MODULI SPACES OF RIEMANN SURFACES AT TSINGHUA

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This is the write up of a course I taught at Tsinghua University in the Fall of 2011, while being supported by the there-based Mathematics Sciences Center.

One can enter the subject via algebraic geometry, complex analysis, (combinatorial) topology and even homotopy theory. My aim was not to confine myself to just one of these approaches, but rather to give the students a sense how they not only supplement, but sometimes also reinforce each other. That sounds rather ambitious and so I hasten to add that the course's content was not only subject to time constraints, but also to my probable bias and the limitations of my knowledge. Having said that, I believe that the implementation of this philosophy not only exhibits the intertwined nature mentioned above, but also allows occasionally for shorter proofs, at least shorter than the ones I found in the literature.

Conspicuously absent in these notes are references to the primary sources, a defect I may remedy in the near future. But whatever I do, the reader is in this regard probably best served by the references list in the recent book by Arbarello-Cornalba-Griffiths: Geometry of Algebraic Curves II (volume 268 of the Springer's Grundlehren series). This magnum opus of almost a thousand pages is also an excellent reference for much of the material that is discussed here and I highly recommend to use it on the side.

I knew from the outset that this 'multidisciplinary approach' would be rather demanding on the students. But my audience, which even counted a few undergraduates, proved to be very motivated and gave me plenty of feedback (which occasionally led to a correction or a more detailed discussion). The course was a joy to give.

NB: In these notes a *surface* always means an oriented 2-manifold.

CONTENTS

1. Mapping class groups and Dehn twists

Mapping class groups. Let S be a surface and $P \subset S$ closed subset. We then denote by $\mathrm{Diff}(S, P)$ the group of diffeomorphisms $S \to S$ that are the identity on P (but usually write $\mathrm{Diff}(S)$ when $P = \emptyset$) and $\mathrm{Diff}^\circ(S, P) \subset \mathrm{Diff}(S, P)$ stands for its identity component–this is a normal subgroup. Two elements $h, h' \in \mathrm{Diff}(S)$ are called *isotopic relative to* P if $h^{-1}h'$ in $\mathrm{Diff}^\circ(S, P)$. This means that they are in the same path component of $\mathrm{Diff}(S, P)$.

We write $\mathrm{Diff}^+(S, P) \subset \mathrm{Diff}(S, P)$ for the group of orientation preserving diffeomorphisms. Note that it contains $\mathrm{Diff}^\circ(S, P)$ as a normal subgroup. We call the quotient $\Gamma(S, P) := \mathrm{Diff}^+(S, P)/\mathrm{Diff}^\circ(S, P)$ (which comes endowed with the discrete topology) the *mapping class group* of (S, P).

Dehn twists. The collection of embedded circles in $S - P$ is acted on by $\mathrm{Diff}(S, P)$. Two embedded circles α, α' are said to be isotopic relative to P is they have the same $\mathrm{Diff}^\circ(S, P)$-orbit. So $\Gamma(S, P)$ acts on the set of isotopy classes of embedded circles in $S - P$.

An embedded circle $\alpha \subset S - P$ leads to a so-called Dehn twist $D_\alpha \in \Gamma(S, P)$: let $\phi : (-1, 1) \times S^1 \to S - P$ be an open, *orientation preserving* embedding such that α is the image of ϕ_0 and let $f : (-1, 1) \to [0, 2\pi]$ be a smooth function which is constant 0 on $(-1, -\frac{1}{2})$ and constant 2π on $(\frac{1}{2}, 1)$. Let $h : S \to S$ be the identity outside the image of ϕ and be such that $h(\phi(t, u)) = \phi(t, ue^{-\sqrt{-1}f(t)})$. Then h is a diffeomorphism and one can check that its image in $\Gamma(S, P)$ only depends on the isotopy class relative to P of α. In particular, it does not depend on an orientation of α. We call it this element of $\Gamma(S, P)$ the *Dehn twist* associated to α and denote it by D_α.

Exercise 1. Prove that D_α is trivial when α bounds a disk in S which meets P in at most one point.

If S is a closed surface and $P \subset S$ is finite, then we call the pair (S, P) a P-*pointed surface*. We then often write S° for $S - P$. If S is connected and g is the genus of S, then we sometimes denote $\Gamma(S, P)$ by $\Gamma_{g,P}$, or if the elements of P have been effectively numbered: $P = \{p_1, \dots, p_n\}$, by $\Gamma_{g,n}$.

It can be shown that $\Gamma_{g,P}$ is finitely presented with Dehn twists as generators. We shall see that $\Gamma_{0,3}$ is trivial and that $\Gamma_{1,1} \cong SL(2, \mathbb{Z})$.

Suppose S closed connected of genus g and P finite. The $\mathrm{Diff}^+(S,P)$-orbits of embedded circles in S° can then be topologically distinguished: A *non-separating* embedded circle $\alpha \subset S^\circ$ is by definition such that its complement $S-\alpha$ connected. That complement is then the interior of a compact surface of genus $g-1$ with two boundary components. Otherwise α is *separating* and splits S into two connected components S', S'', each of which has as its closure a compact subsurface with α as boundary. If the genera of these surfaces are denoted g' and g'', then $g = g' + g''$. The members of P will divide themselves in $P' := P \cap S'$ and $P'' := P \cap S''$. The diffeomorphism type of a compact surface with boundary is completely given by its genus and its number of boundary components and this is still true in the relative situation where the surface has been equipped with a finite subset in hits interior. From this we easily deduce:

Proposition 1.1. *The* nonseparating *embedded circles in* S° *make up a single orbit under* $\mathrm{Diff}^+(S,P)$ *and for a* separating *embedded circle its* $\mathrm{Diff}^+(S,P)$*-orbit is characterized by the unordered pair of pairs* $\{(g',P'),(g'',P'')\}$ *defined above. This gives rise to a corresponding characterization of the* $\Gamma(S,P)$*-conjugacy classes of Dehn twists.*

The notion of a mapping class group does not change if we pass to the topological setting: any closed topological surface admits a differentiable structure, and for (S,P) as above the inclusion $\mathrm{Diff}^+(S,P) \subset \mathrm{Homeo}^+(S,P)$ induces an isomorphism $\Gamma(S,P) \cong \mathrm{Homeo}^+(S,P)/\mathrm{Homeo}^\circ(S,P)$. In fact, this is even true if we descend to the homotopy category: the natural map from $\Gamma(S,P)$ to the group $\mathrm{Htp}(S,P)$ of homotopy equivalences $(S,P) \to (S,P)$ relative to P is an isomorphism. That property amounts to a characterization of $\Gamma(S,P)$ as a group of *outer automorphisms* of the fundamental group of S°, which we describe the next subsection. Before we do so, we mention that there is an intermediate structure which is quite useful in Teichmüller theory, namely that of a quasiconformal structure. We will not give the definition, but just mention that the connected component group of the group of automorphisms of that structure which preserve orientation and fix P pointwise also maps isomorphically to $\Gamma(S,P)$.

Fundamental groups and mapping class groups. We will here describe without proof a characterization of a mapping class group in terms of its (outer) action on the fundamental group. We begin with general discussion of outer actions of groups.

Recall that given a group π, the group $\mathrm{Aut}(\pi)$ of group automorphisms of π contains the group $\mathrm{Inn}(\pi)$ of its inner automorphisms as a normal subgroup. The quotient group $\mathrm{Aut}(\pi)/\mathrm{Inn}(\pi)$ is called the group of its *outer automorphisms* and is denoted $\mathrm{Out}(\pi)$. Since $\mathrm{Inn}(\pi)$ acts trivially on the (co)homology of π, the group $\mathrm{Aut}(\pi)$ acts on this (co)homology always via $\mathrm{Out}(\pi)$. If $B \subset \pi$ is a $\mathrm{Inn}(\pi)$-invariant subset, in other words, a union on conjugacy classes of π, then we can form the subgroup $\mathrm{Aut}(\pi,B) \subset \mathrm{Aut}(\pi)$ of automorphisms which preserve each conjugacy class in B and as this this contains $\mathrm{Inn}(\pi)$, we also have defined the quotient group $\mathrm{Out}(\pi, B) := \mathrm{Aut}(\pi, B)/\mathrm{Inn}(\pi)$.

Let X be a path connected space. Its *fundamental groupoid* π_X is the category whose objects are the points of X and for which a morphism from $p \in X$ to $q \in X$ is a homotopy class of arcs from p to q. It is a groupoid since every morphism is an isomorphism. The fundamental group based at $p \in X$, $\pi(X, p)$, then appears as the group of π_X-endomorphisms of p (this presupposes that elements of $\pi(X, p)$ are read from right to left). Given $p, q \in X$, then by assumption there exists a π_X-morphism $\gamma : \pi(X, p) \cong \pi(X, q)$ and any two such differ by an element of $\pi(X, q)$. So the resulting isomorphism $\mathrm{Out}(\gamma) : \mathrm{Out}(\pi(X, p)) \cong \mathrm{Out}(\pi(X, q))$ is independent of the choice of γ. We denote by $\mathrm{Out}(\pi_X)$ this common group. Or if we insist on treating all the points of X on an equal footing: an element of $\mathrm{Out}(\pi_X)$ is the subgroup of $\prod_{p\in X} \mathrm{Out}(\pi(X, p))$ of elements whose components are related by the isomorphisms $\mathrm{Out}(\gamma)$. This renders evident the observation that a homotopy equivalence $h : X \to X$ induces an element of $\pi(h) \in \mathrm{Out}(\pi_X)$ and that thus is defined a group homomorphism from the group $\mathrm{Htp}(X)$ of homotopy classes of self homotopy equivalences $X \to X$ to $\mathrm{Out}(\pi_X)$, to which one often refers as an *outer action* (of $\mathrm{Htp}(X)$) on the fundamental group.

Let us return to S, a closed connected oriented surface. If $o \in S$ is a base point, then it is well-known that $\pi(S, o)$ has a (standard) presentation with generators $\alpha_{\pm 1}, \dots, \alpha_{\pm g}$, and relation $[\alpha_g, \alpha_{-g}] \cdots [\alpha_1, \alpha_{-1}] \equiv 1$.

The outer action of $\mathrm{Diff}^+(S)$ on π_S defines a group homomorphism

$$\Gamma(S) \to \mathrm{Out}(\pi_S).$$

The theorem alluded to above asserts that for genus $g > 0$ this is an isomorphism onto a subgroup $\mathrm{Out}^+(\pi_S)$ of $\mathrm{Out}(\pi_S)$ of index 2. (The preservation of orientation can be expressed in terms of π_S, because $H^2(S)$ can be understood as group cohomology: $H^2(S) = H^2(\pi_S)$; we give another description below which avoids group cohomology.) For $g = 0$, $\Gamma(S)$ is trivial. The following question was asked by Grothendieck and is still open:

Question 1.2 (Grothendieck). Does there exist for every subgroup $\Gamma \subset \Gamma(S)$ of finite index a $\Gamma(S)$-invariant normal subgroup $\pi \subset \pi(S, o)$ of finite index ($o \in S$ a base point) with the property that every $g \in \Gamma(S)$ whose image in $\mathrm{Out}(\pi(S, o)/\pi)$ is trivial is in Γ? (See a paper by Boggi **[?]** for more on this question.)

Now let $P \subset S$ be finite nonempty. Recall that $S^\circ := S - P$. We also assume that when $g = 0$, $|P| \geq 2$ (we will see that $\Gamma(S, P)$ is trivial otherwise). Choose $o \in S^\circ$. In order to give a presentation of $\pi(S^\circ, o)$, we first number the points of P so that $P := \{p_1, \dots, p_n\}$. Then we choose for $k = 1, \dots, n$, $\beta_k \in \pi(S^\circ, o)$ representing a simple positive loop in S° around p_k based at o such that $\beta_n\beta_{n-1} \cdots \beta_1$ is represented by a positive loop which encircles P and which becomes trivial in S. A presentation $\pi(S^\circ, o)$ has now generators

$\alpha_{\pm 1}, \dots, \alpha_{\pm g}, \beta_1, \dots, \beta_n$ that are subject to the relation

$$\beta_n \beta_{n-1} \cdots \beta_1 [\alpha_g, \alpha_{-g}] \cdots [\alpha_1, \alpha_{-1}] \equiv 1.$$

This relation allows us to eliminate β_n and then we see that this group is in fact freely generated by $\alpha_{\pm 1}, \dots, \alpha_{\pm g}, \beta_1, \dots, \beta_{n-1}$. However it has some additional structure: the conjugacy class $B_k \subset \pi(S^\circ, o)$ of β_k is invariantly defined as it consists of *all* the simple positive loops in S° around p_k based at o. If we divide the group $\pi(S^\circ, o)$ out by the normal subgroup generated by B_k, then we get of course the fundamental group of $S^\circ \cup \{p_k\}$. We put $B := B_1 \cup \cdots \cup B_n$ and observe that we have an evident map

$$\Gamma(S, P) \to \mathrm{Out}(\pi(S^\circ, o), B) = \mathrm{Out}(\pi_{S^\circ}, B).$$

Another basic result asserts that this map is also an isomorphism. (There is no orientation issue, for if a diffeomorphism preserves B_1, then it must be orientation preserving. Why?) Now consider the exact sequence of groups

$$1 \to \mathrm{Inn}(\pi(S^\circ, o)) \to \mathrm{Aut}(\pi(S^\circ, o), B) \to \mathrm{Out}(\pi(S^\circ, o), B) \to 1.$$

Since $\pi(S^\circ, o)$ has trivial center (it is a free group), we may identify it with its group of inner automorphisms. The middle term may be identified with $\Gamma(S, \tilde{P})$, where $\tilde{P} := P \sqcup \{o\}$ and we thus find the *Birman exact sequence*

$$1 \to \pi(S^\circ, o) \to \Gamma(S, \tilde{P}) \to \Gamma(S, P) \to 1.$$

This is in fact also valid in case $P = \emptyset$: $\Gamma(S)$ can be identified with group of outer automorphisms of $\pi(S, o)$, which, in terms of the above presentation, can be lifted to an automorphism of the free group on the generators $\tilde{\alpha}_{\pm 1}, \dots, \tilde{\alpha}_{\pm g}$ that preserves the conjugacy class of $[\tilde{\alpha}_g, \tilde{\alpha}_{-g}] \cdots [\tilde{\alpha}_1, \tilde{\alpha}_{-1}]$.

2. Conformal structures and a rough classification

Conformal structures. Let T be a real vector space of dim 2. A *conformal structure* on T is an inner product on T given up to scalar multiplication. If T is endowed with an orientation, then a conformal structure yields notion of angle (rotation over $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is defined) and turns T into a complex vector space of dim 1: scalar multiplication by $\sqrt{-1}$ is rotation over $\pi/2$. In fact, giving a complex structure on T is equivalent to giving an orientation plus a conformal structure.

The space of positive definite quadratic forms on T makes up a convex cone in the space of quadratic forms of T (if T has coordinates (x, y), then a quadratic form $ax^2 + 2bxy + cy^2$ is positive definite if and only if $b^2 < ac$). So the set of conformal structures on T makes up a disk $\mathrm{Conf}(T)$ in the projective space associated to the space of quadratic forms on T. This disk has a convex structure, but has no natural center (it is rather like the upper half plane which we will later discuss in some detail).

Let S be a surface. The spaces $\mathrm{Conf}(T_pS)$, $p \in S$, make up a disk bundle $\mathrm{Conf}(TS/S)$ over S. A *conformal structure on* S is a C^∞ section of this bundle.

This space of sections has also a convex structure: two conformal structures are naturally joined by an interval. In particular, the space of conformal structures on S, denoted $\mathrm{Conf}(S)$, is a contractible space.

Since S is oriented, a conformal structure on S defines a complex structure J_p on each T_pS. If $U \subset S$ is open, then we say that a C^∞-map $f : U \to \mathbb{C}$ is *holomorphic* if $D_pf : T_pU \to T_{f(p)}\mathbb{C} = \mathbb{C}$ is $\mathbb{C}$-linear for all $p \in U$. If $S = \mathbb{C}$ (so that U open in $\mathbb{C}$), then this means that f satisfies the Cauchy-Riemann equations and so is holomorphic in the usual sense. It is a classical but nontrivial fact that a Riemann metric on a surface is locally conformal to a Euclidean metric. This implies that S admits an atlas consisting of holomorphic charts. Since the coordinate changes of such an atlas are holomorphic, the conformal structure turns S into a Riemann surface. Conversely, a Riemann surface has an underlying orientation and conformal structure.

Let S be closed of genus g and $P \subset S$ finite. The group $\mathrm{Diff}^+(S, P)$ acts on $\mathrm{Conf}(S)$. The *Teichmüller space* of (S, P) is the space of conformal structures on S given up to isotopy relative to P:

$$\mathcal{T}_{g,P} = \mathcal{T}(S, P) = \mathrm{Diff}^\circ(S, P)\backslash \mathrm{Conf}(S).$$

If the elements of P have been numbered: $P = \{p_1, \dots, p_n\}$, then we also write $\mathcal{T}_{g,n}$. We will see that this has the structure of a finite dimensional complex manifold. Notice that $\Gamma(S, P)$ acts on $\mathcal{T}_{g,P}$ and has orbit space

$$\mathcal{M}_{g,P} := \mathrm{Diff}^+(S, P)\backslash \mathrm{Conf}(S)$$

($\mathcal{M}_{g,n}$ if P has been numbered). A point of $\mathcal{M}_{g,P}$ can be understood as an isomorphism class of pairs $(C, \iota : P \to C)$, with C a closed Riemann surface of genus g and ι an embedding of the abstract finite set P in C. We will see that this has structure of an algebraic variety.

The following is a basic result from the theory of Riemann surfaces. We will take it for granted.

Theorem 2.1 (Uniformization theorem). *Every simply connected Riemann surface is isomorphic to the Riemann sphere $\mathbb{P}^1$, the affine line (= complex plane) $\mathbb{C}$ or the upper half plane $\mathbb{H}$.*

The automorphism groups of these surfaces are as follows:

$\mathrm{Aut}(\mathbb{P}^1) = \{z \mapsto \frac{az+b}{cz+d}\}$, where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{C})$. This is $\mathrm{SL}(2, \mathbb{C})/\{\pm 1\} = \mathrm{PSL}(2, \mathbb{C})$. Notice that it is simply transitive on ordered triples of distinct points: if $z_0, z_1, z_\infty \in \mathbb{P}^1$ are distinct, then there is unique automorphism of $\mathbb{P}^1$ taking them to resp. $0, 1, \infty$.

$\mathrm{Aut}(\mathbb{C})$ is the affine group $\mathrm{Aff}(\mathbb{C}) := \{z \mapsto az + b\}$, where $a \in \mathbb{C} - \{0\}$ and $b \in \mathbb{C}$.

$\mathrm{Aut}(\mathbb{H}) = \{z \mapsto \frac{az+b}{cz+d}\}$, where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{R})$. This case is special because $\mathbb{H}$ comes with a metric, the *Poincaré metric* $y^{-2}(dx^2 + dy^2)$, that is invariant under $\mathrm{Aut}(\mathbb{H})$. This metric is also complete and has constant curvature -1 (more about this later).

If C is a connected Riemann surface, then its universal cover $\tilde{C}$ is isomorphic to one of the three above. If Π denotes the covering group of the universal cover $\tilde{C} \to C$, then Π acts freely and properly discretely on $\tilde{C}$. So Π can be identified with a subgroup of $\mathrm{PSL}(2,\mathbb{C})$, $\mathrm{Aff}(\mathbb{C})$ or $\mathrm{PSL}(2,\mathbb{R})$ acting freely and properly discretely on $\mathbb{P}^1$, $\mathbb{C}$, $\mathbb{H}$ respectively, and C may be identified with its orbit space. We get the following rough classification into spherical, flat and hyperbolic cases.

Spherical case. Every element of $\mathrm{SL}(2,\mathbb{C})$ has an eigen vector in $\mathbb{C}^2$ and so every element of $\mathrm{PSL}(2,\mathbb{C})$ has a fixed point in $\mathbb{P}^1$. Hence the only possible $\Pi \subset \mathrm{PSL}(2,\mathbb{C})$ is the trivial group. It follows that $C = \mathbb{P}^1$ in that case. Notice that in this case we can pick 3 distinct points p_0, p_1, p_∞ in C and ask that these are mapped to $0, 1, \infty$ respectively. This makes the isomorphism $f : C \to \mathbb{P}^1$ unique.

Corollary 2.2. *The Teichmüller space $\mathcal{T}_{0,3}$ is a singleton and the mapping class group $\Gamma_{0,3}$ is trivial.*

Proof. Let $S = S_0$ and $P = \{p_0, p_1, p_\infty\}$. The assertion that $\mathcal{T}_{0,3}$ is a singleton means that any two conformal structures on S are isotopic relative to P. To see this, let J be a conformal structure on S. As we observed above, there exists a unique orientation preserving diffeomorphism $f : \mathbb{P}^1 \cong S$ that is conformal and is such that $f(k) = p_k$ for $k = 0, 1, \infty$. Let J' be another conformal structure on S. By the convexity property of conformal structures, we can connect J with J' by a path of complex structures $\{J_t\}_{0\le t\le 1}$ (with $J = J_0$ and $J' = J_1$). This gives a path of isomorphisms $f_t : \mathbb{P}^1 \cong (S, J_t)$ with $f_t(k) = p_k$. Since $h_t := f_t f_0^{-1}$ takes J to J_t and preserves P, the family of diffeomorphisms $\{h_t\}_{0\le t\le 1}$ accomplishes an isotopy between J and J'.

This argument also shows that $\mathrm{Diff}^+(S,P) = \mathrm{Diff}^0(S,P)$. To see this, we may just as well assume that $S = \mathbb{P}^1$ and $p_k = k$ for $k = 0, 1, \infty$. Given $h \in \mathrm{Diff}^+(S,P)$, and denoting by J the standard conformal structure J on $\mathbb{P}^1$, then arguing as above we get a path $\{h_t\}_{0\le t\le 1}$ in $\mathrm{Diff}^+(S,P)$ with $h_0 = 1$ and $h^*J = h_{1*}J$. Since $h_1^{-1}h$ respects J and fixes $0, 1, \infty$, the uniqueness property implies that $h = h_1$. It follows that $h \in \mathrm{Diff}^0(S,P)$. □

Remark 2.3. This implies that for $n = 0, 1, 2$, $\mathcal{T}_{0,n}$ is also a singleton and $\Gamma_{0,n}$ is trivial. For $n \ge 4$ this is no longer the case: $\mathcal{T}_{0,n}$ can then be identified with the configuration space of pairwise distinct $(n-3)$-tuples $(z_4, \dots, z_n)$ in $\mathbb{P}^1 - \{0, 1, \infty\} = \mathbb{C}^\times - \{1\}$ (and the mapping class group $\Gamma_{0,n}$ is the quotient of the colored braid group on $(n-1)$ strands modulo its center).

Flat cases. Every element of $\mathrm{Aff}(\mathbb{C})$ without fixed point must be a translation $z \mapsto z + b$. So $\Pi \subset \mathrm{Aff}(\mathbb{C})$ will be a translation group, that is, we may regard it as a subgroup of $\mathbb{C}$. This subgroup is discrete and hence has rank at most two and we have $C \cong \mathbb{C}/\Pi$. Notice that $\mathbb{C}/\Pi$ is not just a Riemann surface, but also an abelian group. We show that if we choose $p \in C$, then C has naturally the structure of an abelian group with p as unit element.

Let $(\tilde{C}, \tilde{p}) \to (C, p)$ be the universal cover of C defined by p: a point of $\tilde{C}$ is given by a $q \in C$ and a homotopy class of arcs from p to q; $\tilde{p}$ is given by p and the constant homotopy class and we have $\Pi \cong \pi_1(C, p)$. We then get an isomorphism $\tilde{f} : \tilde{C} \cong \mathbb{C}$. After translation in $\mathbb{C}$, we can arrange that $\tilde{f}(\tilde{p}) = 0$. We get an induced isomorphism $f : C \cong \mathbb{C}/\Pi$ with $f(p) = [0]$. If $\tilde{f}' : \tilde{C} \cong \mathbb{C}$ is another such isomorphism with $\tilde{f}'(\tilde{p}) = 0$, then $\tilde{f}^{-1}\tilde{f}'$ is an automorphism of $\mathbb{C}$ which fixes 0. So it is given by scalar multiplication by some $\lambda \in \mathbb{C}^\times$. Hence f' is obtained by composing f with the map $\mathbb{C}/\Pi \cong \mathbb{C}/(\lambda\Pi)$ defined by multiplication by λ. This map is an isomorphism of Riemann surfaces as well as of abelian groups and so (C, p) is a natural manner an abelian group.

If $\Pi = \{0\}$, then we get $(C, p) \cong (\mathbb{C}, 0)$.

If $\Pi = \mathbb{Z}\omega_0$, with $\omega_0 \neq 0$. Then $z \mapsto e^{2\pi i z/\omega_0}$ identifies C with the multiplicative group $\mathbb{C}^\times$ of $\mathbb{C}$.

If Π is of rank 2, then $C \cong \mathbb{C}/\Pi$ is a complex torus ($g = 1$). In this case, it is convenient to fix an oriented basis (e_0, e_1) for $\pi_1(C, p)$ (any orientation preserving diffeomorphism $C \cong S^1 \times S^1$ yields one). If $\varepsilon_i : [0, 1] \to C$ is a loop based at p that represents e_i and $\tilde{\varepsilon}_i : [0, 1] \to \tilde{C}$ the lift of ε_i which begins in $\tilde{p}$, then the end point $\tilde{\varepsilon}(1) \in \tilde{C}$ only depends on e_i and we denote that point by $e_i(1)$. Then under the isomorphism $\tilde{C} \to \mathbb{C}$, $\tilde{p}$ is mapped to 0 and $(e_0(1), e_1(1))$ is mapped to an oriented basis (ω_0, ω_1) of $\mathbb{C}$, when regarded as a real vector space. Then after multiplication by ω_0^{-1}, we get an isomorphism $\tilde{f} : \tilde{C} \cong \mathbb{C}$ with $\tilde{f}(\tilde{p}) = 0$ and $\tilde{f}(e_0(1)) = 1$. This is unique, for any automorphism of $\mathbb{C}$ which fixes both 0 and 1 must be the identity. Notice that $\tau := \tilde{f}(e_1(1))$ lies in $\mathbb{H}$ (for $(1, \tau)$ must be an oriented basis) and so we get a unique isomorphism $(C, p) \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ which on fundamental groups sends $e_0(1)$ to the class represented by the loop defined by $[0, 1] \subset \mathbb{C} \to \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$.

