REVIEW OF SMOOTH MANIFOLDS

This review is rather terse. For a more complete discussion (which includes examples) one might consult the notes to a course on *Smooth manifolds*, available at
`<http://www.math.uu.nl/people/looijeng/coursenotes.html>`.

BASIC NOTIONS OF DIFFERENTIAL TOPOLOGY

Manifold. A *topological m-manifold* is a Hausdorff space M locally isomorphic to $\mathbb{R}^m$. So M can be covered by open subsets U for which there exists a homeomorphism κ of U onto an open subset of $\mathbb{R}^m$. Such a pair (U, κ) is called a *chart* of M and a collection of such pairs $(U_\alpha, \kappa_\alpha)_{\alpha\in\mathcal{A}}$ with $M = \cup_\alpha U_\alpha$ is called an *atlas* for M. In the last situation we have for any pair $\alpha, \beta \in \mathcal{A}$ a homeomorphism between two open subsets of $\mathbb{R}^m$:

$$\kappa_\beta\kappa_\alpha^{-1} : \kappa_\alpha(U_\alpha \cap U_\beta) \to \kappa_\beta(U_\alpha \cap U_\beta).$$

This is called a *coordinate transformation*. We say this atlas is *differentiable* if all its coordinate transformations are differentiable. We then have a notion of differentiability for functions on open subsets of M: if $\subset M$ is open, then we say that $f : U \to \mathbb{R}$ is *differentiable relative to $\mathcal{A}$* if all the functions $f\kappa_\alpha^{-1} : \kappa_\alpha(U \cap U_\alpha) \to \mathbb{R}$ are. Two differentiable atlases are said to be equivalent if they give rise to the same notion of differentiability for functions open open subsets. A *differentiable structure* on M is an equivalence class of differentiable atlasses. A topological m-manifold endowed with a differentiable structure is simply called an *m-manifold*. In this course we exclusively deal with manifolds (rather than topological manifolds) and so an atlas is always supposed to be a differentiable one representing the differentiable structure and a chart is always assumed to be taken from such an atlas.

Given an m-manifold M, then a subset $N \subset M$ is said to be a *submanifold of dimension n* (where $n \le m$) if M admits an atlas of charts (U, κ) such that $U \cap N = \kappa^{-1}(\mathbb{R}^n \times \{0\})$. It is then easy to check that this atlas restricted to N gives N the structure of an n-manifold.

Differentiable map. Let $f : M \to N$ be any map from an m-manifold to an n-manifold. We say that f is *differentiable* if for any pairs of charts (U, κ) of M and (V, λ) of N, the map

$$\lambda f\kappa^{-1} : \kappa(U \cap f^{-1}V) \to \lambda(V)$$

(from an open subset of $\mathbb{R}^m$ to an open subset of $\mathbb{R}^n$) is differentiable. It is not hard to verify that if $f : M \to N$ and $g : N \to P$ is differentiable, then so is $gf : N \to P$. In accordance with our convention all maps between manifolds are supposed to be differentiable unless the contrary is stated.

We say that $f : M \to N$ is a *diffeomorphism* if f is a bijection with the property that both f and f^{-1} are differentiable and is such a diffeomorphism exists, we say that Mmand N are *diffeomorphic*.

Tangent space. Let M be a m-manifold and $p \in M$. A *tangent vector* of M at p is roughly speaking an infinitesimal displacement of p in M. In order to be precise we make an auxiliary definition first. An *arc germ at* p is an arc $\gamma : I \to M$ (where I is an unspecified neighborhood of 0) with $\gamma(0) = p$. We say that two such arc germs $\gamma : I \to M$, $\gamma' : I' \to M$ at p are *first order equivalent* if for some (or equivalently, any) chart (U, κ) of M with $p \in U$, $\kappa\gamma : \gamma^{-1}U \to \mathbb{R}$ and $\kappa\gamma' : \gamma'^{-1}U \to \mathbb{R}$ have the same derivative at 0. A *tangent vector at* p is an equivalence class for this equivalence relation. The set of tangent vectors at p is denoted T_pM. It is clear that for (U, κ) as above, $\gamma \mapsto (\kappa\gamma)\dot{}(0)$ defines an injection $T_pM \to \mathbb{R}^m$. We denote it by $D_p\kappa$. The map $D_p\kappa$ is in fact a bijection, for if $v \in \mathbb{R}^m$, then the arc germ $\gamma_v(t) := \kappa^{-1}(tv)$ has v as image. The reason for denoting this map by $D_p\kappa$ is that if (U', κ') is another such chart, then a chain rule holds: $D_p\kappa' = D_0(\kappa'\kappa{-}1)D_p\kappa$, where $D_0(\kappa'\kappa{-}1)$ is of course the ordinary derivative at 0 of a diffeomorphism from an open neighborhood of 0 in $\mathbb{R}^m$ to $\mathbb{R}^m$ and hence given by a nonsingular matrix. This observation leads to the conclusion that T_pM has in a natural manner the structure of an m-dimensional vector space: if we transfer the vector space structure of $\mathbb{R}^m$ to T_pM via $D_p\kappa$ or via $D_p\kappa'$, we get the same law for adding elements of T_pM or for scalar multiplication. We call T_pM the *tangent space* of M at p. It is now a vector space of dimension m.

