ON THE TAUTOLOGICAL RING OF $\mathcal{M}_g$

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Abstract. We prove that any product of tautological classes of $\mathcal{M}_g$ of degree d (in the Chow ring of $\mathcal{M}_g$ with rational coefficients) vanishes for $d > g-2$ and is proportional to the class of the hyperelliptic locus in degree $g-2$.

In memory of Nicolaas H. Kuiper (1920-1994)

1. The theorem

Fix an integer $g \geq 2$ and denote by $\mathcal{C}_g^n$ the moduli space of tuples $(C, x_1, \dots, x_n)$, where C is a complex smooth connected projective curve of genus g and $x_1, \dots, x_n$ are (not necessarily distinct) points of C; we also write $\mathcal{M}_g$ for $\mathcal{C}_g^0$. Forgetting the ith point defines a morphism $\mathcal{C}_g^n \to \mathcal{C}_g^{n-1}$ whose relatively dualizing sheaf is denoted by ω_i ($i = 1, \dots, n$). We write K_i for the first Chern class of ω_i, considered as an element of the Chow group $A^1(\mathcal{C}_g^n)$ (with rational coefficients); for $n = 1$ we also write K. The direct image of K^{d+1} in $A^d(\mathcal{M}_g)$ is the Mumford–Morita–Miller *tautological class* κ_d. Mumford showed in his fundamental paper [5] that the $\mathbb{Q}$-subalgebra of $A^\bullet(\mathcal{M}_g)$ generated by these classes (the *tautological ring* of $\mathcal{M}_g$) is already generated by $\kappa_1, \dots, \kappa_{g-2}$. On the basis of many calculations Carel Faber has made the intriguing conjecture that this ring has the formal properties of the even-dimensional cohomology ring of a projective manifold of dimension $g-2$, i.e., satisfies Poincaré duality and a Lefschetz decomposition.

It is natural to define the tautological ring of $\mathcal{C}_g^n$ as the $\mathbb{Q}$-subalgebra of $A^\bullet(\mathcal{C}_g^n)$ generated by $K_1, \dots, K_n$, the (pull-backs of the) classes κ_i and the classes of the diagonal divisors on $\mathcal{C}_g^n$ defined by $x_i = x_j$ ($1 \leq i < j \leq n$).

The theorem below supports Faber's conjecture (take $n = 0$) and answers two questions of S. Morita [4] affirmatively (take $n = 1$).

Theorem. *Any element of degree d of the tautological ring of $\mathcal{C}_g^n$ is a linear combination of the classes of the irreducible components of the locus parametrizing tuples $(C, x_1, \dots, x_n)$ admitting a finite morphism $C \to \mathbb{P}^1$ of degree $\leq 2g-2+n$ such that the fiber over 0 (resp. ∞) has at most $g+n-d-1$ points (resp. is a singleton) and $\{x_1, \dots, x_n\}$ is contained in the union of these two fibers. (Hence such a class is zero when $d > g+n-2$.) All such classes of degree $g+n-2$ are proportional to the class of the locus H_g^n parametrizing tuples $(C, x_1, \dots, x_n)$ with C hyperelliptic and $x_1 = \cdots = x_n$ a Weierstraß point.*

According Mumford [5] the classes of H_g^0 and H_g^1 are in the tautological ring (from which it follows that the class of any H_g^n is), but it is not known whether these are actually nonzero. As was pointed out by Frans Oort in [6], (2.2), results

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as the above theorem retain their validity in arbitrary characteristic. This also the case for a theorem of Diaz [3]:

Corollary. *Whatever the characteristic of the base field, there is no complete subvariety of $\mathcal{M}_g$ of dimension $> g-2$.*

Proof. (See Oort [6].) The class κ_1 is ample in arbitrary characteristic. So if S is a complete subvariety of $\mathcal{M}_g$ of dimension d, then the intersection number $\kappa_1^d.S$ is nonzero and hence κ_1^d is numerically nonzero. This property is independent of the base field and so $d \leq g-2$.

