M is easily seen to be smooth. If we blow up \tilde{S} in M , then the total transform of D is still arrangementlike and its abnormal strata are the strict transforms of the abnormal strata of D (where we regard the exceptional divisor as the strict transform of S). The logarithmic orders of (the pull-back of) ω are unaffected and the number of abnormal strata of codimension > 1 has gone down by one. So this establishes the induction step. \square

4. RAMADAS' VANISHING THEOREM

Fix a finite nonempty set T and put $k := |T|$. For a positive integer $\ell \leq k$ denote by $I_\ell(T)$ the ideal in $\mathbb{C}[T]^{S_T}$ generated by the S_T -invariant polynomials that vanish on every codimension ℓ diagonal. It is clear that $I_\ell(T)$ is homogeneous and increases with ℓ . For instance, $I_1(T)$ is the principal ideal generated by the squared discriminant $\prod_{t, t' \in T, t \neq t'} (t - t')$. We denote

by $d_\ell(k)$ the multiplicity of $I_\ell(T)$ at 0 and extend its domain to all pairs of integers by letting it be equal to zero if $\ell > k$, $\ell \geq 0$ and infinity if $\ell < 0$.

Lemma 4.1. *We have $d_\ell(k) \geq \min\{2(k - \ell) + d_\ell(k - \ell), d_{\ell-1}(k)\}$.*

Proof. Let $G \in I_\ell(T) - I_{\ell-1}(T)$ be homogeneous. It is enough to show that $\deg(G) \geq 2(k - \ell) + d_\ell(k - \ell)$. Our assumption implies that for some ℓ -element subset A of T , G is nonzero on the corresponding codimension $(\ell - 1)$ -diagonal Δ_A . If $t_0 \in A$, then for every $t \in T - A$, $G|_{\Delta_A}$ vanishes on the hyperplane section defined by $t = t_0$. Since G is invariant under the transposition of t_0 and t , the order of vanishing there will be at least 2 and so we can write $G|_{\Delta_A} = \prod_{t \in T - A} (t - t_0)^2 G_A$, where G_A is a nonzero homogeneous polynomial. It is clear that G_A is invariant under \mathcal{S}_{T-A} and vanishes on every codimension ℓ diagonal defined by an $(\ell + 1)$ -element subset of $T - A$. So its degree is at least $d_\ell(k - \ell)$. \square

The following corollary is due to Ramadas or rather, it is implied by Theorem 8.1 of [8].

Corollary 4.2. *We have $d_\ell(k) \geq k(k - \ell)/\ell$.*

Proof. We saw that $I_1(T)$ is generated by a polynomial of degree $k(k - 1)$. We proceed with induction on ℓ . For a fixed k , $k(k - \ell)/\ell$ is monotonously nonincreasing in ℓ . So the above lemma implies that in fact

$$d_\ell(k) \geq 2(k - \ell) + \max\{0, (k - 2\ell)(k - \ell)/\ell\} \geq k(k - \ell)/\ell. \quad \square$$

5. SQUARE INTEGRABILITY

We define a rational function on $(\mathbb{P}^1)^N$ by

$$F_z(t_1, \dots, t_N) := \prod_{v=1}^n \prod_{i=1}^N (t_i - z_v)^{m_v} \cdot \prod_{1 \leq i < j \leq N} (t_i - t_j)^{-2}.$$

Note that the order of F_z along the divisor $(t_i = \infty)$ is $-\sum_v m_v + 2(N - 1) = -2$. The support of the divisor of F_z is denoted D_z ; it is the union of hypersurfaces defined by $t_i = z_v$, $t_i = \infty$ and $t_i = t_j$, $i < j$. This union is clearly arrangementlike. Its abnormal strata of codimension k are:

- (i) diagonals in $(\mathbb{P}^1)^N$ defined by letting $k \geq 2$ coordinates coalesce,
- (ii) loci in $(\mathbb{P}^1)^N$ defined by setting $k \geq 1$ coordinates equal to ∞ ,
- (iii) loci in $(\mathbb{P}^1)^N$ defined by setting $k \geq 1$ coordinates equal to some z_v .

Here is the main result of this paper. The ‘only if’ part is the most substantial and is due to Ramadas [8].

