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Lectures on Hodge Theory

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TRENTO NOTES ON HODGE THEORY

Abstract. These are the notes of an introductory course on Hodge theory. The subject matter includes mixed Hodge theory and the period mappings of some special projective manifolds.

Introduction

The present paper is essentially a slightly expanded and revised version of a set of notes, written to accompany a course on Hodge theory given for the Trento Summer School in September 2009. My course alternated with one given by Claire Voisin, who lectured on the interplay between Hodge theory and algebraic cycles. Part of my choice of topics was motivated as to provide some supporting material for her course and this may to some extent account for the fact that these notes are more like a plant with offshoots close to its roots than a tall tree. Let me add however that even without this auxiliary intent, I still would have found it hard to state a single result that could have been the course's final goal. Not that I regretted this, for my aim was simply to make my audience – consisting of mostly graduate students with a background in algebraic geometry – familiar with some of the basic results and techniques of Hodge theory. This can of course be done in more ways than one and I hope the reader will not blame me for letting on occasion my own preferences prevail.

Let me now say a bit about the contents. The first half is primarily concerned with Hodge theory, as understood in a broad sense of the term. We first sum up what the classical theory tells us, and then discuss the relative version of that theory, including the Kodaira–Spencer class and Griffiths transversality. This leads naturally to the notion of a variation of (polarized) Hodge structure. Along the way we give Deligne's argument for the degeneration of the Leray spectral sequence of a smooth projective family. We then introduce mixed Hodge theory, but we limit the concrete description of such structures on the cohomology to relatively simple situations. Next we outline some of the work of Schmid and Steenbrink on degenerations, which culminates in "deriving" the Clemens–Schmid exact sequence.

The second half is mostly about period mappings. It certainly does not presuppose all of the preceding. For example, mixed Hodge structures do not enter here (although they do eventually in the omitted proofs of some of the claims). We begin with Griffiths' beautiful description of the Hodge filtration on the primitive cohomology of smooth hypersurface and then quickly concentrate on period mappings that take their values in a locally symmetric variety (these are period mappings for which the Griffiths tranversality condition is empty). We conclude with discussing a number of examples, including K3 surfaces, quartic curves and cubics hypersurfaces of dimension at most four.

A word about the bibliography is in order. As one glance at it shows, I have made no attempt to be complete. I even admit to having occasionally ignored the original sources. However, among the items listed is the recent book by Peters and Steenbrink [**?**], a comprehensive work on Mixed Hodge Theory that I recommend to any (aspiring) algebraic geometer, and to which I also refer for its extensive bibliography insofar this subject is involved.

It was a joy to give this course for such an interested and responsive audience – I am grateful for its feedback. What made it an even more memorable experience is that it took place near the town of Trento, an architectural gem set in the mountains.

I thank the organizers of this summer school, Gianfranco Casnati and Simon Salamon, for their help and support.

Conventions. For a space we usually write $H^\bullet(X)$ for $H^\bullet(X;\mathbb{Q})$, but often remember that it comes with a lattice $H^\bullet(X)(\mathbb{Z})$ (namely the image of $H^\bullet(X;\mathbb{Z})$ in $H^\bullet(X;\mathbb{Q})$).

For a filtration we use a subscript or a superscript according to whether it is nondecreasing or nonincreasing. Thus, for a vector space V, a nonincreasing filtration is written $F^\bullet V$ (so that $F^pV \supset F^{p+1}V$). We can pass from one type to the other by setting $F_pV := F^{-p}V$. The associated graded vector spaces are denoted

$$\mathrm{Gr}_F^p V := F^pV/F^{p+1}V, \qquad \mathrm{Gr}_p^F V := F_pV/F_{p-1}V.$$

We use this when we denote the filtration on the dual of V: the annihilators $\mathrm{ann}(F^pV) \subset V^*$ satisfy $\mathrm{ann}(F^pV) \subset \mathrm{ann}(F^{p+1}V)$ and so we put $F^pV^* := \mathrm{ann}(F_{1-p}V)$ (with this convention $\mathrm{Gr}_F^{-p}(V^*)$ becomes the dual of $\mathrm{Gr}_F^p(V)$).

The context we work in is that of complex manifolds and so a structure sheaf $\mathscr{O}_M$ is always understood to be a sheaf of holomorphic functions, even if M happens to be an projective manifold.

1. Basic definitions

Let V be a finite dimensional complex vector space defined over $\mathbb{Q}$. So that it has complex conjugation.

DEFINITION 1. *A* Hodge structure *(usually abbreviated HS) on* V *is a decomposition* $V = \oplus_{p,q} V^{p,q}$ *with* $\overline{V^{p,q}} = V^{q,p}$ *such that for every* m, $\oplus_{p+q=m} V^{p,q}$ *is defined over* $\mathbb{Q}$. *Its* Weil element *is the transformation* $C \in \mathrm{GL}(V)$ *which on* $V^{p,q}$ *is multiplication by* i^{p-q}. *If* V *happens to come with a lattice* $V(\mathbb{Z}) \subset V(\mathbb{Q})$ *(the additive subgroup spanned by a basis), then we call this a* $\mathbb{Z}$*-Hodge structure. If* $V^{p,q} = 0$ *whenever* $p+q \neq m$, *then* V *is said to be a* pure HS of weight m *and* $\min\{p \mid h^{p,m-p} \neq 0\}$ *is called its* colevel.

Notice that the Weil element is defined over $\mathbb{R}$.

We have two associated filtrations: the (nonincreasing) *Hodge filtration* defined by $F^pV := \sum_{p' \geqslant p,q} V^{p',q}$, and the (nondecreasing) *weight filtration* defined by

$W_m V = \sum_{p+q \leqslant m} V^{p,q}$. The latter is defined over $\mathbb{Q}$. (It is by now a standard convention to use for a nonincreasing filtration a superscript and for nondecreasing one a subscript.)

EXERCISE 1. Show that a HS is completely determined by its weight and Hodge filtrations.

EXAMPLE 1. Let V be a complex vector space of even dimension $2g$ and defined over $\mathbb{Q}$. A complex structure on $V(\mathbb{R})$ is an operator $C \in GL(V(\mathbb{R}))$ with $C^2 = -1$, which then indeed turns $V(\mathbb{R})$ into a complex vector space. Such a transformation has $\pm i$-eigenspaces in V and if we denote these $V^{1,0}$ resp. $V^{0,1}$, then this defines a HS pure of weight one on V. Conversely, such a Hodge structure defines a complex structure on $V(\mathbb{R})$ by insisting that "taking the real part" gives a complex isomorphism $V^{1,0} \to V(\mathbb{R})$.

EXAMPLE 2. Suppose V is a HS of pure odd weight $2k+1$. Then C acts in V as a complex structure. We call this the *Weil complex structure*; the HS associated to its eigenspaces $V = V_W^{1,0} \oplus V_W^{0,1}$ is characterized by the fact that

$$V_W^{1,0} = \oplus\{V^{p,q} \,|\, p - q \equiv 1 \pmod 4\}.$$

There is an alternate complex structure, the *Griffiths complex structure*, characterized by $V_G^{1,0} = F^{k+1}V$. If a lattice $V(\mathbb{Z}) \subset V(\mathbb{Q})$ is given, then

$$J_W(V) = V/(V(\mathbb{Z}) + V_W^{1,0}) \quad \text{and} \quad J_G(V) = V/(V(\mathbb{Z}) + F^{k+1}V)$$

are complex tori, which in a geometric context are refered to as the *Weil* resp. *Griffiths intermediate Jacobian*.

DEFINITION 2. *The* Serre-Deligne torus *is the algebraic torus* $\mathbb{S}$ *defined over* $\mathbb{Q}$ *characterized by the property that its group of real points,* $\mathbb{S}(\mathbb{R})$*, is* $\mathbb{C}^\times$ *(in other words,* $\mathbb{S} = \mathrm{Res}_{\mathbb{C}|\mathbb{R}}\,\mathbb{G}_m$*). Its (universal)* Weil element $C \in \mathbb{S}(\mathbb{R})$ *corresponds to* $i \in \mathbb{C}^\times$.

We could think of $\mathbb{S}$ as the group of matrices of the form $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ satisfying $a^2 + b^2 \neq 0$ (then $C = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$), but a characterization in terms of its characters is more useful. First observe that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ by definition. This is the restriction of a character $z : \mathbb{S} \to \mathbb{G}_m$, which on real points is the identity. Complex conjugation in $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ is the restriction of another character $\bar{z} : \mathbb{S} \to \mathbb{G}_m$ (the notation is justified by the fact that if $s \in \mathbb{S}(\mathbb{C}) \mapsto \bar{s} \in \mathbb{S}(\mathbb{C})$ is complex conjugation that defines the real structure on $\mathbb{S}$, then $\overline{z(s)} = \bar{z}(\bar{s})$). Together these characters define an isomorphism $(z, \bar{z}) : \mathbb{S} \to \mathbb{G}_m^2$ of groups. We also note the one-parameter subgroup $w : \mathbb{G}_m \to \mathbb{S}$ which on real points is the inclusion $\mathbb{R}^\times \subset \mathbb{C}^\times$. It is essentially the diagonal embedding, for both wz and $w\bar{z}$ are the identity.

To give a Hodge structure on V is equivalent to giving an action of $\mathbb{S}$ on V defined over $\mathbb{R}$ such that w is defined over $\mathbb{Q}$: $V^{p,q}$ is then the eigenspace with character $z^p \bar{z}^q$ and the weight decomposition is the eigenspace decomposition of w. Notice that the Weil element is the image of the universal one.

The real representations of $\mathbb{S}$ (and hence the Hodge structures) form a category (abelian, with dual and $\otimes$). For instance, the dual Hodge structure on V^* has $(V^*)^{p,q} = (V^{-p,-q})^*$ so that the obvious pairing $V^* \otimes V \to \mathbb{C}$ becomes a HS-morphism if we give $\mathbb{C}$ the trivial HS (of bidegree $(0,0)$).

EXERCISE 2. The real structure of $\mathbb{S}(\mathbb{C})$ (which we regard as a complex conjugation map) is transfered via the isomorphism $(z,\bar{z}) : \mathbb{S} \to \mathbb{G}_m^2$ to a complex conjugation map on $\mathbb{C}^\times \times \mathbb{C}^\times$. Exhibit that map.

DEFINITION 3. *The* Tate HS $\mathbb{Z}(1)$ *is the* $\mathbb{Z}$*-Hodge structure on* $\mathbb{C}$ *of bidegree* $(-1,-1)$ *with integral lattice* $2\pi i\mathbb{Z}$*. (So the underlying* $\mathbb{Q}$*-vector space is* $2\pi i\mathbb{Q} \subset \mathbb{C}$*.)*

REMARK 1. This is in fact the natural HS that algebraic geometry puts on $H_1(\mathbb{G}_m(K))$ when K is an algebraically closed subfield of $\mathbb{C}$ (recall that $\mathbb{G}_m(K) = \operatorname{Spec} K[t,t^{-1}]$ is the multiplicative group of K; it has the differential $\frac{dt}{t}$). The map

$$\gamma \in H_1(\mathbb{C}^\times;\mathbb{Z}) \mapsto \int_\gamma \frac{dt}{t} \in \mathbb{C}$$

is an embedding with image $2\pi i\mathbb{Z}$. The factor $2\pi i$ is important when the field of definition of the variety is that of the complex numbers and we want to compare De Rham cohomology with Betti cohomology. Notice by the way that the codifferential in a Mayer-Vietoris sequence for the open cover of $\mathbb{P}^1(\mathbb{C})$ obtained by removing $[0:1]$ resp. $[1:0]$, identifies $H_1(\mathbb{G}_m(\mathbb{C});\mathbb{Z}) = H_1(\mathbb{C}^\times;\mathbb{Z})$ with $H_2(\mathbb{P}^1(\mathbb{C});\mathbb{Z})$; this is why the latter can be identified with $\mathbb{Z}(1)$ as a Hodge structure.

For an integer m, we set $\mathbb{Z}(m) := \mathbb{Z}(1)^{\otimes m}$; it is the Hodge structure on $\mathbb{C}$ of bidegree $(-m,-m)$ with integral lattice $(2\pi i)^m\mathbb{Z}$. If V is a HS, then $V(m)$ stands for the HS-tensor product $V \otimes_{\mathbb{Z}} \mathbb{Z}(m)$. It is clear that if V is a pure HS of colevel c, then $V(c)$ has colevel zero.

DEFINITION 4. *Let* V *be a pure HS of weight* m*. A* polarization *of* V *is a linear map* $Q : V \otimes V \to \mathbb{Z}(-m)$ *defined over* $\mathbb{Q}$ *such that*

$$(v,v') \mapsto (2\pi i)^m Q(Cv,\overline{v'})$$

is hermitian and positive definite.

Notice that to say that $Q : V \otimes V \to \mathbb{Z}(-m)$ is defined over $\mathbb{Q}$ is equivalent to saying that $(2\pi i)^m Q : V \otimes V \to \mathbb{C}$ is defined over $\mathbb{Q}$. We shall write I for this map so that the polarizing condition amounts to: $(v,v') \mapsto I(Cv,v')$ is positive definite. But if we do not want to single out a square root of -1, then Q is the prefered choice. We may of course also give a polarization as a morphism $V \to V^*(-m)$ (which in a sense is closer to its origin in Poincaré duality).

EXERCISE 3. Verify that I is symmetric or alternating according to whether m is even or odd. Assuming that this (anti)symmetry is given, then show that the polarizing property can be stated in terms of the Hodge filtration as follows:

(i) the Hodge filtration is m-*selfdual relative to* I in the sense that F^p and F^{m+1-p} are each others annihilator with respect to I, so that I induces a perfect pairing $I^p : \mathrm{Gr}_F^p \times \mathrm{Gr}_F^{m-p} \to \mathbb{C}$,

(ii) if we identify Gr_F^{m-p} with the complex conjugate of Gr_F^p, then for $0 \neq v \in \mathrm{Gr}_F^p$, $i^{2p-m} I(v, \bar{v}) > 0$.

Notice that (i) is a closed condition on the Hodge filtration, whereas (ii) is an open condition.

EXERCISE 4. Prove that a polarization of a HS of pure odd weight also polarizes the Weil complex structure, but not in general the one of Griffiths.

2. A brief review of classical Hodge theory

Let M be a complex manifold of complex dimension n. Recall that every locally free $\mathscr{O}_M$-module $\mathscr{F}$ of finite rank is resolved by its Dolbeault complex

$$\mathscr{F} \to (\mathscr{E}_M^{0,\bullet}(\mathscr{F}), \bar{\partial}^\bullet).$$

This resolution is fine so that $H^\bullet(M, \mathscr{F})$ is the cohomology of the complex of global sections, $\mathscr{E}^{0,\bullet}(M, \mathscr{F}) := H^0(M, \mathscr{E}_M^{0,\bullet}(\mathscr{F}))$.