We shall see that the theorem is easily reduced to the case of a monomial in the classes $K_1, \dots, K_n$. The proof for such monomials uses a refinement of the flag of subvarieties of $\mathcal{M}_g$ introduced by Arbarello [1] (a variant of which was exploited by Diaz [3] to prove the result that the above corollary generalizes). Our simple key result (2.4) serves as a substitute for Diaz's lemma 2 in [3] and can be used in that paper to eliminate the use of compactifications of Hurwitz schemes (see (2.9)). The proof of the second assertion of the theorem involves an application of the Fourier transform for abelian varieties, due to Mukai and Beauville.

In this paper we only consider Chow groups with respect to rational equivalence, tensorized with $\mathbb{Q}$, and graded by codimension, notation: $A^\bullet$. If X is a variety that is smooth, or more generally, that admits a smooth Galois covering, then there is an intersection product $A^k(X) \otimes A^l(X) \to A^{k+l}(X)$.

I thank Johan de Jong for drawing my attention to the paper by Deninger–Murre [2] and for comments on a first draft, Carel Faber for some helpful observations regarding the proof of the theorem, and Frans Oort for explaining to me the repercussions in positive characteristic. I also thank the referee for useful comments.

2. The proof

We first show that the theorem is implied by:

(2.1) Proposition. *Any monomial of degree d in the classes $K_1, \dots, K_n$ is a linear combination of the classes of the irreducible components of the locus parametrizing tuples $(C, x_1, \dots, x_n)$ admitting a morphism $C \to \mathbb{P}^1$ of degree $\leq g+n$ such that the fiber over 0 (resp. ∞) has at most $g+n-d-1$ points (resp. is a singleton) and $\{x_1, \dots, x_n\}$ is contained in the union of these two fibers. All monomials of degree $g+n-2$ are proportional to the class of the locus parametrizing tuples $(C, x_1, \dots, x_n)$ with C hyperelliptic and $x_1 = \cdots = x_n$ a Weierstraß point.*

Proof that this proposition implies the theorem. For $1 \leq i < j \leq n$, let $\phi : \{1, \dots, n\} \to \{1, \dots, n-1\}$ be a surjection that has $\{i, j\}$ as a fiber, and let $u_\phi : \mathcal{C}_g^{n-1} \to \mathcal{C}_g^n$ be the corresponding morphism. The class of the diagonal defined by $x_i = x_j$ is given by $u_{\phi *}(1)$. Then the class of the normal bundle of u_ϕ is $-K_{\phi(i)}$ so that $u_{\phi *}(1)^{r+1} = u_{\phi *}((-K_{\phi(i)})^r)$. It follows that the the ideal generated by $u_{\phi *}(1)$ in the tautological ring of $\mathcal{C}_g^n$ is the image of the tautological ring of $\mathcal{C}_g^{n-1}$ under $u_{\phi *}$. It is clear that if the theorem holds for an element of the tautological ring of $\mathcal{C}_g^{n-1}$, then it does so for its image under $u_{\phi *}$. With induction we see that it is enough to verify the theorem for monomials that do not involve diagonals.

Let $M = K_1^{d_1}\cdots K_n^{d_n}\kappa_{e_1}\cdots\kappa_{e_m}$ be a monomial of degree d (with each $e_i > 0$). The projection $p_i : \mathcal{C}_g^{n+m} \to \mathcal{C}_g^{n+m-1}$ that forgets the ith point is a pull-back of the projection $\mathcal{C}_g^1 \to \mathcal{M}_g$; since K_1 pulls back to K_i, it follows that $p_{i*}(K_i^{e+1}) = \kappa_e$. So M is the direct image of $\tilde{M} := K_1^{d_1}\cdots K_n^{d_n}K_{n+1}^{e_1+1}\cdots K_{n+m}^{e_m+1}$ on $\mathcal{C}_g^n$ under the projection $\mathcal{C}_g^{n+m}\to\mathcal{C}_g^n$ that forgets the last m points. According to the proposition, $\tilde{M}$ parametrizes tuples $(C, x_1,\dots,x_{n+m})$ admitting a morphism $C\to\mathbb{P}^1$ of degree $\le g+n+m$ such that the fiber over 0 (resp. ∞) has at most $g+(n+m)-(d+m)-1 = g+n-d-1$ points (resp. is a singleton) and $\{x_1,\dots,x_{n+m}\}$ is contained in the union of these two fibers. Applying this to the monomial κ_i, we find that $\kappa_i = 0$ for $i > g-2$. So we may assume that $m \le g-2$. Then $g+n+m \le 2g-2+n$. The theorem follows.