Corollary 2.4. *The Teichmüller space $\mathcal{T}_{1,1}$ can be identified with $\mathbb{H}$ and the mapping class group $\Gamma_{1,1}$ with* $\mathrm{SL}(2, \mathbb{Z})$ *(acting as a group of fractional linear transformations.*

Proof. If $S = S_1$ and $P = \{p\}$, then choose an oriented basis (e_0, e_1) for $\pi_1(S, p)$ as above. The preceding construction finds for every conformal structure J on S a unique $\tau \in \mathbb{H}$ and a *unique* isomorphism

$$f : (S, J) \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$$

with the property that it maps p to 0 and e_0 to the class of the loop $[0, 1] \subset \mathbb{C} \to \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$.

Suppose now J' is another conformal structure structure on S and assume that the above construction produces the same $\tau \in \mathbb{H}$. We show that J' is isotopic to J relative to p. Since the space of conformal structures on S is contractible, there exists a path $\{J_t\}_{0 \leq t \langle 1}$ of such structures with $J_0 = J$ and

$J_1 = J'$. This yields a continuous family of isomorphisms $f_t : (S, J_t) \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau_t)$ as above with $\tau_0 = \tau_1 = \tau$. Let $\tilde{h}_t : \mathbb{C} \to \mathbb{C}$ be the orientation preserving $\mathbb{R}$-*linear* isomorphism defined by $\tilde{h}_t(1) = 1$ and $\tilde{h}_t(\tau_t) = \tau$. This induces an orientation preserving diffeomorphism $g_t : \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau_t) \to \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ that sends 0 to 0. Notice that $g_0 = g_1 = 1$. Consider now the diffeomorphism $f'_t := g_t f_t : S \to \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. If we replace J_t by the image of the conformal structure J'_t on $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ under $f_t'^{-1}$, then we get a new family of conformal structures connecting J with J' for which the corresponding curve τ'_t in $\mathbb{H}$ is constant equal to τ. This means that J'_t is the image of the given complex structure J under $h_t := f'_t f_0'^{-1}$. So $J' = J'_1$ is isotopic to $J = J'_0$ relative to p.

The last part of the argument also shows that τ only depends on the isotopy class of J. So we have a well-defined map $\mathcal{T}_{1,1} \to \mathbb{H}$ and this map is injective. It is also onto: we hit every $\tau \in \mathbb{H}$ by chosing an orientation preserving diffeomorphism $f : S \to \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ with $f(p) = 0$ and which induces on $\pi_1(S, p)$ the map which sends e_0 resp. e_1 to the loops defined by 1 resp. τ.

The isomorphism $\mathcal{T}_{1,1} \cong \mathbb{H}$ depends on the choice of oriented basis (e_0, e_1). Another choice yields in general another isomorphism. Since the oriented bases of $\pi_1(S, p) = \mathbb{Z}e_0 + \mathbb{Z}_1$ are transitively permuted by $SL(2, \mathbb{Z})$, we get an identification of $\Gamma_{1,1}$ with $SL(2, \mathbb{Z})$. We show that this induces the familiar action of $SL(2, \mathbb{Z})$ of $\mathbb{H}$. If (e'_0, e'_1) is another oriented basis of $\pi_1(S, p)$, then write (additive notation) $e'_1 = ae_1 + be_0$ and $e'_0 = ce_1 + de_0$ with $a, b, c, d, \in \mathbb{Z}$. We must have $\sigma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$. This takes the pair $(\tau, 1)$ of generators of $\mathbb{Z}\tau + \mathbb{Z}$ to the pair $(a\tau + b, c\tau + d)$. If we compose with scalar multiplication by $(c\tau + d)^{-1}$ it acquires the desired form $(\sigma(\tau), 1)$. $\square$

Hyperbolic cases. These are the remaining cases. It follows that all such surfaces are naturally endowed with a complete metric of curvature -1 (also called a *hyperbolic structure*). If C is isomorphic to a connected closed Riemann surface of genus g minus an n-element subset, then $(g, n) = (0, 0)$ is excluded for then $C \cong \mathbb{P}^1$ by Remark 2.3. Similarly are excluded $(g, n) = (0, 1), (0, 2), (1, 1)$, for according to Corollary 2.4 we are then in the flat case (C is isomorphic to $\mathbb{C}$, $\mathbb{C}^\times$ and a complex torus respectively). So if $g = 0$, then $n \geq 3$ and if $g = 0$, then $n > 0$. Now notice that the excluded cases are precisely the ones for which the Euler characteristic $\chi(C)$ of C, which is $2 - 2g - n$, is ≥ 0. We conclude that Riemann surfaces of the above type are hyperbolic if and only if $\chi(C) < 0$.

3. Geometry of the upper half plane

We begin with observing that certain subsets of $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ merit to be called a circle: this can either be an ordinary circle in $\mathbb{C} \subset \mathbb{P}^1$ having an equation $|z - a| = r$ with $r > 0$ or the union of a straight line in $\mathbb{C}$ and ∞. The justification is provided by

Exercise 2. Prove that $\mathrm{PSL}(2,\mathbb{C})$ transitively permutes the circles in $\mathbb{P}^1$. (A conceptual proof consists in showing that a circle can also be given as the zero set of a nondegenerate Hermitian form on $\mathbb{C}^2$ of indefinite signature $(1,1)$. Then deduce from the fact that such a Hermitian form can be brought into a standard diagonal form $|z_1|^2 - |z_0|^2$ and that $\mathrm{GL}(2,\mathbb{C})$ transitively permutes such forms in $\mathbb{P}^1$.)

We notice that the boundary $\partial\mathbb{H}$ of $\mathbb{H}$ in $\mathbb{P}^1$ is a circle, namely the real axis together with $\{\infty\}$, or more canonically, the real projective line $\mathrm{P}^1(\mathbb{R})$.

Lemma 3.1. *Let $p_0, p_1, p_\infty \in \partial\mathbb{H}$ be distinct. Then there is a unique $g \in \mathrm{PSL}(2,\mathbb{R})$ with $g(p_0) = 0$, $g(p_1) = 1$ and $g(p_\infty) = \infty$. The group of $g \in \mathrm{PSL}(2,\mathbb{R})$ which fix both 0 and ∞ consists of the homotheties $z \mapsto az$ with $a > 0$. The group of $g \in \mathrm{PSL}(2,\mathbb{R})$ which fix ∞ only is the affine group $z \mapsto az + b$ with $a > 0$ and $b \in \mathbb{R}$.*

The proof is left as an exercise.

We fix some terminology regarding geodesics. Let γ be a map from an interval I to a Riemannian space M. We say that γ is a *shortest geodesic parameterized by arc length* if for every pair $t, t' \in I$, $d(\gamma(t), \gamma(t')) = |t - t'|$. We recall that given a $p \in M$ and a unit vector $\nu \in T_pM$ there is a shortest geodesic parameterized by arc length $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$ and $\dot{\gamma}(o) = \nu$. This is unique in the sense that any other such geodesic coincides with this one where both are defined. The map $\gamma : I \to M$ is called a *geodesic parametrized by arc length* if it is locally a shortest geodesic parametrized by arc length and the image of such a map is called a geodesic. We use the adjective *complete* if $I = \mathbb{R}$. A complete geodesic parametrized by arc length which is periodic (so that it is compact) is said to be *closed*.

Lemma 3.2. *Let $p, q \in \partial\mathbb{H}$ be distinct. Then there is a unique circle in $\mathbb{P}^1$ through p and q which meets $\partial\mathbb{H}$ in these points perpendicularly. The intersection of this circle with $\mathbb{H}$ is a complete geodesic and all complete geodesics are thus obtained.*

Proof. The proof of the first assertion is left to you. To prove that the half circle defined by p and q is a geodesic, it is enough, in view of Lemma 3.1, to do the case $p = 0$ and $q = \infty$. We prove that $\gamma : \mathbb{R} \to \mathbb{H}$, $\gamma(t) := \sqrt{-1}e^t$, is a complete geodesic parametrized by arc length. We have $\dot{\gamma}(t) = \sqrt{-1}e^t = \gamma(t)$ and so this is unit vector for the Poincaré metric. If $\delta(t) = x(t) + iy(t) : [0,1] \to \mathbb{H}$ is a path from $\sqrt{-1}y_0$ to $\sqrt{-1}y_1$ with $y_0 < y_1$, then we have the following lower bound for its length:

$$L(\delta) = \int_0^1 \sqrt{\frac{\dot{x}(t)^2 + \dot{y}(t)^2}{y(t)^2}}\,dt \geq \int_0^1 \frac{|\dot{y}(t)|}{y(t)}\,dt \geq \int_0^1 \frac{\dot{y}(t)}{y(t)}\,dt =$$
$$\log(y(t))\big|_{t=0}^{t=1} = \log(y_0/y_1) = L(\gamma|[\log y_0, \log y_1]).$$

If we have equality, then we must have $x(t) \equiv 0$ and $\dot{y}(t) \geq 0$, which means that δ traverses in a monotonous manner a piece of γ. So γ is a complete geodesic.

The last assertion follows from the fact that every unit tangent vector of $\mathbb{H}$ is tangent to a circle that is perpendicular to $\partial\mathbb{H}$. $\square$

Exercise 3. Prove that the group $\mathrm{PSL}(2,\mathbb{R})$ acts transitively on the set tangent vectors of $\mathbb{H}$ of unit length.

Proposition 3.3. *Any $1 \neq g \in \mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}(2,\mathbb{R})$ has either a fixed point in $\mathbb{H}$ (and called* elliptic*) or two distinct fixed points in $\partial\mathbb{H}$ (and called* hyperbolic*) or a single fixed point in $\partial\mathbb{H}$ with multiplicity 2 (and called* horocyclic or parabolic*) and is then conjugate to one of the following transformations*

- **elliptic:** *g has in $\mathbb{H}$ the fixed point i and is the rotation around $i \in \mathbb{H}$ over an angle θ, $z \mapsto (\cos(\theta/2)z - \sin(\theta/2))/(\sin(\theta/2)z + \cos(\theta/2)) \in \mathrm{SO}(2)/\{\pm 1\}$,*
- **hyperbolic:** *g has on $\partial\mathbb{H}$ the fixed points 0 and ∞ and is given by the homothety $z \mapsto az$ with $1 \neq a > 0$,*
- **parabolic:** *g has in $\partial\mathbb{H}$ the unique fixed point ∞ and is given by the translation $z \mapsto z + 1$.*

Let M be a Riemannian manifold, $N \subset M$ a submanifold and $p \in M - N$. Then it is well-known (and not hard to see) that any geodesic $\gamma : [0, a] \to M$ which realizes the distance between p and N (so $\gamma(0) = p$, $\gamma(a) \in N$ and $a = d(p, M)$) is perpendicular to N ($\dot{\gamma}(a) \perp T_{\gamma(a)}N$).

Lemma 3.4. *Let $\ell \subset \mathbb{H}$ be a complete geodesic which joins $p, q \in \partial\mathbb{H}$. Then each orbit of the group of homotheties in $\mathrm{PSL}(2,\mathbb{C})$ which fix both p and q is the intersection of a circle passing through p and q with $\mathbb{H}$. Such an orbit consists of points having a fixed distance, $d \geq 0$ say, to ℓ and any geodesic which meets ℓ perpendicularly also meets each of these orbits perpendicularly. In fact, if $d > 0$, then it is the intersection of a connected component of $\mathbb{H} - \ell$ with the locus of points in $\mathbb{H}$ having distance d to ℓ.*

Proof. It is clear that if $z \in \mathbb{H} - \ell$, and $g \in \mathrm{PSL}(2,\mathbb{R})$ fixes both p and q, then $d(gz, \ell) = d(z, \ell)$. So the locus in question is invariant under the group of homotheties. For the rest of the assertion we may assume that $p = 0$ and $q = \infty$. The orbit of $\tau \in \mathbb{H}$ is then a ray $\{r\tau\}_{r>0}$ and the distance from τ to ℓ is realized by the circle arc from τ to ℓ that has its center in 0. The assertion easily follows from this. $\square$

For what follows we need the following

Proposition 3.5. *Let $D \subset \mathbb{H}$ be a right-angled hexagon and enumerate its vertices in their natural counterclockwise cyclic order: $\{p_i\}_{i\in\mathbb{Z}/6}$ (this means that D is a compact domain in $\mathbb{H}$ bounded by the 6 geodesic arcs $\{p_ip_{i+1}\}_{i\in\mathbb{Z}/6}$ with $p_{i-1}p_i$ meeting p_ip_{i+1} at a right angle). Then D is determined up to isometry by the lengths of the geodesic arcs p_0p_1, p_2p_3 and p_4p_5.*

Proof. Let us write $d_{i,i+1}$ for the length of p_ip_{i+1}. We show how D can be reconstructed up to isometry from d_{01}, d_{23}, d_{45}. We begin with choosing $p_2, p_3 \in \mathbb{H}$ such that $d(p_2, p_3) = d_{23}$. If p_2', p_3' is another such pair, then there is a unique $\sigma \in PSL(2, \mathbb{R})$ with $\sigma(p_2) = p_2'$ and $\sigma(p_3) = p_3'$. So it is enough to show that from the pair (p_2, p_3) and the positive numbers d_{01} and d_{45} we can uniquely reconstruct the other p_i's.

To see this, we first draw the complete geodesic ℓ_{34} through p_3 which makes a right angle with p_2p_3 and denote by ℓ_{34}^+ the part of ℓ_{34} that begins in p_3 and 'goes to the left'. Denote by ℓ_5' the circular arc of points that have distance d_{45} to ℓ_{34}, lie in the same component of $\mathbb{H} - \ell_{34}$ as p_2p_3 and lie in the same connected component of $\mathbb{H} - \ell_{23}$ as ℓ_{34}^+. Do similarly at the other side and find ℓ_{12}, ℓ_{12}^+ and ℓ_0'. It now suffices to show that there is a unique complete geodesic ℓ_{50} which is tangent to both ℓ_5' and ℓ_0' and is such that both ℓ_5' and ℓ_0' lie in the same connected component of $\mathbb{H} - \ell_{50}$ at p_2p_3. For if ℓ_{50} meets ℓ_5' in p_5 and ℓ_0' in p_0, then we take for p_4 the point of ℓ_{34}^+ that is closest to p_5, and for p_1 the point on ℓ_{21}^+ that is closest to p_0. This then reproduces D.

The existence of the asserted ℓ_{50} is based on elementary planar geometry: consider the locus of points that are the center of a circle which is tangent to ℓ_5' and ℓ_0' and contains both these arcs in its interior. This is part of hyperbola which meets $\partial\mathbb{H}$ in a unique point. So there is a circle centered on $\partial\mathbb{H}$ that is tangent to both ℓ_5' and ℓ_0'. Its intersection with $\mathbb{H}$ is the desired geodesic. □

Remark 3.6. The values of d_{01}, d_{23}, d_{45} can be arbitrary positive numbers. But this proposition also holds (with essentially the same proof) if some of them are zero, although its statement then needs to be properly interpreted. For instance $d_{01} = 0$ must be understood as: $p_0 = p_1$ and this vertex is *improper*, meaning that it lies on $\partial\mathbb{H}$. The extremal case we get when $d_{01} = d_{23} = d_{45} = 0$ is what is called an *ideal triangle*.

Remark 3.7. A well-known formula states that the area of a hyperbolic geodesic triangle (measured with the hyperbolic measure $y^{-2}dxdy$), is given by its angle excess: π minus the sum of its interior angles, regardless whether vertices are proper or not. This implies that the area of any right-angled hexagon is π (which is also the value that we get for an ideal triangle).

4. Hyperbolic surfaces

Closed geodesics on a hyperbolic surface. Let C be a closed connected Riemann surface of genus g and of hyperbolic type: we have obtained C as a quotient of $\mathbb{H}$ by a subgroup $\Pi \subset PSL(2, \mathbb{C})$ which is discrete and acts properly and freely on $\mathbb{H}$ (in other words, $g \geq 2$).

Let $\gamma \subset C$ be a complete geodesic. This lifts to a oriented geodesic $\tilde{\gamma}$ in $\mathbb{H}$, joining $p \in \partial\mathbb{H}$ with $q \in \partial\mathbb{H}$. If γ is closed, i.e., is an embedded circle, then there must exist an element of $g \in \Pi$ which preserves $\tilde{\gamma}$ and which is such that γ is obtained as the quotient of $\tilde{\gamma}$ by the action of g on it. In particular, g is a hyperbolic transformation and hence determines $\tilde{\gamma}$. If $\tilde{p} \in \tilde{\gamma}$, then a parametrization of the arc in $\tilde{\gamma}$ from $\tilde{p}$ to $g\tilde{p}$) projects to a parametrization of the closed geodesic γ. So g is then the covering transformation associated to this loop. Notice that after composing with an element of $\mathrm{PSL}(2,\mathbb{R})$ we may can arrange that $\tilde{\gamma}$ joins 0 with ∞ so that $\tilde{\gamma}(t) = \sqrt{-1}e^t$ is a parametrization by arc length and g is given by multiplication by a scalar $\lambda > 1$. This scalar has the property that if $\tilde{p} \in \tilde{\gamma}$, then the length ℓ of γ must be equal to $d(\tilde{\gamma}(t), g\tilde{\gamma}(t)) = \int_1^\lambda y^{-1}dy$, and so $\lambda = e^\ell$. For later purposes it is useful to have a concrete parametrization of a neighborhood of γ.

Let for $\varepsilon > 0$, $U_\varepsilon(\gamma)$ denote the set of points in C of having distance $< \varepsilon$ to γ. For sufficiently small $\varepsilon > 0$, the preimage $\tilde{U}_\varepsilon(\gamma)$ of $U_\varepsilon(\gamma)$ in $\mathbb{H}$ is the locus of points having distance $< \delta$ to $\tilde{\gamma}$ and hence will be a sector of the form $|\arg(z) - \pi/2| < \delta$ for some $\delta \in (0, \pi/2)$. We then recover $U_\varepsilon(\gamma)$ by taking the g-orbit space of this sector. We put $r := \ell/2\pi$ identify this orbit space using the holomorphic map

$$f : z \in \tilde{U}_\varepsilon(\gamma) \mapsto e^{\sqrt{-1}/r.(\log z)} \in \mathbb{C}.$$

Here we take the logarithm of z with argument in $(0, \pi)$. So if we write $z = \exp(\rho + \sqrt{-1}\theta)$ with $\rho \in \mathbb{R}$ and $|\theta - \pi/2| < \delta$, then $f(z) = \exp(-\theta + \sqrt{-1}\rho)/r)$. We see from this expression that f induces an isomorphism of $U_\varepsilon(\gamma)$ onto the annulus $\exp((-\pi/2 - \delta)/r) < |w| < \exp((-\pi/2 + \delta)/r)$ with γ mapping onto the circle $|w| = \exp(-\pi/2r)$. It sends geodesic arcs perpendicular to γ to rays. It becomes an isometry if we endow this annulus with the metric

$$\frac{r^2|dw|^2}{|w|^2 \sin^2(r \log |w|)}.$$

Lemma 4.1. *There is an isometry of a neighborhood of the closed geodesic γ onto an annulus $\exp((-\pi/2 - \delta)/r) < |w| < \exp((-\pi/2 + \delta)/r)$ with γ mapping to to the circle $|w| = \exp(-\pi/2r)$ (where $r = \ell_\gamma/2\pi$), with the annulus endowed with the metric $\frac{r^2}{|w|^2 \sin^2(r \log |w|)}|dw|^2$. This isometry sends every geodesic perpendicular to γ to an interval on a ray emanating from 0.*

Geodesic shear. In the situation of this lemma we have the notion of *shear* along the closed geodesic γ over an angle $u = e^{\sqrt{-1}\theta} \in S^1$, with $\theta \in \mathbb{R}$. This amounts to cutting C open along γ (in such a manner that we get a surface with two boundary components, each of which lies over γ) and then glueing back the two boundary components after we have displaced the opposite component over a distance $\theta\ell_\gamma$. Let us make this more precise with the help of the model furnished by Lemma 4.1. Denote by A_- resp. A_+ the closed subset of U_γ that maps to the part of the annulus defined by $|w| \leq \exp(-\pi/2r)$ resp. $|w| \geq \exp(-\pi/2r)$. Then we rotate A_+ over an angle u to produce a new Riemann surface $C_\gamma(u)$ with a *new* underlying topological space: an open subset of $C_\gamma(u)$ is given as a pair open open subsets $U_\pm \subset A_\pm$ such that $U_+ \cap \gamma$ is obtained by rotating $U_- \cap \gamma$ over v (= translating along γ over rv). Similarly, a complex valued function on this open set will be holomorphic for $C_\gamma(u)$ it is given as a pair of homolomorphic functions

$\kappa_\pm : U_\pm \to \mathbb{C}$ so that $\kappa_+|U_+ \cap \gamma$ precomposed with rotation over ν yields $\kappa_-|U_- \cap \gamma$. Notice that from whatever side we approach γ on C, if we want to cross γ, but want to stay in $C_\gamma(u)$, then the continuation on the other side must take place over a rotation ν further down on γ 'on the left'. This formulation uses the orientation of C, not a particular one of γ. So $C_\gamma(u)$ only depends on γ as a closed geodesic and not on an orientation of γ.

The surface underlying $C_\gamma(u)$ is diffeomorphic to C, but we have not specified a diffeomorphism. We cannot do this in way that it depends continuously on u, but we can make this depend continuously on ν: if $\phi : [\pi/2-\delta, \pi/2] \to [0, 2\pi]$ is constant 0 near $\pi/2-\delta$ and constant 2π near $\pi/2$, then define $h_\nu : C \to C_\gamma(u)$ by letting it be the identity outside A_- and on A_- be given by

$$h_\nu f(e^{\rho+\sqrt{-1}\theta}) = f(e^{\rho+\sqrt{-1}(\theta-\nu\phi(\rho))}), \quad |w| \le e^{-\pi/2)/r}.$$

This is indeed a diffeomorphism and is even complex-analytic outside A_-. Other choices of parametrization may lead to another diffeomorphism, but that diffeomorphism will always lie in the same connected component of $\mathrm{Diff}(C, C_\gamma(u))$, in other words, will differ from h_θ by an isotopy. Notice that $h_{2\pi}$ is a diffeomorphism of C onto $C(1) = C$ whose isotopy class is the Dehn twist D_γ, hence need not be (and as we shall see, never is) the identity.

So if we pull-back the complex structure on $C_\gamma(u)$ to C, then we get a new complex structure on the underlying surface S that is well defined up to isotopy. The following proposition essentially sums up the preceding discussion, but states the conclusion somewhat differently:

Proposition-definition 4.2 (Shearing action). *Let S be a closed surface of genus $g > 1$ and $\alpha \subset S$ an embedded circle given up to isotopy that is not homotopically trivial. If J is a conformal structure on S (so that we have defined a hyperbolic structure on S, a geodesic representative γ of α and an element $[J] \in \mathcal{T}(S)$), then the pull-back of the conformal structure of $C_\gamma(u)$ along h_θ yields a new element of $\mathcal{T}(S)$ that we will denote by $\theta_\alpha[J]$. This defines an action of the additive group $(\mathbb{R}, +)$ on $\mathcal{T}(S)$ with the property that $2\pi \in \mathbb{R}$ induces the action of the Dehn twist $D_\alpha \in \Gamma(S)$.*

Geodesic representation. Let now, conversely, be given an embedded circle $\alpha \subset C$ and suppose α is not homotopically trivial in C. Choose $p \in \alpha$ and $\tilde{p} \in \mathbb{H}$ over p. Then α is covered by a copy $\tilde{\alpha}$ of $\mathbb{R}$ which passes through $\tilde{p}$. With these choices the loop α defines a covering transformation $g \in \Pi$ which generates the group of covering transformations of the universal covering $\tilde{\alpha} \to \alpha$. If we let α vary in its isotopy class, say by a continuous one parameter family $\{\alpha_t\}_{0\le t\le 1}$, then we can let p and $\tilde{p}$ follow that family and as the corresponding family g_t can only move in the discrete set Π, it will be constant. The isotopy class will contain an embedded circle γ of minimal length. This will be a geodesic which admits a lift $\tilde{\gamma}$ invariant under g so that we are in the situation considered above. In particular, $g \in \mathrm{PSL}(2, \mathbb{R})$ is a hyperbolic transformation and hence determines $\tilde{\gamma}$. It follows that γ is the unique geodesic representative of α.

Disjunction of geodesics. Let $\alpha' \subset C$ be an embedded circle that is not isotopic to α. By the above discussion its associated geodesic γ' is then distinct from γ. We claim that if α' is disjoint with α, then γ' is disjoint with γ. For if that were not the case, you can argue that (using the isotopies that makes them disjoint), then there must exist geodesic intervals γ_1 on γ and γ_1' on γ' with identical end points that are path homotopic. But then these geodesic intervals can be lifted to $\mathbb{H}$ such that they have the same end points. We then end up with two distinct geodesic intervals in $\mathbb{H}$ with the same end points and this is impossible.

Pair of pants. A *pair of pants* is a compact surface with boundary of genus zero whose boundary has three connected components. A *hyperbolic pair of pants* is a pair of pants endowed with a hyperbolic structure for which the three boundary components are geodesics.

Notice that a pair of pants has the homotopy type of a one point union of two circles (a figure eight) and so has Euler characteristic -1.

Lemma 4.3. *A hyperbolic pair of pants M is determined up to isometry by the circumference of its three boundary components: if we are given another hyperbolic pairs of pants M' and a bijection $\pi_0(\partial M) \cong \pi_0(\partial M')$ which preserves circumference, then there is a unique orientation preserving isometry $M \cong M'$ inducing this bijection. In particular (take $M' = -M$), M has a unique involution which reverses orientation; this involution has on each boundary component exactly two fixed points and its orbit space is a right-angled hexagon.*

Proof. Let M be hyperbolic pair of pants and let $\{\partial_i M\}_{i\in\mathbb{Z}/3}$ be a cyclic numbering of the boundary components. The shortest path from $\partial_i M$ to $\partial_{i+1}M$ will be realized by a geodesic arc. It goes from $p_i \in \partial_i M$ say, to $q_{i+1} \in \partial_{i+1}M$ say, and meets these boundary components there perpendicularly. So if we cut M open along these three arcs we get two right-angled hexagons. These two right-angled hexagons have three of their alternating sides in common and so by Proposition 3.5 they must be congruent, or rather (if we take their orientation into account), must be mirror images of each other. This implies that the two arcs on $\partial_i M$ joining its two points p_i and q_i must have the same length. In particular, this length is half the circumference of $\partial_i M$. But then another application of Proposition 3.5 shows that these circumferences completely determine the two hexagons up to congruence and hence determine M.

The last assertion follows essentially from the above construction. $\square$

Pants decompositions. Let S be a closed connected surface of genus $g \geq 2$. A *pants decomposition* of S is the isotopy class of a closed one dimensional submanifold $A \subset S$ (so every connected component of A is then an embedded circle) with the property that each connected component of $S - A$ is diffeomorphic to a sphere with three punctures.