The holomorphic De Rham sheaf complex resolves the constant sheaf $\mathbb{C}_M$ by coherent sheaves, and

$$\mathbb{C}_M \to (\Omega_M^\bullet, d^\bullet)$$

is a quasi-isomorphism so that $H^\bullet(M, \mathbb{C}_M) = H^\bullet(M; \mathbb{C})$ is the hypercohomology of $\Omega_M^\bullet$. This give rise to the (Hodge) spectral sequence

$$E_1^{p,q} = H^q(M, \Omega_M^p) \Rightarrow H^{p+q}(M; \mathbb{C}).$$

(This is also the spectral sequence associated to what is called the *stupid filtration* $\sigma_\geqslant$ on the complex $\Omega_M^\bullet$, i.e., the one for which $\sigma_{\geqslant p}\Omega^\bullet$ is the complex obtained from $\Omega^\bullet$ by replacing all terms on degree $< p$ by zero.) In case M is affine (or more generally, Stein), then $E_1^{p,q} = 0$ if $q \neq 0$, and we find that $H^\bullet(M; \mathbb{C})$ is the cohomology of the complex $(H^0(M, \Omega^\bullet), d^\bullet)$.

Suppose now M is connected and projective. Hodge theory tells us:

(I) The Hodge spectral sequence degenerates at E_1 and we obtain a nonincreasing filtration $F^\bullet H^d(M; \mathbb{C})$ with

$$\mathrm{Gr}_F^p H^d(M; \mathbb{C}) \cong H^{d-p}(M, \Omega_M^p).$$

This puts a Hodge structure on $H^d(M; \mathbb{C})$ of pure weight d having $F^\bullet H^d(M; \mathbb{C})$ as Hodge filtration. We have $H^{2n}(M; \mathbb{C}) \cong \mathbb{Z}(-n)$ canonically.

(II) If $\eta \in H^2(M,\mathbb{Q})$ denotes the hyperplane class, then η is of type $(1,1)$. The *Lefschetz operator* L, defined by cupping with η, is a morphism of bidegree $(1,1)$ and satisfies the *Hard Lefschetz* property: for $k = 0,1,2,\ldots$

$$L^k : H^{n-k}(M) \to H^{n+k}(M)$$

is an isomorphism . If we define the *primitive part* $H_\circ^{n-k}(M) \subset H^{n-k}(M)$ as the kernel of $L^{k+1} : H^{n-k}(M) \to H^{n+k+2}(M)$, then the obvious map

$$\bigoplus_{k=0}^{n} H_\circ^{n-k}(M) \otimes_{\mathbb{R}} \mathbb{C}[L]/(L^{k+1}) \to H^\bullet(M;\mathbb{C})$$

is an isomorphism of Hodge structures.

(III) The map $I : H_\circ^{n-k}(M) \otimes H_\circ^{n-k}(M) \to \mathbb{Q}$ characterized by:

$$I(\alpha \otimes \beta) = (-1)^{(n-k)(n-k-1)/2} \int_M L^k(\alpha \wedge \beta)$$

polarizes the Hodge structure on $H_\circ^{n-k}(M)$.

The preceding holds more generally for a compact connected Kähler manifold, except that then the Lefschetz operator is then only defined over $\mathbb{R}$ (and hence the same is true for the primitive cohomology).

REMARK 2. Note that L has a geometric definition: if cohomology in degree k is represented by (intersection with) cycles of degree $2n-k$, then L amounts to intersecting cycles with a hyperplane that is in general position. Property (II) is equivalent to having defined a representation of the Lie algebra $\mathfrak{sl}(2)$ for which $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ acts on $H^d(M;\mathbb{C})$ as multiplication by $d-k$ and $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ acts as L. (The image of $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ is the so-called Λ-operator. No geometric definition is known for it.)

REMARK 3. Assigning to a projective manifold its HS defines a functor: a morphism $f : M \to M'$ between two projective manifolds clearly sends $\mathscr{E}^{p,q}(M)$ to $\mathscr{E}^{p,q}(M')$, so that the induced map on cohomology preserves the bidgree. This means that f induces a HS-morphism on cohomology. The functoriality is in fact much stronger: one can safely say that any map between cohomology groups of projective manifolds that has a geometric definition is one of Hodge structures. For instance, the Künneth decomposition $H^\bullet(M \times M') = H^\bullet(M) \otimes H^\bullet(M')$ is a HS-isomorphism.

REMARK 4 (On signs). The sign that we encounter in property (III) above originates from the fact that there are two natural ways to orient a complex vector space. Let T be a real vector space of dimension $2n$ endowed with a complex structure given by $C \in GL(V)$ with $C^2 = -1$ (think of a tangent space of a complex manifold). Denote by V the complexification of its dual: $V := \mathrm{Hom}_{\mathbb{R}}(T,\mathbb{C})$. The i-eigenspace of C^* in V is denoted $V^{1,0}$ so that its complex conjugate, $V^{1,0}$, is the $-i$-eigenspace of C^*. Except

for the fact that V is not defined over $\mathbb{Q}$, this amounts to putting a Hodge structure of weight 1 on V. Choose a cobasis of T of the form

$$(x_1, y_1 := -C^* x_1, \dots, x_n, y_n := -C^* x_n).$$

We can either take the orientation defined by this order or by $(x_1, \dots, x_n, y_1, \dots, y_n)$. It is not hard to verify that they differ by the sign $(-1)^{n(n-1)/2}$. The first orientation is the standard one, but the second is more related to the polarization. To see this, put $z_\nu := x_\nu + iy_\nu$. Then $C^* z_\nu = iz_\nu$ and indeed, $(z_1, \dots, z_n)$ is a $\mathbb{C}$-basis of $V^{1,0}$, whereas $(\bar{z}_1, \dots, \bar{z}_n)$ is one of $V^{0,1}$. Now let us focus on the associated Hodge structure of weight n on $\wedge^n V$. A typical generator of $V^{p,q}$ is $\omega_{I,J} := z_I \wedge \bar{z}_J$, where I and J are complementary subsequences of $(1, 2, \dots, n)$. A straightforward computation shows that $C^* \omega_{I,J} \wedge \overline{\omega_{I',J'}} = i^{p-q} \omega_{I,J} \wedge \overline{\omega_{I',J'}}$ equals $2^n x_1 \wedge \dots \wedge x_n \wedge y_1 \wedge \dots \wedge y_n$ when $(I', J') = (I, J)$ and is zero otherwise. So if we orient complex manifolds according to this somewhat unusual orientation convention, then we see that for a nonzero n-form on a complex manifold M of complex dimension n, $C^* \omega \wedge \overline{\omega}$ is $\geqslant 0$ everywhere. This means that, relative to the standard orientation, $(-1)^{n(n-1)/2} C^* \omega \wedge \overline{\omega} \geqslant 0$ everywhere, so that when ω has compact support,

$$(-1)^{n(n-1)/2} \int_M C^* \omega \wedge \overline{\omega} \geqslant 0.$$

This is why the form I that defines the polarization on $H^n_\circ(M)$ is $(-1)^{n(n+1)/2}$ times the intersection pairing.

3. VHS and PVHS

Here S is a smooth connected variety, $\mathscr{M} \subset \mathbb{P}^N \times S$ a closed smooth subvariety for which the projection $f : \mathscr{M} \to S$ is smooth (= a submersion) and of relative dimension n. In this situation, $R^\bullet f_* \mathbb{Z}_{\mathscr{M}}$ is a (graded) local system.

3.1. A degenerating spectral sequence

Before we embark on the theme of this section, we first prove an theorem about f that illustrates the power of the Hard Lefschetz theorem.

THEOREM 1. *For $f : \mathscr{M} \to S$ as above, the rational Leray spectral sequence*

$$E_2^{p,q} := H^p(S, R^q f_* \mathbb{Q}) \Rightarrow H^{p+q}(\mathscr{M})$$

degenerates so that the (Leray) filtration on $H^\bullet(\mathscr{M})$ has $H^\bullet(S, R^\bullet f_* \mathbb{Q})$ as its successive quotients. In particular, the natural map of relative restriction $H^\bullet(\mathscr{M}) \to H^0(S, R^\bullet f_* \mathbb{Q})$ is onto and $f^* : H^\bullet(S) \to H^\bullet(\mathscr{M})$ is injective.

Proof. This is essentially linear algebra. The Lefschetz operator L is given by cupping with the hyperplane class and therefore the Leray spectral sequence is one of $\mathbb{Q}[L]$-modules. In particular, we have $L : R^q f_* \mathbb{Q} \to R^{q+2} f_* \mathbb{Q}$ and the hard Lefschetz theorem

produces the decomposition of local systems

$$\bigoplus_{k=0}^{n} R_\circ^{n-k} f_* \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}[L]/(L^{k+1}) \cong R^\bullet f_* \mathbb{Q},$$

where $R_\circ^{n-k} f_* \mathbb{Q}$ has the obvious meaning. Since the differentials of the spectral sequence are $\mathbb{Q}[L]$-linear, it is enough to prove that they vanish on $H^\bullet(S, R_\circ^{n-k} f_* \mathbb{Q})$. We proceed inductively and assume that we verified that for $r \geqslant 2$, $E_r^{p,q} = E_2^{p,q}$. Consider the commutative diagram

$$\begin{array}{ccc} H^p(S, R_\circ^{n-k} f_* \mathbb{Q}) & \xrightarrow{d_r} & H^{p+r}(S, R^{n-k-r+1} f_* \mathbb{Q}) \\ {\scriptstyle L^{k+1}}\downarrow & & {\scriptstyle L^{k+1}}\downarrow \\ H^p(S, R^{n+k+2} f_* \mathbb{Q}) & \xrightarrow{d_r} & H^{p+r}(S, R^{n+k-r+3} f_* \mathbb{Q}) \end{array}$$

The map $L^{k+r-1} : H^{p+r}(S, R^{n-k-r+1} f_* \mathbb{Q}) \to H^{p+r}(S, R^{n+k+r-1} f_* \mathbb{Q})$ is an isomorphism by the Hard Lefschetz theorem and since $k + r - 1 \geqslant k + 1$, the vertical map on the right must be injective. The vertical map on the left is zero (for the map $L^{k+1} : R_\circ^{n-k} f_* \mathbb{Q} \to R^{n+k+2} f_* \mathbb{Q}$ is) and hence so is the top horizontal map. In other words, d_r vanishes on the primitive part of the spectral sequence. □

3.2. Griffiths transversality

We continue to investigate $f : \mathscr{M} \to S$. The vector bundle over S associated to the local system $R^d f_* \mathbb{Q}_{\mathscr{M}}$ is

$$\mathscr{O}_S \otimes_{\mathbb{Q}_S} R^d f_* \mathbb{Q}_{\mathscr{M}} = R^d f_* f^{-1} \mathscr{O}_S.$$

Hence $R^d f_* f^{-1} \mathscr{O}_S$ has the structure of a local system; it has a flat connection ∇ (the *Gauß-Manin connection*) whose local sections recover the complexified local system $R^d f_* \mathbb{C}_{\mathscr{M}}$.

The *relative De Rham sheaf complex,* $\Omega^\bullet_{\mathscr{M}/S}$, is the quotient of the sheaf of differential graded $\mathscr{O}_{\mathscr{M}}$-algebras $\Omega^\bullet_{\mathscr{M}}$ by the ideal generated by $f^* \Omega^1_S$. Notice that its fiber over $s \in S$ is the De Rham sheaf complex of M_s. This complex resolves the sheaf $f^{-1} \mathscr{O}_S$ by coherent $\mathscr{O}_{\mathscr{M}}$-modules: the inclusion

$$f^{-1} \mathscr{O}_S \subset \Omega^\bullet_{\mathscr{M}/S}$$

is a quasi-isomorphism. This complex has in turn a fine resolution by the double complex of C^∞-relative forms, $(\mathscr{E}^{p,q}_{\mathscr{M}/S}, (-1)^q d^{p,q})$.

The relative analogue of the properties I, II, III of the previous section hold. In particular, the spectral sequence of coherent $\mathscr{O}_S$-modules

$$E_1^{p,q} = R^q f_* \Omega^p_{\mathscr{M}/S} \Rightarrow R^{p+q} f_* f^{-1} \mathscr{O}_S.$$

It degenerates at E_1 and so for every d we get a filtration

$$R^d f_* f^{-1} \mathscr{O}_S = \mathscr{F}^0 R^d f_* f^{-1} \supset \cdots \supset \mathscr{F}^p R^d f_* f^{-1} \mathscr{O}_S \supset \cdots \supset \mathscr{F}^{d+1} R^d f_* f^{-1} \mathscr{O}_S = 0$$

of $R^d f_* f^{-1}\mathscr{O}_S$ by coherent submodules for which $\mathrm{Gr}^p_{\mathscr{F}} R^d f_* f^{-1}\mathscr{O}_S$ is identified with $R^{d-p} f_* \Omega^p_{\mathscr{M}/S}$. These submodules are in fact locally free and specialize to the Hodge filtration on $H^d(M_s;\mathbb{C})$.

The Gauß-Manin connection manifests itself as follows: if X is a holomorphic vector field on S (or an open subset thereof), then choose a C^∞-lift $\tilde{X}$ of X on $\mathscr{M}$. The Lie derivative $\mathscr{L}_{\tilde{X}} = d\iota_{\tilde{X}} + \iota_{\tilde{X}} d$ acts in the sheaf complex $(\mathscr{E}^\bullet_{\mathscr{M}/S}, d)$ of relative C^∞ forms and sends $\mathscr{F}^p\mathscr{E}^\bullet_{\mathscr{M}/S}$ to $\mathscr{F}^{p-1}\mathscr{E}^\bullet_{\mathscr{M}/S}$. The resulting action in $R^d f_* f^{-1}\mathscr{O}_S$ is independent of the lift and equal to the covariant derivative ∇_X with respect to the Gauß-Manin connection. We see from this description that ∇_X need not preserve $\mathscr{F}^p$, but in general maps $\mathscr{F}^p$ to $\mathscr{F}^{p-1}$ (the term $\bar{\partial}\iota_{\tilde{X}}$ in $\mathscr{L}_{\tilde{X}}$ is responsible for that). The resulting map

$$R^{d-p} f_* \Omega^p_{\mathscr{M}/S} = \mathrm{Gr}^p_{\mathscr{F}} \mathscr{V} \to \mathrm{Gr}^{p-1}_{\mathscr{F}} \mathscr{V} = R^{d+1-p} f_* \Omega^{p-1}_{\mathscr{M}/S}$$

is $\mathscr{O}_S$-linear in X. In other words, it defines an $\mathscr{O}_S$-linear map

$$\theta_S \otimes_{\mathscr{O}_S} R^{d-p} f_* \Omega^p_{\mathscr{M}/S} \to \mathbb{R}^{d+1-p} f_*(\Omega^{p-1}_{\mathscr{M}/S}),$$

where θ_S is the sheaf of holomorphic vector fields on S. It is given in terms of the obstruction to lifting X holomorphically: $\bar{\partial}\iota_{\tilde{X}}$ acts as pairing with $\bar{\partial}\tilde{X} \in f_*\mathscr{E}^{0,1}(\theta_{\mathscr{M}/S})$ (here $\theta_{\mathscr{M}/S}$ denotes the $\mathscr{O}_{\mathscr{M}}$-dual of $\Omega^1_{\mathscr{M}/S}$, the sheaf of vertical vector fields). The latter is $\bar{\partial}$-closed and hence defines an element $\mathrm{KS}_f(X) \in R^1 f_* \theta_{\mathscr{M}/S}$, which is called the *Kodaira–Spencer class* of f, and the above map is then just given by contraction with that class. This feature is called *Griffiths tranversality*.