Theorem 5.1. *An invariant tensor $\Phi \in L(\mathbf{m})^{\mathrm{SL}(2)}$ lies in $W(\mathbf{m})_\ell$ if and only if $F_z^{1/(\ell+2)} \Omega_z(\Phi)$ is square integrable.*

Proof. It is clear that the polar divisor of the integrand has its support contained $D_z \subset (\mathbb{P}^1)^N$. Suppose first that $F_z^{1/(\ell+2)} \Omega_z(\Phi)$ is square integrable. In

order to show that that $\Phi \in L(\mathbf{m})^{SL(2)}$, we can assume that $N \geq \ell + 1$ (otherwise $W(\mathbf{m})_\ell = L(\mathbf{m})^{SL(2)}$). The order of $F_z^{1/(\ell+2)}$ at the abnormal stratum defined by $t_{N-\ell} = \cdots = t_N = \infty$ is $-\ell - 1$, and so the square integrability assumption implies that $\Omega_z(\Phi)$ has logarithmic order ≥ 1 at this stratum. According to Proposition 2.3 this implies that $\Phi \in W(\mathbf{m})_\ell$.

We prove the converse via the criterion Proposition 3.2. The following three lemmas establish that $\Omega_z(\Phi)$ has positive logarithmic order along each of the abnormal strata and this will then imply the theorem. \square

We abbreviate $\Omega_z(\Phi)$ by ω and F_z by F .

Lemma 5.2. *The logarithmic order of ω along any partial diagonal defined by letting $k \geq 2$ t_i 's coalesce is $\geq (k - 1) + \max\{0, k(k - \ell)/\ell\}$.*

Proof. Immediate from Corollary 4.2. \square

Lemma 5.3. *The logarithmic order of $F^{1/(\ell+2)}\omega$ along any partial diagonal defined by letting $k \geq 2$ t_i 's coalesce is positive.*

Proof. The order of $F^{1/(\ell+2)}$ along that stratum is $-k(k - 1)/(\ell + 2)$. If $k \leq \ell$, then ω has logarithmic order at least $k - 1$ and so the logarithmic order of $F^{1/\kappa}\omega$ is there $\geq -k(k - 1)/(\ell + 2) + k - 1 = (k - 1)(\ell + 2 - k)/(\ell + 2) > 0$.

Assume now that $k \geq \ell + 1$. Then the logarithmic order of ω is according to Lemma 5.2 there at least $\ell^{-1}k(k - \ell) + (k - 1)$ and so the logarithmic order of $F^{1/\kappa}\omega$ along that diagonal is

$$\geq \frac{-k(k - 1)}{\ell + 2} + (k - 1) + \frac{k(k - \ell)}{\ell} = k \frac{\ell + 2k}{\ell + 2} - 1$$

and since $k > 1$, this is always positive. \square

Lemma 5.4. *The logarithmic order of $F^{1/(\ell+2)}\omega$ along any stratum defined by putting $k \geq 1$ t_i 's equal to ∞ is positive.*

Proof. The order of $F^{1/(\ell+2)}$ along a such a stratum is $-k(k + 1)/(\ell + 2)$. If $k \leq \ell$, then ω has along this codimension k stratum logarithmic order at least k and so the logarithmic order of $F^{1/(\ell+2)}\omega$ is there $\geq -k(k + 1)/(\ell + 2) + k = k(\ell + 1 - k)/(\ell + 2) > 0$.

Suppose now that $k \geq \ell + 1$. In that case, Lemma 5.2 tells us that the logarithmic order of ω is there $\geq k + k(k - \ell)/\ell$. We compute:

$$\frac{-k(k + 1)}{\ell + 2} + k + \frac{k(k - \ell)}{\ell} = \frac{-k(k + 1)}{\ell + 2} + \frac{k^2}{\ell} = k \frac{2k - \ell}{\ell(\ell + 2)},$$

which is also positive. \square

Lemma 5.5. *The logarithmic order of $F^{1/(\ell+2)}\omega$ along any (codimension k) stratum defined by putting k t_i 's equal to a fixed z_ν is positive.*

Proof. We prove this for the first k coordinates and so let S be the stratum in $(\mathbb{P}^1)^N$ defined by $t_1 = \cdots = t_k = z_\nu$. In order to keep notation simple we make the innocent assumption that $z_\nu = 0$, abbreviate m for m_ν .