(2.2) Let C be a smooth projective curve of genus g and let D_0 and D_∞ be positive divisors on C that are linearly equivalent, but whose supports are disjoint. Then there is a finite morphism $\pi : C \to \mathbb{P}^1$ such that $\pi^*(i) = D_i$ $(i = 0,\infty)$. If $p\in C$ occurs in D_i with multiplicity $m_p > 0$, then π determines an isomorphism

$$\mathbb{C} = \mathfrak{m}_{\mathbb{P}^1,i}/\mathfrak{m}^2_{\mathbb{P}^1,i} \to \mathfrak{m}^{m_p}_{C,p}/\mathfrak{m}^{m_p+1}_{C,p} = T_p^*C^{\otimes m_p}.$$

However, π is not unique for it is defined up to an obvious action of $\mathbb{C}^\times$ on $\mathbb{P}^1$. That ambiguity can be eliminated as follows.

Let R denote the part of the ramification divisor of π that lies over $\mathbb{P}^1 - \{0,\infty\}$. If c denotes the number of points of $\operatorname{supp}(D_0+D_\infty)$, then the Riemann-Hurwitz formula implies that the degree r of R is equal to $2g-2+c$. If $\pi_*(R) = \sum_i n_i(z_i)$, then π can be normalized in such a way that $\prod_i z_i^{n_i} = 1$. This normalization is unique up to multiplication by an rth root of unity. So for p and m_p as above, and π normalized, the corresponding generator of $T_p^*C^{\otimes m_p}$ raised to the rth power gives a *canonical* generator of $T_p^*C^{\otimes m_p r}$.

This argument works just as well in families and so we obtain:

(2.3) Proposition. *Let $f : \mathcal{C}\to S$ be a projective family of smooth genus g curves with reduced base. Let D_0 and D_∞ be positive relative divisors on $\mathcal{C}$ whose supports are disjoint and are étale over S. Suppose that their difference is linearly equivalent to the pull-back of a divisor on S. Then for every section $x : S\to\mathcal{C}$ of f with image in the support of D_0+D_∞, a suitable positive tensor power of $x^*\omega_{\mathcal{C}/S}$ is trivial.*

The key result we need is:

(2.4) Lemma. *Let d be a positive integer, Δ a smooth curve germ with closed point 0 and generic point η, $t : \mathcal{C}\to\Delta$ a smooth curve over Δ, $x : \Delta\to\mathcal{C}$ a section, and $\mathcal{P}$ a relative pencil on $\mathcal{C}$ that has $d(x)$ as a member. Assume that $\mathcal{P}_\eta$ has no base points and let R be the part of the ramification divisor on $\mathcal{C}_\eta - x(\eta)$ of the associated morphism $\mathcal{C}_\eta\to\mathbb{P}^1_\eta$. Regard R as a relative divisor on $\mathcal{C}$ without vertical component and let R_0 be its specialization over 0. Then the multiplicity of $x(0)$ in R_0 is also the multiplicity of $x(0)$ as a fixed point of P_0.*

Proof. Extend t to a chart (z,t) of $\mathcal{C}$ at $x(0)$ such that $z=0$ is the image of x. In terms of these coordinates generators of $\mathcal{P}$ can be represented by z^d and a function

$A(z,t)=\sum_{i\neq d} a_i(t)z^i$ which is divisible neither by t nor by z. The first index k for which $a_k(0)\neq 0$ is $< d$ and is equal to the multiplicity of $x(0)$ as fixed point of P_0. In terms of the chart, the divisor R is given by the locus where the z-derivative of $[A : z^d]$ is zero, that is, by the divisor of $\sum_{i\neq d}(i-d)a_i(t)z^i$ $(t\neq 0)$. This expression is not divisible by z or t so that R_0 is given by $\sum_{i\neq d}(i-d)a_i(0)z^i$. So $x(0)$ occurs with multiplicity k in R_0.

An immediate consequence is an amplification of a result due to Arbarello [1] and Diaz [3]:

(2.5) Corollary. *In the situation of (2.4) we have that for every member D of $\mathcal{P}$ that specializes to $d(x(0))$, the degree of the moving part of P_0 is $\leq$ the number of η-valued points of* $\mathrm{supp}(D_\eta)\setminus x(\eta)$.