Notice that the mapping class group $\Gamma(S)$ acts on the collection of pants decompositions of S. There are in general several $\Gamma(S)$-orbits of pants decompositions. This is best understood in terms of a graph associated to the pair (S, A) that is defined as follows.

The Euler characteristic of S is $2-2g$, that of a pair of pants is -1 and that of a circle is 0, and so the additive properties of the Euler characteristic imply that if $A \subset S$ defines a pants decomposition, then $S - A$ must have $2g-2$ connected components. If α is a connected component of A, then for any orientation of α, there is a unique connected component of $S - A$ such that α with this orientation appears as the oriented boundary of that component (with its orientation inherited from S). In this way we hit every connected component of $S - A$ three times and so the number of connected components of A must be $\frac{3}{2}(2g-2) = 3g-3$. The topological type of the pair (S, A) is conveniently encoded by a connected graph $G(S, A)$ with vertex set $X_0(S, A) := \pi_0(S - A)$ and edge set $X_1(S, A) := \pi_0(A)$, with the obvious incidence relation.

Exercise 4. Let $X(S, A)$ be the set of boundary components of the connected components of $S - A$ (so we have $6g - 6$ of these) and let $p_1 : X(S, A) \to X_1(S, A)$ and $p_0 : X(S, A) \to X_0(S, A)$ be the evident forgetful maps whose fibers have cardinality 2 resp. 3. Reconstruct $G(S, A)$ as a quotient of the product $[-1, 1] \times X(S, A)$ (where $X(S, A)$ has the discrete topology) solely in terms of the maps p_0 and p_1.

So we have $2g-2$ vertices, all of degree 3 (one then says that the graph is *trivalent*) and has $3g-3$ edges (in particular, its Euler characteristic of the graph is half that of S, namely $1-g$). Notice that if a connected component of A happens to have the same component of $S - A$ at both sides then it defines a loop.

Clearly, two pants decompositions that lie in the same $\Gamma(S)$-orbit define homeomorphic graphs. In fact, it is not difficult to see that the graph is a complete invariant of the $\Gamma(S)$-orbit and that every connected trivalent graph of Euler characteristic $1 - g$ thus occurs.

Exercise 5. Draw the possible graphs for genus 2. Then try genus 3.

Exercise 6. Let G be the graph attached to a pants decomposition (S, A). By altering just one member of A, we may get a different graph G' whose vertex set may be identified with that of G. Describe the possible graphs G' in terms of G.

Fenchel-Nielsen coordinates. Let us fix a pants decomposition of S, defined by $A \subset S$. A conformal structure J on S is subordinate to a natural hyperbolic structure on S. With respect to that structure every connected component α of A is uniquely represented by a closed geodesic. According to the discussion at the beginning of this section these geodesics are disjoint. Denote by $\ell_\alpha(J)$ the length of the geodesic representing α. Since the connected components of A label the vertices of $G(S, A)$, we may view this also as defining a metric on $G(S,A)$: the edge of $G(S,A)$ labeled by α is then given the same length as its associated geodesic. Since this only depends on the isotopy class of the conformal structure we have thus defined a function

$$\ell_\alpha : \mathcal{T}(S) \to \mathbb{R}_{>0}.$$

Notice that we have attached to J a decomposition into hyperbolic pants and according to Proposition 3.5 the metric $\ell(J) : \alpha \in \pi_0(A) \mapsto \ell_\alpha(J) \in \mathbb{R}_{>0}$ on $G(S,A)$ completely determines each of these hyperbolic pairs of pants up to isometry. However, this does not enable us yet to reconstruct (S,J), for this gives us only the building blocks (namely, the hyperbolic pairs of pants) and although the graph tells us which pairs of boundary components should be welded onto each other, it does not determine (S,J) yet, for any such welding requires an angular parameter. To be precise, two conformal structures J, J' on S for which $\ell(J) = \ell(J')$ will differ by geodesic shears along the geodesic representatives of the connected components of A. This shows that $\mathcal{T}(S)$ is as a topological space locally like $\mathbb{R}^{6g-6}$.

Theorem 4.4 (Fenchel-Nielsen parametrization)**.** *The map ℓ is a principal $\mathbb{R}^{\pi_0(A)}$-fibration: the action of the vector group $\mathbb{R}^{\pi_0(A)}$ on $\mathcal{T}(S)$ by geodesic shears is free, each orbit is a fiber of $\ell : \mathcal{T}(S) \to \mathbb{R}_{>0}^{\pi_0(A)}$ and ℓ admits a section so that we have a homeomorphism of $(\mathbb{R}_{>0} \times \mathbb{R})^{\pi_0(A)}$ onto $\mathcal{T}(S)$. In particular, $\mathcal{T}(S)$ is contractible and the Dehn twists $(D_\alpha)_{\alpha\in\pi_0(A)}$ commute with each other and are independent so that they generate in $\Gamma(S)$ a free abelian subgroup D_A of rank $|\pi_0(A)|$.*

Proof. Let us write X_1 for $X_1(S,A) = \pi_0(A)$. We begin with constructing a section $j : \mathbb{R}_{>0}^{X_1} \to \mathcal{T}(S)$ of ℓ. Let $l \in \mathbb{R}_{>0}^{X_1}$. This gives for us for every connected component P of $S - A$ a hyperbolic pair of pants $(\bar{P}, J_{\bar{P},l})$. Such a hyperbolic pair of pants comes with a natural involution $\iota_{\bar{P},l}$. Choose on every boundary component a fixed point of this involution (that is for every boundary component a choice of one out of two) and then assemble these hyperbolic pair of pants into a hyperbolic surface C_l as prescribed by (S,A), by welding boundary component on boundary component in such a manner that the marked fixed points get identified as well. The resulting hyperbolic surface C_l comes with an evident decomposition Γ_l into hyperbolic pair of pants. It has the the additional remarkable property that the involutions in the parts come from a global involution (it is as if C_l were defined over $\mathbb{R}$), but this interesting feature is not so relevant here: we just wanted to give a construction that is sufficiently rigid so that we can say that we have produced a family of hyperbolic surfaces with hyperbolic pants decompositions that depend continuously on l. There is no obvious diffeomorphism between two incongruent right angled hexagons, so neither between the various C_l's, but our construction is such that they appear as the fibers of a locally trivial fibration $\mathcal{C} \to \mathbb{R}_{>0}^{X_1}$ of manifolds (with local trivializations also respecting the pants decompositions). This local triviality implies that any two fibers can be identified by means of a diffeomorphism which preserves both orientation and decomposition. That diffeomorphism is not unique, but its isotopy class is. So once we have chosen for some $l_o \in \mathbb{R}^{X_1}_{>0}$ an orientation preserving diffeomorphism $S \to C_{l_o}$ such that the pull-back of Γ_{l_o} is in A, then we have a well-defined isotopy class of such diffeomorphisms $S \to C_l$ for every $l \in \mathbb{R}^{X_1}_{>0}$. If we denote the pull-back of the conformal structure on C_l to S along this isotopy class by $j(l) \in \mathcal{T}(S)$, then we have obtained a continuous section $j : \mathbb{R}^{X_1}_{>0} \to \mathcal{T}(S)$ of ℓ.

The shearing actions defined by the given closed geodesics now produce an $\mathbb{R}^{X_1}$-equivariant map

$$J : \mathbb{R}^{X_1} \times \mathbb{R}^{X_1}_{>0} \to \mathcal{T}(S), \quad (t, l) \mapsto t(j(l)),$$

where on the right, $t = (t_\alpha)_\alpha$ acts by shearing the geodesic associated to α over t_α. It is clear that its composite with $\ell : \mathcal{T}(S) \to \mathbb{R}^{X_1}_{>0}$ is the obvious projection $(t, l) \in \mathbb{R}^{X_1} \times \mathbb{R}^{X_1}_{>0} \mapsto l \in \mathbb{R}^{X_1}_{>0}$. Notice that if $J(t, l) = J(t', l')$, then $l = l'$ (clear), but we also have $t_\alpha \cong t'_\alpha \pmod{\pi\mathbb{Z}}$ for every α, because the two fixed point pairs of the involution on the associated boundary components must be in the same relative position. It follows from the preceding that J is a homeomorphism onto its image, when restricted to $I \times \mathbb{R}^{X_1}_{>0}$, where $I \subset \mathbb{R}^{X_1}$ is a product of open intervals of length $\le \pi$. In fact, J is a covering map, for if we let $I \subset \mathbb{R}^{X_1}$ is a product of open intervals of length $\le \pi/2$, then it follows that $J^{-1}J(I \times \mathbb{R}^{X_1}_{>0})$ is a collection of copies of $I \times \mathbb{R}^{X_1}_{>0}$, each of which is obtained by a translating $I \times \mathbb{R}^{X_1}_{>0}$ over a vector, which if not lying in $(2\pi\mathbb{Z})^{X_1}$, then certainly belongs to $(\pi\mathbb{Z})^{X_1}$, and hence are pairwise disjoint.

A covering map going from a connected space to a simply connected space is always a homeomorphism. So the first assertion of the theorem follows if we show that $\mathcal{T}(S)$ is simply connected, i.e., that loop $t \in [0,1] \to u_t \in \mathcal{T}(S)$ is null-homotopic. To see this, lift u to a path $t \in [0,1] \to \tilde{u}_t \in \mathrm{Conf}(S)$. We can arrange $\tilde{u}$ to be a loop: as $\tilde{u}_0$ and $\tilde{u}_1$ are isotopic, there exists a path $t \in [0,1] \mapsto h_t \in \mathrm{Diff}(S)$ with $h_0 = 1$ and such that $h_{1*}\tilde{u}_1 = \tilde{u}_0$, and then $t \mapsto u'_t := h_{t*}\tilde{u}_t$ is a loop that lifts u. Assume now that $\tilde{u}$ is a loop. Since $\mathrm{Conf}(S)$ is contractible, $\tilde{u}$ will be then be null-homotopic and a contraction of $\tilde{u}$ gives one for u.

The last clause of the theorem is now seen as follows: it is clear that under the shearing representation, the subgroup $(2\pi\mathbb{Z})^{X_1} \subset \mathbb{R}^{X_1}$ maps onto D_A. Since the former acts faithfully on $\mathbb{R}^{X_1} \times \mathbb{R}^{X_1}_{>0}$ by translations, the map $(2\pi\mathbb{Z})^{X_1} \to D_A$ is an isomorphism. $\square$

Hyperbolic surfaces with cusps. We here no longer assume that $g \ge 2$, but we suppose given a subset $P \subset S$ of n elements, such that the Euler characteristic of S° is negative. Equivalently: $2g - 2 + n > 0$. In order to deal in similar fashion with the Teichmüller space $\mathcal{T}(S, P)$ we make some observations about quotients of $\mathbb{H}$. We know that an element $g \in \mathrm{PSL}(2, \mathbb{R})$ which has no fixed points in $\mathbb{H}$ is either hyperbolic or parabolic. We show that this dichotomy is still manifest if we pass to the orbit space $\mathbb{H}_g$ of $\mathbb{H}$ by the subgroup of $\mathrm{PSL}(2,\mathbb{R})$ generated by g. First observe that the isomorphism type of $\mathbb{H}_g$ only depends on the conjugacy class of g in $\mathrm{PSL}(2,\mathbb{R})$: if $g' = hgh^{-1}$, then $h : \mathbb{H} \to \mathbb{H}$ induces an isomorphism $\mathbb{H}_g \cong \mathbb{H}_{g'}$. So if g is hyperbolic, then we may assume it is scalar multiplication by a positive real number $\neq 1$, and as we have seen in Lemma 4.1, $\mathbb{H}_g$ is then biholomorphic to an annulus $r_1 < |w| < r_2$ with $0 < r_1 < r_2$. When g is parabolic, then we may assume it is given by $z \to z+1$ and $w := e^{2\pi\sqrt{-1}z}$ then identifies $\mathbb{H}_g$ with the punctured unit disk $0 < |w| < 1$.

An annulus is not isomorphic to a punctured disk: a bounded holomorphic function on a punctured disk extends, by the Riemann extension theorem, holomorphically to the unit disk and so has a well-defined limit in the puncture. But for an annulus as above, the coordinate w is bounded and holomorphic and has the property that if we approach one of the boundary circles, it does *not* go to a constant value. We also notice a metric difference: for g hyperbolic, the evident isotopy class of embedded circles on $\mathbb{H}_g$ has a unique geodesic representative which minimizes length. But if $g(z) = z+1$, then $t \in [0,1] \mapsto t + \sqrt{-1}y$ defines a loop on $\mathbb{H}_g$ around the missing point of length $1/y$, and so this length can be made arbitrarily small in its isotopy class.

Suppose we are given a conformal structure J on S. Since $2g-2+n > 0$, its restriction to S° underlies a complete hyperbolic metric, so that we can identify S° with $\Pi\backslash\mathbb{H}$ for a subgroup $\Pi \subset \mathrm{PSL}(2,\mathbb{R})$. Let $p \in P$. Choose a closed disk $D \subset S$ having p in its interior such that D meets no other point of P. So $D - \{p\}$ is closed in S°.

Let $q \in \partial D$ and let α the loop based at q which traverses in ∂D. We identify Π with $\pi_1(S^\circ, q)$, so that α determines an element $g \in \Pi$. We begin with observing that α cannot be homotopically trivial. For then the function $w := \frac{z-\sqrt{-1}}{z+\sqrt{-1}}$ (which maps on $\mathbb{H}$ isomorphically onto the unit disk $\mathbb{D}$) can be regarded as a holomorphic embedding of $D - \{p\}$ in $\mathbb{D}$. It extends holomorphically to $w : D \to \mathbb{C}$ (by the Riemann extension theorem) and the maximum principle forces $w(p) \in \mathbb{D}$. But since $D - \{p\}$ is closed in S°, its image in $\mathbb{D}$ must be closed as well and we get a contradiction.

In a similar fashion we can exclude that g is hyperbolic. For then $D - \{p\}$ would embed in an annulus $r_1 < |w| < r_2$. Such an embedding extends holomorphically to D and the maximum principle then forces p taking a value in the interior of D. But then we contradict the fact the image of $D - \{p\}$ in the annulus must be closed.

We conclude that g is parabolic so that $\mathbb{H}_g$ is isomorphic to the punctured disk $\mathbb{D} - \{0\}$. We have a holomorphic embedding $D - \{p\} \to \mathbb{D} - \{0\}$ and the Riemann extension theorem plus the maximum principle imply that this must extend to a holomorphic function $D \to \mathbb{D}$ which takes p to 0. This function is injective and so cannot have p as a multiple zero. This implies that it is in fact a holomorphic embedding on D. We conclude:

Proposition-definition 4.5 (Cusps on a hyperbolic surface)**.** *The hyperbolic structure on* (S°, J) *makes of any* $p \in P$ *a* cusp*, by which we mean that a neighborhood of* p *is isomorphic to a neighborhood of* 0 *in the unit disk* $\mathbb{D}$ *(with* p *mapping to* 0*) in such a manner that it becomes an isometry where this makes sense. Here* $\mathbb{D} - \{0\}$ *has been endowed with the hyperbolic metric it receives as the orbit space of* $\mathbb{H}$ *with respect to the group of integral translations.*

Pants decomposition in the presence of cusps. We have a notion of a pants decomposition of S° as before: it is given by closed compact submanifold of S which does not meet P and is given up to isotopy relative to P, and is such that each connected component of $S - A - P$ is diffeomorphic to the interior of a pair of pants (a thrice punctured sphere). A connected component of the boundary of $S - A - P$ is now a connected component of A or a singleton in P. We can still encode this in terms of a graph $G(S, P; A)$: the vertex set is $\pi_0(S - A - P)$ as before, but edges come in two types: those indexed by $\pi_0(A)$ (and called interior edges) and those indexed by P (the exterior edges). Edges of the last kind appear as 'loose ends': they are connected with just one vertex.

Exercise 7. Prove that the graph $G(S, P; A)$ is trivalent, has $2g-2+n$ vertices and $3g - 3 + n$ interior edges.

If we are given a conformal structure J on S, then for the hyperbolic structure on S°, connected components of of $S - A - P$ are hyperbolic pairs of pants, except that some pants have become cusps. These are obtained as glueing a degenerate hyperbolic hexagon (an $(6 - n)$-gon with n improper points, where $n = 1, 2, 3$) onto its mirror image. By allowing this possibility, the arguments in the proof of Theorem 4.4 subsist and we find that this theorem continues to hold for $\mathcal{T}(S, P)$ as stated:

Theorem 4.6 (Fenchel-Nielsen parametrization in the punctured case)**.** *The action of* $\mathbb{R}^{\pi_0(A)}$ *on* $\mathcal{T}(S, P)$ *by geodesic shears makes* $\ell : \mathcal{T}(S, P) \to \mathbb{R}_{>0}^{\pi_0(A)}$ *a principal* $\mathbb{R}^{\pi_0(A)}$*-fibration. This bundle admits a section so that we have a homeomorphism of* $(\mathbb{R}_{>0} \times \mathbb{R})^{\pi_0(A)}$ *onto* $\mathcal{T}(S)$*. The Dehn twists around the connected components of* A *generate a free abelian subgroup of* $\Gamma(S, P)$ *of rank* $\pi_0(A)$*.*

Remarks 4.7. An embedded circle on S° which does not bound a disk that contains at most one point of P can always be made part of a pants decomposition. So this theorem implies the converse of what you proved in Exercise 1: a Dehn twist defined by such an embedded circle is never the identity.

The Fenchel-Nielsen parametrization makes $\mathcal{T}(S, P)$ homeomorphic to $(\mathbb{R}_{>0} \times \mathbb{R})^{3g-3+n}$. This homeomorphism gives it even the structure of a smooth manifold, although this a priori depends on the pants decomposition. But one can show that given this decomposition, then for any other isotopy class of embedded circles, the associated length function and shearing action is differentiable in terms of our $6g-6+2n$ Fenchel-Nielsen parameters. This implies that the manifold structure is independent of the pants decomposition. We shall see that $\mathcal{T}(S,P)$ has even a natural structure of a complex manifold of dimension $3g-3+n$.

5. Quadratic differentials

Local normal form. Let C be a Riemann surface. A *holomorphic quadratic differential* on C is a complex valued function on the total space of the tangent bundle TC which is quadratic, i.e., a function $\eta : TS \to \mathbb{C}$ such that for $v \in TC$ and $\lambda \in \mathbb{C}$ we have $\eta(\lambda v) = \lambda^2\eta(v)$ and such that in terms of any local coordinate z, η takes the form $f dz^2$, with f holomorphic in z. (This is the same thing as a holomorphic section of the sheaf $\Omega_C^{\otimes 2} := \Omega_C \otimes_{\mathcal{O}_C} \Omega_C$.) We have a similar notion of a *meromorphic quadratic differential* (f is then meromorphic).

Lemma 5.1. *Let η be a meromorphic quadratic differential at $p \in C$ and let $k \in \mathbb{Z}$ be its order at p.*

If $k \geq -1$, then for some local coordinate z at p, $\eta = (k/2+1)^2 z^k dz^2$ (which we can write, at least formally, as $(d(z^{k/2+1}))^2$) and this coordinate is unique up to multiplication by a $(k+2)$th root of unity.

If $k = -2$, then there is a nonzero constant $c \in \mathbb{C}$ and a coordinate z at p such that $\eta = c(d\log z)^2 = cz^{-2}(dz)^2$ and c is unique (and called the squared residue *of η at p, denoted* $\mathrm{Resq}_p\,\eta$*) and z is unique up to a scalar.*

Proof. We may as well assume that C is open in $\mathbb{C}$ and $p = 0$. So we have $\eta = f(z)dz^2$ with f meromorphic at $0 \in \mathbb{C}$ and of order k at 0. Suppose first $k \geq -1$. Then $\sqrt{f}$ is $z^{k/2}$ times a holomorphic function which is nonzero in 0. Since $k/2 > -1$, we can integrate $\sqrt{f}$ in the sense that there exists a g such that $g' = \sqrt{f}$ with g of the form $z^{(k+2)/2}u(z)$, with u holomorphic and $u(0) \neq 0$. So if $\tilde{z} = zu(z)^{2/(k+2)}$, then $\eta = (d(\tilde{z}^{k/2+1}))^2$.

If z and w are local coordinates such that $z^k dz^2 = w^k dw^2$, then it follows that $(w/z)^{k/2}w' = 1$. If we substitute for w/z a power series in z, then we see that its constant term c is a $(k+2)$th root of unity. If we then write $w/z = c(1 + a_r z^r + a_{r+1}z^{r+1} + \cdots)$, we see that $a_r = 0$. Hence $w/z = c$.

Now assume $k = -2$. Then we may write $f(z) = z^{-2}u(z)$, where u is holomorphic with $u(0) \neq 0$. Let $c := u(0)$. Any other coordinate $\tilde{z}$ at 0 can be written as ze^{ϕ}, with ϕ holomorphic in 0. Then

$$c\Big(\frac{d\tilde{z}}{\tilde{z}}\Big)^2 = c(1+z\phi')^2\Big(\frac{dz}{z}\Big)^2.$$

So we must solve the equation $c(1+z\phi')^2 = u$ for ϕ. Now u/c takes the value 1 in 0 and so it has a square root v taking the value 1 in 0. If we write this square root as $1 + zg$ with g holomorphic so that $u = c(1+zg)^2$, then we observe that the equation is solved by taking for ϕ an integral of g. You may check the uniqueness assertion yourself. □

Exercise 8. The preceding lemma does not extend as such to meromorphic quadratic differentials with a pole of even order ≥ 4: give a meromorphic quadratic differential at $0 \in \mathbb{C}$ which has in 0 a pole of order 4, but can not be brought into the simple form $z^{-4}dz^2$. (Hint: use a notion of residue.)

Structure defined by a holomorphic quadratic differential. Let η be a meromorphic quadratic differential on C. Suppose first it has neither poles nor zeroes. According to Lemma 5.1 we can find at any point of C a local coordinate z, unique up to sign and an additive constant, such that $\eta = dz^2$. This means that $|\eta|$ defines on C a flat Euclidean metric. The conformal structure defined by this metric is clearly the given one. So if S is the surface underlying C, then the function $\eta : TS \to \mathbb{C}$ allows us to reconstruct the conformal structure.

Exercise 9. Prove that this is even true if η is meromorphic and not identically zero on any connected component of C. (Hint: show that a conformal structure on the unit disk which coincides with the usual one outside the origin, must be equal to the usual conformal structure.)

But η gives us more: we find an atlas whose transition maps are of the form $w \mapsto \pm w + \text{constant}$. In terms of a local coordinate z as above, the foliation defined by $\mathrm{Im}(z) = \text{constant}$ is therefore independent of that local coordinate and is hence globally defined. Alternatively, a tangent vector $v \in T_pC$ is tangent to this foliation precisely when $\eta_p(v)$ is real and ≥ 0 (we here think of η_p as a quadratic function $\eta_p : T_pC \to \mathbb{C}$). Similarly, the foliation defined by $\mathrm{Re}(z) = \text{constant}$ is globally defined and its tangent vectors are those on which η is real and ≤ 0. These two foliations are perpendicular with respect to the metric $|\eta|$. We call these the *horizontal* and the *vertical* foliations attached to η.

Now consider the case when η has order $k \geq -2$ at $p \in C$. Then the Euclidean structure and the two foliations are defined on a punctured neighborhood of p and we wish to understand what happens to that structure at p. According to Lemma 5.1 we can a local coordinate as z at p such that it acquires the standard form $\eta = (d(z^{k/2+1}))^2$. Let us first assume $k = -1$, so that $\eta = (d(z^{1/2}))^2$. The root $z^{1/2}$ is of course multivalued. It is best to understand this locally as a passage to the double cover defined by $w^2 = z$ so that the pull-back of η to this double cover becomes dw^2, i.e., the case we just analyzed. In particular, the horizontal and the vertical foliations pull back to the obvious ones. We recover geometrically the z-plane from the w-plane by taking the closed upper half plane in $\mathbb{C}_w$ defined by $\mathrm{Im}(w) \geq 0$ and identifying opposite points on its boundary (the real axis). This helps us to understand what happens to the two foliations. For instance, the horizontal foliation has as a leaf the positive real axis, the origin is a singular point, and its other leaves go almost around this half line. Similarly, the vertical foliation has as a leaf the negative real axis.

For general $k \geq -1$ we must take $k+2$ copies $(H_i)_{i\in\mathbb{Z}/(k+2)}$ of the closed upper half plane $\mathrm{Im}(w) \geq 0$ (with its two foliations and its Euclidean structure) and identify for $x \geq 0$, $x \in H_i$ with $-x \in H_{i-1}$. So the final result consists of $k+2$ sectors whose gluing rays are leaves of the horizontal foliation (except that when $k=0$ the two rays join to form a single leaf). The same is true for vertical foliation (whose rays bisect the sectors). As a Euclidean metric $|\eta|$ is degenerate at such a point, but the notion of distance survives (so that we have a metric space in the usual sense). This is because for $a > 0$, $\int_0^a t^{k/2}dt$ converges and is finite for $k \geq -1$.

The case $k = -2$ is special. It then best to use the coordinate $w := \log z$ and to think of z as obtained from the strip $0 \leq \mathrm{Im}(w) \leq 2\pi$ by identifying its two boundary lines by means of a shift over $2\pi\sqrt{-1}$. Since the pull back of η is now $c\,dw^2$, this helps us understand the structure near p. In particular, we see that a punctured neighborhood of p is a metric product of an interval (a,∞) and a circle so that a neighborhood basis of p consists of the $\{(a',\infty)\times \text{circle}\}_{a'>a}$.

The case $c \in \mathbb{R}$ is of special interest. If $c > 0$, then in terms of the above metric product, the horizontal foliation is radial and the vertical foliation it circular: it has closed leaves only. For $c < 0$ the roles of horizontal and vertical foliation are interchanged. Notice that in either case the length of circular leaf is that of the circumference of the circle (measured by means of $|\eta|$, of course), which is here $2\pi\sqrt{|c|}$.

Quadratic differentials and the Teichmüller flow. Let S be a surface. Suppose we are given a conformal structure J on S and a holomorphic quadratic differential η for (S,J) without zeroes.