Here is a more abstract definition of the Gauß-Manin connection, due to Katz-Oda. A straightforward check in terms of local analytic coordinates shows that that we have the short exact sequence of complexes:

$$0 \to f^{-1}\Omega^1_S \otimes_{f^{-1}\mathscr{O}_S} \Omega^{\bullet-1}_{\mathscr{M}/S} \to \Omega^\bullet_{\mathscr{M}/S} / \big(f^{-1}\Omega^2_S \otimes_{f^{-1}\mathscr{O}_S} \Omega^{\bullet-2}_{\mathscr{M}/S}\big) \to \Omega^\bullet_{\mathscr{M}/S} \to 0.$$

If we apply the functor $\mathbb{R}f_*$, we get a long exact sequence and if we unravel the definition of its codifferential,

$$\delta : \mathbb{R}^d f_* \Omega^\bullet_{\mathscr{M}/S} \to \mathbb{R}^{d+1} f_*\big(f^{-1}\Omega^1_S \otimes_{f^{-1}\mathscr{O}_S} \Omega^\bullet_{\mathscr{M}/S}\big) = \Omega^1_S \otimes \mathbb{R}^d f_* \Omega^\bullet_{\mathscr{M}/S},$$

we find that up to a sign, it is just ∇, where we should remember that $\mathbb{R}^d f_*(\Omega^\bullet_{\mathscr{M}/S}) = R^d f_* f^{-1}\mathscr{O}_S$. If we pass to the grading relative to the stupid (= Hodge) filtration, then we obtain a $\mathscr{O}_S$-linear map

$$\mathrm{Gr}(\delta) : R^{d-p} f_* \Omega^p_{\mathscr{M}/S} \to \Omega^1_S \otimes R^{d+1-p} f_*(\Omega^{p-1}_{\mathscr{M}/S}),$$

which is also the codifferential of the long exact sequence associated to the short exact sequence

$$0 \to f^{-1}\Omega^1_S \otimes_{f^{-1}\mathscr{O}_S} \Omega^{p-1}_{\mathscr{M}/S} \to \Omega^p_{\mathscr{M}/S} / \big(f^{-1}\Omega^2_S \otimes_{f^{-1}\mathscr{O}_S} \Omega^{p-2}_{\mathscr{M}/S}\big) \to \Omega^p_{\mathscr{M}/S} \to 0.$$

This is essentially the map found above, as it is given by the natural pairing with Kodaira–Spencer class $\mathrm{KS}_f \in \Omega^1_S \otimes_{\mathscr{O}_S} R^1 f_* \theta_{\mathscr{M}/S}$.

This discussion suggests the following definition.

DEFINITION 5. *A* variation of Hodge structure *(VHS) over the smooth variety* S *is a local system* $\mathbb{V}$ *on* S *of finite dimensional complex vector spaces defined over* $\mathbb{Q}$*, equipped with a nondecreasing (weight) filtration* $W_\bullet\mathbb{V}$ *defined over* $\mathbb{Q}$ *(by local subsystems) and a nonincreasing (Hodge) filtration* $\mathscr{F}^\bullet\mathscr{V}$ *of the underlying vector bundle* $\mathscr{V} := \mathscr{O}_S \otimes_{\mathbb{C}_S} \mathbb{V}$ *(by subbundles) such that*

(i) we have a Hodge structure in every fiber and

(ii) covariant derivation maps $\mathscr{F}^p\mathscr{V}$ *to* $\mathscr{F}^{p-1}\mathscr{V}$*.*

If the VHS is of pure weight m*, then a* polarization *of it is a linear map* $\mathbb{V} \otimes_{\mathbb{C}_S} \mathbb{V} \to \mathbb{Z}_S(-m)$ *with the expected property (the resulting structure is refered to as a PVHS over* S*).*

Notice that there is an associated $\mathscr{O}_S$-linear map of graded vector bundles

$$KS(\mathscr{V}) \in \mathrm{Hom}(\mathrm{Gr}^\bullet_{\mathscr{F}}\mathscr{V}, \Omega^1_S \otimes_{\mathscr{O}_S} \mathrm{Gr}^{\bullet-1}_{\mathscr{F}}\mathscr{V}).$$

Summing up: a family $f : \mathscr{M} \to S$ as above determines a VHS over S and the primitive part in any prescribed degree defines a PVHS over S.

4. Mixed Hodge structures

4.1. A 19th century MHS

Let C° be a smooth connected complex curve of finite type. We can complete it to a smooth projective curve C by filling in a finite set S of points: $C^\circ = C - S$. Let us assume $S \neq \emptyset$, so that C° is affine and hence Stein. First note that $H^i_S(C;\mathbb{Z}) = H^i(C, C^\circ;\mathbb{Z}) \cong H_{2-i}(S;\mathbb{Z})$ by duality. If we substitute this in the exact (Gysin) sequence for the pair (C, C°) we get a short exact sequence

$$0 \to H^1(C) \to H^1(C^\circ) \to \tilde{H}_0(S) \to 0.$$

Denote by W_1 the image of $H^1(C) \to H^1(C^\circ)$. Since C° is a Stein manifold, its complex cohomology is that of its De Rham complex. In fact, it is well-known that if D is an effective divisor on C of degree $2g-1$ with support in S, then we can restrict to meromorphic differentials having a polar divisor $\leqslant D + S$. Precisely, the sequence

$$H^0(C, \mathscr{O}(D)) \xrightarrow{d} H^0(C, \Omega_C(D+S)) \to H^1(C^\circ;\mathbb{C}) \to 0$$

is exact. The composite map $H^0(C, \Omega_C(D+S)) \to H^1(C^\circ;\mathbb{C}) \to \tilde{H}_0(S)$ is given by taking residues. The differential of a meromorphic function cannot have a pole of order one and so the subspace $H^0(C, \Omega_C(S)) \subset H^0(C, \Omega_C(D+S))$ embeds in $H^1(C;\mathbb{C})$. Let us denote the image of this embedding by F^1. A basic theorem in the theory of Riemann surfaces states that the residues of an element of $H^0(C, \Omega(S))$ can be prescribed arbitrarily provided their sum is zero (as is required by the residue theorem). In other

words, $F^1 \to \tilde{H}_0(S;\mathbb{C})$ is surjective. The kernel consists evidently of the regular differentials on C, so that $F^1 \cap W_1 = H^0(C,\Omega_C) = H^{1,0}(C)$. Hence $\bar{F}^1 \cap W_1 = H^{0,1}(C)$. It follows that $F^1 \cap \bar{F}^1$ is a supplementary summand for $H^1(C;\mathbb{C})$ in $H^1(C^\circ;\mathbb{C})$.

If we put $W_0 = 0$, $W_2 = F^0 = H^1(C)$ and $F^2 = 0$, then the pair $(W_\bullet, F^\bullet)$ is a mixed Hodge structure as defined below.

REMARK 5. If S is a singleton $\{s\}$, then $H^1(C) \to H^1(C^\circ)$ is an isomorphism. The above discussion is therefore also useful for describing $H^1(C;\mathbb{C})$: we find that it is the space of holomorphic differentials on C with pole order at most $2g$ at s modulo the differentials of homomorphic functions with a pole of order $\leqslant 2g-1$ at s.

4.2. The notion of a MHS

DEFINITION 6. *Let V be a finite dimensional complex vector space defined over $\mathbb{Q}$. A* Mixed Hodge structure (MHS) *on V consists of two filtrations: a nondecreasing (weight) filtration $W_\bullet V$ defined over $\mathbb{Q}$ and a nonincreasing (Hodge) filtration $F^\bullet V$ so that for every m, the image of $F^\bullet \cap W_m$ in $\mathrm{Gr}^W_m V$ defines a HS of pure weight m in $\mathrm{Gr}^W_m V$. We say that the MHS is* polarized *if each $\mathrm{Gr}^W_m V$ is. A* MHS-morphism *from a MHS $(V, F^\bullet V, W_\bullet V)$ to a MHS $(V', F^\bullet V', W_\bullet V')$ is a linear map $\phi : V \to V'$ defined over $\mathbb{Q}$ which sends $F^p V$ to $F^p V'$ and $W_m V$ to $W_m V'$.*

REMARK 6. The two filtrations of a MHS can be split simultaneously in a canonical manner: if we set

$$I^{p,q} := F^p \cap W_{p+q} \cap (\bar{F}^q \cap W_{p+q} + \bar{F}^{q-1} \cap W_{p+q-2} + \bar{F}^{q-2} \cap W_{p+q-3} + \cdots),$$

then $V = \oplus_{p,q} I^{p,q}$, $W_m = \oplus_{p+q\leqslant m} I^{p,q}$ and $F^p = \oplus_{p'\geqslant p,q} I^{p',q}$. But it is in general not true that $I^{q,p}$ is the complex conjugate of $I^{p,q}$, we only have that $\overline{I^{p,q}} \equiv I^{q,p}$ $(\mathrm{mod}\ W_{p+q-2})$. Yet this shows that that the MHS category becomes a category of bigraded vector spaces. This implies that it is an abelian category, which here comes down to the assertion that a MHS-morphism $\phi : V \to V'$ is necessarily strict for both filtrations: $\phi(V) \cap W_m = \phi(W_m)$ and $\phi(V) \cap F^p = \phi(F^p)$.

We also note that this category has duals and tensor products and that (consequently) the space of all linear maps between two MHS's has a natural MHS.

The raison d'être of this definition is the following fundamental result.

THEOREM 2 (Deligne). *If X is a complex variety of finite type, then $H^d(X)$ comes with a functorial MHS, whose weights range in the interval $0, 1, \ldots, 2d$; if X is smooth resp. complete, then these weights are even further restricted to being $\geqslant d$ resp. $\leqslant d$. It is compatible with Künneth decomposition and cup product.*

By functoriality we mean that we thus have defined an extension of the functor from the category of projective manifolds to the category of Hodge structures as a functor from the category of complex varieties of finite type to the graded MHS-category (we have a MHS in every degree).

This theorem has later been extended in several ways. We mention here the case of cohomology with supports: if $Y \subset X$ is closed, then then $H^d_Y(X)$ comes with a natural MHS with the property that the Gysin sequence

$$\cdots \to H^d_Y(X) \to H^d(X) \to H^d(X-Y) \to H^{d+1}_Y(X) \to \cdots$$

is an exact sequence of MHS's. Also, if U_Y is a regular neighborhood of Y (so that Y is a deformation retract), then $H^d(U_Y - Y)$ carries a MHS for which the Gysin sequence of th pair (U, Y) is an exact sequence of MHS's.

Deligne's theorem is in fact somewhat stronger in the sense that it identifies for any X objects in suitable (filtered) derived categories of sheaf complexes on X whose hypercohomology produces the cohomology with the filtrations that yield the mixed Hodge structures. We illustrate this with two simple but basic cases.

4.3. The MHS for a normal crossing union of smooth varieties

Let $\mathscr{I}$ be a finite set which effectively labels the irreducible components of a variety D. We assume that each irreducible component D_i is a smooth projective variety of dimension n and that they meet each other with normal crossings. We denote for $r = 1, 2, \cdots$, by D^r the locus of points contained in at least r irreducible components and by $\nu^r : \hat{D}^r \to D^r$ its normalization. Notice that $\hat{D}^r$ is smooth of dimension $n-r$. The natural sheaf homomorphism $\mathbb{Z}_D \to \nu^1_* \mathbb{Z}_{\hat{D}^1}$ is injective with cokernel supported by D^2. If we put a total order on $\mathscr{I}$, then this cokernel maps naturally to $\nu^2_* \mathbb{Z}_{\hat{D}^2}$. Continuing in this manner, we obtain the exact sequence

$$0 \to \mathbb{Z}_D \to \nu^1_* \mathbb{Z}_{\hat{D}^1} \to \nu^2_* \mathbb{Z}_{\hat{D}^2} \to \cdots \nu^n_* \mathbb{Z}_{\hat{D}^n} \to 0.$$

We consider this a resolution of the constant sheaf $\mathbb{Z}_D$. This sets up a spectral sequence $E_1^{p,q} = H^q(D, \nu^{p+1}_* \mathbb{Z}_{\hat{D}^p}) = H^q(\hat{D}^{p+1}; \mathbb{Z}) \Rightarrow H^{p+q}(D; \mathbb{Z})$. Notice that $E_1^{p,q}$ has a HS pure of weight q and so has

$$E_2^{p,q} = H\big(H^q(\hat{D}^p) \to H^q(\hat{D}^{p+1}) \to H^q(\hat{D}^{p+2})\big).$$

If we can show that this is a spectral sequence of Hodge structures, then it will degenerate at this point, for the differentials $d_r : E_r^{p,q} \to E_r^{p+r,q+1-r}$, $r \geqslant 2$, being HS-morphisms between pure HS's of different weight, must be zero. Our weight filtration is then the filtration defined by this spectral sequence, renumbered to make into a nondecreasing one: the superscript k becomes the subscript $d-k$: $\mathrm{Gr}^W_m H^d(D) = E_2^{d-m,m}$. Evidently, this is defined over $\mathbb{Q}$ and zero for $m > d$. It is also clear that $H^d(D) = W_d H^d(D)$ and that the obvious map $H^d(D) \to H^d(\hat{D})$ factors through an embedding of $\mathrm{Gr}^W_d H^d(D)$ in $H^d(\hat{D})$.

In order to verify that this a HS spectral sequence, we identify the Hodge filtration on $H^\bullet(D; \mathbb{C})$: it is in fact the one coming from the stupid filtration on the double complex whose columns are holomorphic De Rham complexes of the $\hat{D}^r$'s (for which $F^p\Omega^k = \sigma_{\geqslant p}\Omega^k$ equals Ω^k when $k \geqslant p$ and is zero otherwise). This indeed makes the above spectral sequence one of Hodge structures.