The function $F^{1/(\ell+2)}$ has order along S equal to $k(1-k+m)/\kappa$. Since the logarithmic order of ω is ≥ 0 , the logarithmic order of $F^{1/(\ell+2)}\omega$ along S is positive if $k \leq m$. For $k > m$, we write

$$\omega = \sum_{I=(1 \leq i_1 < i_2 \dots < i_m \leq k)} G_I \frac{dt_I}{t_I} \wedge dt_{K-I} \wedge dt_{K'}.$$

Here $K = (1, 2, \dots, k)$ and $K' = (k+1, \dots, N)$. The function G_I is regular in the generic point of S . The logarithmic order of ω is at least $k-m$ and so the logarithmic order of $F^{1/(\ell+2)}\omega$ at least

$$\frac{k(1-k+m)}{\ell+2} + k-m = (k-m)\left(1 - \frac{k}{\ell+2}\right) + \frac{1}{\ell+2},$$

which is positive as long as $k \leq \ell+2$.

Finally assume $k \geq \ell+1$. Observe that the restriction of $G_I dt_{K-I}$ to the stratum S_I (defined by setting $t_i = 0$ for all $i \in I$) is the residue of ω on S_I . Let $J = (1 \leq j_1 < j_2 < \dots < j_s \leq k)$ be a maximal subset of K such that

$$\text{Res}_{(t_{j_1}=0)} \cdots \text{Res}_{(t_{j_s}=0)} \omega \neq 0.$$

So $s \leq m$ and by the S_N -invariance of ω , this then holds for all s -element subsets of $\{1, \dots, N\}$ (in terms of the representation this means that $e_{\mathcal{V}}^s \Phi \neq 0 = e_{\mathcal{V}}^{s+1} \Phi$). So if we write $\text{Res}_{(t_{j_1}=0)} \cdots \text{Res}_{(t_{j_s}=0)} \omega = \tilde{G}_J dt_J$, then \tilde{G}_J is regular on S_J and vanishes on the codimension ℓ diagonals in S_J . The codimension of S in S_J is $k-s$ and so this implies that \tilde{G}_J vanishes along $S \subset S_J$ of order at least $(k-s)(k-s-\ell)/\ell$. The logarithmic order of ω along S is at least that of $\tilde{G}_J dt_J$ and so $\geq (k-s) + (k-s)(k-s-\ell)/\ell = (k-s)^2/\ell \geq (k-m)^2/\ell$. It follows that the logarithmic order of $F^{1/(\ell+2)}\omega$ is at least

$$\frac{k(1-k+m)}{\ell+2} + \frac{(k-m)^2}{\ell} = (k-m)\left(\frac{k-m}{\ell} - \frac{1}{\ell+2}\right) + \frac{k}{\ell+2} > 0,$$

(for $k-m \geq 1$) □

6. AN INTERPRETATION IN TERMS OF HODGE THEORY

It will be convenient to write G for the cyclic group of $(\ell+2)$ th roots of unity as an abstract group. Put $U_{\mathbf{z}} := (\mathbb{P}^1)^N - D_{\mathbf{z}}$ and consider the G -covering of $\pi : \hat{U}_{\mathbf{z}} \rightarrow U_{\mathbf{z}}$ defined by the equation $w^{\ell+2} = F_{\mathbf{z}}(t)$. The action of G is on the coordinate w with (antitautological) character $\chi : G \rightarrow \mathbb{C}^{\times}$ and clearly commutes with the action of S_N on the t -coordinates. The direct image of the De Rham complex $\pi_* \Omega_{\hat{U}_{\mathbf{z}}}^{\bullet}$ decomposes under the characters of G . It is easy to see that multiplication by w identifies $\Omega_{U_{\mathbf{z}}}^k$ with $(\pi_* \Omega_{\hat{U}_{\mathbf{z}}}^k)^{\chi}$ and that under this isomorphism the De Rham differential on \hat{U} corresponds

to the De Rham differential on U_z plus wedging with

$$\eta := \frac{dw}{w} = \frac{1}{\ell+2} \sum_{i=1}^N \left(\sum_{\nu=1}^n m_\nu \frac{1}{t_i - z_\nu} - 2 \sum_{j \neq i}^N \frac{1}{t_i - t_j} \right) dt_i$$