Proof. If $\mathrm{supp}(D_\eta)\setminus x(\eta)$ has $d-r$ η-valued points, then $\mathrm{supp}(D_\eta)$ contributes to R with degree r. Since D specializes to $d(x(0))$, it follows that $x(0)$ has multiplicity $\geq r$ in R_0. By the lemma, $x(0)$ is then a fixed point in P_0 of order $\geq r$, so that the moving part of P_0 has degree $\leq d-r$.

(2.6) If d is a positive integer, then we have a moduli space $P(d)$ of triples (C,x,P) with C a smooth projective curve of genus g, $x\in C$ and P a pencil on C containing $d(x)$. The existence of this is clear if $d>2g-2$, for then this is just a bundle of projective spaces of dimension $d-g-1$ over $\mathcal{C}_g$; the remaining cases $d\leq 2g-2$ follow from this by simply viewing $P(d)$ as the locus in $P(2g-1)$ parametrizing triples (C,x,P) for which x is a fixed point in P of multiplicity $2g-1-d$. This implies that we also have defined a moduli space Z of tuples $(C,x_1,\dots,x_n,x,D,P)$ with C a smooth projective curve of genus g, $x_1,\dots,x_n,x\in C$, P a pencil on C containing $(n+g)x$, D a degenerate member of P (that is, a member with less than $n+g$ points in its support) and $\{x_1,\cdots,x_n\}\subset \mathrm{supp}(D)$. Notice that D and x determine P unless $D=(n+g)(x)$. The forgetful morphism $f: Z\to \mathcal{C}_g^n$ is clearly proper.

The tuples for which $\mathrm{supp}(D)$ has at most $g+n-k-1$ points outside x define a closed subvariety Z^k of Z. It is clear that Z^{n+g-1} can be identified with the set of tuples $(C,x,\dots,x,x,(n+g)x,P)$ with P a pencil through $(n+g)(x)$.

(2.7) Lemma. *For $k<g+n-1$, Z^k-Z^{k+1} is quasi-affine of pure dimension $3g-3+n-k$ and $f^*K_i|Z^k-Z^{k+1}=0$ $(i=1,\dots,n)$.*

Proof. Let $k<g+n-1$ and let W be a connected component of Z^k-Z^{k+1}. If $(C,x_1,\dots,x_n,x,D,P)$ represents an element of W, then write $D=m(x)+D'$ with $x\notin\mathrm{supp}(D')$ so that $\mathrm{supp}(D')$ has exactly $g+n-k-1$ points. There is a finite morphism $\pi: C\to\mathbb{P}^1$ with $\pi^*(0)=D'$ and $\pi^*(\infty)=(g+n-m)(x)$. The part of the ramification divisor R of π over $\mathbb{P}^1-\{0,\infty\}$ has by Riemann-Hurwitz degree $2g-2+(g+n-k)=3g-2+n-k$.

The multiplicity m, the multiplicity of x_i in D, and the stratum of the diagonal stratification of $\mathcal{C}_g^{n+1}$ containing $(C,x_1,\dots,x_n,x)$ only depend on W. So assigning to $(C,x_1,\dots,x_n,x,P)$ the $\mathbb{C}^\times$-orbit of π_*R defines a flat, quasi-finite morphism from W to the quotient of a $(3g-3+n-k)$-dimensional torus by an action of the symmetric group. So W is quasi-affine and pure of dimension $3g-3+n-k$. Proposition (2.3) implies that $f^*K_i|W$ is trivial.

We shall also use the following simple fact:

(2.8) Lemma. *Let $L_1, \dots, L_d$ be line bundles on a variety V and let $V = V^0 \supset V^1 \supset \cdots \supset V^d$ be a chain of closed subvarieties such that L_k is trivial on $V^{k-1} - V^k$. Then $c_1(L_1) \cdots c_1(L_d)$ has support in V^d.*

Proof of the first clause of (2.1).