4.4. The MHS for a smooth variety

Let $\hat{M}$ be a projective manifold, $D \subset \hat{M}$ a normal crossing divisor whose irreducible components D_i, $i \in \mathscr{I}$, are smooth and $j : M := \hat{M} - D \subset \hat{M}$. We have a Gysin resolution of the total direct image $Rj_*\mathbb{Z}_M$ of $\mathbb{Z}_M$ as follows. Let $\mathbb{Z}_{\hat{M}} \to \mathscr{I}^\bullet$ be an injective resolution of the constant sheaf on $\hat{M}$. Then the sequence of complexes

$$\cdots \to \nu^2_*\nu^{2!}\mathscr{I}^\bullet \to \nu^1_*\nu^{1!}\mathscr{I}^\bullet \to \mathscr{I}^\bullet \to j_*j^*\mathscr{I}^\bullet \to 0$$

is exact. The Thom isomorphism shows that $\nu^{r!}\mathscr{I}^\bullet$ is quasi-isomorphic to the constant sheaf on $\hat{D}^r$ put in degree $2r$, but if we want this to stay in the HS-category we must make an appropriate Tate twist: the Thom isomorphism requires an orientation of the normal bundle and so $\nu^{r!}\mathscr{I}^\bullet$ is quasi-isomorphic to the constant sheaf $\mathbb{Z}_{\hat{D}^r}(-r)$ put in degree $2r$: $\mathbb{Z}_{\hat{D}^r}(-r)[-2r]$ (we recall that $[k]$ denotes the shift over k, so for a complex $\mathscr{C}^\bullet$, $\mathscr{C}[k]^n := \mathscr{C}^{k+n}$). Since $\mathbb{Z}_{\hat{D}^r}$ is injective resolved by $\mathscr{I}^\bullet_{\hat{D}^r} := \nu^{r*}\mathscr{I}^\bullet$, we find that $Rj_*\mathbb{Z}_M$ is injectively resolved by the complex

$$\cdots \to \nu^2_*\mathscr{I}^\bullet_{\hat{D}^2}(-2)[-4] \to \nu^1_*\mathscr{I}^\bullet_{\hat{D}^1}(-1)[-2] \to \mathscr{I}^\bullet \to 0.$$

So if we write $\hat{D}^0$ for $\hat{M}$, then we obtain a spectral sequence

$$E_1^{-p,q} = H^{-2p+q}(\hat{D}^p;\mathbb{Z})(-p) \Rightarrow H^{-p+q}(M;\mathbb{Z}).$$

Notice that $E_1^{-p,q}(-p)$ is pure of weight q. The resulting spectral sequence is one of pure Hodge structures and hence must degenerate (after complexification) at the E^2-term by the same reasoning as above. The corresponding filtration is on $H^\bullet(M)$ is our weight filtration: $\mathrm{Gr}^W_{d+r}H^d(M) = E_2^{-r,r+d}$ is the central cohomology of the short complex

$$H^{d-r-2}(\hat{D}^{r+1})(-r-1) \to H^{d-r}(\hat{D}^r)(-r) \to H^{d-r+2}(\hat{D}^{r-1})(-r+1).$$

which is one of Hodge structures pure of weight $d+r$. It is evidently defined over $\mathbb{Q}$. Notice that for $r = 0$, we get the short exact sequence

$$H^{d-2}(\hat{D}^1)(-1) \to H^d(\hat{M}) \to W_dH^d(M) \to 0,$$

which shows in particular that $W_dH^d(M)$ is the image of $H^d(\hat{M}) \to H^d(M)$.

We may resolve the above sheaf complex after complexification by a double complex of logarithmic De Rham complexes of C^∞-forms with the horizontal (Gysin) maps being given wedging with certain $(1,1)$-forms. To be precise: choose for every irreducible component E of D tubular neighborhood data $(\pi_E : U_E \to E, \rho : U_E \to \mathbb{R}_{\geqslant 0})$, where π_E is a C^∞ retraction, $\rho_E^{-1}(0) = E$ and $(\pi_E, \rho_E) : U_E \to E \to \mathbb{R}_{\geqslant 0}$ is locally trivial over $E \times \mathbb{R}_{>0}$). Let for $t \geqslant 0$, $\phi(t)$ be constant one near $t = 0$ and have support in $[0,1]$. Then $\eta_E := -\pi\partial\bar{\partial}(\phi r_E)$ is a $(1,1)$-form that can be used to represent the Gysin map by the rule $\alpha \in \mathscr{E}_E^{p,q} \mapsto \eta_E \wedge \pi_E^*\alpha \in \mathscr{E}_{\hat{M}}^{p+1,q+1}$. The Hodge filtration is then obtained as before from the stupid filtration on this double complex.

One can indeed show that this MHS is independent of the simple normal crossing compactification $M \subset \hat{M}$. The characterization of $W_dH^d(M)$ holds for any smooth compactification:

PROPOSITION 1. *Let $\hat{M}$ be a projective manifold and $M \subset \hat{M}$ a Zariski open subset. Then $W_d H^d(M)$ is the image of $H^d(\hat{M}) \to H^d(M)$.*

EXERCISE 5. Suppose we are in the situation of Proposition 1 and let $f : X \to M$ be a morphism with X projective. Prove that the image of $f^* : H^\bullet(M) \to H^\bullet(X)$ does not change if we compose it with the restriction map $H^\bullet(\hat{M}) \to H^\bullet(M)$.

As a corollary we have the theorem of the fixed part. Let $\hat{M}$ be a projective manifold, $M \subset \hat{M}$ open and $f : M \to S$ a smooth projective morphism with S connected. This implies that S is smooth and that f is C^∞ locally trivial. For every $s \in S$, the fundamental group $\pi_1(S, s)$ acts on $H^\bullet(M_s)$ by monodromy. Notice that the part fixed by this action, $H^\bullet(M_s)^{\pi_1(S,s)}$, is naturally identified with $H^0(S, R^\bullet f_* \mathbb{Q}_M)$.

COROLLARY 1 (Theorem of the fixed part). *Any $\pi_1(S, s)$-invariant cohomology class on M_s is the restriction of a class on $\hat{M}$: the composite map $H^\bullet(\hat{M}) \to H^\bullet(M) \to H^0(S, R^\bullet f_* \mathbb{Q}_M)$ is onto.*

Proof. According to Theorem 1, $H^\bullet(M) \to H^0(S, R^\bullet f_* \mathbb{Q}_M)$ is onto. By exercise 5, $H^\bullet(\hat{M}) \to H^\bullet(M) \to H^\bullet(M_s)$ is then also onto. □

4.5. The MHS on some local cohomology

The two cases we discussed combine in an interesting manner when we put a MHS on $H^k_D(\hat{M})$ (which is just $H^k(\hat{M}, M)$). This is up to a Tate twist Poincaré dual to $H^{2n-k}(U - D)$, where U is a regular neighborhood of D in $\hat{M}$ and we prefer to concentrate on the latter. The left resolution we used for the total direct image of $\mathbb{Z}_M$ in $\mathbb{Z}_{\hat{M}}$ must now be restricted to D. This complex then ends with $\mathbb{Z}_D$ and we have seen how to right resolve that. So if $i : D \subset \hat{M}$ is the inclusion, then $i^* R j_* \mathbb{Z}_M$ is quasi-isomorphic to the complex

$$0 \to \nu^n_* \mathscr{I}^\bullet_{\hat{D}^n}(-n)[-2n] \to \cdots \to \nu^2_* \mathscr{I}^\bullet_{\hat{D}^2}(-2)[-4] \to \nu^1_* \mathscr{I}^\bullet_{\hat{D}^1}(-1)[-2] \to \\ \to \nu^1_* \mathscr{I}^\bullet_{\hat{D}^1} \to \nu^2_* \mathscr{I}^\bullet_{\hat{D}^2} \to \cdots \to \nu^n_* \mathscr{I}^\bullet_{\hat{D}^n} \to 0,$$

where we should put the complex $\nu^1_* \mathscr{I}^\bullet_{\hat{D}^1}$ in the zero spot. The corresponding spectral sequence tensored with $\mathbb{Q}$ has

$$E_1^{p,q} = \begin{cases} H^q(\hat{D}^{p+1}) & \text{if } p \geqslant 0; \\ H^{2p+q}(\hat{D}^{-p})(p) & \text{if } p < 0. \end{cases}$$

(which we see is of pure weight q) and converges to $H^\bullet(U - D)$. As before, a C^∞ De Rham resolution of the individual terms makes it a spectral sequence of Hodge structures and hence degenerates at the E_2-term. The definition shows that the long exact sequence

$$\cdots \to H^{k-1}(U - D) \to H^k_D(U) \to H^k(U) \to H^k(U - D) \to \cdots$$

(in which we may substitute $H^k(D)$ for $H^k(U)$) lives in the MHS category.

REMARK 7. We remark that this puts a MHS on $H^\bullet(U - Y)$ whenever Y is a projective subvariety of an analytic variety U for which $U - Y$ is smooth and that has Y as a deformation retract: just modify U over Y so that Y gets replaced by a simple normal crossing divisor; the cohomology in question does not change so that the preceding applies.

5. Degenerating Hodge structures and classifying spaces

5.1. A classifying space for polarized Hodge structures

We fix an integer m and a set of Hodge numbers belonging to a Hodge structure of weight m, i.e., a symmetric map $(p,q) \in \mathbb{Z} \times \mathbb{Z} \mapsto h^{p,q} \in \mathbb{Z}_{\geqslant 0}$ with finite support and such that $h^{p,q} = 0$ if $p + q \neq m$. Let V be a finite dimensional complex vector space defined over $\mathbb{Q}$ endowed with a bilinear form $I : V \times V \to \mathbb{C}$ defined over $\mathbb{Q}$ which is nondegenerate $(-1)^m$-symmetric. Notice that the group $G := \mathrm{Aut}(V, I)$ of linear automorphisms of V that preserve I is an algebraic group defined over $\mathbb{Q}$ and orthogonal or symplectic according to whether m is even or odd.

We want that V supports Hodge structures with the given Hodge numbers that are polarized by $Q = (2\pi i)^{-m} I$. So $\dim V = \sum_{p,q} h^{p,q}$ and when m is even, I must have index $(-1)^{m/2} \sum_p (-1)^p h^{p,m-p}$. The space of filtrations $F^\bullet$ on V for which (i) $\mathrm{Gr}^p_{F^\bullet}(V)$ has dimension $h^{p,m-p}$ and (ii) $F^\bullet$ is m-selfdual relative to I, defines a projective submanifold $\check{D}$ in a product of Grassmannians of V. It is easily seen to be an $G(\mathbb{C})$-orbit. To require that $F^\bullet$ defines a Hodge filtration polarized by Q amounts to:

$$v \in \mathrm{Gr}^p_{F^\bullet}(V) \mapsto i^{2p-m} I(v, \bar{v})$$

is positive definite. This defines an open subset D of $\check{D}$ that is also an orbit for the group of real points $G(\mathbb{R})$. The stabilizer of $G(\mathbb{R})$ of a point of D is a unitary group and hence compact. In other words, $G(\mathbb{R})$ acts properly on D. We may say that D is a classifying space for polarized Hodge structures with the given Hodge numbers.

The trivial bundle V_D over D with fiber V comes with a (tautological) Hodge filtration $\mathscr{F}^\bullet$ that gives every fiber a Hodge structure. It is universal in the sense that any Hodge structure of the prescribed type on V appears here. But this need not be a VHS as the Griffiths tranversality condition in general fails to hold. A remarkable consequence of this fact is that as soon as the Hodge numbers force the Griffiths tranversality condition to be nontrivial, most Hodge structures of that type do not appear in algebraic geometry, for those that do are contained in a countable union of lower dimensional analytic subvarieties.

EXERCISE 6. First notice that the Lie algebra of G is the space of $A \in \mathrm{End}(V)$ that satisfy $I(Av, v') + I(v, Av') = 0$. Let now be given a filtration $F^\bullet V$ on V that defines $z \in D$. This results in a filtration $F^\bullet \mathfrak{g}$ of $\mathfrak{g}$. Prove that the tangent space $T_z D$ can be identified with $\mathfrak{g}/F^0\mathfrak{g}$. Prove that if $S \subset D$ is an analytic submanifold through p such that V_S is a PVHS, then we must have $T_p S \subset F^{-1}\mathfrak{g}/F^0\mathfrak{g}$.

5.2. The monodromy theorem

Now let Δ° stand for the punctured unit disk and let $\mathbb{V}$ be a PVHS over Δ° with the property that it contains a lattice $\mathbb{V}(\mathbb{Z})$ as a sublocal system. (This is always the case in geometric situations.) The monodromy operator in a fiber $\mathbb{V}_q$ of $\mathbb{V}$ preserves both the polarizing form I_q and the lattice $\mathbb{V}_q(\mathbb{Z})$.

We avoid the choice of a base point in Δ° by using its universal cover by the upper half plane: $\pi : \tau \in \mathbb{H} \mapsto e^{2\pi i\tau} \in \Delta^\circ$. The local system underlying $\pi^*\mathbb{V}$ is trivial and $V_\infty := H^0(\mathbb{H}, \pi^*\mathbb{V})$ is a vector space with a lattice $V_\infty(\mathbb{Z})$ and endowed with a form I as above. The group of covering transformations is $\mathbb{Z}$ and the monodromy is a homomorphism

$$\rho : \mathbb{Z} \to \mathrm{Aut}(V_\infty, I) = G.$$

The image of ρ is discrete as it preserves $V_\infty(\mathbb{Z})$ and this makes $\rho(\mathbb{Z})\backslash D$ a normal analytic variety, where $D \subset \check{D}$ is as above. The PVHS we started out with is given by an analytic $\mathbb{Z}$-equivariant map $\tilde{\phi} : \mathbb{H} \to D$ which lifts the Griffiths transversal analytic map $\phi : \Delta^\circ \to \rho(\mathbb{Z})\backslash D$. A fundamental fact due to Borel states that for any such map we have the following:

THEOREM 3 (Monodromy theorem). *The eigenvalues of the monodromy operator* $\rho(1)$ *are roots of unity and its Jordan blocks have size at most* $d := \max\{p - q \mid h^{p,q} \neq 0\}$.

5.3. The limit MHS

Let us write T for $\rho(1)$ and denote by T_s resp. T_u its semisimple resp. unipotent part. These two transformations commute, lie both in $G(\mathbb{Q})$ and have product T. The monodromy theorem asserts that T_s is of finite order and that $(T_u - 1)^{d+1} = 0$. After a finite base change we can make T_s the identity. This amounts to replacing the coordinate q on Δ by the coordinate $\sqrt[e]{q}$ of a finite cover (where the order of T_s divides e). From now on we assume this to be the case, so that $T = T_u$ is unipotent. We put

$$N := \log(T-1) = -\sum_{k\geqslant 1} \frac{(1-T)^k}{k} = (T-1)\sum_{k\geqslant 0} \frac{(1-T)^k}{k+1}.$$

Notice that $\sum_{k\geqslant 0} \frac{(1-T)^k}{k+1}$ is unipotent (hence invertible) and commutes with N. So we have $N^{d+1} = 0$ and N and $T-1$ have the same kernel and cokernel. The advantage of N over $T-1$ is that it is an element of the Lie algebra of $G(\mathbb{R})$. We now recall a rather subtle linear algebra lemma, which can be understood as saying that every nilpotent transformation in a finite dimensional vector space lies in the image of a representation of $SL(2)$.

LEMMA 1 (Jacobson-Morozov Lemma). *If* N *is a nilpotent transformation in a finite dimensional vector space* V*, then there is a* unique *nondecreasing filtration* $W_\bullet(N)$ *of* V *with the property that* N *sends* $W_\bullet(N)$ *to* $W_{\bullet-2}(N)$ *and the induced maps*

$$[N]^k : \mathrm{Gr}_k^{W(N)}(V) \to \mathrm{Gr}_{-k}^{W(N)}(V), \quad k = 1, 2, \ldots$$

are isomorphisms. If N *respects a nondegenerate (anti-)symmetric form on* V*, then this filtration is self-dual.*

Proof. Just verify that the unique solution is given by the formulae

$$\begin{aligned} W_k(N) =& \{v \in V | N^{k+i} v \in N^{k+2i} V \text{ for all } i \geqslant 0\}, \quad k = 0, 1, 2, \ldots \\ W_{-k}(N) =& \{v \in V | N^{i} v \in N^{k+2i} V \text{ for all } i \geqslant 0\}, \quad k = 0, 1, 2, \ldots. \end{aligned}$$

□

REMARK 8. Observe that $\ker(N) \subset W_0(N)$ and $N(V) \supset W_{-1}(N)$.