Multiplication by w identifies the space of regular N -forms on U_z with the space of regular N -forms on \hat{U}_z that transform according to the character χ . The ‘normalization’ of the unramified G -covering $\pi : \hat{U}_z \rightarrow U_z$ over $(\mathbb{P}^1)^N$ produces a canonical extension of this covering to a G -covering $X_z \rightarrow (\mathbb{P}^1)^N$ (with possible ramification along D_z) such that X_z is normal and \hat{U}_z is dense in X_z . (This covering also depends on \mathbf{m} and ℓ , but we do not want to bring that in the notation here.) The variety X_z is projective, but need not be smooth. If we choose a resolution its singularities $\tilde{X} \rightarrow X_z$, then the G action need not extend to \tilde{X} , but we don’t care: it acts on \tilde{X} as a group of birational transformations and hence will act on its space of its regular N -forms, $H^{N,0}(\tilde{X})$. According to Deligne, the map $H^N(X_z; \mathbb{Q}) \rightarrow H^N(\tilde{X}; \mathbb{Q})$ is a morphism of Hodge structures and factors through an injective map $\mathrm{Gr}_W^N H^N(X_z; \mathbb{Q}) = H^N(X_z; \mathbb{Q}) / W_{N-1} H^N(X_z; \mathbb{Q}) \rightarrow H^N(\tilde{X}; \mathbb{Q})$. In particular, it is an isomorphism on the $H^{N,0}$ -part so that we can identify: (i) the bidegree $(N, 0)$ -summand of $H^N(\tilde{X}; \mathbb{C})$, (ii) idem of $H^N(X_z; \mathbb{C})$ (in the sense of mixed Hodge theory), and (iii) the space $H_{(2)}^{N,0}(X_z)$ of meromorphic forms on X_z that are square integrable. This renders not only evident the G -action, but also one of the permutation group S_N . Hodge theory provides $H^N(\tilde{X}; \mathbb{C})$ with the Hermitian form

$$\langle \omega_1, \omega_2 \rangle := (\sqrt{-1})^{n^2} \int_{X_z} \omega_1 \wedge \bar{\omega}_2.$$

The coefficient has been chosen as to make it positive definite on the bidegree $(N, 0)$ -part.

Theorem 6.1. *If $\varepsilon : S_N \rightarrow \mathbb{C}^\times$ denote the sign character, then the map $\Phi \mapsto wP(\Phi)$ defines an isomorphism of $W(\mathbf{m})_\ell$ onto $H^{N,0}(X_z; \mathbb{C})^{\chi, \varepsilon}$.*

Proof. Proposition 3.2 and Theorem 5.1 imply that we have a well-defined linear map $W(\mathbf{m})_\ell \rightarrow H^{N,0}(X_z)^{\chi, \varepsilon}$. This is clearly an embedding.

If $A^k(U_z)$ denotes the space of logarithmic k -forms on U_z , then according to [9], the map Ω_z defines an isomorphism

$$L(\mathbf{m})_\ell \xrightarrow{\cong} (H^N(A^\bullet(U_z), \eta \wedge)^\varepsilon = A^N(U_z)^\varepsilon / \eta \wedge A^{N-1}(U_z)^\varepsilon.$$

The surjectivity property now follows from Corollary 8.3 of the appendix. \square

Note that this corollary implies that a regular N -form on U_z which transforms under S_N according to the sign character and becomes square integrable after multiplication by $F_z^{1/(\ell+2)}$ is in the image of P .

We remark that Ramadas [8] already makes the connection with an isotypical component of the cohomology of a cyclic cover.

Question 6.2. Is $H^{N,0}(X_{\mathbf{z}})^{\chi,\varepsilon}$ the (χ, ε) -eigenspace of a polarized pure weight N Hodge structure? (This is almost equivalent to asking whether that image is topologically defined.) If $\ell \geq N$, then we have $W(\mathbf{m})_{\ell} = L(\mathbf{m})_{\ell}$ and it follows from Corollary 8.3 that the answer is yes. A somewhat weaker question is whether the image of P is rigid in the sense that its image is defined over a number field (say, over $\mathbb{Q}(\mu_{\ell+2})$).

7. UNITARITY

The justification for introducing the map P lies in its behavior in families. To be precise, if we work universally in the sense that we let \mathbf{z} vary in the open subset of $Z_n \subset \mathbb{C}^n$ of distinct n -tuples, then we obtain

- (i) a vector bundle $\mathcal{W}(\mathbf{m})_{\ell} \rightarrow Z_n$ of conformal blocks,
- (ii) a projective G -covering $\mathcal{X} \rightarrow (\mathbb{P}^1)^N \times Z_n$ with a lift of the \mathcal{S}_N -action on $(\mathbb{P}^1)^N$ that commutes with the G -action,
- (iii) if $\pi : \mathcal{X} \rightarrow Z_n$ denotes the evident projection, then the square integrable meromorphic relative N -forms define a vector bundle $\pi_* \omega_{\mathcal{X}/Z_n, (2)}^N$ over Z_n with a fiberwise $G \times \mathcal{S}_N$ -action, and we have an embedding of vector bundles over Z_n :

$$\Omega : \mathcal{W}(\mathbf{m})_{\ell} \xrightarrow{\cong} (\pi_* \omega_{\mathcal{X}/Z_n, (2)}^N)^{\chi,\varepsilon} \subset (R^N \pi_* \pi^* \mathcal{O}_{Z_n})^{\chi,\varepsilon}.$$