Let X^k be the union of irreducible components of Z^k that are distinct from Z^{n+g-1}. (It can be shown that Z^{n+g-1} is actually an irreducible component of Z and so $X^0 \neq Z$.) The restriction $f : X^0 \to \mathcal{C}_g^n$ is clearly proper. It is also surjective, because for given $(C, x_1, \dots, x_n)$, the morphism

$$(y, y_1, \dots, y_{g-1}) \in C^g \mapsto [-(n+g)y + 2(x_1) + \sum_{i=2}^{n} (x_i) + \sum_{j=1}^{g-1} (y_j)] \in J(C)$$

is onto. Observe that $X^{n+g-1} = \emptyset$.

We claim that $f(X^k \cap Z^{n+g-1}) \subset f(X^{k+1})$. For if $(C, x, \dots, x, x, (n+g)(x), P)$ represents an element of $X^k \cap Z^{n+g-1}$, then by (2.5), the moving part of P will have degree $\leq n+g-k-1$. So P has a member $\neq (n+g)(x)$ with at most $n+g-k-2$ points.

It follows that the pre-image U^k of $f(X^k) - f(X^{k+1})$ in X^k is contained in $Z^k - Z^{k+1}$. In particular, $f^*K_i|U^k = 0$ for $i = 1, \dots, n$. Since $f : U^k \to f(X^k) - f(X^{k+1})$ is proper and onto, we also have $K_i|f(X^k) - f(X^{k+1}) = 0$. So by (2.8), a monomial of degree k in $K_1, \dots, K_n$ is a linear combination of irreducible components of $f(X^k)$ of codimension k. One easily checks that these components are as described in the proposition.

(2.9) Since $f(X_k) - f(X_{k+1})$ admits a finite covering that is Zariski-open in an affine variety, it is quasi-affine. So it cannot contain a complete curve. From this we recover Diaz's theorem which asserts that $\mathcal{C}_g^n$ does not contain a complete subvariety of dimension $> g+n-2$.

In order to complete the proof of (2.1) we need two more results, one algebraic, one topological.

(2.10) Lemma. *Let $f : \mathcal{A} \to S$ be a family of abelian varieties of dimension g and let d be a positive integer. Then the class of the locus $\mathcal{A}\langle d \rangle$ of points of order d is a positive multiple of the class of the zero section in $A^g(\mathcal{A})$. (The coefficient is the number of elements in $(\mathbb{Z}/d)^{2g}$ of order d.)*

Proof. We use the Fourier transform for abelian varieties introduced by Mukai, developed by Beauville and extended to abelian schemes by Deninger–Murre [2]. Mukai's transform gives an (in general inhomogeneous) isomorphism $\mathcal{F} : A(\mathcal{A}) \to A(\hat{\mathcal{A}})$, where $\hat{\mathcal{A}} \to S$ is the dual family. We shall compare the images of the two classes in $A(\hat{\mathcal{A}})$ under $\mathcal{F}$.

Let k be an positive integer relatively prime to d. Multiplication by k in $\mathcal{A}$ maps $\mathcal{A}\langle d \rangle$ isomorphically onto itself. So the class of $\mathcal{A}\langle d \rangle$ in $A^g(\mathcal{A})$ is fixed under k_*. Lemma (2.18) of [2] implies that then $\mathcal{F}([\mathcal{A}\langle d \rangle]) \in A^0(\hat{\mathcal{A}})$. Since the projection induces an isomorphism $A^0(S) \to A^0(\hat{\mathcal{A}})$, the lemma follows.

(2.11) Lemma. *Let $\pi : C \to \mathbb{P}^1$ be a covering of degree d by a smooth connected curve that is totally ramified over 0 and ∞ such that the part D of the discriminant in $\mathbb{P}^1 - \{0, \infty\}$ is reduced. Then there exists a disk neighborhood B of* $\operatorname{supp}(D)$ *in $\mathbb{P}^1 - \{0, \infty\}$ such that for $p \in \partial B$, the monodromy group of π over $(B - \operatorname{supp}(D), p)$ is a single transposition (a', a''). Moreover, if σ is the monodromy of a simple loop in $\mathbb{P}^1 - \operatorname{int}(B)$ around 0 based at p, then $a'' = \sigma^r(a')$ for some divisor r of d and π factorizes through the covering $z \in \mathbb{P}^1 \mapsto z^r \in \mathbb{P}^1$.*

Proof. We choose a base point $p \in \mathbb{P}^1$ outside the discriminant and we put $F := \pi^{-1}(p)$. By a *simple arc* we shall mean an embedded interval connecting p with a point of the discriminant that does not meet the discriminant along the way. A simple arc α determines up to isotopy (relative p and the discriminant) a simple loop based at p around a point of the discriminant and hence a monodromy transformation $\tau_\alpha \in \operatorname{Aut}(F)$. A collection of simple arcs that do not meet outside p shall be called an *arc system.* Notice that the directions of departure of the members of such a collection determine a cyclic ordering (our preference is clockwise) of these.