In the present case we need a shift of the J-M filtration of $N \in \mathrm{End}(V_\infty)$:

$$W_k^\infty := W_{k-m}(N).$$

Since $\tau \in \mathbb{C} \mapsto e^{N\tau} \in G(\mathbb{C})$ is a homomorphism (of complex Lie groups), the map

$$\tau \in \mathbb{H} \mapsto e^{-N\tau} \phi(\tau)$$

takes its values in $\check{D}$. We forced it to be constant under translation over $\mathbb{Z}$ so that it factors through a map

$$\psi : \Delta^\circ \to \check{D}.$$

It can be shown that the map ψ is meromorphic. This implies that it is in fact holomorphic: since $\check{D}$ is a closed submanifold of a flag space (which in turn can be embedded as a closed subset of a high dimensional projective space), such a map can be given as the mutual ratio of a finitely many of meromorphic functions, and we can then clear denominators by dividing by the function with the lowest order. Denote by $F_\infty^\bullet$ the filtration of V_∞ defined by $\psi(0) \in \check{D}$.

THEOREM 4 (Schmid). *The pair* $(W_\bullet^\infty, F_\infty^\bullet)$ *defines a MHS on* V_∞*. Moreover,* N *is for this structure a morphism of bidegree* $(-1,-1)$ *and polarizes the pure subquotients in the same way as the Lefschetz operator: if* $(\mathrm{Gr}_{m+k}^W)_\circ$ *denotes the kernel of* $[N]^{k+1}$*, then*

$$(v, v') \in (\mathrm{Gr}_{m+k}^{W^\infty})_\circ \times (\mathrm{Gr}_{m+k}^{W^\infty})_\circ \mapsto I(v, [N]^k v')$$

is a polarization. In particular, $\ker(N) = \ker(T-1)$ *is contained in* W_m^∞ *and* $N(V) = (T-1)(V)$ *contains* W_{m-1}^∞

This is however part of a larger package. Among its contents are (what are called) the nilpotent orbit theorem and the $SL(2)$-orbit theorem [**?**]. We state neither, but just remark that the former amounts to saying that the nilpotent orbit $\tau \in \mathbb{C} \mapsto e^{\tau N}\psi(0)$ takes values in D when $\mathrm{Im}(\tau) >> 0$ and that on such a shifted half plane it very well approximates $\tilde{\phi}$.

6. One parameter degeneration of manifolds

Here $f : \mathscr{M} \to \Delta$ is a projective morphism over a punctured unit disk with connected fibers of dimension n. We suppose $\mathscr{M}$ smooth and we assume that f is smooth over Δ°. So

$$f^\circ : \mathscr{M}^\circ := f^{-1}\Delta^\circ \to \Delta^\circ$$

is a C^∞-fiber bundle. Since $\mathscr{M}$ is a regular neighborhood of M_0 it has the same cohomology as M_0 and hence this cohomology comes with a MHS. The same is true for the cohomology $\mathscr{M}^\circ = \mathscr{M} - M_0$ by Remark 7 (because the latter is smooth). But for $s \in \Delta^\circ$, the inclusions $M_s \subset \mathscr{M}^\circ$ and $M_s \subset \mathscr{M}$ will not in general induce on cohomology a MHS morphism (after all, this is not even true in case f is smooth), but we will see that in a sense this is true in the limit if we let $s \to 0$.

As before we use the universal cover $\pi : \tau \in \mathbb{H} \mapsto e^{2\pi i\tau} \in \Delta^\circ$ with covering group the integral translations. Since $\mathbb{H}$ is contractible, the pull-back

$$\mathscr{M}_\infty := \mathscr{M}^\circ \times_{\Delta^\circ} \mathbb{H} \to \mathbb{H}$$

is C^∞-trivial and hence has the same homotopy type as a fiber of f°. The group of covering transformations, $\mathbb{Z}$, acts on $\mathscr{M}_\infty$ via the second factor. Denoting the image of $1 \in \mathbb{Z}$ by T, then $(\mathscr{M}_\infty, T)$ is a good homotopical incarnation of the general fiber endowed with its monodromy mapping class.

The results of Section 5 can be invoked:

COROLLARY 2. *The eigenvalues of* T *acting on* $R^k f_*\mathbb{Q}$ *are roots of unity and its Jordan blocks have size at most* k. *Moreover* $H^k(\mathscr{M}_\infty) = H^0(\mathbb{H}, \pi^* R^k f^\circ\mathbb{C})$ *acquires a MHS with weights a priori ranging from* 0 *to* $2k$, *with* $\ker(N)$ *contained in* $W_k H^k(\mathscr{M}_\infty)$ *and the image of* N *containing* $W_{k-1}H^k(\mathscr{M}_\infty)$.

Proof. Since everything commutes with L, it is enough to prove this for the primitive parts $H^k_\circ(\mathscr{M}_\infty) = H^0(\mathbb{H}, \pi^* R^k_\circ f^\circ\mathbb{C})$. The primitive cohomology of the fibers define (for $k \leqslant n$) a PVHS $R^k_\circ f^\circ\mathbb{C}$ to which the discussion of Section 5 applies and we find that the assertions then holds. □

This corollary has the following important supplement, which we state without proof.

PROPOSITION 2 (Schmid [**?**], Steenbrink [**?**]). *The mapping* $\tilde{j} : \mathscr{M}_\infty \to \mathscr{M}^\circ$ *induces a MHS-morphism* $H^\bullet(\mathscr{M}^\circ) \to H^\bullet(\mathscr{M}_\infty)$.

Because M_0 is projective, the weights of $H^k(\mathscr{M}) \cong H^k(M_0)$ are $\leqslant k$.

LEMMA 2. *The dual in the MHS-category of the diagram*

$$H^\bullet(\mathscr{M}) \to H^\bullet(\mathscr{M}^\circ) \to H^\bullet(\mathscr{M}_\infty)$$

twisted with $\mathbb{Z}(-n-1)$ *can be identified up to signs with the diagram*

$$H^\bullet_{M_0}(\mathscr{M}) \leftarrow H^{\bullet-1}(\mathscr{M}^\circ) \leftarrow H^{\bullet-2}(\mathscr{M}_\infty)(-1),$$

where the map on the left is a coboundary. Furthermore, the weights on $H^k_{M_0}(\mathscr{M})$ *are* $\geqslant -(2n+2-k)+(2n+2)=k$.

Proof. The Poincaré dual of the first diagram twisted with $\mathbb{Z}(-n-1)$ is

$$H^\bullet_c(\mathscr{M}) \leftarrow H^\bullet_c(\mathscr{M}^\circ) \leftarrow H^\bullet_c(\mathscr{M}_\infty)$$

We identify the terms. The map

$$H^\bullet_{M_0}(\mathscr{M}) \to H^\bullet_c(\mathscr{M})$$

is an isomorphism. Since $H^{2n+2-k}(\mathscr{M})$ has weights $\leqslant 2n+2-k$, it follows that $H^k_{M_0}(\mathscr{M})$ has weights $\geqslant -(2n+2-k)+(2n+2)=k$.

If $\Sigma \subset \Delta^\circ$ is the circle of radius $\frac{1}{2}$, then the diffeomorphism $\Delta^\circ \cong \Sigma \times (0,1)$ lifts to a diffeomorphism $\mathscr{M}^\circ \cong \mathscr{M}_\Sigma \times (0,1)$. Since $\mathscr{M}_\Sigma$ is an compact oriented manifold and a deformation retract of $\mathscr{M}^\circ$, we have $H^k_c(\mathscr{M}^\circ) \cong H^{k-1}(\mathscr{M}_\Sigma) \cong H^{k-1}(\mathscr{M}^\circ)$.

Similarly, $\mathscr{M}_\infty$ has a general fiber M_s (a projective manifold) as deformation retract and hence has the same cohomology as M_s. It follows $H^k_c(\mathscr{M}_\infty)$ is identified with $H^{k-2}(\mathscr{M}_\infty)$. □

Some of these maps enter in a modified Wang sequence:

PROPOSITION 3 (Wang sequence in the MHS setting). *The maps appearing in the above proposition make up an exact sequence of MHSs that is selfdual with respect to Poincaré duality:*

$$\cdots \to H^{k-1}(\mathscr{M}_\infty)(-1) \to H^k(\mathscr{M}^\circ) \to H^k(\mathscr{M}_\infty) \xrightarrow{N} H^k(\mathscr{M}_\infty)(-1) \to \cdots.$$

In particular, the image of $H^{k-1}(\mathscr{M}_\infty)(-1) \to H^k(\mathscr{M}^\circ)$ *supplements the weight space* $W_k H^k(\mathscr{M}^\circ)$*, so that the latter maps isomorphically onto the kernel of* N.

Proof. The Leray spectral sequence $E_2^{p,q} = H^p(\Delta^\circ, R^q f^\circ_* \mathbb{Z}_{\mathscr{M}^\circ}) \Rightarrow H^{p+q}(\mathscr{M}^\circ)$ degenerates (for $E_2^{p,q}=0$ unless $q=0,1$). Furthermore, $E_2^{0,q}$ resp. $E_2^{1,q}$ is the module of monodromy invariants resp. co-invariants. If we want to be precise, then

$$E_2^{0,q} = H^0(\Delta^\circ, R^q f^\circ_* \mathbb{Z}_{\mathscr{M}^\circ}) \cong H^q(\mathscr{M}_\infty)^T = \operatorname{Ker}\big(N : H^q(\mathscr{M}_\infty) \to H^q(\mathscr{M}_\infty)\big),$$

$$\begin{aligned} E_2^{1,q} = H^1(\Delta^\circ, R^q f^\circ_* \mathbb{Z}_{\mathscr{M}^\circ}) &\cong H^1(\Delta^\circ) \otimes H^q(\mathscr{M}_\infty)_T \\ &= \operatorname{coker}\big(N : H^q(\mathscr{M}_\infty) \to H^q(\mathscr{M}_\infty)\big)(-1). \end{aligned}$$

which yields the long exact sequence. All the maps are known to be MHS morphisms.

The last assertion follows from the fact that MHS morphisms are strict for the weight filtration: the image of $H^k(\mathscr{M}^\circ) \to H^k(\mathscr{M}_\infty)$ is the kernel of $N : H^k(\mathscr{M}_\infty) \to H^k(\mathscr{M}_\infty)(-1)$, hence has weights $\leqslant k$ and the image of $H^{k-1}(\mathscr{M}_\infty)(-1) \to H^k(\mathscr{M}^\circ)$ is the cokernel of $N : H^{k-1}(\mathscr{M}_\infty) \to H^{k-1}(\mathscr{M}_\infty)(-1)$, hence has weights $\geqslant (k-1)+2 = k+1$. □

COROLLARY 3. *The local invariant cycle theorem holds: the image of the map* $H^\bullet(\mathscr{M}) \to H^\bullet(\mathscr{M}_\infty)$ *is just the monodromy invariant part of* $H^\bullet(\mathscr{M}_\infty)$. *In fact, more is true: we have a long exact sequence (known as the* Clemens–Schmid exact sequence*) of MHS's:*

$$\cdots \to H^k_{M_0}(\mathscr{M}) \to H^k(\mathscr{M}) \to H^k(\mathscr{M}_\infty) \xrightarrow{N} H^k(\mathscr{M}_\infty)(-1) \to H^{k+1}_{M_0}(\mathscr{M}) \to \cdots$$

Proof. Consider the following piece of the MHS Wang sequence

$$H^k(\mathscr{M}^\circ) \to H^k(\mathscr{M}_\infty) \xrightarrow{N} H^k(\mathscr{M}_\infty)(-1).$$

We found that $W_k H^k(\mathscr{M}^\circ) \to \ker(N)$ is an isomorphism. Our earlier observation that $H^{k+1}_{M_0}(\mathscr{M})$ has only weights $\geqslant k+1$ implies that in the exact sequence

$$H^k_{M_0}(\mathscr{M}) \to H^k(\mathscr{M}) \to H^k(\mathscr{M}^\circ) \to H^{k+1}_{M_0}(\mathscr{M})$$

the image of the middle map equals $W_k H^k(\mathscr{M}^\circ)$ and so we have an exact MHS-sequence

$$H^k_{M_0}(\mathscr{M}) \to H^k(\mathscr{M}) \to H^k(\mathscr{M}_\infty) \xrightarrow{N} H^k(\mathscr{M}_\infty)(-1).$$

The sequence dual to this is

$$H^\bullet(\mathscr{M}_\infty) \xrightarrow{N} H^\bullet(\mathscr{M}_\infty)(-1) \to H^{\bullet+1}(\mathscr{M}) \to H^{\bullet+1}_{M_0}(\mathscr{M}).$$

and therefore also an exact MHS-sequence. The two combine to give the Clemens–Schmid sequence as a long exact MHS sequence. □

REMARK 9. We may here substitute $H_{2n+2-k}(M_0)(-n-1)$ for $H^k_{M_0}(\mathscr{M})$ and $H^k(M_0)$ for $H^k(\mathscr{M})$, but the resulting map between the two,

$$H_{2n+2-k}(M_0)(-n-1) \to H^k(M_0),$$

involves the ambient $\mathscr{M}$. In geometric terms, this map takes a singular $2n+2-k$-cycle on M_0 and brings it then inside the smooth $\mathscr{M}$ in general position with M_0 (i.e., transversal to the strata of Whitney stratification of M_0). Its trace on M_0 is more than just a $2n-k$-cycle: it has the interpretation of a k-cocycle (defined by "intersecting with it'). It follows that N will be nonzero precisely when there is a pair of cycles on $\mathscr{M}$ with nonzero intersection number (and then no matter how we modify $\mathscr{M}$ over M_0, this property will not go away).

7. Hodge structures attached to a hypersurface

We observed in Remark 5 that the middle cohomology of a compact Riemann surface can be neatly described in terms of meromorphic differentials. This was (in part) generalized by Griffiths to hypersurfaces.

7.1. The Hodge filtration of a hypersurface

Let $M \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d. The weak Lefschetz theorem asserts that this map on integral cohomology $H^k(\mathbb{P}^{n+1}) \to H^k(M)$ is an isomorphism for $k \neq n, 2n+2$ and is surjective for $k = n$ (and dually, that the Gysin map $H^k(M) \to H^{k+2}(\mathbb{P}^{n+1})(-1)$ is an isomorphism for $k \neq n, 0$ and is surjective for $k = n$. So the primitive cohomology resides in the (uninteresting) degree 0 and in degree n. We therefore focus on degree n:

$$H^n_\circ(M) = \ker\Big(H^n(M) \to H^{n+2}(\mathbb{P}^{n+1})(1) \cong H^{n+2}(M)(1)\Big).$$

Griffiths recovers the Hodge filtration as follows. It begins with

LEMMA 3. *The map* $H^{n+1}(\mathbb{P}^{n+1} - M) \to H^{n+2}_M(\mathbb{P}^{n+1}) \cong H^n(M)(-1)$ *that appears in the Gysin sequence of* $(\mathbb{P}^{n+1}, M)$ *(and which is in fact a cohomological residue) is injective with image* $H^n_\circ(M)(-1)$.