This generates a remarkable one-parameter family of pairs $(J_t,\eta_t)_{t\in\mathbb{R}}$, where J_t is a conformal structure on S and η_t is a holomorphic quadratic differential for J_t such that $(J_0,\eta_0) = (J,\eta)$ and η_t defines the same horizontal resp. vertical foliation as η. The definition is as follows: let $\sigma_t : \mathbb{C} \to \mathbb{C}$ be the $\mathbb{R}$-linear transformation defined by

$$\sigma_t(x + y\sqrt{-1}) = e^{-t/2}x + e^{t/2}y\sqrt{-1}.$$

Notice that $\sigma_{t'}\sigma_t = \sigma_{t'+t}$ (it is in fact a geodesic in $\mathrm{SL}(2,\mathbb{R})$). Now (J,η) defines an atlas of charts (U open in $\mathbb{C}, z : U \to \mathbb{C}$) in which η takes the form dz^2 and the horizontal resp. vertical foliations are given as the level sets of $\mathrm{Im}(z)$ resp. $\mathrm{Re}(z)$. For this atlas, any coordinate change is of the form $w \mapsto \pm w + c$), where $c \in \mathbb{C}$. If we replace (U,z) by $(U, z_t := \sigma_t z)$, then the corresponding coordinate change will be $w \mapsto \pm w + \sigma_t(c)$ and hence will still be of this form. This shows that we have thus defined a new conformal structure J_t as well as a holomorphic quadratic differential η_t for (S,J_t) which in terms of the chart z_t is given as $(dz_t)^2$. The definition is canonical in the sense that if we start doing this for $(J_{t'},\eta_{t'})$ instead of

(J, η), then after time t we get $(J_{t+t'}, \eta_{t+t'})$. This has all the features of a geodesic in a Riemannian manifold (and indeed, it can be shown that its projection in $\mathcal{T}(S)$ is one relative to a Finsler metric).

The assumption that the quadratic differential be without zeroes or poles can be eliminated: all we need is that η is not identically zero on every connected component of S: the zeroes are then isolated. But then we must be willing to adapt the differentiable structure of S at such points, for the conformal structure (and the quadratic differential) need not extend for the given differentiable structure: the glueing of the $k+2$ copies of the upper half planes after applying σ_t to each copy will induce in the z-coordinate a homeomorphism that is no longer a diffeomorphism at its origin. Fortunately, there is a path of homeomorphisms connecting this homeomorphism with a diffeomorphism and the isotopy class of that diffeomorphism is unique. The details are not so hard, but tedious and so we omit them.

Exercise 10. Let $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$. Prove that for (U, z) as above, the charts $(U, {}^{\sigma}z := \sigma z)$ define a pair $({}^{\sigma}J, {}^{\sigma}\eta)$ consisting of a conformal structure plus a quadratic differential that is holomorphic for that structure. Prove that ${}^{\sigma}J = J$ if and only if $\sigma \in SO(2)$. Bearing in mind that the stabilizer of $\sqrt{-1} \in \mathbb{H}$ in $SL(2, \mathbb{R})$ is $SO(2)$, conclude now that ${}^{\sigma}J$ only depends on $\sigma(\sqrt{-1}) \in \mathbb{H}$, so that we have a well-defined continuous map $\gamma : \mathbb{H} \to \mathcal{T}(S)$. Prove also that $\gamma(e^t\sqrt{-1}) = J_t$ as defined above.

We will see that if S is a connected surface and $P \subset S$ is finite such that S° has negative Euler characteristic, then for any conformal structure J on S, the cotangent space of $\mathcal{T}(S, P)$ at $[J]$ can be identified with the space of quadratic differentials on the Riemann surface (S, J) that have a pole of order ≤ 1 at the points of P and are holomorphic elsewhere. But we first show how certain meromorphic quadratic differentials can be used to almost triangulate $\mathcal{T}(S, P)$.

6. Ribbon graphs and ideal triangulation of Teichmüller space

Jenkins-Strebel differentials. Let us first observe that given a foliation on a surface, then a closed leaf of it either has a neighborhood consisting of closed leaves or it lies in the closure of an infinite number of (necessarily nonclosed) leaves. With this in mind we make the following definition.

Definition 6.1. A *Jenkins-Strebel differential* on a closed Riemann surface C is a meromorphic quadratic differential η on C which has everywhere order ≥ -2 and is such that all but a finite number of the leaves of its horizontal foliation are closed and encircle a point where η has order -2 with real negative squared residue.

By the above observation, the finitely many exceptional leaves can not be closed and will not have an accumulation point on a closed leaf. So they will join points where η has a zero or a pole of order one and are therefore the edges of an embedded graph $G(\eta) \subset C$ whose vertices are the aforementioned points. So if η has at $q \in C$ order $k \in \{-1, 1, 2, 3, \dots\}$, then q is a vertex of order $k+2 \in \{1, 3, 4, 5, \dots\}$. Each leaf has a length that can be measured be means of $|\eta|$: a closed leaf will encircle a point $p \in C$ where η has a pole of (exact) order 2, and then the preceding discussion shows that its length is $\ell_p(\eta) := 2\pi\sqrt{-\operatorname{Resq}_p \eta}$. A leaf that is an edge of the graph and has a simple pole of η as a vertex has still finite length because $\int_0^1 t^{-1/2}dt$ exists and is finite. Thus $G(\eta)$ is endowed with a metric. Notice that the orientation of C also determines a cyclic order of the leaves emanating from a given vertex.

Exercise 11. Verify that the quadratic differential $\eta^o := (z^2 - z)^{-1}dz^2$ has no zeroes, has a simple pole in 0 and 1, and an order two pole at ∞ with squared residue -1. Prove that its horizontal foliation has the real interval $(0, 1)$ as its only nonclosed leaf and that all the other leaves must be closed, encircle the real interval $[0, 1]$ and fill up all of $\mathbb{C} - [0, 1]$. Conclude that η^o is a Jenkins-Strebel differential on $\mathbb{P}^1$.

We will now give a converse construction that starts out from a metrized graph that for each vertex has a cyclic order on its emanating edges. This will lead in a surprising manner to an ideal triangulation of the Teichmüller space $\mathcal{T}(S, P)$.

Combinatorial description of a graph. We will only be concerned with finite graphs. We prefer to give a graph in combinatorial terms, as we did for pants decompositions. That is, a graph consist of a finite set X (that we call the *set of oriented edges*, sometimes also called *half edges*), a fixed point free involution σ_1 of X and an equivalence relation $\sim_0$ on X. We denote by $x \in X \mapsto x_1 \in X_1$ the formation of the set of orbits of σ_1 (so this is a surjective 2-to-1 map) and by $x \in X \mapsto x_0 \in X_0$ the formation of the set of equivalence classes of $\sim_0$. We refer to X_0 resp. X_1 as the *vertex set* resp. *edge set* of the graph, in view of the fact that they are the thus named items of the geometric realization $G(X; \sim_0, \sigma_1)$ of the graph: this is by definition the quotient of a disjoint union of copies of $[-1, 1]$ indexed by X: $\coprod_{x\in X}[-1, 1]_x$, by the following two sets of relations: $t_x \sim -t_{\sigma_1 x}$ and $-1_x \sim -1_{x'}$ if $x \sim_0 x'$. We see here X_0 appear as the set of vertices (and realized as the image of $\{\pm 1_x\}_{x\in X}$) and X_1 as the set of midpoints of the edges (and realized as the image of $\{0_x\}_{x\in X}$). A *half edge* is the image of $[-1, 0]_x$ for some $x \in X$. These cover the graph, for then $[0, 1]_x = [-1, 0]_{\sigma_1 x}$). Half edges are effectively labeled by X, with the members of an equivalence class of $\sim_0$ yielding the half edges (or equivalently, the oriented edges) coming out of the associated vertex. If we drop the condition that σ_1 be fixed point free, then this produces a graph 'with loose ends' such as one which describes a pants decomposition in the presence of cusps.

Ribbon graphs. If we are also given a permutation σ_0 of X that induces the equivalence relation $\sim_0$ (this means that $x \sim_0 x'$ if and only if $x' = \sigma_0^r x$ for some r), then we call the triple $(X; \sigma_0, \sigma_1)$ (often abbreviated by $\sigma_\bullet$) a *set* *of ribbon data*. Notice that this amounts to giving a cyclic structure on the set of half edges emanating from a vertex: σ_0 then assigns to every such half edge a successor. We are going to show that then the above geometric realization canonically expands to an embedding of $G(\sigma_\bullet) = G(X; \sigma_0, \sigma_1)$ in a closed combinatorial surface $S(\sigma_\bullet)$ (oriented, as usual) in such a manner that the cyclic structure on the half edges emanating from a vertex comes from the orientation of $S(\sigma_\bullet)$. In fact, we will combinatorially reproduce almost everything that we attached to a Jenkins-Strebel differential.

To this end, we define another permutation σ_∞ of X by the property $\sigma_0\sigma_1\sigma_\infty = 1$. So if $e \in X$ represents an oriented edge, then $\sigma_\infty e = \sigma_1^{-1}\sigma_0 e$ represents the oriented edge that begins in the end point of e and precedes the oppositely oriented $\sigma_1 e$ for the cyclic ordering. We denote the formation of its orbit set by $x \in X \mapsto x_\infty \in X_\infty$ and for reasons that become clear in a moment, we sometimes call a point of X_∞ a *boundary cycle*. Consider now the half strip $D \subset \mathbb{R}^2$ consisting of the $(t, u) \in \mathbb{R}^2$ with $|t| \leq 1$ and $u \geq 0$. Its one point compactification D^+ (whose underlying set is the disjoint union of D and ∞ and for which a neighborhood basis of ∞ meets D in the subsets defined by $u > N$) is homeomorphic to a solid triangle: for example,

$$(t, u) \mapsto (a, b) := \frac{(t, u)}{u + 1}, \quad \infty \mapsto (0, 1),$$

is a homeomorphism of D^+ onto the solid triangle defined by $|a| + b \leq 1$, $b \geq 0$. Consider the disjoint union copies of D^+ indexed by X, $\coprod_{x\in X} D^+_x$, and let $S(\sigma_\bullet)$ be the quotient of that union by the set of relations $(t, 0)_x \sim (-t, 0)_{\sigma_1 x}$ and $(1, u)_x \sim (-1, u)_{\sigma_\infty x}$. Let us write

$$\pi : \coprod_{x\in X} D^+_x \to S(\sigma_\bullet)$$

for the quotient map. The π-images of the solid triangles D^+_x cover $S(\sigma_\bullet)$ and define a triangulation of $G(\sigma_\bullet)$, except that in case $\sigma_1 x = \sigma_0 x$, the two vertical sides get identified via $(1, u)_x \sim (-1, u)_{\sigma_\infty x} = (-1, u)_x$. The quotient $S(\sigma_\bullet)$ is a closed combinatorial surface with an orientation inherited from the copies of D. The sets $X_1 = \{(\pm 1, 0)_x\}_{x\in X}$, $X_1 = \{(0, 0)_x\}_{x\in X}$ and $X_\infty = \{\infty_x\}_{x\in X}$ are effectively labeled by the orbit set of σ_0, σ_1 and σ_∞ in X respectively. We sometimes refer to elements of X_∞ as *cusps*. Notice that in the presence of the first relation, the last one implies $(-1, 0)_x \sim (-1, 0)_{\sigma_1 x}$ and so we see $G(\sigma_\bullet)$ embedded in $S(\sigma_\bullet)$ as the image of $\cup_{x\in X}([-1, 1] \times \{0\})_x$.

We have arranged things in such a manner that the cyclic ordering on the emanating edges of a vertex of $G(\sigma_\bullet)$ that comes from the orientation of $S(\sigma_\bullet)$ coincides with the one defined by σ_0. We also note that the projection $(t, u) \in D_x \mapsto t \in [-1, 1]_x$ is a deformation retract, which descends to one of $S(\sigma_\bullet) - X_\infty$ onto $G(\sigma_\bullet)$. Therefore, $S(\sigma_\bullet) - X_\infty$ may be thought of as a thickened version of $G(\sigma_\bullet)$ which implements in a geometric (ribbonlike) manner the ribbon structure, whence the name. The obvious horizontal and vertical foliations on D determine (in a combinatorial sense) a horizontal and a vertical foliation of $S(\sigma_\bullet) - X_0 - X_\infty$. The horizontal foliation has all its leaves closed except the leaves that make up the edges of $G(\sigma_\bullet)$.

Notice that $S(\sigma_\bullet)$ is connected precisely if the group generated by σ_0 and σ_1 acts transitively on X. In that case we can compute the genus g by means of an Euler characteristic count: we have as many 2-cells as there are 1-cells having a point of X_∞ in their boundary, the remaining 1-cells are indexed by X_1 and the 0-cells are indexed by $X_0 \sqcup X_\infty$. It follows that

$$2 - 2g = |X_0| - |X_1| + |X_\infty|.$$

We estimate the number of edges in terms of g:

Lemma 6.2. *Put $n_\infty := |X_\infty|$ and denote by n_1 resp. n_2 the number of vertices of degree 1 resp. 2. Then $G(\sigma_\bullet)$ has at most $6g - 6 + 3n_\infty + n_2 + 2n_1$ edges and we have equality precisely when there are no vertices of degree > 3.*

Proof. If $X_0(r)$ denotes the set of vertices of degree r, then the cycle decomposition of σ_0 shows that $\sum_r r|X_0(r)| = |X|$. It follows that $3|X_0| - n_2 - 2n_1 \le |X| = 2|X_1|$. On the other hand, the formula for the Euler characteristic yields $2g-2+n_\infty = |X_1| - |X_0|$. Hence $6g-6+3n_\infty = 3|X_1| - 3|X_0| \ge |X_1| - n_2 - 2n_1$, with equality holding only if $X_0(r) = \emptyset$ for $r > 3$. □

Exercise 12. The minimal (and degenerate) ribbon graph is defined by a singleton $X = \{*\}$ and so with $\sigma_0, \sigma_1, \sigma_\infty$ all trivial (we should think here of a graph that consists of a single half edge). Although σ_1 is no longer fixed point free, the above construction still makes sense. Prove that $S^o := S(1_\bullet)$ can be identified with $\mathbb{P}^1$ in such a manner that $G^o := G(1_\bullet)$ is mapped to $[0, 1]$ and the three special points x_0, x_1, x_∞ are mapped to their namesakes $0, 1, \infty$. Try also to argue that this can even be done in such a way that this also respects the horizontal and vertical foliations associated to the quadratic differential η^o that we encountered in Exercise 11.

Exercise 13. Now construct for any set of ribbon data $(X; \sigma_0, \sigma_1)$ a natural (combinatorially defined) map $\pi : S(\sigma_\bullet) \to S^o$ with the property that $G(\sigma_\bullet) = \pi^{-1}G^o$ and the horizontal resp. vertical foliation is the preimage of that of S^o.

Remark 6.3. The map in the preceding exercise is a covering projection over the thrice punctured sphere $S^o - \{x_0, x_1, x_\infty\}$. If we choose a base point on the interior of the half line G^o, then the fiber over it is effectively indexed by X in such a manner that the monodromy around x_0 resp. x_1 is given by σ_0 resp. σ_1. This construction leads to a realization of the Galois group of $\mathbb{Q}$ as a permutation group of the collection of isomorphism types of ribbon graphs as follows: if $S(\sigma_\bullet) \to S^o$ is as in Exercise 13 and S^o has been identified with $\mathbb{P}^1$ as in Exercise 12, then there is a unique conformal structure on $S(\sigma_\bullet)$ for which the map $S(\sigma_\bullet) \to S^o \cong \mathbb{P}^1$ is holomorphic; the resulting Riemann surface $C(\sigma_\bullet)$ then naturally comes as a branched covering of $\mathbb{P}^1$ with branching in $\{0, 1, \infty\}$ only, where over 1 we only have ramification of order 2. Such a Riemann surface is in fact an algebraic curve that is defined over a finite field extension of $\mathbb{Q}$: it can be given by homogeneous polynomial equations with coefficients in a finite extension K of $\mathbb{Q}$ contained in $\mathbb{C}$.

If τ is a field automorphism of K, then applying τ to the coefficients of the equations that define $C(\sigma_\bullet)$ yields a new variety $C(\sigma_\bullet)^\tau$. Algebraic geometry tells us that this variety is again a smooth Riemann surface which still comes with a holomorphic map to $\mathbb{P}^1$ that is unramified over $\mathbb{P}^1 - \{0, 1, \infty\}$ with over 1 only ramification of order 2. The preimage of $[0, 1]$ in $C(\sigma_\bullet)^\tau$ yields a ribbon graph $(X, \sigma_\bullet)^\tau$ and $C(\sigma_\bullet)^\tau$ is the Riemann surface associated to the graph. We have thus defined an action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ of $\mathbb{Q}$ on the set of isomorphism types of ribbon graphs. The group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is elusive and hard to understand as a group, whereas the set of isomorphism classes of ribbon graphs is rather concrete. Unfortunately no general method is known that can tell us whether two given ribbon graphs lie in the same Galois orbit.

The dual ribbon graph and the arc complex. The identity $\sigma_0\sigma_1\sigma_\infty = 1$ implies $\sigma_\infty^{-1}\sigma_1\sigma_0^{-1} = 1$ and so the triple $(\sigma_0^* := \sigma_\infty^{-1}, \sigma_1^* := \sigma_1, \sigma_\infty^* := \sigma_0^{-1})$ defines another ribbon graph $G(\sigma_\bullet^*)$, called the *dual* of $G(\sigma_\bullet)$. The associated surface $S(\sigma_\bullet^*)$ is homeomorphic with $S(\sigma_\bullet^*)$ under an orientation reserving homeomorphism that sends $X_0(\sigma_\bullet)$ to $X_\infty(\sigma_\bullet^*)$, $X_\infty(\sigma_\bullet)$ to $X_0(\sigma_\bullet^*)$ and $X_1(\sigma_\bullet)$ to $X_1(\sigma_\bullet^*)$. In particular, it sends sends the left half of $D_x^+ \subset S(\sigma_\bullet)$ with vertices $(-1, 0)_x$, $(0, 0)_x$, ∞_x to the right half of $(D_x^*)^+ \subset S(\sigma_\bullet^*)$ with vertices ∞_x^*, $(0, 0)_x^*$, $(1, 0)_x$ respectively. So under this homeomorphism, $G(\sigma_\bullet^*)$ is embedded in $S(\sigma_\bullet)$ as the union of the symmetry axes of the D_x^+ which connect $(0, 0)_x$ with ∞_x. We denote the latter by $G^*(\sigma_\bullet) \subset S(\sigma_\bullet)$.

Definition 6.4. An *arc* of (S, P) is an isotopy class relative to P of an embedded interval which connects two points of P. We also allow the two points to coincide (so that we have a loop), but then require the loop not to bound a disk in S°. The *arc complex* $\mathrm{Arc}(S, P)$ is the simplicial complex whose vertex set is the set of arcs of (S, P) with a finite nonempty set of arcs representing a simplex precisely if that set can be simultaneously be represented by pairwise disjoint arcs. If we can do this in such a manner that their union $A \subset S$ has the property that every connected component of $S - A$ is homeomorphic to an open disk which meets P in at most one point, then we say that the simplex is *proper*.

If a simplex has a proper face, then it is proper and so the nonproper simplices define a subcomplex $\mathrm{Arc}_\infty(S, P)$ of $\mathrm{Arc}(S, P)$. The point is that a pair $(S(\sigma_\bullet), X_\infty(\sigma_\bullet))$, where $\sigma_\bullet$ defines a ribbon graph, naturally is endowed with a proper simplex of its arc complex, namely $G^*(\sigma_\bullet)$. Indeed, has $G^*(\sigma_\bullet)$ has vertex set $X_\infty(\sigma_\bullet)$ and each connected component of $S(\sigma_\bullet) - G^*(\sigma_\bullet)$ is an open cell with center a point of $X_0(\sigma_\bullet)$. This observation will be helpful in understanding the following definition:

Definition 6.5. An (S, P)-*marked ribbon graph* is a ribbon graph $(X, \sigma_\bullet)$ endowed with an orientation preserving piecewise-linear homeomorphism $h : S \to S(\sigma_\bullet)$ which maps P to $X_\infty \cup X_0$ with $h(P)$ containing X_∞ and all vertices of degree ≤ 2 and that is given up to an isotopy relative to P.

Notice that in terms of the notation of Lemma 6.2 we then have that $n := |P| \geq n_\infty + n_1 + n_2$. So the lemma in question shows that the number of edges $|X_1|$ is bounded by $3(2g-2+n)$.

Observe that the mapping class group $\Gamma(S,P)$ acts on the class of (S,P)-marked ribbon graphs by letting $[g] \in \Gamma(S,P)$ send $(X,\sigma_\bullet,[h])$ to the triple $(X,\sigma_\bullet,[hg^{-1}])$.

Given an (S,P)-marked ribbon graph $(\sigma_\bullet,[h])$, then h^{-1} maps an edge of $G^*(\sigma_\bullet)$ to an arc connecting two distinct points of P or to a loop at p when $h(p)$ is a vertex of degree one. We see that thus is defined a proper simplex $\Delta(\sigma_\bullet,[h])$ of $\mathrm{Arc}(S,P)$ which only depends on $(\sigma_\bullet,[h])$. Conversely, if we are given a proper simplex of $\mathrm{Arc}(S,P)$, then a representative set of pairwise disjoint arcs define a ribbon graph. If the dual of this ribbon graph is defined by $\sigma_\bullet$, then it is easily seen that we have naturally defined isotopy class of orientation reserving homeomorphisms relative to P, $S \cong S(\sigma^*_\bullet)$, which sends P to a subset of $X_0(\sigma^*_\bullet)\cup X_\infty(\sigma^*_\bullet)$. If we compose the latter with the homeomorphism of $S^*(\sigma^*_\bullet)$ onto $S(\sigma_\bullet)$ discussed above, then we have an (S,P)-marked ribbon graph.

Notice that we have an evident simplicial map from $\mathrm{Arc}(S,P)$ to the barycentric subdivision of the simplex $\Delta(P)$ spanned by P: if an arc connects p and p' (with possibly $p=p'$), then we assign to that arc the barycenter of the face spanned by p,p' (this is p if $p'=p$). We denote its geometric realization by $\ell/2 : |\mathrm{Arc}(S,P)| \to |\Delta(P)|$ (the funny choice of notation will be explained in a moment).

Metrized ribbon graphs. Here is a beautiful construction that starts out with a set ribbon data $(X,\sigma_\bullet)$ and a metric on $G(\sigma_\bullet)$, i.e., a function $e \in X_1 \mapsto l(e) \in \mathbb{R}_{>0}$, and leads to a Riemann surface and a Jenkins-Strebel differential on that surface. Consider on D_x the complex coordinate $z_x := \frac{1}{2}l(x_1)dt + \sqrt{-1}du$. This identifies D_x with the half strip in $\mathbb{C}$ defined by $|\mathrm{Re}(z)| \leq \frac{1}{2}l(x_1)$, $\mathrm{Im}(z) \geq 0$ and the associated holomorphic quadratic differential dz_x^2 yields the given horizontal and vertical foliation on D_x. The point is now that the holomorphic quadratic differentials $\eta_x := dz_x^2$ fit together nicely to form one on $S(\sigma_\bullet) - X_0 - X_\infty$, turning the latter into a Riemann surface. Indeed, $\pi(D_x)$ and $\pi(D_{\sigma_\infty x})$ are adjacent in the sense that have in common a vertical boundary piece and in the way they are fitted together, we see that their union has a natural conformal structure for which the differentials dz_x and $dz_{\sigma_\infty x}$ are restrictions of a holomorphic differential on the interior of their union (which is identified with an open half strip of width $l(x_1) + l((\sigma_\infty x)_1)$). The interior of $\pi(D_x \cup D_{\sigma_1 x})$ can be identified with the strip $|\mathrm{Re}(z)| < \frac{1}{2}l(x_1)$, at least when x and y lie not in the same σ_0-orbit (analyze that situation yourself), and we then see that z_x continues as a holomorphic function to $-z_{\sigma_1 x}$. So η_x then extends by $\eta_{\sigma_1 x}$. We thus end up with conformal structure $J(l)$ on $S(\sigma_\bullet) - X_0 - X_\infty$ plus a quadratic differential $\eta(l)$ on it. Both extend across $X_0 \cup X_\infty$: at a vertex in X_0 the quadratic differential will have a zero of order equal to the degree of that vertex less 2 (so a pole of order one if the vertex is an end vertex and no zero if it is of degree 2) and at any cusp $p \in X_\infty$, $\eta(l)$ will have a pole of order 2 with a real, negative squared residue.

We have set up things in such a manner that the following is true:

Observation 6.6. *Let* C *be a closed connected Riemann surface endowed with a Jenkins-Strebel differential* η*. Then the nonclosed leaves of its horizontal foliation define an abstract graph and the orientation of* C *endows this graph with a ribbon structure so that we have a ribbon graph* $G(\sigma_\bullet)$*. If* l *is the metric on this graph defined by* $|\eta|$*, then there is a natural piecewise linear homeomorphism of* $S(\sigma_\bullet)$ *onto* C *such that* $\eta(l)$ *is the pull-back of* η*.*

Let us return to the metrized ribbon graph $(S(\sigma_\bullet), l)$ above. We assume however that the metric is *unital*: the total length of the graph is 1. This means that l is in the relative interior $|\Delta(X_1)|^\circ$ of the simplex $|\Delta(X_1)|$ spanned by the edge set of σ.

As we know, the circumference of the closed leaves near p of the horizontal foliation is equal to $\ell_p := 2\pi\sqrt{-\operatorname{Resq}_p \eta}$. This is also the length of a boundary cycle:

$$\ell_p = \sum_{\{x|x_\infty = p\}} l(x_1).$$

Since $x \mapsto x_1$ is a 2-to-1 map, it follows that

$$\sum_{p\in X_\infty} \ell_p = \sum_{p\in X_\infty} \sum_{\{x|x_\infty=p\}} l(x_1) = \sum_{x\in X} l(x_1) = 2\sum_{e\in X_1} l(e) = 2.$$

So the functions $(\ell_p/2)_{p\in P}$ define a map $\Delta^\circ(X_1) \to \Delta(P)$. Observe that if we identify X_1 with X_1^* and regard the latter as the vertex set of a simplex of $\mathrm{Arc}(S, P)$, then this is precisely the map we earlier denoted by $\ell/2$. The conformal structures thus defined have a common piecewise linear structure on $S(\sigma_\bullet)$ and so this defines a continuous map

$$\Phi^\circ_{\sigma_\bullet} : \Delta^\circ(X_1) \to \mathcal{T}(S(\sigma_\bullet), X_\infty)) \times |\Delta(X_\infty)|,$$

where the first map is clear and the second has barycentric coordinates $(\ell_p/2)_{p\in P}$. In fact the construction does more: it endows each $S(\sigma_\bullet)$ with a Jenkins-Strebel differential.

Degenerating the metric on a ribbon graph. By definition the open simplex $\Delta^\circ(X_1)$ parametrizes the unital metrics on $G(\sigma_\bullet)$. There is a straightforward way to interpret some of its relatively open faces as parametrizing the metrics on a ribbon graph that is a quotient of the given one.