We begin by fixing a simple arc ω connecting p with 0. We write σ for τ_ω; this is a d-cycle in $\operatorname{Aut}(F)$. Any transposition τ in $\operatorname{Aut}(F)$ can be written $(a, \sigma^l(a))$ for some $l \in \{0, 1, \dots, \frac{1}{2}d\}$; this means that $\sigma\tau$ is the product of two disjoint cycles of length l and $d - l$. Let us call l the *mesh* of τ.

Let α_1 be an simple arc to a point of $\operatorname{supp}(D)$ that forms with ω an arc system and is such that $\tau := \tau_{\alpha_1}$ has minimal mesh r. Write $\sigma\tau = \sigma'\sigma''$ with σ' and σ'' disjoint cycles of length r resp. $d - r$ and denote by F' and F'' the corresponding parts of F. Notice that τ_{α_1} interchanges some $a' \in F'$ with some $a'' \in F''$.

Let β be another simple arc to a point of $\operatorname{supp}(D)$ such that $(\omega, \alpha_1, \beta)$ is a clockwise oriented arc system. Then τ_β cannot commute with σ'': if it did, then it would interchange two points of F' and would therefore have a mesh $< r$. It may happen that τ_β commutes with σ'. But not every choice for β can be like this, for then σ' would occur in the cycle decomposition of the monodromy around ∞ and this is impossible as the latter is a d-cycle.

So for some β, τ_β interchanges some $b' \in F'$ with some $b'' \in F''$. If we modify β by letting it first wind k times around the union of ω and α_1, then its monodromy gets conjugated by $(\sigma'\sigma'')^{\pm k}$. In this way we can arrange that $b'' = a''$. If $b' \neq a'$, then a straightforward verification shows that τ_β would have a smaller mesh than r. So $b' = a'$ and hence $\tau_\beta = \tau$. This argument proves more: the fact that for every integer k the mesh of the $(\sigma'\sigma'')^k$-conjugate of τ_β is $\geq r$ implies that r divides d. We put $\alpha_2 := \beta$.

We now prove with induction on l that for $l \leq \deg(D)$ there is an arc system $(\alpha_1, \alpha_2, \dots, \alpha_l)$ in clockwise cyclic order such that $\tau_{\alpha_i} = \tau$ for $i = 1, \dots, l$. The lemma then follows: we already showed that r divides d, and it is easy to see that the asserted factorization exists. So suppose we found such an arc system $(\alpha_1, \alpha_2, \dots, \alpha_l)$ for some $2 \leq l < \deg(D)$.

First assume l even. Then the monodromy around the union of these arcs is equal to σ and so the above argument yields simple arcs β_1, β_2 such that $\tau_{\beta_1} = \tau_{\beta_2}$ and $(\omega, \alpha_1, \dots, \alpha_l, \beta_1, \beta_2)$ is an arc system in clockwise order. Since τ_{β_i} does not commute with σ'', we can modify β_1 and β_2 by letting both go round the union of $(\omega, \alpha_1, \dots, \alpha_l)$ the same number of times first, to ensure that $\tau_{\beta_1} = \tau_{\beta_2}$ moves a''.

If τ_{β_i} does not commute with σ', then the argument above shows that in fact $\tau_{\beta_i} = \tau$ and so we managed to take two induction steps.

If τ_{β_i} does commute with σ', then let β_i' be obtained from β_i by going round α_l first. Then $(\omega, \alpha_1, \dots, \alpha_{l-1}, \beta_1', \beta_2', \alpha_l)$ is in clockwise order and $\tau_{\beta_1'} = \tau_{\beta_2'}$ interchanges an element of F' with an element of F''. Next modify the β_1' and β_2' by letting them first encircle $(\omega, \alpha_1, \dots, \alpha_{l-1})$ the same number of times as to arrange that $\tau_{\beta_i'}$ moves a'' (this might cause them to meet α_l in a point $\neq p$). Then $\tau_{\beta_i'} = \tau$ and hence we have made the induction step.