Tangent bundle and derivative. If $f : M \to N$ is a (differentiable) map between manifolds with $p \in M$ and $q := f(p)$, then an arc germ at p composed with f yields an arc germ at q. This induces a map between tangent spaces $T_pM \to T_qN$ that we shall call the *derivative of* f *at* p and denote by D_pf. If (U, κ) is a chart at p and (V, λ) is a chart at q, then

$$D_q\lambda \circ D_pf \circ (D_p\kappa)^{-1} = D_0(\lambda f\kappa^{-1}),$$

where on the right we have an ordinary derivative of a map from a neighborhood of 0 in $\mathbb{R}^m$ to $\mathbb{R}^n$. Since the latter is a linear map (it is given by the matrix of partial derivatives of $\lambda f\kappa^{-1}$ at 0), it follows that D_pf is linear with respect to the vector space structures on T_pM and T_qN that we just introduced.

If $g : N \to P$ is another map between manifolds, then $D_p(gf) = D_qgD_pf$.

Let TM denote the disjoint union of the tangent spaces T_pM. There is an obvious map $\pi : TM \to M$ (whose fiber over p is T_pM). We give TM the structure of a $2m$-manifold such that π is differentiable as follows. If (U, κ) is a chart for M, then we combine the maps $\{D\kappa_p : T_pM \to \mathbb{R}^m\}_{p\in M}$ can be combined to form a bijection $D\kappa : TU = \pi^{-1}U \to \mathbb{R}^m \times \kappa(U)$. Since $\mathbb{R}^m \times \kappa(U)$ is open in $\mathbb{R}^{2m}$ we think of $D\kappa$ as a chart of TM (although TM

has not been given a topology yet). We determine its coordinate changes. If (U', κ') is another chart, then the bijection

$$D\kappa' \circ (D\kappa)^{-1} : \mathbb{R}^m \times \kappa(U \cap U') \to \mathbb{R}^m \times \kappa'(U \cap U')$$

is given by the familiar derivative of $\kappa'\kappa^{-1}$:

$$(v, x) \in \mathbb{R}^m \times \kappa(U \cap U') \mapsto (D_x(\kappa'\kappa^{-1})(v), \kappa'\kappa^{-1}(x)) \in \mathbb{R}^m \times \kappa'(U \cap U').$$

This map, which we simply denote by $D\kappa'\kappa^{-1}$, is clearly differentiable. It is now easy to show that the collection of subsets $\Omega \subset TM$ with the property that $D\kappa(\Omega \cap U)$ is open in $\mathbb{R}^{2m}$ for all charts (U, κ) of M defines a topology on TM. This topology is Hausdorff (easy) and hence the collection $(TU, D\kappa)$, with (U, κ) running over all charts of M, is an atlas for TM. Since, as we just saw, this atlas is differentiable, TM is in fact in a natural manner a manifold. The map $\pi : TM \to M$ is in terms of the chart (U, κ) given by the projection $\mathbb{R}^m \times \kappa(U) \to \kappa(U)$. In particular, π is differentiable. We call TM (or rather the map $TM \to M$) the *tangent bundle of* M.

If $f : M \to N$ is as above, then the derivatives $\{D_pf : T_pM \to T_{f(p)}N\}_{p\in M}$ combine to a map $Df : TM \to TN$. If (U, κ) is a chart for M and (V, λ) is one for N, then $D\lambda \circ Df \circ (D\kappa)^{-1}$ equals the ordinary derivative

$$D(\lambda f \kappa^{-1}) : \mathbb{R}^m \times \kappa(U \cap f^{-1}V) \to \mathbb{R}^n \times \lambda(V).$$

The latter is differentiable and hence so is $Df : TM \to TN$.

Submersion, immersion, embedding. A map $f : M \to N$ between manifolds is called a *submersion* at $p \in M$, if $D_pf : T_pM \to T_{f(p)}N$ is onto. If that last property is satisfied we have $n \le m$ and it follows from the implicit function theorem that there exist charts (U, κ) at p and (V, λ) at $f(p)$, such that $\lambda f \kappa^1$ is the restriction of the projection $(x^1, \dots, x^m) \mapsto (x^1, \dots, x^n)$. In particular, $f^{-1}f(p)$ is given by putting $\kappa_{n+1}, \dots, \kappa_m$ equal to zero so that $f^{-1}f(p)$ is a submanifold near p. It follows that if $q \in N$ is such f is a submersion along $f^{-1}(p)$, then $f^{-1}(p)$ is a submanifold of M of dimension $m - n$.

A map $f : M \to N$ between manifolds is called an *immersion* at $p \in M$, if $D_pf : T_pM \to T_{f(p)}N$ is injective. If that last property is satisfied we have $n \ge m$ and it follows from the implicit function theorem that there exist charts (U, κ) at p and (V, λ) at $f(p)$, such that $\lambda f \kappa^1$ is the restriction of the inclusion $(x^1, \dots, x^m) \mapsto (x^1, \dots, x^m, 0, \dots, 0)$. In particular, f maps a neighborhood of p onto a submanifold near p.

We say that f is an *embedding* if it is an injective immersion that is also a homeomorphism onto its image. This implies that $f(M)$ is a submanifold of N and that f is a diffeomorphism of M onto this submanifold.