Proof. The Gysin sequence in question is

$$\begin{aligned} H^{n-1}(M)(-1) \to H^{n+1}(\mathbb{P}^{n+1}) \to \\ \to H^{n+1}(\mathbb{P}^{n+1} - M) \to H^n(M)(-1) \to H^{n+2}(\mathbb{P}^{n+1}) \end{aligned}$$

The kernel of the last map is $H^n_\circ(M)(-1)$ and $H^{n-1}(M)(-1) \to H^{n+1}(\mathbb{P}^{n+1})$ is bijective. □

Our gain is that, in contrast to M, $\mathbb{P}^{n+1} - M$ is affine, so that its complex cohomology is that of its De Rham complex of regular forms; in particular, we have an exact sequence

$$H^0(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(*M)) \xrightarrow{d} H^0(\mathbb{P}^{n+1}, \Omega^{n+1}_{\mathbb{P}^{n+1}}(*M)) \to H^n_\circ(M; \mathbb{C}) \to 0,$$

where $(*M)$ means we allow poles of arbitrary but finite order along M. We describe the Hodge filtration on $H^{n+1}(\mathbb{P}^{n+1} - M)$ via this isomorphism. Let $z_0, \dots, z_{n+1}$ be the coordinates of $\mathbb{C}^{n+2}$ whose projective space "at infinity" is $\mathbb{P}^{n+1}$ and let $F \in \mathbb{C}[z_0, \dots, z_{n+1}]$ be a homogeneous equation of M. We make the preceding even more concrete by passing to the complement U_F of the affine cone defined by $F = 0$ in $\mathbb{C}^{n+2}$. The latter is a $\mathbb{C}^\times$-bundle over $\mathbb{P}^{n+1} - M$ (which is in fact trivial). This is reflected by the $\mathbb{C}^\times$-action on the De Rham complex of U_F which makes $\mathbb{P}^{n+1} - M$ appear as the subcomplex of the De Rham complex of U_F of $\mathbb{C}^\times$-invariants, i.e., forms that are homogenenous of degree zero. We conclude:

COROLLARY 4. *Letting* $\Omega^k[M]$ *denote the space of rational* k*-forms on* $\mathbb{C}^{n+2}$*, regular on* U_F *and homogeneous of degree zero, then* $H^n_\circ(M; \mathbb{C})$ *may be identified with* $\Omega^{n+2}[M]/d\Omega^{n+1}[M]$.

According to Griffiths, the Hodge filtration on $H^n_\circ(M; \mathbb{C})$ can be recovered as a filtration by pole order on $\Omega^{n+2}[M]$. Denote by $\Omega^k_r[M] \subset \Omega^k[M]$ the subspace

spanned by forms of with pole order at most r along $F=0$. In particular, $\Omega^{n+2}_{r+1}[M]$ is spanned by forms of the type

$$A\frac{dz_0\wedge\cdots\wedge dz_{n+1}}{F^{r+1}}$$

with $A\in\mathbb{C}[z_0,\dots,z_{n+1}]_{d(r+1)-(n+2)}$ and $\Omega^{n+1}_r[M]$ is spanned by forms of the type

$$\sum_{k=0}^{n+1}(-1)^k A_k\frac{dz_0\wedge\cdots\widehat{dz_k}\cdots\wedge dz_{n+1}}{F^r}.$$

with $A_k\in\mathbb{C}[z_0,\dots,z_{n+1}]_{dr-(n+1)}$. It is clear that the exterior derivative maps the space $\Omega^{n+1}_r[M]$ to $\Omega^{n+2}_{r+1}[M]$. We can now state the main result of Griffiths in this context.

THEOREM 5. *The Hodge filtration on* $H^n_\circ(M;\mathbb{C})$ *is in terms of the preceding identification given by pole order. More precisely,* $F^{n-r}H^n_\circ(M;\mathbb{C})$ *corresponds to* $\Omega^{n+2}_{r+1}[M]/d\Omega^{n+1}_r[M]$ *so that*

$$\mathrm{Gr}^{n-r}_F H^n_\circ(M;\mathbb{C})\cong\Omega^{n+2}_{r+1}[M]/\big(\Omega^{n+2}_r[M]+d\Omega^{n+1}_r[M]\big).$$

This implies for instance that every member of $\Omega^{n+2}[M]/d\Omega^{n+1}[M]$ is representable by a form with pole order at most $n+1$.

Let us make this concrete: the map

$$d:\Omega^{n+1}_r[M]\to\Omega^{n+1}_{r+1}[M]/\Omega^{n+1}_r[M]$$

is essentially given by

$$\big(A_k\in\mathbb{C}[z_0,\dots,z_{n+1}]_{dr-(n+1)}\big)_{k=0}^{n+1}\mapsto\sum_k F^{r+1}\frac{\partial(F^{-r}A_k)}{\partial z_k}\equiv$$
$$-r\sum_k A_k\frac{\partial F}{\partial z_k}\in\mathbb{C}[z_0,\dots,z_{n+1}]_{d(r+1)-(n+2)}\quad(\mathrm{mod}\ F\mathbb{C}[z_0,\dots,z_{n+1}]_{dr-(n+2)}).$$

In view of the Euler formula $F=\sum_k z_k\frac{\partial F}{\partial z_k}$, it follows that that the cokernel of this map can be identified with a subspace of the jacobian algebra of F

$$J(F):=\mathbb{C}[z_0,\dots,z_{n+1}]/\big(\frac{\partial F}{\partial z_0},\dots,\frac{\partial F}{\partial z_{n+1}}\big).$$

If we regard $J(F)$ as a graded vector space, then the subspace in question is the homogeneous summand of degree $d(r+1)-(n+2)$. We write this most naturally as

$$\mathrm{Gr}^{n-r}_F H^n_\circ(M;\mathbb{C})\cong J(F)_{d(r+1)-(n+2)}\otimes\frac{dz_0\wedge\cdots\wedge dz_{n+1}}{F^{r+1}}.$$

EXERCISE 7. Show that when M is a nonsingular quadric (in other words, $d = 2$), then $H^n_\circ(M;\mathbb{C}) = 0$ for odd n, and isomorphic to $\mathbb{Z}(-\frac{1}{2}n)$ for even n.

The dimension of such a homogeneous summand is computed as follows. Recall that the Poincaré series of a graded vector space $V_\bullet$ (with $\dim V_k$ finite and zero for $k \ll 0$) is defined as the Laurent series $P(V_\bullet) = \sum_k \dim(V_k)t^k$.

LEMMA 4. *We have* $P(J(F)) = (1 + t + \cdots + t^{d-2})^{n+2}$. *In particular, the Hodge number* $h^{p,n-p}_\circ(M) := \dim \mathrm{Gr}^p_F H^n_\circ(M)$ *is the coefficient of* $t^{d(n-p)}$ *in the polynomial* $t^{n+2-d}(1+t+\cdots+t^{d-2})^{n+2}$.

Proof. This jacobian algebra $J(F)$ can be understood as the local ring of the fiber over zero of the map

$$\Big(\frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_{n+1}}\Big) : \mathbb{C}^{n+2} \to \mathbb{C}^{n+2}.$$

This map is finite. The Cohen-Macaulay property of $\mathbb{C}[z_0, \ldots, z_{n+1}]$ then ensures that the map on coordinate rings, $\mathbb{C}[w_1, \ldots, w_{n+1}] \to \mathbb{C}[z_0, \ldots, z_{n+1}]$ (which substitutes $\frac{\partial F}{\partial z_\nu}$ for w_ν) makes $\mathbb{C}[z_0, \ldots, z_{n+1}]$ a free $\mathbb{C}[w_1, \ldots, w_{n+1}]$-module. The Poincaré series of $\mathbb{C}[z_0, \ldots, z_{n+1}]$ is $(1-t)^{-(n+2)}$ and that of $\mathbb{C}[w_1, \ldots, w_{n+1}]$, when regarded as a subalgebra of $\mathbb{C}[z_0, \ldots, z_{n+1}]$, equals $(1-t^{d-1})^{-(n+2)}$. A monomial basis of $\mathbb{C}[z_0, \ldots, z_{n+1}]$ as a $\mathbb{C}[w_1, \ldots, w_{n+1}]$-module maps to a basis of $J(F)$. So if $V_\bullet$ denotes the $\mathbb{C}$-span of this basis, then

$$P(J(F)) = P(V_\bullet) = \frac{P(\mathbb{C}[z_0, \ldots, z_{n+1}])}{P(\mathbb{C}[w_0, \ldots, w_{n+1}])} = \frac{(1-t^{d-1})^{n+2}}{(1-t)^{n+2}}$$

and the lemma follows. □

REMARK 10. In case n is even, this allows us to compute the signature of the intersection pairing on $H^n_\circ(M)$, as that equals $(\sum_{p \text{ even}} h^{p,n-p}_\circ, \sum_{p \text{ odd}} h^{p,n-p}_\circ)$ or $(\sum_{p \text{ odd}} h^{p,n-p}_\circ, \sum_{p \text{ even}} h^{p,n-p}_\circ)$, depending on whether $n \equiv 2 \pmod 4$ or $n \equiv 0 \pmod 4$.

EXAMPLE 3. (i) Let us take the case of cubics: $d = 3$.

For $n = 1$ we get elliptic curves and indeed we find $H^{1,0}(M)$ represented by $F^{-1}\,dz_0 \wedge dz_1 \wedge dz_2$.

The case of cubic surfaces is almost as classical, and we find that $H^2_\circ(M,\mathbb{C})$ is pure of bidegree $(1,1)$ and is represented by the rational forms $AF^{-2}dz_0 \wedge \cdots \wedge dz_3$ with A homogeneous of degree 2 (a space of dimension $\binom{5}{2} = 10$), but taken modulo the 4-dimensional subspace spanned by the partial derivatives of F. This yields a space of dimension 6.

For cubic threefolds, we get $H^3(M,\mathbb{C}) = H^{2,1}(M) \oplus H^{1,2}(M)$ with $H^{2,1}(M)$ represented by the forms $AF^{-2}dz_0 \wedge \cdots \wedge dz_4$ with A of degree 1 (the jacobian ideal of F is generated elements of degree 2). So $h^{2,1}(M) = 5$.

A similar computation shows that the nonzero Hodge numbers of cubic fourfolds are $h^{3,1} = h^{1,3} = 1$ and $h^{2,2}_\circ = 20$ with a generator of $H^{3,1}(M)$ represented by $F^{-2}dz_0 \wedge \cdots \wedge dz_5$.

(ii) If we take $d = n+2$, we get a Calabi–Yau manifold: $H^{n,0}(M)$ is spanned by $F^{-1}dz_0 \wedge \cdots \wedge dz_{n+1}$ and the n-form in question appears as a double residue: first at the $\mathbb{P}^{n+1}$ at infinity; this form has nonzeroes and a pole of order one along M and so its residue along M is a regular n-form without zeroes. (The Calabi–Yau property is usually stated as the vanishing of the $H^{k,0}$ for $0 < k < n$ and the triviality of the canonical sheaf Ω^n_M.) For $n = 1$, we get elliptic curves, for $n = 2$ we get K3 surfaces. These are somewhat special. The case of quintic 3-folds is more typical.

EXERCISE 8. Show that for $d < n+2$, $H^{n,0}(M) = 0$ and conclude that the Hodge structure on $H^{n,0}_\circ(M)$ has Hodge colevel > 0.

7.2. The cyclic covering trick

To the n-fold $M \subset \mathbb{P}^{n+1}$ as above we can associate the the cyclic degree d cover $M' \to \mathbb{P}^{n+1}$ which totally ramifies along M. To be precise, if $F \in \mathbb{C}[z_0, \ldots, z_{n+1}]$ is a defining equation for M, then $F' := F + z^d_{n+2}$ is a defining equation for M' as a hypersurface in $\mathbb{P}^{n+2}$. So it is also a smooth hypersurface of degree d and the covering group is the group of dth roots of unity, μ_d, which acts as the tautological character (i.e., as scalar multiplication) on the added coordinate z_{n+2}. Instead of considering the Hodge structure on $H^n_\circ(M)$, we may focus on the Hodge structure on $H^{n+1}_\circ(M')$ with its μ_d-action. Sometimes this modified period map for hypersurfaces of degree d is better behaved; we will meet instances where it is a local isomorphism to a classifying space of Hodge structures with μ_d-symmetry, whereas the usual period map fails to be a local isomorphism. Notice that the Griffiths description allows us to make the μ_d-action on $H^{n+1}_\circ(M')$ explicit. We have

$$J(F') = J(F) \otimes \mathbb{C}[z_{n+2}]/(z^{d-1}_{n+2}),$$

as algebra with μ_d-action and so we find that the eigenspace of the group μ_d acting in $\mathrm{Gr}_F^{(n+1)-r} H^{n+1}_\circ(M'; \mathbb{C})$ that belongs to the jth power of the tautological character can be identified with

$$J(F)_{d(r+1)-(n+2)-j} \otimes z^j_{n+2} \frac{dz_0 \wedge \cdots \wedge dz_{n+1}}{F'^{r+1}} \wedge \frac{dz_{n+2}}{z_{n+2}},$$

where $j = 1, \ldots, d-1$. Notice that the case $j = 0$ does not occur, so this Hodge structure uses different graded subspaces of $J(F)$ than the Hodge structure on $H^n_\circ(M)$.

There is of course nothing to prevent one from iterating this construction. In this way all the graded pieces of $J(F)$ acquire a Hodge theoretic interpretation.