Suppose $x \in X$ is not a fixed point of σ_∞ and put $e := \pi_1(x)$. Note that this allows e to be a loop: we then have $\sigma_\infty\sigma_1(x) = x$. We can collapse e inside $G(\sigma_\bullet)$ and still get a ribbon graph with $X - \{x, \sigma_1 e\}$ as its set of oriented edges (in case e happens to be a loop, then the boundary cycle disappears.) We call this an *edge contraction* and denote the resulting graph by $G(\sigma_\bullet)/e$. So if we denote its combinatorial data by $(X/e, (\sigma/e)_\bullet)$, then

$G(\sigma_\bullet)/e = G((\sigma/e)_\bullet)$. In term of the dual graph, this simply amounts to removal of the edge e^* corresponding to e.

Exercise 14. If the edge e is defined by a σ_1-orbit $\{x, y\}$, show that $X/e := X - \{x, y\}$, $(\sigma/e)_1 = \sigma_1|X - \{x, y\}$ and that $(\sigma/e)_0$ is on $X - \{x, y\} - \sigma_0^{-1}\{x, y\}$ equal to σ_0 and on $\sigma_0^{-1}\{x, y\} \setminus \{x, y\}$ equal to $\sigma_0\sigma_1\sigma_0$. (Beware of the special case when an end vertex of e is of degree one, i.e., when x or y is a fixed point of σ_0.) Verify that $(X/e)_1 = X_1 - \{e\}$ and $(X/e)_\infty = X_\infty$.

A metric on $G(\sigma_\bullet)/e$ can be understood as a degenerate metric on $G(\sigma_\bullet)$, namely one which gives e zero length. We can also obtain the associated surface $S((\sigma/e)_\bullet)$ as a quotient of $S(\sigma_\bullet)$: if $\{x, y\}$ is the σ_1-orbit representing e, then in D_x resp. D_y, or rather its π-image in $G(\sigma_\bullet)$, we identify points if they have the same imaginary part. You may verify that $\pi(D_x^+ \cup D_y^+)$ is homeomorphic to a disk and that the quotient map $S(\sigma_\bullet) \to S((\sigma/e)_\bullet)$ contracts this disk to an interval (distinguish the two special cases when $\{x, y\}$ is a σ_0-orbit and when one of $\{x, y\}$ is fixed point of σ_0). Although this is clearly not a homeomorphism, such a contraction determines a natural isotopy class of homeomorphisms between them relative to any prescribed closed subset that is disjoint with the disk $\pi(D_x^+ \cup D_y^+)$. Let us now write P for X_∞. The maps $\Phi^\circ_{\sigma_\bullet}$ and $\Phi^\circ_{(\sigma/e)_\bullet}$ combine to define a continuous map defined on the union of the simplicial cell $|\Delta(X_1)|^\circ$ and its face $|\Delta(X_1 - e)|^\circ$ to $\mathcal{T}(S(\sigma_\bullet), P)$.

We may iterate this construction and the order in which we do so does not matter. Thus lower dimensional faces of $[0, \infty)^{X_1}$ may also parametrize metrized ribbon graphs of the same genus. A proper subset $E \subsetneq X_1$ arises in this manner (in the sense that successive contraction of its elements in some order is of the above type) if and only if the subgraph of $G(\sigma_\bullet)$ with edge set E has the property that each connected component contains at most one boundary cycle (we must avoid that two elements of X_∞ have the same image). Let us say that such a subset E is *allowed*. If we identify X_1 with the vertex set of simplex of $\mathrm{Arc}(S(\sigma_\bullet), X_\infty)$, then note that a subset $E \subset X_1$ is allowed if only if the simplex of $\mathrm{Arc}(S(\sigma_\bullet), X_\infty)$ defined by $X_1 - E$ is proper.

For an allowed E, we get a contraction $S(\sigma_\bullet) \to S((\sigma/E)_\bullet)$ which is injective on $P = X_\infty$. Although the contraction is not a homeomorphism, it singles out an isotopy class of piecewise-linear homeomorphisms between $S(\sigma_\bullet)$ and $S((\sigma/E)_\bullet)$ that enables us to regard $\Phi^\circ_{(\sigma/E)_\bullet}$ as a map taking its values in $\mathcal{T}(S(\sigma_\bullet), P) \times |\Delta(P)|$. And if we think of its domain as a simplicial cell of $|\Delta(X_1)|$, then it yields a continuous extension of $\Phi^\circ_{\sigma_\bullet}$ to that cell.

Ideal triangulation of thickened Teichmüller space. Let us phrase the preceding discussion somewhat differently. We fix a closed connected surface S of genus g and $P \subset S$ a finite nonempty subset. We write n for $|P|$ and assume that $2g - 2 + n > 0$. We have defined the arc complex $\mathrm{Arc}(S, P)$ and a closed subcomplex $\mathrm{Arc}_\infty(S, P)$. We write $|\mathrm{Arc}(S, P)|$ resp. $|\mathrm{Arc}_\infty(S, P)|$ for their geometric realization so put $|\mathrm{Arc}(S, P)|^\circ := |\mathrm{Arc}_\infty(S, P)| - |\mathrm{Arc}_\infty(S, P)|$.

If we are given a proper simplex Δ of $\mathrm{Arc}(S,P)$ such that $\sigma^*_\bullet$ is the associated ribbon graph, then this defines an (S,P)-marking of the ribbon graph defined by $\sigma_\bullet$ (with $\Delta = \Delta(X_1^*) = \Delta(X_1)$) and the allowed faces of Δ are its proper faces. So $\Phi^\circ_{(\sigma/E)_\bullet}$ the above define a continuous map $\Phi_\Delta : |\Delta| \cap |\mathrm{Arc}(S,P)|^\circ \to \mathcal{T}(S,P)$. By construction this map is canonically defined in the sense that if Δ' is a proper face of Δ, then $\Phi_{\Delta'}$ is the restriction of Φ_Δ. We therefore obtain a continuous map $\Phi_{(S,P)} : |\mathrm{Arc}(S,P)|^\circ \to \mathcal{T}(S,P)\times\Delta(P)$.

One of the main results of the Jenkins-Strebel theory states:

Theorem 6.7. *The map $\Phi_{(S,P)} : |\mathrm{Arc}(S,P)|^\circ \to \mathcal{T}(S,P) \times |\Delta(P)|$ constructed above is a $\Gamma(S,P)$-equivariant homeomorphism and (hence) drops to a homeomorphism of $\Gamma(S,P)\backslash|\mathrm{Arc}(S,P)|^\circ$ onto $\mathcal{M}_{g,P} \times |\Delta(P)|$.*

This says essentially the following: if we are given a closed connected Riemann surface of genus g and a nonempty finite subset $P \subset C$ of n elements, then for every $\ell \in \Delta(P)$ there is a unique Jenkins-Strebel differential η_ℓ on C whose polar set is contained in P and with $\mathrm{Resq}_p(\eta_\ell) = \ell_p^2/\pi$ for all $p \in P$.

We will not prove this theorem. One approach bears some formal similarity to the proof of the Fenchel-Nielsen parametrization and consists in showing that $\Phi_{(S,P)}$ is a covering projection; the simple connectivity of the range then implies that we must have a homeomorphism. Notice that the simplices of $\mathbb{T}^\circ(S,P)$ are effectively indexed by the (S,P)-marked ribbon graphs. (It is not so clear how to do this for the simplicial cells of $|\mathrm{Arc}_\infty(S,P)|$.)

Example 6.8. Let us do the case $(g,n) = (1,1)$. Following Lemma 6.2, a trivalent ribbon graph with just one boundary cycle that yields a genus 1 surface, has at 3 edges. The Euler characteristic of the graph will be $2-2g-1 = -1$ and so we have two vertices. We then easily see that this ribbon graph is unique. We can contract one edge of this trivalent ribbon graph and still have a ribbon graph, but we cannot contract two. So we have 2-simplices and 1-simplices (each make up a single $\Gamma_{1,1}$-orbit), but no 0-simplices: $\mathrm{Arc}_{1,1}$ is a simplicial complex of dimension two and $(\mathrm{Arc}_\infty)_{1,1}$ is simply the set of its vertices. We identified in Corollary 2.4 the pair $(\Gamma_{1,1}, \mathcal{T}_{1,1})$ with $(\mathrm{SL}(2,\mathbb{Z}), \mathbb{H})$. In terms of this identification, $|\mathrm{Arc}_{1,1}|^\circ$ and its ideal triangulation is familiar from the theory of automorphic forms: $|\mathrm{Arc}_{1,1}|$ amounts to adding to $\mathbb{H}$ the set $\mathbb{Q}\cup\{\infty\}$ (or more canonically, $\mathbb{P}^1(\mathbb{Q})$) and a typical simplex is the ideal hyperbolic triangle with vertices $-1, 1, \infty \in \mathbb{P}^1(\mathbb{Q})$.

Lemma 6.9. *The mapping class group $\Gamma(S,P)$ has only a finite number of orbits in the collection (S,P)-marked ribbon graphs. Furthermore, the star of a proper simplex is finite: such a simplex is a face of only finitely many simplices of $|\mathrm{Arc}(S,P)|$.*

Proof. Two (S,P)-marked ribbon graphs lie in the same $\Gamma(S,P)$-orbit if and only if their underlying ribbon graphs are isomorphic. So the $\Gamma(S,P)$-orbits in question are in bijective correspondence with isomorphism classes of sets of ribbon data $(X, \sigma_\bullet)$ which define a surface of genus g with n boundary cycles and have no vertices of degree ≤ 2. As we have seen, the number of edges of such a ribbon graph is at most $3(2g-2+n)$ and so their number of isomorphism classes is finite.

A proper simplex of $\mathrm{Arc}(S,P)$ is given by a (S,P)-marked ribbon graph $(X, \sigma_\bullet)$. Any simplex that contains this simplex in its as a face of codimension one, is by definition represented by a (S,P)-marked ribbon graph $(X', \sigma'_\bullet, [h'])$ and an allowable edge $e \in X'_1$ such that $(X, \sigma_\bullet, [h])$ is obtained by contraction of e. From the perspective of $(X, \sigma_\bullet, [h])$, this amounts to the 'inflation' of $(X, \sigma_\bullet, [h])$ by insertion an edge. Clearly, this can be done in only finitely many ways. Since the dimension of such a simplex is bounded (by $6g-7+3n$), we see that $(X, \sigma_\bullet, [h])$ can in only finitely many ways be obtained as the edge-contraction of an (S,P)-marked ribbon graph and so the last assertion follows. □

This Lemma 6.9 shows that if Δ is a proper simplex of $\mathrm{Arc}(S,P)$, then the the union $U(\Delta)$ of the simplicial cells of $|\mathrm{Arc}(S,P)|$ that have $|\Delta|$ in their closure is an open subset of $|\mathrm{Arc}(S,P)|^\circ$. Notice that these open subsets cover $|\mathrm{Arc}(S,P)|^\circ$.

Corollary 6.10. *The mapping class group $\Gamma(S,P)$ acts properly discontinuously on $|\mathrm{Arc}(S,P)|^\circ \cong \mathcal{T}(S,P) \times \Delta(P)$ and $\mathbb{M}_{g,P} := \Gamma(S,P)\backslash|\mathrm{Arc}(S,P)|$ is compact.*

Proof. We prove the first assertion by showing that for any proper simplex Δ, the set of $g \in \Gamma(S,P)$ with $U(\Delta) \cap gU(\Delta) \neq \emptyset$ is finite. Any element of $\Gamma(S,P)$ sends a simplicial cell to a simplicial cell and clearly, a simplicial cell has a finite number of symmetries. So the collection of such $g \in \Gamma(S,P)$ is finite.

The last assertion follows from the first clause of Lemma 6.9. □

Barycentric subdivision. Consider the barycentric subdivision $b\,\mathrm{Arc}(S,P)$ of $\mathrm{Arc}(S,P)$. Recall that the barycenter b_Δ of a simplex Δ is the point of $|\Delta|$ where all its barycentric coordinated are equal. The set of barycenters of $\mathrm{Arc}(S,P)$ is partially ordered by incidence: $b_\Delta \leq b_{\Delta'}$ if and only if $\Delta \subset \Delta'$. The barycenters of the simplices of $\Delta(S,P)$ are the vertices of $\Delta(S,P)'$ and a finite nonempty subset of such barycenters spans a simplex of $\mathrm{Arc}(S,P)'$ if and only if it is totally ordered by the partial ordering: so if $\Delta_\bullet = (\Delta_0 \supsetneq \Delta_1 \supsetneq \cdots \supsetneq \Delta_k)$ is a sequence of faces of $\mathbb{T}(S,P)$, then $b_{\Delta_0}, \dots, b_{\Delta_k}$ spans a simplex of $b\,\mathrm{Arc}(S,P)$ and these are all. Let us verify that this makes sense:

Lemma 6.11. *If $\Delta_\bullet = (\Delta_0 \supsetneq \Delta_1 \supsetneq \cdots \supsetneq \Delta_k)$ is a decreasing sequence of faces of Δ, then the associated barycenters $b_{\Delta_0}, \dots, b_{\Delta_k}$ are linearly independent so that they span a k-simplex $|\Delta_\bullet|$. The simplices so obtained define a simplicial complex $b\Delta$, called the* barycentric subdivision *of Δ, whose geometric realization can be identified $|\Delta|$.*

Proof. There are essentially two things to be verified: that the collection of simplices $|\Delta_\bullet|$ is closed under taking intersection and that their union is $|\Delta|$.

The last property is seen as follows. If $z \in |\Delta|^\circ$, then choose a ray starting at b_Δ and passing through z: this ray will meet $|\Delta| - |\Delta|^\circ$ in some point z_1, with z_1 in a proper face $|\Delta_1|$. So then $z = (1-t)b_\Delta + tz_1$ for some $t \in [0,1]$. By induction on $\dim \Delta$, we may assume there exists a flag $(\Delta_1 \supsetneq \cdots \supsetneq \Delta_k)$ such that $z_1 = \sum_{k=1}^r s_k b_{\Delta_k}$ for some $s_k \geq 0$ and $\sum_k s_k = 1$. Then $z = (1-t)b_\Delta + \sum_{k=1}^r s_k t b_{\Delta_k} \in |(\Delta, \Delta_1, \dots, \Delta_k)|$.

In order to see that the collection of simplices $|\Delta_\bullet|$ is closed under taking intersection, it suffices to show that for any $\Delta_\bullet = (\Delta_0 \supsetneq \Delta_1 \supsetneq \cdots \supsetneq \Delta_k)$, $|\Delta'| \cap |\Delta_\bullet|$ is a simplex defined by a subsequence of $\Delta_\bullet$. Let f be an affine-linear function on the affine-linear span of Δ with the property that $f|\Delta \geq 0$ and whose zero set meets Δ in Δ'. So a convex linear combination $\sum_i t_i b_{\Delta_i}$ lies in $|\Delta'|$ precisely when $\sum_i t_i f(b_{\Delta_i}) = 0$. Since $f(b_{\Delta_i}) \geq 0$ with equality if and only if $b_{\Delta_i} \in \Delta'$ (which just means that $\Delta_i \subset \Delta'$), it follows that $|\Delta'| \cap |\Delta_\bullet|$ is indeed a simplex of our collection, namely the one defined by the subsequence of $\Delta_\bullet$ of terms contained in Δ'. □

Fix a nonempty subset $Q \subset P$. This defines a face $\Delta(Q) \subset \Delta(P)$. We denote by $b = b_{\Delta(Q)}$ its barycenter and denote by $\mathbb{T}_b(S,P)$ the fiber of $|\operatorname{Arc}(S,P)| \to |\Delta(P)|$ over b. Since the projection $\operatorname{Arc}(S,P) \to b\Delta(P)$ is simplicial, we may view as $\mathbb{T}_b(S,P)$ as the geometric realization of a subcomplex of $\operatorname{Arc}(S,P)$. We put $\partial\mathbb{T}_b(S,P) := \mathbb{T}_b(S,P) \cap |\operatorname{Arc}_\infty(S,P)|$ and $\mathbb{T}_b(S,P)^\circ := \mathbb{T}_b(S,P) \cap |\operatorname{Arc}(S,P)|^\circ$.

Corollary 6.12. *The map $\Phi_{(S,P)}$ restricts to a $\Gamma(S,P)$-equivariant homeomorphism of $\mathbb{T}_b(S,P)^\circ$ onto $\mathcal{T}(S,P)$ and (hence) drops to a homeomorphism of $\Gamma(S,P)\backslash\mathbb{T}_b(S,P)^\circ$ onto $\mathcal{M}_{g,P}$.*

The orbit space $\Gamma(S,P)\backslash\mathbb{T}_b(S,P)$ need not be a simplicial complex, for passing the orbit space can cause faces of a polyhedron to get identified, or cause a polyhedron to be divided out by a finite group of symmetries. But apart from that, the given structure survives.

Application to the homotopy type of $\mathcal{M}_{g,P}$. We first describe a subcomplex $\mathbb{K}_b(S,P)$ of $\mathbb{T}_b(S,P)$ that is contained in $\mathbb{T}_b(S,P)^\circ$ and invariant under $\Gamma(S,P)$. It will have the property that is an $\Gamma(S,P)$-equivariant deformation retract. This will make the image of $\mathbb{K}_b(S,P)$ in $\mathcal{M}_{g,P}$ a deformation retract of the latter. We denote by $\mathbb{K}_b(S,P)$ the full subcomplex of the $\mathbb{T}_b(S,P)$ spanned by the barycenters b_Δ with $|\Delta|^\circ \subset \mathbb{T}_b(S,P)^\circ$. Clearly, $\mathbb{K}_b(S,P)$ is invariant under $\Gamma(S,P)$ and contained in $\mathbb{T}_b(S,P)^\circ$.

Proposition 6.13. *The subspace $\mathbb{K}_b(S,P) \subset \mathbb{T}_b^\circ(S,P)$ is a $\Gamma(S,P)$-equivariant deformation retract. In particular, $\mathbb{K}_b(S,P)$ is contractible, and the orbit space $\Gamma(S,P)\backslash\mathbb{K}_b(S,P)$ is a deformation retract of $\Gamma(S,P)\backslash\mathbb{T}_b(S,P)^\circ \cong \mathcal{M}_{g,P}$.*

Proof. We lighten the notation a bit and simply write $\mathbb{T}, \Gamma, \mathbb{K}, \dots$ for $\mathbb{T}_b(S,P)$, $\Gamma(S,P)$, $\mathbb{K}_b(S,P)$, $\dots$.

We must construct a map $h : [0,1]\times\mathbb{T}^\circ \to \mathbb{T}^\circ$ with $h_t|\mathbb{K}$ the identity for all t, h_1 the identity, and h_0 having image $\mathbb{K}$. We define h per simplex of of the barycentric subdivision. A simplex of $\mathbb{T}$ that is not contained in $\partial\mathbb{T}$ is of the form $|\Delta_\bullet|$ with $\Delta_\bullet = (\Delta_0 \supsetneq \Delta_1 \supsetneq \cdots \supsetneq \Delta_k)$ such that Δ_0 not contained in $\partial\mathbb{T}$. Let $k' \in \{0,\dots,k\}$ be the maximal for the property that $\Delta_{k'}$ is not contained in $\partial\mathbb{T}$. If $k' = k$, then $|\Delta_\bullet| \subset \mathbb{K}$ and so we must leave $|\Delta_\bullet|$ untouched: $h_t||\Delta_\bullet|$ is the identity for all t. If $k' < k$, then put $\Delta'_\bullet := (\Delta_0 \supsetneq \Delta_1 \supsetneq \cdots \supsetneq \Delta_{k'})$ and $\Delta''_\bullet := (\Delta_{k'+1} \supsetneq \cdots \supsetneq \Delta_k)$. Every $z \in |\Delta_\bullet|$ can be written as $sz' + (1-s)z''$ with $z' \in |\Delta'_\bullet|$ and $z'' \in |\Delta''_\bullet|$. We have $z \notin \partial\mathbb{T}$ if and only if $s \neq 0$ and in that case z' is unique. Put $h_t(z) := (1-t)z' + tz$. The continuity of $h|[0,1]\times(|\Delta_\bullet| - |\Delta_\bullet| \cap \partial\mathbb{T})$ is clear. It is also clear that the restriction of h_t to a face of $|\Delta_\bullet|$ yields the map h_t attached to that face. This proves that h_t is well-defined and that h is continuous. Its uniqueness guarantees it is Γ-equivariant. □

Corollary 6.14. *For $b = p \in P$, the dimension of $\mathbb{K}_p(S,P)$ is bounded by $4g-3$ for $n=1$ and $g>0$ and by $4g-3+n$ otherwise. In particular, $\mathcal{M}_{g,P}$ has the homotopy type of a finite CW-complex whose dimension satisfies the same bound.*

Proof. We have $\dim\mathbb{T}^\circ = \dim\mathcal{T}(S,P) = 6g-6+2n$ and so any simplex of $\mathbb{T}$ has dimension $\leq 6g-6+2n$. On the other hand, a simplex of $\dim\mathbb{T}^\circ$ is spanned by the edge set X_1 of an (S,P)-marked ribbon graph $(X,\sigma_\bullet,[h])$ which has at least $\max\{0,n-1\}$ vertices and one boundary cycle. So the graph has Euler characteristic $1-2g$. The corresponding simplex of $\mathbb{T}$ has dimension $|X_1|-1$ and so this is at least $|X_0|+2g-2 \geq 2g-1$ for $n=1$ and at least $|X_0|+2g-2 \geq 2g-3+n$ for $n>1$. It follows that the a simplex of $\mathbb{K}$ has dimension $\leq (6g-6+2)-(2g-1) = 4g-3$ for $n=1$ resp. $\leq (6g-6+2n)-(2g-3+n) = 4g-3+n$ for $n>1$. □

Remark 6.15. This result is optimal only for $n=1$ or $g=0$. But the $n=1$ case can be exploited to prove that we can do one unit better for $g>0$, $n>1$: $\mathcal{M}_{g,P}$ has for $g>0$ the homotopy type of a finite CW-complex of dimension $\leq 4g-4+n$.

It is a general fact that if a group of automorphisms of a simply-connected simplicial complex has the property that it has finitely many orbits in the set of vertices, then it is finitely presented. Applying this to $\Gamma(S,P)$ acting on $\mathbb{K}_b(S,P)$ yields:

Corollary 6.16. *The mapping class group $\Gamma(S,P)$ is finitely presented.*

The moduli space of curves as a virtual classifying space. If Γ is a discrete group and M an abelian group on which Γ acts by automorphisms, then one has defined the cohomology groups $H^k(\Gamma;M)$. These can be defined directly as follows: regard $\mathbb{Z}$ as a (trivial) $\mathbb{Z}[\Gamma]$-module and choose a resolution $\cdots \to F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$ by free $\mathbb{Z}[\Gamma]$-modules. Write $F_\bullet$ for the complex $\cdots \to F_2 \to F_1 \to F_0 \to 0$ and consider the (co)complex

$\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(F_\bullet, M)$. Then the kth cohomology group of this complex is independent of the resolution and is denoted $H^k(\Gamma, M)$: it is the kth cohomology group of Γ with values in M. The automorphism group $\mathrm{Aut}(\Gamma)$ of Γ acts on each of these groups, but as interior automorphisms are easily seen to act trivially, that is an action is via $\mathrm{Out}(\Gamma)$.

We can reproduce these groups topologically as the cohomology of a local system: let E be a contractible space on which Γ acts properly and freely (the space and the action must be decent, for instance, the space has admits the structure of a CW complex which is preserved by Γ) and denote by B the orbit space. Then the homotopy type of B only depends on Γ and B is called a *classifying space* of Γ. Moreover, the constant sheaf on E defined by M comes with a Γ-action and so determines a locally constant sheaf $\underline{M}$ on B; it has the property that $H^\bullet(B, \underline{M}) = H^\bullet(\Gamma, M)$. If we drop the freeness of the action of Γ on E, but retain its properness (so that Γ will act with finite stabilizers), then this last assertion is still true if M is a $\mathbb{Q}$-vector space. In particular, $H^\bullet(B, \mathbb{Q}) \cong H^\bullet(\Gamma, \mathbb{Q})$. This is precisely the situation that we have here for the action of the mapping class group $\Gamma(S, P)$ on the Teichmüller space $\mathcal{T}(S, P)$. This action is proper and the Teichmüller space $\mathcal{T}(S, P)$ is contractible and so we have:

Proposition 6.17. *The rational cohomology of $\mathcal{M}_{g,P}$ can be identified with the rational cohomology of $\Gamma_{g,P}$.*

Actually one can do better and show that there is normal subgroup $N \subset \Gamma(S, P)$ of finite index which acts freely on $\mathcal{T}(S, P)$. Then $N\backslash\mathcal{T}(S, P)$ is a classifying space of N and $M_{g,P}$ is its G-quotient. This property makes $M_{g,P}$ what is called a *virtual classifying space* for $\Gamma(S, P)$.

7. Dolbault cohomology and Serre duality for Riemann surfaces (review)

With in mind applications to deformation theory, we briefly review Dolbault cohomology, Riemann-Roch and Serre duality for a Riemann surface.

Dolbault cohomology of a coherent sheaf. For a manifold M we denote by $\mathcal{E}^k_M$ the sheaf of its (C^∞) k-forms and by $\mathcal{E}^k_{M,\mathbb{C}} := \mathbb{C} \otimes_\mathbb{R} \mathcal{E}^k_M = \mathcal{E}^k_M + \sqrt{-1}\mathcal{E}^k_M$ its complexification. The exterior derivative extends complex-linearly to $d_\mathbb{C} : \mathcal{E}^k_{M,\mathbb{C}} \to \mathcal{E}^{k+1}_{M,\mathbb{C}}$.

We begin with a linear discussion. Let T be an oriented real vector space of dimension 2. If J is a conformal structure on T (understood here as a complex structure on T, i.e., as a transformation of T whose square is -1), then the complexification $T_\mathbb{C} := T \otimes_\mathbb{R} \mathbb{C}$ decomposes according to the eigen values of $J_\mathbb{C}$: the $\sqrt{-1}$-eigen space is $T^{1,0} := \{v - \sqrt{-1}Jv\}_{v\in T}$ and the $-\sqrt{-1}$-eigen space is $T^{0,1} := \{v + \sqrt{-1}Jv\}_{v\in T}$. Notice that these complex lines are complex conjugates of each other. Likewise, the 2-dimensional complex vector space $\mathrm{Hom}_\mathbb{R}(T, \mathbb{C})$ splits into two one-dimensional complex vector spaces: $(T^*)^{1,0}$ which consists of the maps ϕ that are $\mathbb{C}$-linear: $\phi(Jv) = \sqrt{-1}\phi(v)$, and its complex conjugate $(T^*)^{0,1}$, consisting of those that are antilinear: $\phi(Jv) = -\sqrt{-1}\phi(v)$.