One of the main results of [9] says that in the present case Ω is flat if we equip left hand side with the KZ connection and the right hand side with the Gauß-Manin connection. In particular, the image of Ω (which is purely of type $(N, 0)$) is flat for the Gauß-Manin connection. In view of the Griffiths transversality theorem, this suggests that Ω maps to a $G \times \mathcal{S}_N$ -invariant Hodge subbundle for which the character pair (χ, ε) does not occur in its bidegree $(N-1, 1)$ -summand.

The right hand side comes with the Hodge Hermitian form defined earlier. It is defined in topological terms and hence flat. Since it is positive on the image of Ω , we get

Corollary 7.1 (Ramadas). *The KZ-Hitchin connection on a bundle of conformal blocks in genus zero with group $SL(2)$ is unitary.*

8. APPENDIX: THE TOPOLOGICAL INTERPRETATION OF A TWISTED ORLIK-SOLOMON COMPLEX

Let U be the complement of a finite union of hyperplanes in complex affine N -space. We have (as on any quasi-projective manifold) naturally defined the graded space $A^{\bullet}(U)$ of logarithmic forms on U . It consists of d -closed forms and according to a theorem of Brieskorn, the natural map $A^{\bullet}(U) \rightarrow H^{\bullet}(U; \mathbb{C})$ is an isomorphism.

We now suppose given a logarithmic differential $\eta \in A^1(U)$ and consider the complex defined by wedging with η , $(A^{\bullet}(U), \eta \wedge)$. We recall from [6] how the cohomology of this complex can be interpreted topologically. To

this end we choose a simple normal crossing compactification of U , that is, a connected projective manifold P and a simple normal crossing hypersurface E in P such that $U := P - E$. We view η as a connection on \mathcal{O}_P with a logarithmic pole along E . The rank one local system on U thus defined is denoted by \mathbb{L} . We denote the irreducible components of E by E_α (these are smooth by assumption) and write a_α for the residue of η along E_α . The following proposition extends a theorem in Esnault-Schechtman-Viehweg [3] the case where ‘resonances’ are allowed.

Proposition 8.1. *Let U' be the complement in P of a union of the irreducible components of E such that for any E_α with $a_\alpha \in \mathbb{Z}$, E_α is a removed component if and only if $a_\alpha \leq 0$. Denote by $j : U \rightarrow U'$ the inclusion. Then $H^\bullet(A^\bullet(U), \eta \wedge)$ is naturally isomorphic to $H^\bullet(U'; j_! \mathbb{L})$.*

For its proof we need the following lemma that is implicit in Prop. 3.13 of Deligne [1]:

Lemma 8.2. *Let X be a complex manifold, $Y \subset X$ a normal crossing divisor, and let $\eta \in H^0(X, \Omega_X^1(\log Y))$ be closed. Then the residue of η along every irreducible component of Y is constant and if $U \subset U' \subset X$ is defined as above, with $U \subset U'$ denoted j and $U' \subset X$ denoted k , then $(\Omega_X^\bullet(\log Y), d + \eta \wedge)$ represents $R^\bullet k_* j_! \mathbb{L}$, where \mathbb{L} denotes the local system over $X - Y$ defined by the structure sheaf with flat connection η .*

Proof. For the proof we limit ourselves here to the simple but basic case that X is the unit disk in \mathbb{C} and $Y = \{0\}$, so that η has the form $\phi(z)dz$, with $\phi \in \mathcal{O}(X)$. If $u : X - Y \subset X$ is the inclusion, then the stalk at 0 of $R^\bullet u_* \mathbb{L}$ is represented by the meromorphic De Rham complex

$$0 \rightarrow \mathbb{C}\{\{z\}\} \xrightarrow{d + \eta \wedge} \mathbb{C}\{\{z\}\} \frac{dz}{z} \rightarrow 0.$$