8. Special Hodge structures

We first address the question:

8.1. When is the Griffiths tranversality condition empty?

We return to the classifying space D of polarized Hodge structures with given Hodge numbers $h^{p,q}$. If Tate twisting is regarded as a harmless operation, then we may here just as well assume that $h^{p,q} = 0$ for $q < 0$ and $h^{m,0} > 0$. The possible nonzero Hodge numbers are then $(h^{m,0}, h^{m-1,1}, \dots, h^{0,m})$. If one of them is zero, say $h^{p,m-p} = 0$ for some $\frac{1}{2}m \leqslant p < m$, then for any VHS with these Hodge numbers, the Hodge bundle F^{p+1} will be flat as it cannot deform inside $F^p = F^{p+1}$, and so the Griffiths condition is quite restrictive in that case. Therefore the above question is answered by

PROPOSITION 4. *Let* $(h^{m,0}, h^{m-1,1}, \dots, h^{0,m})$ *be* positive *integers such that* $h^{p,m-p} = h^{m-p,p}$. *Then for a PVHS having these as Hodge numbers the Griffiths transversality condition is empty if and only if either* $m = 1$ *(so that the two Hodge numbers are* $h^{1,0} = h^{0,1}$*) or* $m = 2$ *and* $h^{2,0} = h^{0,2} = 1$ *(which then only leaves* $h^{1,1}$ *unspecified).*

Proof. Let V be a complex vector space defined over $\mathbb{Q}$ of dimension $\sum_p h^{p,q}$ endowed with a nondegenerate $(-1)^m$-symmetric form $I : V \otimes V \to \mathbb{C}$, also defined over $\mathbb{Q}$. Then D is open in the flag space parameterizing filtrations $F^\bullet = (V = F^0 \supset F^1 \supset \cdots F^m \supset F^{m+1} = 0)$ with $\dim F^p/F^{p+1} = h^{p,m-p}$ and F^{m+1-p} the I-annililator of F^p. Let $F^\bullet$ represent an element of D. Choose $\frac{1}{2}m < p \leqslant m$. Then $p \geqslant m+1-p$ and so $F^p \subset F^{m+1-p}$. This implies that F^p is I-isotropic and this is the only restriction on F^p in a neighborhood. In other words, the tangent space $T_{[F^\bullet]}D$ maps onto the tangent space of F^p in the isotropic Grassmannian, i.e., the set of linear maps $\phi : F^p \to V/F^p$ with the property that $I(v, \phi(v'))$ is $(-1)^{m+1}$-symmetric. This last expression only depends on v and the image of $\phi(v')$ in $V/F^{m+1-p} \cong (F^p)^*$. This shows that the tangent space of D at $[F_\bullet]$ maps onto $(-1)^{m+1}$-symmetric forms on F^p (with kernel $\mathrm{Hom}(F^p, F^{m+1-p}/F^p)$). On the other hand, we want $\phi(v)$ to lie automatically inside F^{p-1}/F^p. When m is odd, this means that $F^{p-1} = V$ and hence $p = 1$. As this must hold for all $\frac{1}{2}m < p \leqslant m$, it follows that $m = 1$ also. When m even, this means that $F^p \wedge F^p = 0$, i.e., that $\dim F^p = 1$. This holding for all $\frac{1}{2}m < p \leqslant m$ implies that $m = p = 2$. □

The two types of PVHS that appear in this proposition have classifying spaces that are symmetric domains. We recall some pertinent facts concerning such domains.

8.2. Symmetric domains

It is well-known that a connected complex manifold D that can be embedded in $\mathbb{C}^n$ as an open bounded subset, has a natural Kähler metric, the *Bergmann metric*, one that is therefore invariant under all of its biholomorphic automorphisms. We say that D is a *symmetric domain* if it is a symmetric space with respect to its Bergmann metric. This means that $\mathrm{Aut}(D)$ is semisimple, is transitive on D and that a (or equivalently, any) stabilizer $\mathrm{Aut}(D)_z$, $z \in D$, is a maximal compact subgroup of $\mathrm{Aut}(D)$. This stabilizer acts faithfully in the tangent space T_zD and contains the circle group of complex the

unit scalars as a central subgroup. A converse is also true if we restrict to the irreducible case: if G is a connected (real) simple Lie group with the property that a maximal compact subgroup K contains in its center a circle group, then G/K has the structure of a bounded symmetric domain. (Note that G need not act faithfully on G/K; the kernel of this action is the center of G. In the theory of Shimura varieties one allows G to be reductive: D is then the manifold of compact subgroups K of G modulo its center, $G/Z(G)$, where we require that the image of $Z(K)$ in every simple factor of $G/Z(G)$ is positive dimensional.)

The isomorphism types of irreducible bounded symmetric domains have been classified by Élie Cartan, who labeled these by roman numerals.

If $\Gamma \subset \mathrm{Aut}(D)$ is a discrete subgroup, then Γ acts properly discontinuously on D so that $\Gamma\backslash D$ will have the structure of a normal analytic space. The more interesting case is when we start out from a G as above, assume it is defined over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Q})$ is an arithmetic group. Then the Baily–Borel theory endows $\Gamma\backslash D$ with the structure of a normal quasi-projective variety. Here an important role is played by the canonical bundle of D, or rather the maximal G-equivariant root thereof: a G-equivariant line bundle $\mathscr{L}$ on D of which a positive tensor power has been identified G-equivariantly with the canonical bundle of D and such that this power is maximal for that property. We shall refer to this bundle as the *basic automorphic line bundle for* G and will describe it in the cases of interest to us. The Baily–Borel theory is a package of which a major ingredient is the *graded algebra* $A^{\Gamma}_{\bullet}$ *of automorphic forms*, a subalgebra of

$$\bigoplus_{k=0}^{\infty} H^0(D, \mathscr{L}^{\otimes k})^{\Gamma}$$

defined by certain growth conditions (which are often empty). This subalgebra is of finite type and integrally closed so that $\mathrm{Proj}(A^{\Gamma}_{\bullet})$ defines a normal projective variety. This variety contains $\Gamma\backslash D$ as Zariski open-dense subset; it is the famous *Baily–Borel compactification* of $\Gamma\backslash D$ that gives the latter its quasiprojective structure.

We return to the two cases appearing in Proposition 4.

8.3. First case: symmetric domains of type III

In the first case, we have only two nonzero Hodge numbers: $h^{1,0} = h^{0,1}$. So I is a symplectic form on V and if we write g for $h^{1,0}$, then $\dim V = 2g$ and D parametrizes the g-dimensional subspaces $F \subset V$ that are isotropic for I and for which the hermitian form $H(v, v') := -iI(v, \bar{v}')$ (whose signature is (g, g)) is positive on F. Since taking the real part defines an $\mathbb{R}$-linear isomorphism of F onto $V(\mathbb{R})$, F defines a complex structure J_F on $V(\mathbb{R})$. This gives another interpretation of D: it is the space of complex structures J on $V(\mathbb{R})$ that preserve I (so $I(Jv, Jv') = I(v, v')$) and for which $(v, v') \mapsto I(v, Jv')$ is a positive definite hermitian form on $V(\mathbb{R})$. The group $G(\mathbb{R})$ is the group of symplectic transformations of $V(\mathbb{R})$. We already observed that it acts transitively on D. The stabilizer $G(\mathbb{R})_F$ of F is just the the group of unitary transformations of $V(\mathbb{R})$ relative the associated complex structure with inner product. This group is maximal compact

in $G(\mathbb{R})$ and that makes D a bounded symmetric domain isomorphic to the Siegel half space of genus g (of type III in É. Cartan's classification). If we choose a basis on which I takes the standard form, then we get the identification of D with $Sp(2g,\mathbb{R})/U(g)$.

The family of Hodge structures parameterized by D arises in an evident manner: if we fix a lattice $V(\mathbb{Z}) \subset V(\mathbb{Q})$, then the associated Jacobian $J_F := V/(F+V(\mathbb{Z}))$ is complex torus on which I defines a polarization. Beware however that the Hodge structure we get is that on $H_1(J_F)$ (and so must apply a Tate twist to get the weights right). If we want the Hodge structure to arise as a H^1, we proceed as follows: if we put $V(\mathbb{Z})^\vee := \mathrm{Hom}(V(\mathbb{Z}),\mathbb{Z})$, then for any $F \in D$ the inclusion $F \subset V = \mathbb{C} \otimes_{\mathbb{Z}} V(\mathbb{Z})$ dualizes as a group homomorphism $V(\mathbb{Z})^\vee \to F^*$. Its image is lattice and if we let $J_F^\vee$ be its cokernel (an abelian variety polarized by the inverse of I), then $H^1(J_F^\vee)$ with its polarization Hodge structure is as desired. If $V(\mathbb{Z}) \subset V(\mathbb{R})$ is a lattice such that I is integral and has determinant 1 on $V(\mathbb{Z})$, then G is defined over $\mathbb{Q}$. The subgroup $G(\mathbb{Z}) = \mathrm{Aut}(V(\mathbb{Z}), I)$ is arithmetic in $G(\mathbb{Q})$ (and isomorphic to $Sp(2g,\mathbb{Z})$), acts properly discontinuously on D and its orbit space $Sp(2g,\mathbb{Z})\backslash D$ can be identified with the moduli space $\mathscr{A}_g$ of principal polarized abelian varieties of genus g.

The natural automorphic line bundle over D has fiber over $[F]$ the line $\det(F) = \wedge^g F$, whereas the canonical bundle of D is the $(g+1)$st tensor power of this bundle.

8.4. Second case: symmetric domains of type IV

In the second case, the Hodge numbers that are nonzero are $h^{2,0} = h^{0,2} = 1$ and $h^{1,1} > 0$. Hence I is a symmetric form on $V(\mathbb{R})$ of signature $(2,d)$ and D parametrizes the one-dimensional subspaces $F \subset V$ that are isotropic for I and for which a generator $v \in F$ satisfies $I(v,\bar{v}) > 0$. Notice that taking the real part defines a real embedding of F in $V(\mathbb{R})$. The image $P_F \subset V(\mathbb{R})$ is a plane with a complex structure J_F inherited from F. This plane is positive definite and the complex structure is orthogonal with respect to I. Any real plane with inner product admits just two orthogonal transformations of order four and these define opposite orientations. So we can recover F from the plane P_F together with its orientation.

The group $G(\mathbb{R})$ is the group of orthogonal transformations of $V(\mathbb{R})$. The stabilizer $G(\mathbb{R})_F$ of F is just the group of orthogonal transformations of $V(\mathbb{R})$ which preserve the plane P_F as well as its orientation. The maximal compact subgroup of $G(\mathbb{R})$ containing the latter is the group which ignores this orientation and that makes that D has two components which are interchanged by complex conjugation, each of which is a bounded symmetric domain (of type IV in Cartan's classification). If we choose a basis of $V(\mathbb{R})$ on which I takes the standard form, then we have identified D with $O(2,d)/SO(2) \times O(d)$ and a connected component of D with

$$O^+(2,d)/(SO(2) \times O(d)) \cong O(2,d)^\circ/(SO(2) \times SO(d)),$$

where $O^+(2,d)$ stands for the group of orthogonal transformations of spinor norm 1 relative to $-I$ (this the group characterized by the fact that they can be written as the composition of orthogonal reflections with only an even number reflection vectors v with $I(v,v) > 0$; for $d > 0$ it is union of two connected components of $O(2,d)$), and

$O(d,2)^\circ$ is the identity component of $O(d,2)$. The natural automorphic line bundle over D has as fiber over $[F]$ the line F; the canonical bundle of D is the dth tensor power of this bundle.

A Clifford construction (due to Kuga and Satake) embeds a type IV domain totally geodesically in a type III domain. The details of this construction are discussed in the lectures of Christian Schnell.

8.5. Complex balls

Let W be a complex vector space of dimension $r+1$ endowed with a hermitian form $H: W \times W \to \mathbb{C}$ of hyperbolic signature $(1,r)$. Then the open cone defined by $H(z,z) > 0$ defines an open subset $B \subset \mathbb{P}(W)$. If we choose coordinates $(z_0,\dots,z_r)$ for W on which $h(z,z)$ takes the form $|z_0|^2 - \sum_{k=1}^r |z_k|^2$, then $(\zeta_k := z_k/z_0)_{k=1}^r$ is a system of affine coordinates for $\mathbb{P}(W)$ that identifies B with the open complex unit ball $\sum_k |\zeta_k|^2 < 1$. The special unitary group $\mathrm{Aut}(W,H)$ acts transitively on B and the stabilizer of $[z] \in B$ is identified with the unitary group of the negative definite subspace perpendicular to z. This is a maximal compact subgroup and that makes B a symmetric domain isomorphic to $SU(1,r)/U(r)$. (It is a domain of type $I_{1,r}$.) The natural automorphic line bundle over B has as fiber over $[F]$ the line F and the canonical bundle of B is its $r+1$st tensor power.

8.6. Complex balls in type IV domains

We may encounter such domains as subdomains of type IV domains as follows. Returning to the situation discussed there, assume that we are given a $\sigma \in G(\mathbb{Q})$ of order $\nu \geqslant 3$ and assume that its eigenvalues have the same order, i.e. are primitive νth roots if unity. So we have a decomposition $V = \oplus_\zeta V_\zeta$, with ζ running over the primitive roots of unity. For $v \in V_\zeta$, $v' \in V_{\zeta'}$ we have $I(v,v') = I(\sigma v, \sigma v') = \zeta\zeta' I(v,v')$ and from this see that each V_ζ is isotropic relative to I and that the eigen decomposition is perpendicular with respect to the hermitian form $I(v,\bar{v})$. Since the latter has signature $(2,d)$, we see that there must be a primitive root of unity ζ_0 for which V_{ζ_0} has hyperbolic signature $(1,r)$. Then its complex conjugate $\bar{V}_{\zeta_0} = V_{\bar{\zeta}_0}$ has also this signature and the remaining summands will be negative definite. We thus find that $\mathbb{P}(V_\zeta)$ does not meet D unless $\zeta = \zeta_0^{\pm 1}$ and in that case the intersection is a complex ball (it is in fact a totally geodesic subdomain). The union of these two subdomains (which is evidently disjoint) is totally geodesic in D and parameterizes the polarized HS's on V invariant under σ, more precisely, with the property that $H^{2,0}$ lies in the σ-eigenspace with eigenvalue $\zeta_0^{\pm 1}$. A case of particular interest is when $\nu = 3$ or $\nu = 4$, for then there are only two primitive νth of unity roots whose eigenspaces must be the ones singled out above. So then d is even and either ball has dimension $\frac{1}{2}d$.

8.7. Complex balls in type III domains

In a similar fashion complex balls can occur in Siegel upper half spaces. Let (V, I) be as in that case, so that I is a symplectic form on the $2g$-dimensional V. Let $\sigma \in G(\mathbb{Q})$ have finite order ν with eigenvalues primitive νth roots of unity such that for some such eigenvalue ζ_0, the hermitian form H has on V_{ζ_0} hyperbolic signature $(1, r)$. Then $V_{\bar{\zeta}_0}$ has hyperbolic signature $(r, 1)$. Assume that on the other eigen spaces H is definite (these then come in complex conjugate pairs: one positive, the other negative). Any $[F] \in D$ is fixed by σ must be such that F is a direct sum of eigen spaces. Since H is positive on F, we see F contains all the positive definite eigen spaces (whose sum will have dimension $g - r - 1$), must meet V_{ζ_0} in a positive definite line and $V_{\bar{\zeta}_0}$ in a positive definite hyperplane. This shows that D^σ can be identified with the hyperbolic space V_{ζ_0}. It lies as a totally geodesic complex submanifold in D.

8.8. CM fields

The preceding embeddings hardly involved σ, but rather the field extension of $\mathbb{Q}$ gotten by adjoining to $\mathbb{Q}$ the roots of the minimal polynomial of σ. Indeed, this generalizes to the case where we have a finite Galois extension $K/\mathbb{Q}$ which admits no real embedding (such a field is called a *CM-field*); the algebra homomorphisms $K \to \mathbb{C}$ (in other words, the field embeddings of K in $\mathbb{C}$) then come in complex conjugate pairs and the eigenspaces in V are of course defined by such algebra homomorphisms.

The imaginary quadratic fields then play the special role we attributed to the cases $\nu = 3, 4$ above.