Let C be a Riemann surface. Then $\mathcal{E}^1_{C,\mathbb{C}}$ decomposes accordingly:

$$\mathcal{E}^1_{C,\mathbb{C}} = \mathcal{E}^{1,0}_C \oplus \mathcal{E}^{0,1}_C$$

and the two summands are interchanged by complex conjugation. We write ∂ for the first component of $d_{\mathbb{C}} : \mathcal{E}^0_{C,\mathbb{C}} \to \mathcal{E}^1_{C,\mathbb{C}} = \mathcal{E}^{1,0}_C \oplus \mathcal{E}^{0,1}_C$ so that $d_{\mathbb{C}} = \partial + \bar\partial$. If C happens to be an open subset U of $\mathbb{C}$, then $\bar\partial f = \frac{\partial f}{\partial \bar z} d\bar z$ and so this is the form that $\bar\partial$ will take in terms of a local coordinate. We thus see that complex valued function on C is in the kernel of $\bar\partial$ precisely, if in terms of any local coordinate its $\frac{\partial}{\partial \bar z}$-derivative is zero, and this is just saying that f is holomorphic.

This generalizes in a neat manner to holomorphic vector bundles over C. A holomorphic vector bundle over a Riemann surface is a complex vector bundle for which we are given a (maximal) atlas of charts for which the transition functions are holomorphic. But for our purpose a better way to introduce this notion is by singling out the class of its holomorphic local sections. If E/C be a complex vector bundle of rank r over a Riemann surface, then we denote by $\mathcal{E}_C(E)$ the sheaf of its smooth sections; this is a locally free $\mathcal{E}^0_{C,\mathbb{C}}$-module of rank r. But for the vector bundle $(T^*C)^{0,1} \otimes E$ we prefer to write $\mathcal{E}^{0,1}_C(E)$ instead of $\mathcal{E}_C((T^*C)^{0,1} \otimes E)$; if $s_1, \dots, s_r$ is a basis of sections of E over an open $U \subset C$, then a section $\omega \in \mathcal{E}^{0,1}(U, E)$ of $\mathcal{E}^{0,1}_C(E)$ over U has the form $\omega = \sum_{i=1}^r \omega_i \otimes s_i$, where each ω_i is a $(0,1)$-form on U (so if U happens to be the domain of a chart z, then $\omega_i = \phi_i d\bar z$ for some C^∞-function $\phi_i : U \to \mathbb{C}$).

Definition 7.1. A *holomorphic structure* on E/C amounts to giving a $\mathbb{C}$-linear sheaf map $\bar\partial_E : \mathcal{E}_C(E) \to \mathcal{E}^{0,1}_C(E)$ satisfying the following properties:

(i) if $s \in \mathcal{E}_C(E)$ and $f \in \mathcal{E}_{C,\mathbb{C}}$, then $\bar\partial_{E|U}(fs) = f\bar\partial_{E|U}(s) + \bar\partial f \otimes s$ and
(ii) $\mathcal{E}_C(E)$ is generated by $\mathrm{Ker}(\bar\partial_E)$.

If E/C has been equipped with such a structure, then we say that it is a *holomorphic vector bundle*. A local section $s \in \mathcal{E}_C(E)$ is then called *holomorphic* if $\bar\partial_E(s) = 0$.

Property (ii) says that we can cover C by open subsets U such that $E|U$ admits a basis of holomorphic sections. If $s_1, \dots, s_r$ is such a basis, then an arbitrary section s of $E|U$ can be written $s = f_1 s_1 + \cdots f_r s_r$ and so $\bar\partial_{E|U}(s) = \sum_i \bar\partial f_i \otimes s_i$. We see that s is holomorphic if and only if the functions $f_1, \dots, f_r$ are. So the local holomorphic sections of E form a sheaf of $\mathcal{O}_C$-modules locally free of the same rank as E/C. We denote that sheaf $\mathcal{O}(E)$. An integration argument (which we will not reproduce here) shows that $\bar\partial_E : \mathcal{E}_C(E) \to \mathcal{E}^{0,1}_C(E)$ is locally surjective, so that we have in fact a short exact sequence of abelian sheaves on C:

$$0 \to \mathcal{O}(E) \to \mathcal{E}(C, E) \xrightarrow{\bar{\partial}_E} \mathcal{E}^{0,1}(C, E) \to 0.$$

Conversely, every sheaf of locally free $\mathcal{O}_C$-modules of finite constant rank comes from a vector bundle. This discussion also shows how we can pass from this definition to the atlas definition and vice versa.

Important examples are the holomorphic cotangent bundle of C, whose sheaf of sections is the sheaf of holomorphic differentials Ω_C, and its holomorphic dual, the sheaf of holomorphic vector fields θ_C. Notice that the standard ways of producing out of vector bundles a new one (like 'taking the dual' or 'tensor product') respect the category of holomorphic vector bundles.

Given a space X, then for every abelian sheaf $\mathcal{F}$ over X and every $k \in \mathbb{Z}$ is defined its cohomology group $H^k(X, \mathcal{F})$. This is zero for $k < 0$ and $H^0(X, \mathcal{F})$ is the space of its sections. These groups have among other things the property that a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of such sheaves gives rise to a long exact sequence

$$\cdots \to H^k(X, \mathcal{F}') \to H^k(X, \mathcal{F}) \to H^k(X, \mathcal{F}'') \to H^{k+1}(C, \mathcal{F}') \to \cdots$$

which is canonical (relative to morphisms of such short exact sequences). Once one knows of a sufficiently large class of abelian sheaves that they do not have any cohomology in nonzero degree, then one is often able to compute these groups using the mentioned properties only (so without knowing the actual definition). An example of such a class consists of the sheaves that arise from differentiable vector bundles: the sheaf of C^∞-sections of a vector bundle over a manifold has no cohomology in nonzero degree. This can be used to show that the cohomology of the manifold M with coefficients in the constant sheaf $\underline{\mathbb{R}}_M$ is the same as its De Rham cohomology.

A similar reasoning shows that for a holomorphic vector bundle E/C, $H^k(C, \mathcal{O}(E))$ is the cohomology of the *Dolbeault complex*

$$0 \to \mathcal{E}(C, E) \xrightarrow{\bar{\partial}_E} \mathcal{E}^{0,1}(C, E) \to 0,$$

meaning that

$$H^k(C, \mathcal{O}(E)) = \begin{cases} \mathrm{Ker}(\bar{\partial}_E) = H^0(C, \mathcal{O}(E)) & \text{if } k = 0, \\ \mathrm{Coker}(\bar{\partial}_E) & \text{if } k = 1, \\ 0 & \text{if } k \neq 0, 1. \end{cases}$$

In case C is connected, but not compact, then a fundamental theorem asserts that $H^1(C, \mathcal{O}(E)) = 0$.

Riemann-Roch and Serre duality for a Riemann surface. In this subsection we assume that C is compact connected of genus g. We mention without proof:

Theorem 7.2. *The Dolbeault cohomology group* $H^k(C, \mathcal{O}(E))$ *and more generally,* $H^k(C, \mathcal{F})$*, where* $\mathcal{F}$ *is a coherent* $\mathcal{O}_C$*-module, is finite dimensional.*

The somewhat elusive $H^1(C, \mathcal{O}(E))$ can often be understood via Serre duality: denoting by $\mathcal{E}^{1,0}(C, E^*)$ resp. $\Omega(E^*)$ the sheaf of C^∞ resp. holomorphic sections of the holomorphic cotangent bundle of C tensored with E, then the duality pairing $E \otimes E^* \to \mathbb{C}_C$ determines a natural wedge product

$$\wedge : (\alpha, \beta) \in \mathcal{E}^{0,1}(C, E) \times \mathcal{E}^{1,0}(C, E^*) \to \alpha \wedge \beta \in \mathcal{E}^{1,1}(C) = \mathcal{E}^2(C)_{\mathbb{C}}$$

Notice that if $\alpha = \bar\partial \alpha'$ with $\alpha' \in \mathcal{E}(C, E)$ and β is holomorphic (so $\beta \in H^0(C, \Omega(E^*))$), then $\alpha \wedge \beta = \bar\partial(\alpha' \wedge \beta) = d(\alpha' \wedge \beta)$. By Stokes theorem this gets killed after integration over C. Hence $(\alpha, \beta) \mapsto \int_C \alpha \wedge \beta$ induces a pairing

$$H^1(C, \mathcal{O}(E)) \times H^0(C, \Omega(E^*)) \to \mathbb{C}.$$

The *Serre duality theorem* states that this pairing is nondegenerate.

Example 7.3. Serre duality identifies the dual of $H^1(C, \mathcal{O}_C)$ with $H^0(C, \Omega_C)$; in case C is connected, their common dimension is the genus of C (this is in fact the way to define the genus in Algebraic Geometry, where one cannot resort to topology). We also note the duality between $H^1(C, \Omega_C)$ and $H^0(C, \mathcal{O}_C) = \mathbb{C}$ and between $H^1(C, \theta_C)$ and $H^0(C, \Omega_C^{\otimes 2})$ (which is the space of holomorphic quadratic differentials).

For a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_C$-modules, one defines its *Euler characteristic* $\chi(C, \mathcal{F})$ by $\chi(C, \mathcal{F}) := \dim H^0(C, \mathcal{F}) - \dim H^1(C, \mathcal{F})$. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of coherent $\mathcal{O}_C$-modules, then its associated long exact sequence shows that $\chi(C, \mathcal{F}) := \chi(C, \mathcal{F}') + \chi(C, \mathcal{F}'')$. The Riemann-Roch theorem tells us that this Euler characteristic is of a topological nature. The formalism used here makes it easy to compute it from knowing $\chi(\mathcal{O}_C)$ only. In fact, put $\deg(\mathcal{F}) := \mathrm{rk}(\mathcal{F})(g-1) + \chi(\mathcal{F})$ so that

$$\chi(\mathcal{F}) = \mathrm{rk}(\mathcal{F})(1-g) + \deg(\mathcal{F}).$$

We show how to compute $\deg(\mathcal{F})$. With that recipe, the above definition gets mathematical content and is then called the *Riemann-Roch theorem*. We first observe that since χ and rk are additive for short exact sequences, deg is additive as well. The formula for the genus asserts that $\chi(\mathcal{O}_C) = 1-g$ and so $\deg(\mathcal{O}_C) = 0$. If on the other hand $\mathcal{F}$ has finite support, then $H^1(C, \mathcal{F}) = 0$ and $\mathrm{rk}(\mathcal{F}) = 0$, so that $\deg(\mathcal{F}) = \dim H^0(C, \mathcal{F})$. Now let $D = \sum_{i=1}^k n_i(p_i)$ be a divisor on C, with $p_1, \dots, p_k$ distinct and with $n_i \geq 0$. Then we have a short exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(D) \to \mathcal{O}_C(D)/\mathcal{O}_C \to 0.$$

Notice that $\mathcal{O}_C(D)/\mathcal{O}_C$ has finite support and that $H^0(C, \mathcal{O}_C(D)/\mathcal{O}_C) \cong \oplus_{i=1}^k z^{-n_i}\mathbb{C}\{z\}/\mathbb{C}\{z\}$. It follows that the dimension of $H^0(C, \mathcal{O}_C(D)/\mathcal{O}_C)$ equals $\sum_{i=1}^k n_i$ and so $\deg(\mathcal{O}_C(D)/\mathcal{O}_C) = \dim H^0(C, \mathcal{O}_C(D)/\mathcal{O}_C) = \sum_{i=1}^k n_i$. Since $\deg(\mathcal{O}_C) = 0$, it then follows that $\deg(\mathcal{O}_C(D)) = \sum_{i=1}^k n_i$.

Exercise 15. Prove in a similar fashion that the same formula holds if more generally $n_i \in \mathbb{Z}$.

Exercise 16. Prove that $\deg(\Omega_C) = 2g-2$.

8. Deformation theory of Riemann surfaces

Orbifolds. Let G be a Lie group which acts smoothly a manifold M and denote by $\pi : M \to G\backslash M$ the formation of the quotient space. We assume that G acts properly discontinuously (this means that $(g,p) \in G \times M \mapsto (gp,p) \in M \times M$ is proper) and that the G-stabilizer G_p of every $p \in M$ is finite.

Example 8.1. Let $k, l \in \mathbb{Z} - \{0\}$ and let the group $\mathbb{C}^\times$ act on $\mathbb{C}^2 - \{(0,0)\}$ by assigning to $\lambda \in \mathbb{C}^\times$ the diagonal matrix with entries (λ^k, λ^l). This action is properly continuous. The stabilizer of (z_1, z_2) is a group of roots of unity of order $\gcd(k,l)$, k, l depending on whether $z_1 z_2 \neq 0$, $z_1 \neq 0 = z_2$ or $z_1 = 0 \neq z_2$.

Exercise 17. Prove that if k and l have the same sign and $d := \gcd(k,l)$, then $(z_1, z_2) \in \mathbb{C}^2 - \{(0,0)\} \mapsto [z_1^{l/d} : z_1^{k/d}] \in \mathbb{P}^1$ defines the orbit space. Show that if k and l have different sign, then the orbits of $(1,0)$ and $(0,1)$ cannot be separated in the orbit space.

Definition 8.2. A *slice* to the G-action on M is a submanifold $S \subset M$ such that the evaluation map $(g,q) \in G \times S \mapsto gq \in M$ sends a neighborhood of $\{1\} \times S$ diffeomorphically onto an open subset of M. We call it a *good slice* if in addition every $g \in G$ with $gS \cap S \neq \emptyset$ stabilizes S: $gS = S$.

Exercise 18. Prove that for any slice $S \subset M$, $\pi|S : S \to G\backslash M$ is open. Prove that if S is a good slice and G_S is the group of $g \in G$ with $gS = S$, then $\pi|S$ factors through a homeomorphism of $G_S\backslash S$ onto an open subset of $G\backslash M$.

Lemma 8.3. *Through every point $p \in M$ passes a good slice S whose G-stabilizer is the finite group G_p. If S' is another good slice through p, then after shrinking both S and S', there is a natural G_p-equivariant diffeomorphism $S \cong S'$ which commutes with π.*

Proof. Choose a Riemannian metric on S and make it G_p-invariant by replacing it by the sum of its G_p-transforms. The open unit ball B in the orthogonal complement of $T_p Gp$ in $T_p M$ is clearly invariant under G_p. For $\varepsilon > 0$ small, the exponential map will map εB diffeomorphically onto a submanifold S_ε of M which meets every orbit transversally. Clearly, S_ε wil be G_p-invariant.

The derivative of the map $(g,q) \in G \times S_\varepsilon \mapsto gq \in M$ in $(1,p)$ is a linear isomorphism and so by the inverse function theorem there exists a neighborhood U of $1 \in G$ and an $\varepsilon > 0$ such that the evaluation map $U \times S_\varepsilon \to M$ is a diffeomorphism onto an open subset of M.

Since G acts properly discontinuously on M we can take U and $\varepsilon > 0$ so small that $gS_\varepsilon \cap S_\varepsilon \neq \emptyset$ implies that $g \in UG_p$. So if $g \in G$ and $q \in S_\varepsilon$

are such that $gq \in S_\varepsilon$, then $g = g''g'$ with $g' \in G_p$ and $g'' \in U$. But then $g'q \in S_\varepsilon$ (for S_ε is G_p-invariant) and since $U \times S_\varepsilon \to M$ is injective, it follows that $g'' = 1$. Hence $S := S_\varepsilon$ is as desired.

If S' is another good slice, then after shrinking S, there is for every $q \in S$ a unique $g \in U$ with $gq \in S$. Then $q \in S \mapsto gq \in S'$ maps the good slice S diffeomorphically onto a good slice contained in S'. □

If S is a slice at p, then T_pS is a supplement in T_pM of the tangent space $T_p(Gp)$ of the G-orbit of p and so we may identify T_pS with the normal space $T_pM/T_p(Gp)$, or equivalently, with the cokernel of the map $\mathrm{Lie}(G) \to T_pM$ obtained from the derivative of the G-action (where $\mathrm{Lie}(G)$ denotes the Lie algebra of G).

Notice that if G_p is trivial, then a good slice maps homeomorphically onto an open subset of $G\backslash M$. This exhibits a C^∞-structure on that open subset. Lemma 8.3 shows that this structure is independent of the slice. The tangent space of $G\backslash M$ at $\pi(p)$ may be identified with $T_pM/T_p(Gp)$.

When the stabilizers G_p are nontrivial one still feels that there some structure on $G\backslash M$ worth remembering (and elucidating) and whose complexity does not go beyond that of the case when G is finite. A satisfactory way to nail this down is by regarding the good slices, or rather, their maps to open subsets of $G\backslash M$, as charts of $G\backslash M$, just as one may define a differentiable structure on a manifold. We will here loosely refer to this structure as an *orbifold*.

Remark 8.4. The actual definition of an orbifold does not assume that everything comes from a Lie group acting on a manifold. An *orbifold atlas* for a Hausdorff space X is a collection $\mathfrak{S}$ of triples (S, G_S, π_S), where S is a smooth manifold, G_S a discrete group acting properly discontinuously on S and $\pi_S : S \to X$ a continuous open map which is constant on the G_S-orbits and which induces a homeomorphism of $G_S\backslash S$ onto an open subset of X. In addition one assumes that there are the analogues of inclusions between open subsets that turn $\mathfrak{S}$ into a category: a morphism $(S', G_{S'}, \pi_{S'}) \to (S, G_S, \pi_S)$ is given by a group monomorphism $\phi : G_{S'} \to G_S$ and an open $G_{S'}$-equivariant embedding $f : S' \to S$ over X (i.e., such that $\pi_S f = \pi_{S'}$ and $fg = \phi(g)f$ for all $g \in G_{S'}$) which identifies $G_{S'}$ with the G-stabilizer of $f(S')$. This is set up in such a manner that the morphisms with fixed target (S, G_S, π_S) render the same service for $G_S\backslash S$ as the whole structure is supposed to provide for the original context (i.e., $G\backslash M$): we demand that any open covering of S can be refined as a covering by the images of $\mathfrak{S}$-morphisms.

An *orbifold structure* on X is simply an orbifold atlas that is not contained in a strictly bigger one.

An example to keep in mind is $M = \mathrm{Conf}(TS/S)$ and $G = \mathrm{Diff}^+(S, P)$. Its sheer infinite dimensionality makes this fall outside the scope of the discussion above. Yet the associated orbifold exists and is here representable by a single triple, namely $(\mathcal{T}(S, P), \Gamma(S, P), \mathcal{T}(S, P) \to \mathcal{M}_{g,P})$. The notion of a good slice will here only serve as a useful heuristic that helps us to justify the notion of universal deformation that we will introduce in order to put more structure on $\mathcal{M}_{g,P}$.

Heuristic approach towards the tangent bundle of Teichmüller space. Here S is a closed connected surface of genus g and $P \subset S$ is an n-element subset. Let J be a conformal structure on S, so that we have a Riemann surface C. The stabilizer of J in $\mathrm{Diff}^+(S,P)$ is the group of automorphisms of C that are the identity on P. In the nonhyperbolic cases $(g,n) = (0,0)$, $(0,1)$, $(0,2)$, $(1,0)$, this group is infinite, but otherwise it acts on C° as a group of isometries and can easily be shown to be finite. We therefore assume that $2g-2+n > 0$. Let J be a conformal structure on S, so that we have a Riemann surface C. We want to compute the tangent space of $\mathcal{T}(S,P)$ at the point defined by $[J]$. We take a naive approach: we first determine the space of first order deformations of J in $\mathrm{Conf}(S)$ and divide out by the space of first order deformations that come from $\mathrm{Diff}^o(S,P)$ (of which we think as a vector fields on S that are zero on P).

Let T be an oriented real vector space of dimension 2 as before. If J is a conformal structure on T, then we have defined $T_J^{1,0} \subset T_\mathbb{C}$ and the composite map $T \subset T_\mathbb{C} \to T_\mathbb{C}/T_J^{0,1}$ has the property that this yields a $\mathbb{C}$-linear isomorphism of (T,J) onto $T_\mathbb{C}/T_J^{0,1}$. We see from this description that J is completely determined by $T_J^{0,1}$. Conversely, if $L \subset T_\mathbb{C}$ is such that the $\mathbb{R}$-linear map $T \subset T_\mathbb{C} \to T_\mathbb{C}/L$ is orientation preserving (where we give $T_\mathbb{C}/L$ the orientation defined by its complex structure), then we can transfer the complex structure of $T_\mathbb{C}/L$ to T and get an element of $\mathrm{Conf}(T)$. We cannot have $L = T_J^{1,0}$ (for $T \subset T_\mathbb{C} \to T_\mathbb{C}/T_J^{1,0}$ is orientation reversing) and so L can be given as the graph of a $\mathbb{C}$-linear map $\phi : T_J^{0,1} \to T_J^{1,0}$. Which ϕ occur this way? If $e \in T$ is nonzero, then $e - \sqrt{-1}Je \in T_J^{1,0}$ resp. $e + \sqrt{-1}Je \in T_J^{0,1}$ are generators and so if $\phi(e+\sqrt{-1}Je) = \lambda(e-\sqrt{-1}Je)$, with $\lambda \in \mathbb{C}$, then

$$e+\sqrt{-1}Je+\lambda(e-\sqrt{-1}Je) = (1+\lambda)e+\sqrt{-1}(1-\lambda)Je = (1+\lambda)\Big(e+\sqrt{-1}\frac{1-\lambda}{1+\lambda}Je\Big)$$

is a generator of L. Since L cannot be real, it follows that $\mathrm{Re}(\frac{1-\lambda}{1+\lambda}) \neq 0$, or equivalently, $|\lambda| \neq 1$. We need to be in the part where the complex structure is compatible with the given orientation and this amounts to the condition $|\lambda| < 1$. We thus find:

Lemma 8.5. *The space of conformal structures on* T, $\mathrm{Conf}(T)$, *can be identified with the open subset of the complex projective line* $\mathbb{P}(T_\mathbb{C})$ *parametrizing the complex lines* $L \subset T_\mathbb{C}$ *for which* $T \subset T_\mathbb{C} \to T_\mathbb{C}/L$ *is an orientation preserving* $\mathbb{R}$*-linear isomorphism, hence has a natural complex structure (it is in fact a copy of the complex unit disk). Its tangent space* $T_{[J]}\,\mathrm{Conf}(T)$ *is as a complex vector space naturally isomorphic* $\mathrm{Hom}_\mathbb{C}(T_J^{0,1}, T_J^{1,0}) = (T^*)_J^{0,1} \otimes_\mathbb{C} T_J^{1,0}$.

We return to $C = (S,J)$. Lemma 8.5 shows that the tangent space of $\mathrm{Conf}(TS/S)$ at $[J]$ is the space of sections of the complex vector bundle $\mathrm{Hom}(T^{0,1}C, T^{1,0}C)$. But this is just $\mathcal{E}^{0,1}(T^{1,0}C)$.

Suppose we have a curve $t \in (-\varepsilon,\varepsilon) \mapsto h_t \in \mathrm{Diff}^0(S)$ with $h_0 = 1$. Then this defines a vector field $\frac{d}{dt}|_{t=0}h_t$ on S. It is a real section of the complexified tangent bundle $T^{1,0}C \oplus T^{0,1}C$: $\frac{d}{dt}|_{t=0} = (X, \bar{X})$, with $X \in \mathcal{E}(C, T^{1,0}C)$. We thus regard $\mathcal{E}(C, T^{1,0}C)$ as the Lie algebra of $\mathrm{Diff}^0(S)$.

The tangent vector of $\mathrm{Conf}(TS/S)$ defined by the curve $t \in (-\varepsilon, \varepsilon) \mapsto h_t^* J$ is given as $\frac{d}{dt}|_{t=0} h_t^* J \in \mathrm{Hom}(T^{0,1}C, T^{1,0}C)$ and we want to understand this as an element of $\mathcal{E}^{0,1}(T^{1,0}C)$. If (U, z) is a coordinate chart for C, then

$$h_t^* dz = \frac{\partial h_t}{\partial z} dz + \frac{\partial h_t}{\partial \bar{z}} d\bar{z}.$$

We have $h_t^* dz \equiv dz \pmod t$ (for h_0 is the identity) and $\frac{\partial}{\partial t}|_{t=0} \frac{\partial h_t}{\partial \bar{z}} d\bar{z} = \frac{\partial}{\partial \bar{z}}(\frac{\partial h_t}{\partial t}|_{t=0}) d\bar{z} = \bar{\partial} X$. Hence the tangent vector of $\mathrm{Conf}(TS/S)$ is on U given by $\frac{\partial}{\partial \bar{z}} \mapsto \frac{\partial X}{\partial \bar{z}}$ or equivalently, by $\bar{\partial} X$. Thus the tangent space of the $\mathrm{Diff}^0(S)$-orbit of $[J]$ can be identified with $\bar{\partial}\mathcal{E}(C, T^{1,0}C) \subset \mathcal{E}^{0,1}(C, T^{1,0}C)$.

This suggests that the tangent space $T_{[J]}\mathcal{T}(S)$ of $\mathcal{T}(S)$ at $[J]$ can be identified as the Dolbeault cohomology space $H^1(C, \theta_C)$. In particular, that it has a natural complex structure. So by Serre duality its complex dual (the cotangent space $T^*_{[J]}\mathcal{T}(S)$) should be identified with the space of quadratic differentials $H^0(C, \Omega_C^{\otimes 2})$.

A similar heuristic reasoning shows that the tangent space $T_{[J]}\mathcal{T}(S, P)$ of $\mathcal{T}(S)$ at $[J]$ can be identified with $H^1(C, \theta_C(P))$, where $\theta_C(-P)$ is the sheaf of holomorphic vector fields which vanish in P.

Deformation theory. The Ehresmann fibration theorem says that a proper submersion $f : \mathcal{M} \to B$ between manifolds is C^∞-locally trivial. So for any $o \in B$ there exist an open neighborhood U of o in B and a differentiable retraction $r : \mathcal{M}_U \to M_o$ such that $(f_U, r) : \mathcal{M}_U \to U \times M_o$ is a diffeomorphism. Notice that then for every $b \in U$, r restricts to a diffeomorphism $r_b : M_b \cong M_o$. This diffeomorphism varies smoothly with b and since r is a retract, r_o is the identity. If U is path connected, then the isotopy class of r_b is independent of the choice of r, for if r' is another differentiable retraction $r : M_U \to M_o$ as above, then choose a path $\gamma : [0, 1] \to U$ from o to b. The map $t \in [0, 1] \mapsto h_t := r'_{\gamma(t)} r^{-1}_{\gamma(t)} \in \mathrm{Diff}(M_o)$ is a path with h_0 the identity and so $h_t r_o$ connects $h_0 r_b = r_b$ with $h_1 r_b = r'_b r_b^{-1} r_b = r'_b$. In other words, we have a natural identification of M_b with M_o up to isotopy. The Ehresmann fibration theorem has also relative version: if in the situation above, $\mathcal{N} \subset \mathcal{M}$ is a closed submanifold such that $f|\mathcal{N}$ is also a submersion, then we may choose U and r in such a manner that (f_U, r) maps $\mathcal{N}_U$ onto $U \times N_o$. We find that for U path connected, we have for every $b \in U$ a well-defined isotopy class of diffeomorphisms of (M_b, N_b) onto (M_o, N_o). This makes the Ehresmann fibration theorem a convenient tool for the study of complex structures up to isotopy. In particular, it suggests:

Definition 8.6 (A family of pointed Riemann surfaces)**.** Let B be a complex manifold. A *family of* P*-pointed Riemann surfaces of genus* g consists a proper holomorphic submersion of complex manifolds $f : \mathcal{C} \to B$ endowed with a set x_P pairwise disjoint holomorphic sections $(x_p : B \to \mathcal{C})_{p \in P}$ such that every fiber $C_b := f^{-1}(b)$ is a connected Riemann surface of genus g. We call the domain and range of f the *total space* and the base respectively.