Notice that the middle map sends z^k to $(k + \phi(0))z^k \frac{dz}{z}$ plus higher order terms. If $\phi(0) \notin \mathbb{Z}$, then this complex is exact (implying that $R^\bullet u_* \mathbb{L} = u_! \mathbb{L}$). If $\phi(0) \in \mathbb{Z}$, then we have a kernel and cokernel of dimension one, both of which have (logarithmic) order $-\phi(0)$. The stalk at 0 of the logarithmic De Rham complex $(\Omega_X^\bullet(\log Y), d + \eta \wedge)$ is the subcomplex of the above complex in which $\mathbb{C}\{\{z\}\}$ has been replaced by $\mathbb{C}\{z\}$. This inclusion is a quasi-isomorphism unless $\phi(0)$ is a positive integer: in that case, the logarithmic De Rham complex is exact and so represents the stalk of $u_! \mathbb{L}$ at 0. It also follows that $(\Omega_X^\bullet(\log Y), d + \eta \wedge)$ represents $u_! \mathbb{L}$ unless $\phi(0)$ is a nonpositive integer. \square

Proof of Proposition 8.1. Let $\Omega_P^\bullet(\log E)$ be the logarithmic holomorphic De Rham complex of (P, E) . This is a complex of free \mathcal{O}_P -modules which represents the total direct image of the sheaf of complex constants on U under $U \subset P$. Notice that $A^p(U) = H^0(P, \Omega_P^\bullet(\log E))$. Following Deligne [2], the spectral sequence defined by the logarithmic De Rham resolution

$$E_1^{p,q} = H^q(P, \Omega_P^p(\log(E))) \Rightarrow H^{p+q}(U; \mathbb{C})$$

degenerates at the E_1 -term. This means that we have a decreasing filtration F^\bullet (the Hodge filtration) on $H^\bullet(U; \mathbb{C})$ such that

$$\mathrm{Gr}_F^p H^n(U; \mathbb{C}) = H^{n-p}(P, \Omega_P^p(\log E)).$$

By a theorem of Brieskorn, $A^p(U)$ maps isomorphically onto $H^p(U; \mathbb{C})$. As

$$H^0(P, \Omega_P^p(\log E)) = H^0(P, \Omega_P^p(\log E)) = A^p(U)$$

it follows that $H^q(P, \Omega_P^p(\log E)) = 0$ for $q \neq 0$. We apply this to the pair (P, E) . So if $j : U \subset U'$ and $k : U' \subset P$ denote the inclusions, then we have a spectral sequence whose E^1 -term is the same as in the constant case, but whose differentials are different: $E_1^{p,q} = H^q(P, \Omega_P^p(\log E)) \Rightarrow H^q(P, R^\bullet k_* j_! \mathbb{L})$. The preceding calculation shows that this sequence degenerates at the E_2 -term and that $E_2^{0,q} = H^q(A^\bullet(U), \eta \wedge) = H^q(P, R^\bullet k_* j_! \mathbb{L}) = H^q(U'; j_! \mathbb{L})$. The proposition follows. \square

We now make the assumption that the residues a_α are all rational. Let κ be the smallest positive integer such that all κa_α are integral. Then we have a regular function F on U whose logarithmic differential is $\kappa \eta$. This function is rational on P . Consider the cyclic unramified covering $\hat{U} \rightarrow U$ defined by $w^\kappa = F$ and let $\pi : \hat{P} \rightarrow P$ be its normalization over P . The latter is not smooth, but its (toric) singularities are well understood. We denote the covering group G and the character by which it acts on w by $\chi : G \rightarrow \mathbb{C}^\times$. Multiplication by w identifies \mathbb{L} with $(\pi_* \mathbb{C}_P)^\chi|_U$. It also identifies $(A^\bullet(U), \eta)$ with a subcomplex of the χ -eigenspace of the De Rham complex of \hat{U} . Let \hat{U}' and \hat{E} have the obvious meaning.

The following corollary is applicable to the general situation considered by Schechtman-Varchenko in [9] and thus provides a purely topological interpretation of the KZ solution space.

Corollary 8.3. *Multiplication by w defines an isomorphism of $H^\bullet(A^\bullet(U), \eta)$ onto $H^\bullet(\hat{U}', \hat{U}' \cap \hat{E}; \mathbb{C})^\chi$. Moreover, if $A^N(U)_{(2)} \subset A^N(U)$ denotes the subspace of logarithmic N -forms ω which on each component E_α of E vanish of order $> -a_\alpha$ then $A^N(U)_{(2)}$ is identified with $H^{N,0}(\hat{X}; \mathbb{C})^\chi$.*

Proof. The first assertion follows from Proposition 8.1 and the second from Proposition 3.2. \square

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