It remains to do the induction step for l odd. That is handled in the same way as the case $l = 1$.

Proof of the second clause of (2.1).

First notice that every component of X^{g+n-2} parametrizes triples (C, x, y), where C is smooth of genus g, $x, y \in C$ are distinct and $d(x) \equiv d(y)$ for some $d \in \{2, \dots, n+g\}$. By our previous discussion this defines a closed subvariety Y of $\mathcal{C}_g^2$ of pure codimension g. We first show that the classes of the irreducible components of Y are proportional in $A^g(\mathcal{C}_g^2)$. Our first business is therefore to describe these irreducible components.

For $d \geq 2$, let $Y_d \subset \mathcal{C}_g^2$ be the locus parametrizing triples (C, x, y) for which $(x) - (y)$ has order d in $J(C)$. For such (C, x, y) we have a morphism $\pi : C \to \mathbb{P}^1$ of degree d such that $\pi^*(0) = d(x)$, $\pi^*(\infty) = d(y)$ and π does not factor through a cover $z \in \mathbb{P}^1 \mapsto z^r \in \mathbb{P}^1$ for some $r > 1$. The triples (C, x, y) for which π has reduced discriminant outside $\{0, \infty\}$ make up a Zariski open-dense subset of Y_d and according to lemma (2.11) the corresponding family of coverings are of the same topological type. This implies that Y_d is irreducible. So every irreducible component of Y is equal to some Y_d.

Let $\mathcal{J}_g \to \mathcal{M}_g$ be the universal Jacobian and let $q : \mathcal{C}_g^2 \to \mathcal{J}_g$ be the Abel-Jacobi map $(C, x, y) \mapsto (x) - (y) \in J(C)$. Then $Y_d = q^{-1}\mathcal{J}_g\langle d\rangle$. Since Y_d has the correct codimension g in $\mathcal{C}_g^2$, it follows that $[Y_d]$ is a positive multiple of $q^*[\mathcal{J}_g\langle d\rangle]$. According to (2.10), $[\mathcal{J}_g\langle d\rangle]$ is a positive multiple of the class of the zero section in $A^g(\mathcal{J}_g)$. So the class of every Y_d is proportional to the class of Y_2.

For every partition (P, Q) of $\{1, \dots, n\}$ into two subsets, we denote by $H_g^{P,Q} \subset \mathcal{C}_g^n$ the locus parametrizing tuples $(C, x_1, \dots, x_n)$ with C hyperelliptic, each x_i a Weierstraß point, and $x_i = x_j$ if and only if i and j belong to the same part. The previous discussion shows that every component of $f(X^{g+n-2})$ is proportional to the class of some $H_g^{P,Q}$ and also that the latter is irreducible. It therefore remains to see that the classes of $H_g^{P,Q}$ and H_g^n (which corresponds to $Q = \emptyset$) are proportional.

For this we use the following argument suggested by Carel Faber. Let C be hyperelliptic and let y_0, y_1, y_∞ Weierstraß points of C that are mutually distinct. Then there is a unique hyperelliptic covering $\pi : C \to \mathbb{P}^1$ that maps y_i to i. Hence, if $U \subset \mathcal{C}_g^3$ is the locus defined these properties and $\mathcal{C}_U \subset \mathcal{C}_g^4$ is the universal curve over it, then there is a natural morphism $\Pi : \mathcal{C}_U \to \mathbb{P}^1$. The fibers over 0 and ∞ have the same class by definition. The map $(C, y_0, y_1, y_\infty, x) \in \mathcal{C}_U \mapsto (C, y_0, x) \in \mathcal{C}_g^2$ is finite and maps $\Pi^{-1}(0)$ resp. $\Pi^{-1}(\infty)$ onto H_g^2 resp. Y_2. It follows that the class of Y_2 is proportional to the class of H_g^2. Repeated application of this argument shows that the class of $H_g^{P,Q}$ is proportional to H_g^n. This concludes the proof.

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