So if $U \subset B$ is a path connected open subset such that f is C^∞-locally trivial over U relative to the sections indexed by P with fiber (S, P), then we have define a map from to the Teichmüller space $\mathcal{T}(S, P)$. We will also need a local version:

Definition 8.7 (Deformation of a pointed Riemann surface)**.** Let C be a closed Riemann surface and $P \subset C$ a finite subset. A *deformation* of (C, P) consists of a holomorphic submersion of complex manifolds $f : \mathcal{C} \to B$, a set x_P of holomorphic sections $(x_p : B \to \mathcal{C})_{p \in P}$, a point $o \in B$ and an isomorphism of complex manifolds $\iota : C \cong C_o$ such that $\iota(p) = x_p(o)$ for all p. It is here understood that for any neighborhood U of o in B, the system $(f_U : \mathcal{C}_U \to U, x_P|U, o, \iota)$ defines the same deformation.

A deformation is therefore given as a system $\mathcal{D} = (f, x_P, \iota)$ where $f : (\mathcal{C}, C_o) \to (B, o)$ is a *germ* of a submersion, $\iota : C \cong C_o$ an isomorphism, and $x_p : (B, o) \to (\mathcal{C}, C_o)$ a germ of a section with $\iota(p) = x_p(o)$. We need not require that the sections are pairwise disjoint: since their values in o are distinct, so will be their values in a neighborhood of o. The preceding shows that if S is the surface underlying C, then for B connected but so small that the sections x_p are pairwise disjoint, then such a deformation yields a continuous map $B \to \mathcal{T}(S, P)$. We call this the *classifying map* of the deformation (although strictly speaking this is only defined as map-germ $(B, o) \to \mathcal{T}(S, P)$). Note also that f then defines a deformation of (C_b, P) for every $b \in B$.

The deformations are the objects of a category $\mathfrak{Def}_{C,P}$ for which a morphism $\mathcal{D}' \to \mathcal{D}$ is given by a Cartesian diagram

$$\begin{array}{ccc} (\mathcal{C}', C'_o) & \xrightarrow{\Phi} & (\mathcal{C}, C_o) \\ {\scriptstyle f'}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ (B', o') & \xrightarrow{\phi} & (B, o) \end{array}$$

of holomorphic maps such that $\iota = \Phi\iota'$ and $\Phi x'_p = x_p\phi$ for all $p \in P$. We recall that the Cartesian property means that the diagram is commutative and that the resulting map from $(\mathcal{C}', C'_o)$ to the fiber product $(B', o') \times_{(B,o)} (\mathcal{C}, C_o)$ is an isomorphism, where

$$B' \times_B \mathcal{C} = \{(b', q) \in B' \times \mathcal{C} \mid \phi(b') = f(q)\}.$$

The Cartesian property implies that Φ maps $C'_{b'}$ isomorphically onto $C_{\phi(b')}$, taking $x'(p)$ to $x(p)$. So the mere existence of a morphism $\mathcal{D}' \to \mathcal{D}$ implies that $\mathcal{D}$ is richer than $\mathcal{D}'$ in the sense that every deformed complex structure appearing in the family $\mathcal{D}'$ appears in $\mathcal{D}$.

Remark 8.8. One actually wants to allow the base germ (B, o) to be singular and asks that f is local-analytically trivial on $\mathcal{C}$: at every point $x \in C_o$, f is the last component of an isomorphism of germs $(\mathcal{C},x) \cong (C_o,x)\times(B,o)$ (this is usually phrased by asking that f be *flat*), but the more restrictive definition above is good enough for our purposes.

Definition 8.9. We say that a deformation $\mathcal{D}$ is *universal* if it is a final object of the deformation category $\mathfrak{Def}_{C,P}$.

We recall that a universal object of a category must be unique up to unique isomorphism. The notion of a universal deformation will serve as a workable substitute for our previously defined notion of a good slice. Let us show for instance, that if $\mathcal{D}=(f,x_P,\iota)$ is a universal deformation, then the stabilizer of the point of $\mathcal{T}(S,P)$ defined by the conformal structure defined by J acts on it. To see this, note that if $h\in \mathrm{Aut}(C,P)$, then $\mathcal{D}_h=(f,x_P,\iota h^{-1})$ is also a deformation (C,P) and so there is a unique morphism $(\Phi_h,\phi_h):\mathcal{D}_h\to\mathcal{D}$. This means that the triple $(h,\Phi_,\phi_h)$ defines an endomorphism of the diagram $C\xrightarrow{\iota}(\mathcal{C},C_o)\xrightarrow{f}(B,o)$ in such a way that it commutes with the sections $(x_p)_{p\in P}$. If $h'\in\mathrm{Aut}(C,P)$ is another automorphism, then the uniqueness property of the universal deformation implies that $\Phi_h\Phi_{h'}=\Phi_{hh'}$ and $\phi_h\phi_{h'}=\phi_{hh'}$. It then follows that $h\mapsto (h,\Phi_h,\phi_h)$ defines an action of $\mathrm{Aut}(C,P)$ on the universal deformation with $(h^{-1},\Phi_{h^{-1}},\phi_{h^{-1}})$ giving the two-sided inverse of $(h,\Phi_,\phi_h)$.

The Kodaira Spencer map. We define for a (not necessarily universal) deformation $\mathcal{D}=(f,x_P,\iota)$ a linear map $T_oB\to H^1(C,\theta_C)$ that can be understood as the derivative of the associated map-germ $(B,o)\to\mathcal{T}(S,P)$. Since f is a submersion and ι an isomorphism onto its fiber over o, we have for every $q\in C$ an exact sequence

$$0\to T_qC\xrightarrow{D_q\iota}T_{\iota(q)}\mathcal{C}\xrightarrow{D_{\iota(q)}f}T_oB\to 0.$$

This leads to the following exact sequence of vector bundles

$$0\to\theta_C\to\iota^*\theta_{\mathcal{C}}\to\mathcal{O}_C\otimes T_oB\to 0$$

Let $\theta_{\mathcal{C}}(\log x_P)\subset\theta_{\mathcal{C}}$ denote the sheaf of vector fields on $\mathcal{C}$ that are tangent to the sections $(x_p)_{p\in P}$. So $\theta_{\mathcal{C}}(\log x_P)$ equals $\theta_{\mathcal{C}}$ away from $\cup_{p\in P}x_p(B)$. At a point $x_p(b)$ we can describe it as follows: choose local coordinates $(w_1,\dots,w_k)$ on a neighborhood V of $b\in B$ and a holomorphic function z_0 on a neighborhood U of $x_p(b)$ in $f^{-1}V$ which is zero on $x_p(B)$ and is such that $(z_0,z_1:=f^*w_1,\dots,z_k:=f^*w_k)$ is a coordinate system on U. Then the sheaf $\theta_{\mathcal{C},x_P}|U$ has as a $\mathcal{O}_U$-module the basis $z_0\frac{\partial}{\partial z_0},\frac{\partial}{\partial z_1},\dots\frac{\partial}{\partial z_k}$. It follows that $\iota^*\theta_{\mathcal{C}}(\log x_P)$ is equal to $\iota^*\theta_{\mathcal{C}}$ away from P and if $p\in P$, then for a coordinate system as above with $b=o$ (so that $x_p(o)=\iota(p)$), $\iota^*\theta_{\mathcal{C}}|U_o$ has the $\mathcal{O}_{U_o}$-basis $z_0\frac{\partial}{\partial z_0},\frac{\partial}{\partial z_1},\dots\frac{\partial}{\partial z_k}$, where $U_o=\iota^{-1}U$. We now see that the above sequence becomes the exact sequence

$$0\to\theta_C(-P)\to\iota^*\theta_{\mathcal{C}}(\log x_P)\to\mathcal{O}_C\otimes T_oB\to 0.$$

Its long exact cohomology sequence begins as

$$0 \to H^0(C, \theta_C(-P)) \to H^0(C, \iota^*\theta_{\mathcal{C}}(\log x_P)) \to T_oB \xrightarrow{\delta} H^1(C, \theta_C(-P)) \to \dots$$

and the map $\delta : T_oB \to H^1(C, \theta_C(-P))$ is the one are after. It is called the *Kodaira Spencer* map.

Lemma 8.10. *We have* $H^0(C, \theta_C(-P)) = 0$ *if and only if we are in the hyperbolic case (*$2g-2+n > 0$*), and then* $H^1(C, \theta_C(-P))$ *is of dimension* $3g-3+n$*.*

Proof. Observe that $H^0(C, \theta_C(-P))$ is the space of holomorphic vector fields on C that are zero in P. Any nontrivial vector field on C has as many zeroes (counted with multiplicity) as its Euler characteristic $2-2g$. So for $g > 1$, there is no such vector field, for $g = 1$ it is without zeroes, and for $g = 0$ it has two zeroes. These exceptions correspond to the nonhyperbolic cases.

Assuming that $H^0(C, \theta_C(-P)) = 0$, then the Riemann-Roch formula implies that $\dim_{\mathbb{C}} H^1(C, \theta_C(-P)) = g - 1 - \deg(\theta_C(-P))$. We have $\deg(\theta_C) = -\deg(\Omega_C) = 2-2g$ and hence $\deg(\theta_C(-P)) = 2-2g-n$. The lemma follows. □

Here is the central result:

Theorem 8.11. *Assume that we are in the hyperbolic case:* S° *has negative Euler characteristic. Then a deformation* (C, P) *is universal if and only if its Kodaira-Spencer map is an isomorphism. If these equivalent conditions are satisfied, then the deformation can be represented by a family which defines a universal deformation of each of its fibers and whose classifying map is homeomorphism onto an open subset of* $\mathcal{T}(S, P)$*.*

The proof, which we omit, involves a fair amount of work.

The classifying maps of universal deformations as above now provide an atlas for $\mathcal{T}(S, P)$. Since a universal deformation is unique up to unique (holomorphic) isomorphism, we conclude:

Corollary 8.12. *Suppose* S° *has negative Euler characteristic. Then* $\mathcal{T}(S, P)$ *has the structure of a complex manifold that is characterized by the property that the classifying maps of universal deformations are local isomorphisms. In fact,* $\mathcal{T}(S, P)$ *supports the* universal Teichmüller family*: it is the base of a* P*-pointed family* $(\mathcal{C}(S, P) \to \mathcal{T}(S, P), x_P)$ *of genus* g *Riemann surfaces in such a manner that for every* $b \in \mathcal{T}(S, P)$*, the germ of this family at* b *is a universal deformation of the fiber* (C_b, P)*. The complex tangent space of* $\mathcal{T}(S, P)$ *at* b *is naturally identified with* $H^1(C_b, \theta_{C_b}(P))$*.*

The mapping class group $\Gamma(S, P)$ *acts on this family as a group of biholomorphic transformations. The* $\Gamma(S, P)$*-stabilizer of any* $b \in \mathcal{T}(S, P)$ *is just the group of automorphisms of* C_b *that leave every* $x_p(b)$ *fixed.*

So the group $\Gamma(S, P)$ acts faithfully on $\mathcal{C}(S, P)$. It need not be so on $\mathcal{T}(S, P)$ as the two examples below will show.

Example 8.13. The case $(g, n) = (1, 1)$. We know that $(\Gamma_{1,1}, \mathcal{T}_{1,1})$ can be identified with $(\mathrm{SL}(2, \mathbb{Z}), \mathbb{H})$ and so $\mathrm{SL}(2, \mathbb{Z})$ acts on $\mathbb{H}$ with kernel its center $\{\pm 1\}$. The universal family pointed family $(f : \mathcal{C} \to \mathbb{H}, x_0)$ is then obtained as the quotient of $\mathbb{H} \times \mathbb{C}$ by the lattice of translations acting by $(k, l) \in \mathbb{Z}^2 : (\tau, z) \mapsto (\tau, z + k + l\tau)$ and $x_0 : \mathbb{H} \to \mathcal{C}$ is given by $\tau \mapsto (\tau, 0)$. The mapping class group action on this pointed family is then defined by letting $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ act on $\mathcal{C}$ as

$$(\tau, z + \mathbb{Z} + \tau\mathbb{Z}) \in \mathcal{C} \mapsto \Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} + \mathbb{Z} + \frac{a\tau + b}{c\tau + d}\mathbb{Z}\Big) \in \mathcal{C}.$$

This is best seen by regarding $\mathbb{H}$ as the space of oriented $\mathbb{R}$-bases of $\mathbb{C}$ up to multiplication by a complex nonzero scalar: then every basis is uniquely represented by one of the form $(1, \tau)$ with $\tau \in \mathbb{H}$. The group $\mathrm{SL}(2, \mathbb{Z})$ acts by letting $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ send the basis $(1, \tau)$ to the basis $(c\tau + d, a\tau + b)$, which is a basis of the same lattice $\mathbb{Z} + \tau\mathbb{Z}$. But that basis is represented by $(1, \frac{a\tau+b}{c\tau+d})$. Thus $(1, \tau; z + \mathbb{Z} + \tau\mathbb{Z}) \in \mathcal{C}$ goes to $(c\tau + d, a\tau + b; z + \mathbb{Z} + \tau\mathbb{Z})$, and this is represented by $(1, \frac{a\tau+b}{c\tau+d}; \frac{z}{c\tau+d} + \mathbb{Z} + \frac{a\tau+b}{c\tau+d}\mathbb{Z}) \in \mathcal{C}$.

The element $-1 \in \mathrm{SL}(2, \mathbb{Z})$ leaves every fiber C_τ of $\mathcal{C} \to \mathbb{H}$ fixed and acts there as minus the identity: $z + \mathbb{Z} + \tau\mathbb{Z} \mapsto -z + \mathbb{Z} + \tau\mathbb{Z}$. That action has in C_τ as its fixed points the subgroup of elements of order ≤ 2: the image of $(\frac{1}{2}\mathbb{Z})^2/\mathbb{Z}^2$ in $C_\tau = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. The orbit space $\{\pm 1\}\backslash C_\tau$ has the structure of a compact Riemann surface. Since -1 has 4 fixed points, we have the following identity for the Euler characteristic $2\chi(\{\pm 1\}\backslash C_\tau) - 4 = 2\chi(C_\tau) = 0$, and so $\chi(\{\pm 1\}\backslash C_\tau) = 2$. This implies that $\{\pm 1\}\backslash C_\tau \cong \mathbb{P}^1$. So if we divide out $\mathcal{C} \to \mathbb{H}$ by $\{\pm 1\}$, we find a family $\overline{\mathcal{C}} := \{\pm 1\}\backslash\mathcal{C} \to \mathbb{H}$ whose fibers are isomorphic to $\mathbb{P}^1$. It has four pairwise disjoint sections indexed by $P := \mathbb{Z}/2 \times \mathbb{Z}/2$, three of which can be used to trivialize this bundle.

In fact, we can say a bit more. Since $\overline{\mathcal{C}} \to \mathbb{H}$ with its four sections indexed by $P = (\frac{1}{2}\mathbb{Z})^2/\mathbb{Z}^2 \cong (\mathbb{Z}/2)^2$ is a family of P-pointed genus zero curves, we have a classifying map $\mathbb{H} = \mathcal{T}_{1,1} \to \mathcal{T}_{0,P}$ covered by a map $\overline{\mathcal{C}} \to \mathcal{C}_{0,P}$. Both are actually isomorphisms, for an inverse is defined as follows: given an embedding $i : P \hookrightarrow \mathbb{P}^1$, then we can make a double cover $C(i) \to \mathbb{P}^1$ of $\mathbb{P}^1$ ramified in the image of P and this produces a genus one Riemann surface for which the covering transformation acts as minus the identity relative to one of the ramification points (recall that an automorphism of a genus one surface can be linearized). If $g \in \mathrm{PSL}(2, \mathbb{C})$, then g lifts to an isomorphism $\tilde{g} : C(i) \cong C(gi)$ and $\tilde{g}$ is unique up to sign. This defines the inverses $\mathcal{T}_{0,P} \to \mathbb{H}$ and $\mathcal{C}_{0,P} \to \overline{\mathcal{C}}$.

Thus $\overline{\mathcal{C}} \to \mathbb{H}$ acquires the interpretation of $\mathcal{C}_{0,P} \to \mathcal{T}_{0,P}$. Under this identification, the mapping class group $\Gamma_{0,P}$ becomes a subgroup of $\mathrm{PSL}(2, \mathbb{Z})$: $\mathrm{SL}(2, \mathbb{Z})$ acts on $\mathbb{Z}^2$ hence on the three element set $(\mathbb{Z}/2)^2 - \{(0,))\} = P - \{(0, 0)\}$ (which we might think of as the projective line over $\mathbb{Z}/2$) and that last action is via $\mathrm{PSL}(2, \mathbb{Z})$. The image is the full symmetric group of these three points. The subgroup of $\mathrm{PSL}(2, \mathbb{Z})$ which leaves each of these fixed, i.e., the kernel of $\mathrm{PSL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{Z}/2)$, is denoted $P\Gamma(2) \subset \mathrm{PSL}(2,\mathbb{Z})$ and appears here as $\Gamma_{0,P}$.

This discussion shows that $\mathcal{T}_{1,1}$ and $\mathcal{T}_{0,P} \cong \mathcal{T}_{0,4}$ are isomorphic as complex manifolds, but not as orbifolds: this only happens after we have divided out the orbifold $\mathcal{T}_{1,1}$ by the trivially acting group $\{\pm 1\}$; we therefore better write this as $\overline{\mathcal{T}}_{1,1} \cong \mathcal{T}_{0,4}$.

Example 8.14. The case $(g,n) = (2,0)$ is similar to the preceding one. It is known that a closed connected Riemann surface C of genus 2 is hyperelliptic: it has an involution ι_C with the property that the orbit space $\iota_C\backslash C$ of ι_C is isomorphic to the Riemann sphere $\mathbb{P}^1$. This involution is unique and has 6 fixed points. Conversely, given six distinct points in $\mathbb{P}^1$, then the double cover of $\mathbb{P}^1$ ramified in these six points is of genus two. We have Γ_2 acting on $\mathcal{T}_2$ with a kernel $\{1,\iota\}$. The quotient $\overline{\Gamma}_2 := \Gamma_2/\{1,\iota\}$ acts on $\mathcal{T}_{0,6}$, but permutes the six points and its image is in fact the full permutation group of these points. So we get an identification of $\overline{\mathcal{T}}_2 := \{1,\iota\}\backslash\mathcal{T}_2$ with $\mathcal{T}_{0,6}$ and an identification of $\Gamma_{0,6}$ with the kernel of a surjection $\overline{\Gamma}_2 \to \mathfrak{S}_6$.

At this point it is clear that if divide out the universal Teichmüller family by $\Gamma(S,P)$, then we do not get a family $\mathcal{C}_{g,P} \to \mathcal{M}_{g,P}$ of P-pointed Riemann surfaces over $\mathcal{M}_{g,P}$ when orbifold type structure is ignored. For a fibre is always the quotient of a Riemann surface by its group of automorphisms which leave P pointwise fixed. But if we take into account the orbifold structure, then it becomes legitimate to think of this as universal family. The notation $\mathcal{C}_{g,P} \to \mathcal{M}_{g,P}$ is then understood accordingly.

Exercise 19. The elliptic curve defined by the Gauß lattice $\mathbb{Z} + \sqrt{-1}\mathbb{Z}$, i.e., $C := \mathbb{C}/(\mathbb{Z}+\sqrt{-1}\mathbb{Z}$ (with the image of 0 as origin) has automorphism group of order 4. Prove this and analyze how the group acts on the universal deformation of C. Try to determine what kind of singularities appear on the total space if we divide out the universal deformation by this group.

The Weil-Petersson metric. Recall that if $b \in \mathcal{T}(S,P)$, then the complex tangent space $T_b\mathcal{T}(S,P)$ can be identified with $H^1(C_b, \theta_{C_b}(-P)$. Let us write C for C_b. By Serre duality, the complex dual $T_b\mathcal{T}(S,P)$ is then identified with $H^0(C,\Omega^2_C(P))$. We also have a hyperbolic metric on C_P. We show how this leads to a Hermitian inner product on $H^0(C,\Omega^2_C(P))$. Such an inner product determines one on its dual and so $T_b\mathcal{T}(S,P)$ receives an inner product as well. This is what is called the *Weil-Petersson metric* g_{WP} on $\mathcal{T}(S,P)$.

Let η_1,η_2 be sections of $\Omega^2_C(P)$. In terms of a local coordinate z on C° they are written $\eta_i = f_i dz^2$ and so $\eta_1 \otimes \bar\eta_2$ then takes the form $f_1\overline{f_2}|dz|^4$. If we write the hyperbolic metric g as $\phi|dz|^2$ (with ϕ real valued and positive) ,then $\eta_1 \otimes \bar\eta_2/g$ is given by $\phi^{-1}f_1\overline{f_2}|dz|^2$. We may interpret this as a 2-form by replacing $|dz|^2 = dx^2 + dy^2$ by $dx \wedge dy = \frac{1}{2}\sqrt{-1}dz \wedge d\bar z$. We denote this 2-form $\eta_1 \star_{WP} \eta_2$ and then define the *Weil-Petersson inner product* by

$$\langle \eta_1,\eta_2\rangle_{WP} := \int_{C^\circ} \eta_1 \star_{WP} \eta_2.$$

It is however not yet clear that this integral converges, for C° is not compact. Let us see what the situation is near a point $p \in P$. Then f_i may have a pole of order one. On the other hand, by Proposition 4.5 we have a local coordinate z at p such that for $0 < |z| < \varepsilon$ such that if we write $z = \exp(2\pi\sqrt{-1}\tau)$, then the metric pulls back to $(\operatorname{Im}\tau)^{-2}|d\tau|^2$. We have $z^{-1}dz^2 = -4\pi^2\exp(2\pi\sqrt{-1}\tau)d\tau^2$ and so if we write $\tau = s + \sqrt{-1}t$, and $f_i = z^{-1}\phi_i$ with ϕ_i holomorpic and bounded by $C_i > 0$, say, then

$$|f_1 dz^2|.|f_2 dz^2|/g \leq C_1C_2.|z^{-1}dz^2|^2/g = 16\pi^4 C_1C_2t^2\exp(-4\pi t)ds\wedge dt.$$

and this is clearly absolutely integrable over the strip $0 \leq t \leq 1$, $s \geq \frac{-1}{2\pi}\log\varepsilon$. Hence $\int_{C^\circ}\eta_1 \star_{WP} \eta_2$ converges absolutely.

The Weil-Petersson metric identifies $T_b\mathcal{T}(S,P)$ and $H^0(C,\Omega^2_C(P))$ as real vector spaces, but under this identification their complex structures are opposite: we really have an identification of $T_b^{0,1}\mathcal{T}(S,P) \cong H^0(C,\Omega^2_C(P))$ as complex vector spaces. Via this identification we may understand the Teichmüller flow defined in Section 6 as one which is generated by a tangent vector of $\mathcal{T}(S,P)$. It turns out that the Weil-Petersson metric is not complete.

Let T be a real vector space endowed with complex structure J. If $H : T\times T \to \mathbb{C}$ is a Hermitian form on (T,J) then its imaginary part $A := \operatorname{Im} h : T\times T \to \mathbb{R}$ is antisymmetric (for $\operatorname{Im} H(v',v) = \operatorname{Im}\overline{H(v,v')} = -\operatorname{Im} H(v,v')$) and invariant under the complex structure in the sense that $H(Jv,Jv') = \operatorname{Im} H(v,v')$. The identity $H(v,v') = A(Jv,v') + \sqrt{-1}A(v,v')$ shows that H is determined by A and J and that H is nondegenerate if and only if A is. So a Hermitian metric h on a complex manifold M has its imaginary part $\operatorname{Im}(h)$ a nondegenerate 2-form. If that 2-form is closed, we say that h is a *Kähler* metric. Such then the underlying C^∞-manifold endowed with the symplectic form $\operatorname{Im}(h)$ is a symplectic manifold.

Ahlfors proved that the Weil-Petersson metric is a Kähler metric. Wolpert [] found a nice relation with the shearing action that we mention without proof:

Theorem 8.15. *Let α be an isotopy class of embedded circles in S° which do not bound an open disk or an open cylinder and let $\ell_\alpha : \mathcal{T}(S,P) \to \mathbb{R}_{>0}$ be the corresponding length function. Then half the associated Hamiltonian action on $\mathcal{T}(S,P)$ is the shearing action defined by α on $\mathcal{T}(S,P)$: if ∂_α is the vector field on $\mathcal{T}(S,P)$ defined by the derivative of that action, then $d\ell_\alpha = \frac{1}{2}\operatorname{Im} h(-,\partial_\alpha)$.*

If we let α run over the $3g-3+n$ connected components of a pants decomposition A of S° , then the corresponding shearing actions commute and since $3g-3+n = \frac{1}{2}\dim_{\mathbb{R}}\mathcal{T}(S,P)$, we thus get a *completely integrable system*. This can be simpler stated in terms of the diffeomorphism $(\ell_\alpha, t_\alpha)_{\alpha\in\pi_0(A)} : \mathcal{T}(S,P) \to (\mathbb{R}\times\mathbb{R}_{>0})^{\pi_0(A)}$ found in Theorem 4.6:

$$\operatorname{Im} g_{WP} = \sum_{\alpha\in\pi_0(A)} \tfrac{1}{2}d\ell_\alpha\wedge dt_\alpha.$$

The quasi-projective structure on the universal family. The construction of the universal orbifold family $\mathcal{C}_{g,P} \to \mathcal{M}_{g,P}$ can also be carried out in the context of algebraic geometry. This can be done over any algebraically closed field and it is then constructed as a *Deligne-Mumford stack*, which is the algebro-geometric analogue of an orbifold. As this requires a substantial background in algebraic geometry, we only sketch this approach and use freely the language of algebraic geometry. Assume $2g-2+n>0$ and let K be an algebraically closed field. We construct a vector space V, a representation W of $\mathrm{SL}(V)$ and $\mathrm{SL}(V)$-invariant subvarieties $Z \subset \mathbb{P}(W)$ and $\mathcal{C}_Z \subset \mathbb{P}(V) \times Z \subset \mathbb{P}(V) \times \mathbb{P}(W)$ with $\mathrm{SL}(V)$-equivariant sections $(x_p : Z \to \mathcal{C}_Z)_{p\in P}$ of the projection $f_Z : \mathcal{C}_Z \to Z$ (all defined over K) such that the system $(f_Z, (x_p)_{p\in P})$ is a family of P-pointed nonsingular connected curves of genus g on which $\mathrm{GL}(V)$ acts properly with finite stabilizers. It will have the property that the associated orbifold, or rather its algebro-geometric counterpart, is a K-model of $\mathcal{C}_{g,P} \to \mathcal{M}_{g,P}$.