9. Special families

9.1. K3 surfaces

The best known class of examples of Hodge structures of type IV are provided by polarized K3 surfaces. We recall that a complex compact connected surface is called a *K3 surface* if it has vanishing first Betti number and trivial canonical bundle. Examples are obtained from a complex torus A as follows, if $\tilde{A} \to A$ blows up each of its (16) points of order $\leqslant 2$, then we have an involution of $\tilde{A}$ induced by "minus identity" in A and the quotient of $\tilde{A}$ by this involution is a K3 surface. Other examples are smooth quartic surface. It is known (but not an easily obtained fact) that the defining properties of a K3 surface determine the underlying surface up to diffeomorphism. It is also known and highly nontrivial that such a surface admits a Kähler metric. Moreover, its second cohomology group with integral coefficients is free abelian of rank 22, and the $\mathbb{Z}$-valued bilinear form on that group defined by its intersection pairing is even, unimodular (i.e., the pairing is perfect) and of signature $(3, 19)$. The theory of quadratic forms tell us that these properties fix its isomorphism type as a lattice: it must be isomorphic to

$$\Lambda := 2E_8(-1) \perp 3U,$$

where E_8 stands for the E_8-lattice (a positive definite unimodular even lattice of rank 8) and U for the hyperbolic lattice (the rank two lattice with generators e, f satisfying $e \cdot e = f \cdot f = 0$ and $e \cdot f = 1$); $E_8(-1)$ means that we multiplied the pairing by -1).

The Torelli theorem for Kähler K3 surfaces states that the isomorphism type of a pair (S,η), where $\eta \in H^{1,1}(S;\mathbb{R})$ is the Kähler class, is completely defined by the isomorphism type of the triple $H^2(S;\mathbb{Z})$ (with its intersection pairing), $\eta \in H^{1,1}(S;\mathbb{R})$, and the Hodge structure on $H^2(S;\mathbb{C})$. We focus here on primitively polarized K3 surfaces: these are those for which η is integral (this is equivalent to: η is ample) and primitive in the sense that it is not a proper multiple of an integral class. We then call

$$g := 1 + \tfrac{1}{2}\eta \cdot \eta \in \{2,3,4,\cdots\}$$

the *genus* of S and we say that (S,η) is a *primitively polarized K3 surface of genus* g (it is called that way because the linear system defined by such a polarization consists of curves of arithmetic genus g).

We fix an integer $g \geqslant 2$ and a primitive vector $\eta_g \in \Lambda$ with $\eta_g \cdot \eta_g = 2g-2$. Since $O(\Lambda)$ acts transitively on such vectors, it does not matter which one we choose, but for definiteness let us take $\eta_g = (g-1)e + f$, where (e,f) is the standard basis of the last U-summand. Its orthogonal complement Λ_g has signature $(2,19)$ and so we have defined $D_g \subset \mathbb{P}(\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_g)$. The stabilizer of η_g in the orthogonal group of Λ is an arithmetic group $\Gamma_g = O(\Lambda_g)_{\eta_g}$ in the corresponding algebraic group $O(\mathbb{Q} \otimes \Lambda_g)_{\eta_g}$ so that the orbit variety $\Gamma_g \backslash D_g$ has naturally the structure of a quasiprojective variety by the Baily–Borel theory (the disconnectedness of D_g is counteracted by the fact that Γ_g contains an element that exchanges the two components). If (S,η) is primitively polarized K3 surface of genus g, then $(H^2(S),\eta)$ is isometric to (Λ,η_g) and such an isometry carries $[H^{2,0}(S)]$ to an element of D. These isometries are simply transitively permuted by Γ_g and thus we obtain a well-defined element of $\Gamma_g \backslash D_g$. There is a coarse moduli space $\mathscr{M}_g(K3)$ of primitively polarized K3 surfaces of genus g and the map

$$\mathscr{M}_g(K3) \to \Gamma_g \backslash D_g$$

just defined is in fact a morphism. The Torelli theorem states that this morphism is an open embedding. As is well-known, this map is not onto, but becomes so if we allow η_g to be semi-ample; a multiple η_g then defines a morphism to a projective space which contracts some curves to create rational double point singularities.

9.2. An example: Kondō's period map for quartic curves

A nonsingular quartic plane curve $C \subset \mathbb{P}^3$ has genus 3 and is canonically embedded: the linear system of lines in $\mathbb{P}^3$ is canonical. The converse is also true: if C is a nonhyperelliptic curve of genus 3, then its canonical system embeds it in a projective plane. The Hodge structure on $H^1(C)$ is of type III and principally polarized and the period map from the moduli space of nonsingular quartic plane curves to $\mathscr{A}_3$ is in this case an open embedding. It is not onto, for we miss the locus (of codimension one) that parameterizes the hyperelliptic curves as well as the locus (of codimension two) parameterizing decomposable abelian varieties. The cyclic cover construction of

Subsection 7.2 yields another period map. So if $F \in \mathbb{C}[z_0, z_1, z_2]$ is an equation for C, then $F' := F + z_3^4$ is an equation for a cyclic cover X of $\mathbb{P}^2$ totally ramified along C. This is in fact a K3-surface that comes the holomorphic 2-form

$$\mathrm{Res}_X \mathrm{Res}_{\mathbb{P}^3} \frac{dz_0 \wedge dz_1 \wedge dz_2}{F'} \wedge dz_3,$$

written this way to as make evident that the line $H^{2,0}(X)$ it spans is acted on by μ_4 with the tautological character χ. This is indeed the form furnished by the Griffiths theory that we described earlier. The variation we described tells us how $H^{2,0}(X;\mathbb{C})$ decomposes according to the characters of μ_4: we get $1+7(\chi+\chi^2+\chi^3)$. The eigen space $H^{2,0}(X;\mathbb{C})_1 \oplus H^{2,0}(X;\mathbb{C})_{\chi^2}$ is the complexification of the fixed point lattice of $-1 \in \mu_4$ in $H^2(X;\mathbb{Z})$; it is of rank 8 and of signature $(1,7)$. The orthogonal complement of this lattice in $H^2(X;\mathbb{Z})$ has complexification $H^2(X;\mathbb{C})_\chi \oplus H^2(X;\mathbb{C})_{\bar{\chi}}$; it is of rank 14 and has signature $(2,12)$. As we have seen, the line $H^{2,0}(X)$ lies in $H^2(X;\mathbb{C})_\chi$. It follows from our previous discussion that the corresponding point in $\mathbb{P}(H^2(X;\mathbb{C})_\chi)$ lies in the ball of complex dimension 6 defined by the hermitian form. We thus get a period map from the moduli space of quartic curves to a ball quotient of dimension 6. Kondō deduces from the injectivity of the period map for K3 surfaces that this period map is an open embedding. The image misses two divisors: one parameterizing hyperelliptic curves and another parameterizing singular curves.

9.3. Cubic fourfolds

Let $M \subset \mathbb{P}^5$ be a nonsingular cubic fourfold. The intersection pairing defines on $H^4(M;\mathbb{Z})$ is symmetric bilinear form. It is nondegenerate unimodular (by Poincaré duality), odd (because the self-intersection of the square of the hyperplane class is 3) and has signature $(21,2)$. The theory of quadratic forms tells us that these properties imply that the form can be diagonalized: there is an isomorphism ϕ of $H^4(M;\mathbb{Z})$ onto $L := 21I \perp 2I(-1)$. The theory also shows that the orthogonal group of the latter acts transitively on the vectors of self-product 3. So if we fix one such vector $\eta \in L$, and $L_\circ \subset L$ denotes its orthogonal complement, then we can arrange that ϕ takes the square of the hyperplane class to η and hence $H^4_\circ(M;\mathbb{Z})$ to $L_\circ$. As we observed in Examples 3, the nonzero primitive Hodge numbers of $H^4(M;\mathbb{Z})$ are $h^{3,1}_\circ(M) = h^{1,3}_\circ(M) = 1$ and $h^{2,2}_\circ(M) = 20$. So after a Tate twist: $H^4(M)(1)$, then we get a Hodge structure of type IV. Voisin [**?**] proved that the resulting period map is an open embedding. Laza [**?**] and the author [**?**] independently determined the image of the map (the complement of two irreducible hypersurfaces, one corresponding to mildly singular cubic fourfolds, another represents a single highy degenerate cubic fourfold).

9.4. Cubic threefolds

Let $M \subset \mathbb{P}^4$ be a nonsingular cubic threefold. The cyclic covering construction yields a nonsingular cubic fourfold M' with μ_3-action. We find that μ_3 acts in $H^4_\circ(M;\mathbb{C})$ with character $11(\chi+\bar{\chi})$, where χ is the tautological character. The line $H^{3,1}_\circ(M')$ is

spanned by (an iterated resudue of)

$$\frac{dz_0 \wedge \cdots \wedge dz_5}{F'^2} = z_5 \frac{dz_0 \wedge \cdots \wedge dz_4}{F'^2} \wedge \frac{dz_5}{z_5}$$

and so lies in the χ-eigenspace. We conclude that the period map takes its values in a ten dimensional ball quotient. This period map was shown by Allcock-Carlson-Toledo [**?**] and Looijenga-Swierstra [**?**] to be an open embedding whose image misses two hypersurfaces, one representing mildly singular cubics and the other being represented by a single very singular cubic threefold. Allcock-Carlson-Toledo have also shown that by iterating this construction downward one can give the moduli space of cubics surfaces the structure of a ball quotient.

9.5. Holomorphically symplectic manifolds

A higher dimensional generalization of the K3-surfaces are the holomorphically symplectic manifolds. We say that a compact complex manifold M is *(holomorphically) symplectic* if it admits a holomorphic 2-form ω which is nondegenerate everywhere. This means that M has even complex dimension $2r$ and that the r-fold wedge $(\omega)^{\wedge r}$ is nowhere zero. We shall also require that that $b_1(M) = 0$ (in order to avoid having complex tori as factors) and that ω is unique up to scalar (in order to avoid almost decomposable manifolds). We shall further assume that M is Kähler, more precisely, we assume M endowed with a *Kähler ray*, that is the oriented real line K_M in $H^{1,1}(M)(\mathbb{R})$ spanned by a Kähler class. A famous theorem of S.T. Yau asserts that any Kähler class on M is uniquely represented by a Ricci flat Kähler metric.

This class of Kähler manifolds, introduced by Beauville and Bogomolov, provides examples of Hodges structures of type IV. Beauville proves that for such a manifold M the $\mathbb{C}$-algebra $H^{\bullet,0}(M) = H^0(M, \Omega^\bullet_M)$ of its holomorphic forms is generated by ω. There is a natural nondegenerate symmetric bilinear form, the Beauville–Bogomolov form

$$b_M : H^2(M) \times H^2(M) \to \mathbb{Q}$$

that is integral (but not necessarily unimodular), and has the property that $b_M(x,x)^r = c\int_M x^{2r}$ for some constant c and which (indeed) only depends on the underlying oriented manifold. Its signature is $(3, h^2(M) - 3)$ and has the span of ω, $\bar{\omega}$ and a Kähler class as a maximal positive definite 3-dimensional subspace. These properties imply that $H^2_\circ(M)$ is a Hodge structure of type IV (except that this space need not be defined over $\mathbb{Q}$ unless M is projective).

EXAMPLE 4. The best known classes of examples (due to Beauville [**?**]) are obtained as follows. If S is a Kähler K3 surface, then the Hilbert scheme $S^{[r]}$ of length r subschemes of S is holomorphically symplectic of dimension $2r$. If T is a complex torus of dimension 2, then T acts on $T^{[r+1]}$ by translation and the orbit space has the structure of a holomorphically symplectic manifold of dimension $2r$. In either case, the symplectic manifold admits deformations that do not arise form deformations of S resp. T. Examples that are known to be irreducible and *not* deformation equivalent to one these classes are still very few in number and are due to O'Grady [**?**], [**?**].

EXAMPLE 5. Beauville and Donagi observed that for a smooth cubic fourfold $M \subset \mathbb{P}^5$, its variety $F_1(M)$ of lines on M (a subvariety of the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^6)$) is smooth and is a holomorphically symplectic manifold. It is a deformation to a $S^{[2]}$. If $\tilde{F}_1(M)$ denotes the set of pairs $(p, \ell) \in M \times F_1(M)$ with $p \in \ell$, then the incidence correspondence

$$M \xleftarrow{\pi_M} \tilde{F}_1(M) \xrightarrow{\pi_F} F_1(M)$$

has the property that

$$H^4(M)(1) \xrightarrow{\pi_M^*} H^4(\tilde{F}_1(M))(1) \xrightarrow{\pi_{F*}} H^2(F_1(M))$$

defines an isomorphism of polarized Hodge structures when restricted to the primitive parts. This in a sense gives a geometric explanation for the type IV that we encountered on the primitive cohomology of a smooth cubic fourfold $M \subset \mathbb{P}^5$.

QUESTION 1. This example raises the question whether any type IV Hodge structure that appears in the Hodge structure of a projective manifold M comes from a holomorphically symplectic manifold F and more specifically, whether it can be "explained" by a correspondence involving M and F. This is of course a question impossible to answer in general, but it suggests at the very least to look for an F whenever we have an M.

We define a period for a holomorphically symplectic manifold as we did for K3 surfaces. Let us fix the oriented diffeomorphism type underlying M and choose a complex vector space V with symmetric bilinear form $I : V \times V \to \mathbb{C}$ and a lattice $\Lambda \subset V$ on which this form is integral and isomorphic to the Beauville–Bogomolov form b_M of M. Consider the space Ω_{kr} of pairs (F, K) where $F \subset V$ is a complex line, $K \subset V(\mathbb{R})$ is an oriented real line and such that $F + I$ is contained in the I-annihilator of F^2 and is positive definite relative to the Hermitian form defined by I. The group $O(I)(\mathbb{R})$ acts transitively on Ω; the action is also proper for the stabilizer of a pair (F^1, K) can be identified with the compact subgroup $SO(2) \times O(h^2 - 3)$. If we ignore K, then we obtain the space Ω of I-isotropic complex lines $F \subset V$ that are positive definite relative to the Hermitian form defined by I. We have an obvious forgetful map $\Omega_{\mathrm{kr}} \to \Omega$. The space Ω is open in the quadric defined by I in $\mathbb{P}(V)$ and has hence the structure of a complex manifold. The group $O(I)(\mathbb{R})$ acts transitively on it, but the stabilizer of a point can be identified with the noncompact Lie group $SO(2) \times O(1, h^2 - 3)$ and the action is not proper.

For every M as above, there exists by construction an isometry

$$\phi : (H^2(M; \mathbb{Z}), b_M) \cong (\Lambda, I).$$

We call this a *marking* of M. There is a moduli space of $\mathscr{M}(I)_{\mathrm{kr,mkd}}$ marked symplectic manifolds endowed with a Kähler ray. This space has *not* a complex structure, but does have the structure of a real-analytic separated manifold. Assigning to (M, K_M, ϕ) the pair $(\phi(H^{2,0}(M)), \phi(K_M))$ defines a map $\mathscr{M}(I)_{\mathrm{kr,mkd}} \to \Omega_{\mathrm{kr}}$. This map is a local isomorphism of real-analytic manifolds. There is also a moduli space of $\mathscr{M}(I)_{\mathrm{mkd}}$

marked symplectic manifolds (so with no Kähler ray given), which has the structure of a complex manifold that a priori could be nonseparated.

The preceding map factors through a local isomorphism $\mathscr{M}(\mathrm{I})_{\mathrm{mkd}} \to \Omega$. An argument based on the twistor construction (in this case due to Huybrechts) shows that every connected component of $\mathscr{M}(\mathrm{I})_{\mathrm{mkd}}$ maps surjectively to Ω.

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