We proceed as follows. Let (C, P) be a P-pointed nonsingular connected projective curve over K. The Riemann-Roch theorem implies that a line bundle on C of degree $d \geq 2g+1$ is very ample and that its linear system embeds C in a projective space of dimension $d-g$. The degree of $\Omega_C(P)$ is $2g-2+n$ and so for every $r \geq 3$, $\Omega_C(P)^{\otimes r}$ is very ample and embeds C in $\mathbb{P}(H^0(C, \Omega_C(P)^{\otimes r})^*)$. The image of C is then the common intersection of all the degree d hypersurfaces passing through C, at least for d large enough (and depending only on r, g and n). Their defining equations make up a subspace of $\mathrm{Sym}^d\left(H^0(C, \Omega_C(P)^{\otimes r})\right)$ that can be identified with the kernel of the evident linear map

$$\mathrm{Sym}^d\left(H^0(C, \Omega_C(P)^{\otimes r})\right) \to H^0(C, \Omega_C(P)^{\otimes rd}).$$

With the help of Riemann-Roch one can show that this map is also surjective. We put $N_r := \chi(\Omega_C(P)^{\otimes r}) = r(2g-2+n)-g+1$ so that $\dim H^0(C, \Omega_C(P)^{\otimes r}) = N_r$ and the above kernel has codimension N_{rd}. Now let V be a fixed K-vector space of dimension N_r, e.g., K^{N_r}. Choose an isomorphism of V^* onto $H^0(C, \Omega_C(P)^{\otimes r})$. Then the above kernel determines a linear subspace $E \subset \mathrm{Sym}^d V^*$ of codimension N_{rd} and if we think of this subspace as a linear space of homogeneous polynomials of degree d on V, then the projective variety it defines in $\mathbb{P}(V)$ is a copy C_E of C. The subspace $E \subset \mathrm{Sym}^d V^*$ of codimension N_{rd} corresponds to a subspace $E^\perp \subset \mathrm{Sym}^d V$ of dimension N_{rd}. Notice that $E^\perp$ is completely given by its determinant line $\wedge^{N_{rd}} E^\perp \subset \wedge^{N_{rd}} \mathrm{Sym}^d V$, for it is the set of vectors $u \in \mathrm{Sym}^d V$ with $u \wedge (\wedge^{N_{rd}} E^\perp) = 0$. Each $p \in P$ also defines a line $L_p \in V$ and so we have a line $\wedge^{N_{rd}} E^\perp \otimes \bigotimes_{p\in P} L_p$ in the vector space $W := \wedge^{N_{rd}} \mathrm{Sym}^d V \otimes \bigotimes_{p\in P} V_p$, or equivalently a point $z := [\wedge^{N_{rd}} E^\perp \otimes \bigotimes_{p\in P} L_p] \in \mathbb{P}(W)$. We claim that z determines C_E as well as a P-pointing of C_E, so that that (C_E, P) is isomorphic to (C, P). Observe first that from z we can completely recover E and the L_p's: any generator v of the line defined by z can be written $u_E \otimes \bigotimes_{p\in P} v_p$ with $u_E \in \wedge^{N_{rd}} E^\perp$,

$v_p \in V_p$ all nonzero and these vectors are unique up to scalar. We thus recover each L_p and $E^\perp$. Hence we recover E and C_E. The point defined by L_p lies on C_E and defines a P-pointing of C_E as asserted.

Notice that W can be viewed as a representation of $\mathrm{GL}(V)$. Another choice of isomorphism may determine another such line in W but always lies in the $\mathrm{GL}(V)$-orbit of L. In $\mathbb{P}(W)$ this amounts to a $\mathrm{PSL}(V)$-orbit, or, as we prefer, a $\mathrm{SL}(V)$-orbit. So we have found a $\mathrm{SL}(V)$-invariant subset $Z \subset \mathbb{P}(W)$ whose $\mathrm{SL}(V)$-orbits are in bijective correspondence with the isomorphism types of pairs (C, P). It can be shown that this subset is in fact a (not necessarily closed) subvariety of $\mathbb{P}(W)$ (the more precise statement is that this an open subset of a Hilbert scheme of $\mathbb{P}(W)$) and that the group $\mathrm{SL}(V)$ acts properly on W with finite stabilizers: the map $(g, z) \in \mathrm{SL}(W) \times Z \mapsto (g(z), z) \in Z \times Z$ is a finite morphism. Since $z \in Z$ determines a P-pointed curve in $\mathbb{P}(V)$, we see that we have a family of P-pointed curves $(f_Z : \mathcal{C}_Z \subset \mathbb{P}(V) \times Z \to Z, x_P)$. It has the property that the $\mathrm{SL}(V)$-orbits in Z are in bijective correspondence with the isomorphism types of P-pointed nonsingular connected genus g curves over K. Dividing out this family by the $\mathrm{SL}(V)$-action yields the desired K-model of $\mathcal{C}_{g,P} \to \mathcal{M}_{g,P}$.

We will see that this construction also leads to projective compactification of $\mathcal{M}_{g,P}$ via geometric invariant theory.

9. Harvey bordification and Deligne-Mumford compactification

In this section S is a closed connected surface of genus g and $P \subset S$ a finite subset of n elements with $2g - 2 + n > 0$. We retain our convention to abbreviate $S - P$ by S°.

Our goal is to describe two related compactifications of $\mathcal{M}_{g,P}$, the *Harvey bordification* $\mathcal{M}^+_{g,P}$, and the *Deligne-Mumford compactification* $\overline{\mathcal{M}}_{g,P}$. Both are geometrically meaningful (they have a modular interpretation, one might say) in the sense that the boundary parameterizes degenerate objects. In the former case, we allow the conformal structure (or rather the hyperbolic metric) to degenerate on a fixed surface of genus g. The boundary will then have real codimension one and $\mathcal{M}^+_{g,P}$ becomes an orbifold with corners. On the other hand, the Deligne-Mumford compactification allows the topology of S to degenerate so that it becomes a singular surface and this can be carried out in the complex-analytic (even algebro-geometric) category so that $\overline{\mathcal{M}}_{g,P}$ is a complex-analytic orbifold. The boundary is then a normal crossing divisor. The two compactifications are related through a natural map $\mathcal{M}^+_{g,P} \to \overline{\mathcal{M}}_{g,P}$, which can be understood as a real oriented blowup. Both compactifications can be obtained as $\Gamma(S, P)$-quotients of extensions $\mathcal{T}^+_{g,P}$ resp. $\overline{\mathcal{T}}_{g,P}$ of $\mathcal{T}_{g,P}$.

The curve complex. Consider the collection $\mathcal{A}(S^\circ)$ of isotopy classes of embedded circles in S° that do not bound a disk in S which meets P in at most one point. We make this collection the vertex set of a simplicial complex, called the *curve complex*, as follows: we agree that a nonempty finite subset $\mathbf{a} \subset \mathcal{A}(S^\circ)$ spans a simplex if and only if its elements can be represented by pairwise disjointly embedded circles. So the simplex $\mathbf{a}$ of $\mathcal{A}(S^\circ)$ can be thought of as an isotopy class of closed 1-dimensional submanifolds $A \subset S^\circ$ with the property that every connected component of $S^\circ - A$ has negative Euler characteristic. Observe that $\Gamma(S, P)$ acts on this complex.

Exercise 20. Prove that a pants decomposition of S° defines a maximal simplex of $\mathcal{A}(S^\circ)$ and that every maximal simplex is of this form. Conclude that the geometric realization $|\mathcal{A}(S^\circ)|$ of $\mathcal{A}(S^\circ)$ is everywhere of dimension $3g - 4 + n$.

Example 9.1 (The case of a punctured torus)**.** We illustrate the preceding with the case $(g, n) = (1, 1)$. Let us first observe that the curve complex of the once pointed torus $(S := (\mathbb{R}^2/\mathbb{Z}^2, P = \{o\})$ is discrete (there are no 1-simplices), where o is the origin. If $\alpha \subset S^\circ$ is an embedded circle which does not bound a disk in S, then cutting S open along α is diffeomorphic to a cylinder that has the image of o a distinguished point in its interior. Any two such pointed cylinders are diffeomorphic to each other and from this we can easily deduce that there the group $\mathrm{Diff}^+(S, P)$ acts transitively the set of such embedded circles. So $\Gamma_{1,1}$ acts transitively on $\mathcal{A}(S^\circ)$.

We claim the identification of $(\mathcal{T}_{1,1}, \Gamma_{1,1})$ with $(\mathbb{H}, \mathrm{SL}(2, \mathbb{Z}))$ identifies of the $\Gamma_{1,1}$-set $\mathcal{A}(S^\circ)$ with the projective line $\mathbb{P}^1(\mathbb{Q})$ with its evident $\mathrm{SL}(2, \mathbb{Z})$-action. The correspondence goes like this: Let $q \in \mathbb{P}^1(\mathbb{Q})$ and represent q by a pair $(a, b) \in \mathbb{Z}^2$ with a, b relatively prime. This representation is unique up to sign. Then the map $t \in \mathbb{R} \mapsto (a, b) \pmod{\mathbb{Z}^2}$ is periodic modulo $\mathbb{Z}$ and its image is an embedded circle in S which passes through p. The embedded circles parallel to this circle not passing through o are parametrized by an open interval and lie in a single $\mathrm{Diff}^0(S, P)$-orbit. If we orient any such circle by its parametrization, then it defines a homology class in $H_1(S^\circ)$ and that class is under the obvious isomorphism $H_1(S^\circ) \cong \mathbb{Z}^2$ equal to (a, b) (the opposite orientation yields $(-a, -b)$). This shows that q determines an element $a_q \in \mathcal{A}(S, P)$ and that $q \in \mathbb{P}^1(\mathbb{Q}) \mapsto \mathcal{A}(S^\circ)$ is injective. This map is also equivariant if we identify $\mathrm{SL}(2, \mathbb{Z})$ with $\Gamma_{1,1}$. Since the latter is transitive on $\mathcal{A}(S^\circ)$, it follows that the map $\mathbb{P}^1(\mathbb{Q}) \to \mathcal{A}(S^\circ)$ is also surjective.

Let $A \subset S^\circ$ represent a simplex $\mathbf{a}$ of $\mathcal{A}(S^\circ)$, i.e., a closed one-dimensional submanifold of S° such that every connected component of $S^\circ - A$ has negative Euler characteristic, with the connected components of A effectively indexed by the finite set $\mathbf{a}$. So if S' is such a connected component, then it is of the same type as S°: we may make it compact by adding a finite set, namely $S' \cap P$ and a point for every oriented connected component of A which appears as an oriented boundary component of S'. If we do this for every connected component of $S^\circ - A$ we get a (possibly disconnected surface S^A and a finite subset $P^A \subset S^A$ with $S^A - P^A$ identifiable with $S^\circ - A$. Notice that $|P^A| = n + 2|\mathbf{a}|$.

Another way of arriving at the pair (S^A, P^A) is by first collapsing in S each connected component of A to a point. The resulting quotient space S_A of S is singular, for at the image p_α of a connected component α of A, S_A

has two branches: a neighborhood is homeomorphic to the union two disks having their center in common. We then obtain S^A from S_A as its topological normalization: there is natural map $S^A \to S_A$ which simply separates the two branches at each p_α.

The smooth part of S_A can be identified with $S - A$. Since the latter has the same Euler characteristic (namely $2-2g$) as S, it follows that g can be read off from S_A. Following established terminology, we may refer to g as the *arithmetic genus* of S_A. (The *geometric genus* of S_A is the sum of the genera of the connected components of its normalization S^A.)

Let us point out however that we have not specified a differentiable structure on S^A near the points lying over the p_α's. We can always choose one, but it will not be unique. It will be so however up to an isotopy relative to P^A that is allowed to be nondifferentiable at the points over the p_α's.

Harvey's bordification. Recall that the punctured disk $0 < |z| < 1$ is universally covered by the upper half plane $\mathbb{H}$ by the map $\tau \in \mathbb{H} \mapsto z = \exp(2\pi\sqrt{-1}\tau)$. Since $z^{-1}dz = 2\pi\sqrt{-1}d\tau$ and $|z| = \exp(-2\pi \operatorname{Im}\tau)$, we find that the corresponding hyperbolic metric (often called *Poincaré metric*) is given by $(-|z|\log|z|)^{-2}|dz|^2$. Now consider the map $\phi : (-1,1) \times S^1 \to \mathbb{C}$ defined by $\phi(t,u) := tu$. Since both the restrictions $\phi\big|(-1,0) \times S^1$ and $\phi\big|(0,1) \times S^1$ parametrize the punctured disk, we get a hyperbolic metric on $((-1,1)-\{0\})\times S^1$. We think of this as a degenerate metric on $(-1,1)\times S^1$ and will refer to it as the *standard degenerate hyperbolic metric* on $(-1,1) \times S^1$.

Definition 9.2. Let $A \subset S^\circ$ represent a simplex $\mathbf{a}$ of $\mathcal{A}(S^\circ)$ or be empty. We say that a complete hyperbolic metric on $S^\circ - A$ is *permissible with respect to the embedding* $S^\circ - A \subset S$ if every connected component of A admits a tubular neighborhood parametrized by the cylindrical coordinates: $f : (-r,r) \times S^1 \to S^\circ$ for some $0 < r < 1$ (with the central circle $t = 0$ mapping to A) in which the hyperbolic metric takes the standard form above. We shall call a complete hyperbolic metric of this form (so that it is only defined an open subset of the form $S^\circ - A$ for some A as above) a *hyperbolic quasi-metric* on S° (since we allow A to be empty, this includes the case of a complete hyperbolic metric on S°).

Exercise 21. Prove that for a metric on $S^\circ - A$ permissible with respect to $S^\circ - A \subset S$ the embedding and a connected component α of A, a parametrization of a tubular neighborhood of α, $f : (-r,r) \times S^1 \hookrightarrow S$, as in the definition is unique up a rotation in the second coordinate and the inversion $(t,u) \mapsto (-t,u^{-1})$).

A hyperbolic metric on $S^\circ - A$ permissible with respect to $S^\circ - A \subset S$ defines the structure of a Riemann surface on $S - A$ or rather, we get on S^A the structure of a Riemann surface C^A. Similarly, S_A can be given an analytic structure C_A for which its only singularities are *nodes*, that is, have as their local-analytic model, the union of the two coordinate lines $(z_1z_2 = 0)$ in a neighborhood of the origin of $\mathbb{C}^2$. Then the evident map $C^A \to C_A$ is complex-analytic and can be understood as the normalization of C_A. The space of permissible hyperbolic metrics $S^\circ - A$ up to $\mathrm{Diff}^+(S, P \cup A)$-isotopy only depends on $\mathbf{a}$ and so we feel justified in denoting it by $\mathcal{T}^{\mathbf{a}}(S, P)$. For a similar reason the Teichmüller space $\mathcal{T}(S^A, P^A)$ (which is actually a product of Teichmüller spaces, with one factor for every connected component of S^A) may also be denoted $\mathcal{T}(S^{\mathbf{a}}, P^{\mathbf{a}})$. It is clear that this construction defines a map from $\mathcal{T}^{\mathbf{a}}(S, P)$ to the Teichmüller space $\mathcal{T}(S^{\mathbf{a}}, P^{\mathbf{a}})$. We also have a shearing action of $\mathbb{R}^{\mathbf{a}}$ on $\mathcal{T}^{\mathbf{a}}(S, P)$: if we are given a permissible hyperbolic metric on $S^\circ - A$ and $f : (-r, r) \times S^1 \hookrightarrow S$ parametrizes in an orientation preserving manner a tubular neighborhood of the connected component α of A as in the definition, then we can do the shearing in terms of this parametrization: for every $t_\alpha \in \mathbb{R}$ we have defined the sheared surface $S(t_\alpha)$ obtained from the union of $S - \alpha$ and $(-r, r) \times S^1$ by identifying (t, u) with $f(t, u)$ for $t < 0$ and with $f(t, e^{-\sqrt{-1}t_\alpha}u)$ for $t > 0$. It has the property that $C(t_\alpha)^A = C^A$. Notice that thus is defined an action of $\mathbb{R}^{\mathbf{a}}$ on $\mathcal{T}^{\mathbf{a}}(S, P)$ which preserves the fibers of $\mathcal{T}^{\mathbf{a}}(S, P) \to \mathcal{T}(S^{\mathbf{a}}, P^{\mathbf{a}})$. In fact, it is not hard to see that we have:

Lemma 9.3. *The shearing action of* $\mathbb{R}^{\mathbf{a}}$ *on* $\mathcal{T}^{\mathbf{a}}(S, P)$ *turns the map* $\mathcal{T}^{\mathbf{a}}(S, P) \to \mathcal{T}(S^{\mathbf{a}}, P^{\mathbf{a}})$ *into a principal* $\mathbb{R}^{\mathbf{a}}$*-bundle in the differentiable category.*

Corollary 9.4. *The manifold* $\mathcal{T}^{\mathbf{a}}(S, P)$ *is diffeomorphic to an open cell of dimension* $6g - 6 + 2n - |\mathbf{a}|$.

A hyperbolic quasi-metric arises as a limit of hyperbolic metrics on S° if we let the length of the geodesics in the isotopy classes of the connected components of A tend to zero (so that some of the Fenchel-Nielsen coordinates ℓ_α, $\alpha \in \mathbf{a}$, assume the value zero). Let us make this explicit with the help of Lemma 4.1. Scalar multiplication by $\exp(2\pi r)$ acts on $\mathbb{H}$ and leaves the first and second quadrant, denoted here by $\mathbb{H}^+$ and $\mathbb{H}^-$ respectively, invariant. The common boundary of $\mathbb{H}^+$ and $\mathbb{H}^-$ in $\mathbb{H}$ is the positive imaginary axis and is a geodesic. The function $\exp(\sqrt{-1}/r.(\log z)$ identifies the orbit space C_r of $\mathbb{H}$ by this transformation with the annulus $\exp(-\pi/r) < |w| < 1$ with metric

$$\frac{r^2|dw|^2}{|w|^2 \sin^2(r \log |w|)}.$$

The image of $\mathbb{H}^\pm$ is a open half annulus $C_r^\pm \subset C_r$ whose boundary is the unique closed geodesic α_r of C_r defined by $|w| = \exp(-\pi/2r)$. The length of that geodesic is $2\pi r$. This models the general situation, for according to Lemma 4.1 a neighborhood of a closed geodesic of length $2\pi r$ on any hyperbolic surface is isometric to a neighborhood of α_r in C_r. This justifies that we focus on this case. We want to produce a sensible geometric limit if this length r goes to zero. Now note that for $r \downarrow 0$, the outer annulus $\exp(-\pi/2r) < |w| < 1$ tends to the punctured unit disk $0 < |w| < 1$ with the metric tending to the Poincaré metric $(|w| \log |w|)^{-2}|dw|^2$. This makes the whole inner annulus go to the origin. In order to preserve that annulus in the limit, we invert with respect to the central geodesic: if we write w^+ for

w and define w^- by $w^+w^- = \exp(-\pi/r)$, then the pair (w^+, w^-) maps C_r isomorphically onto the curve in (w^+, w^-)-space defined by $|w^\pm| < 1$ and $w^+w^- = \exp(-\pi/r)$. If we let $r \downarrow 0$, then the limit is the union of the two unit disks in the coordinate axes, each endowed with the Poincaré metric. Notice that the closed geodesic α_r on C_r is given on C_r as the locus where $|w^+| = |w^-|$ and that it tends to the origin of $\mathbb{C}^2$ as $t \downarrow 0$ (algebraic geometers recognize this as a *vanishing cycle* of the function w^+w^-).

Let $\mathbf{a}$ be a simplex of $\mathcal{A}(S^\circ)$ that defines a pants decomposition of S° (according to Exercise 20 this means that $\mathbf{a}$ is a maximal simplex). We then have an associated Fenchel-Nielsen parametrization of the Teichmüller space $\mathcal{T}(S, P)$ of hyperbolic metrics on S° given up to isotopy relative to P by the product of open half planes $(\mathbb{R}_{>0} \times \mathbb{R})^{\mathbf{a}}$. If we now allow the metrics to be permissible with respect to $S^\circ - A \subset A$, where A is a closed submanifold which represents $\mathbf{a}$, then we see that all the coordinates (ℓ_α, t_α) retain their meaning, except that the ℓ_α's can take the value zero. The resulting map from the space of A-permissible hyperbolic metrics to the product of closed half planes $([0, \infty) \times \mathbb{R})^{\mathbf{a}}$ is surjective and constant on the $\mathrm{Diff}^0(S, P)$-orbits. It is not hard to see that these $\mathrm{Diff}^0(S, P)$-orbits are just the fibres of this map. In other words, $([0, \infty) \times \mathbb{R})^{\mathbf{a}}$ parametrizes the $\mathrm{Diff}^0(S, P)$-orbits of such quasi-metrics. We then see that the $\mathrm{Diff}^0(S, P)$-orbit space of quasi-metrics has the structure of a manifold with corners:

Proposition-definition 9.5. *The* Harvey bordification $\mathcal{T}^+(S, P)$ *of* $\mathcal{T}(S, P)$ *is the* $\mathrm{Diff}^+(S, P)$*-orbit space of hyperbolic quasi-metrics on* S° *(with respect to the inclusion* $S^\circ \subset S$*). It is in a natural manner a manifold with corners with* $\Gamma(S, P)$*-action. The corner strata of its boundary* $\partial\mathcal{T}^+(S, P)$ *are the* $\mathcal{T}^{\mathbf{a}}(S, P)$*, where* $\mathbf{a}$ *runs over the simplices of the curve complex* $\mathcal{A}(S^\circ)$ *and this indexing is order reversing:* $\mathcal{T}^{\mathbf{a}}(S, P)$ *is in the closure of* $\mathcal{T}^{\mathbf{b}}(S, P)$ *if and only if* $\mathbf{a} \supset \mathbf{b}$*.*

Moreover, any Fenchel-Nielsen parametrization $(\mathbb{R}_{>0} \times \mathbb{R})^{\mathbf{a}} \cong \mathcal{T}(S, P)$ *extends to a diffeomorphism of* $([0, \infty) \times \mathbb{R})^{\mathbf{a}}$ *onto an open subset of* $\mathcal{T}^+(S, P)$*.*

Example 9.6 (The case of a punctured torus continued)**.** We take up again the case $(\mathcal{T}_{1,1}, \Gamma_{1,1})$. Recall that in Example 9.1 the triple $(\mathcal{T}_{1,1}, \mathcal{A}(S^\circ), \Gamma_{1,1})$ was identified with $(\mathbb{H}, \mathbb{P}^1(\mathbb{Q}), \mathrm{SL}(2, \mathbb{Z}))$. The Harvey bordification corresponds under this identification to what it known as the *Borel-Serre compactification* $\mathbb{H}^{BS}$ of $\mathbb{H}$, which in this case is a manifold with boundary (there are here no corners) whose connected components are effectively indexed by $\mathbb{P}^1(\mathbb{Q})$. In fact, the boundary component indexed by $q \in \mathbb{P}^1(\mathbb{Q})$ can be identified with the set of complete *oriented* geodesics in $\mathbb{H}$ which tend to q. As such a geodesic is given by its 'source', a point of $\mathbb{P}^1(\mathbb{R})$ distinct from q, we see that as a topological space, this boundary component can be identified with the real affine line $A_q := \mathbb{P}^1(\mathbb{R}) - \{q\}$. For instance, $A_\infty = \mathbb{R}$. The topology on the disjoint union of $\mathbb{H}$ and the A_q's is essentially specified if require that it be $\mathrm{SL}(2, \mathbb{Z})$-invariant and that A_∞ is glued onto $\mathbb{H}$ as to form in an evident manner $\mathbb{R} \times (0, \infty]$ (so that A_∞ corresponds to $y = \infty$).

The group $\mathrm{SL}(2, \mathbb{Z})$ acts properly discontinuously on $\mathbb{H}^{BS}$, the stabilizer of A_q being the $\mathrm{SL}(2, \mathbb{Z})$-stabilizer of q. The latter is infinite cyclic and acts faithfully as a translation group in A_q. A fundamental domain of the $\mathrm{SL}(2,\mathbb{Z})$-action on $\mathbb{H}^{BS}$ is the closure D^+ of its standard fundamental domain D in $\mathbb{H}$ defined by $|\mathrm{Re}(\tau)| \le \frac{1}{2}$, $|\tau| \ge 1$ in $\mathbb{H}^{BS}$: the set of added points is the interval in A_∞ defined by $|x| \le \frac{1}{2}$. This makes D^+ compact. This helps us to understand the $\mathrm{SL}(2,\mathbb{Z})$-orbit space $\Gamma_{1,1}\backslash\mathbb{H}^{BS}$, for we then find it to be a quotient of D^+ where any identification takes place on the boundary. It adds to the J-line a circular boundary (namely the quotient of A_∞ by its stabilizer in $\mathrm{SL}(2,\mathbb{Z})$, i.e., the group of integral translations) so that the result is homeomorphic to a closed disk.

This example also illustrates a more general fact:

Stable pointed curves. Let C be a compact *nodal curve*, i.e., a complex-analytic curve whose only singularities are nodes. The normalization $\hat{C} \to C$ is then the smooth curve (Riemann surface) obtained by separating the branches of the nodes. Every node defines a pair in $\hat{C}$ and we can reconstruct C from $\hat{C}$ and this collection of pairs.

We recall that its arithmetic genus g is the Euler characteristic of its smooth part C_{reg}: $\chi(C_{\mathrm{reg}}) = 2-2g$. We want to deal with the pointed situation as well. We stipulate that this means here that we are given a finite subset P *of the smooth part* C_{reg} of C. So P should not contain a node.

Definition 9.7. Let (C,P) be a compact nodal curve. We say that (C,P) is *stable* (more precisely, *Deligne-Mumford stable*) if every connected component of $C^\circ_{\mathrm{reg}} := C_{\mathrm{reg}} - P$ has negative Euler characteristic.

As we know, this implies that the conformal structure on C°_{reg} comes from a complete hyperbolic metric, that is in fact unique. We actually have encountered the stable pointed curves already via the converse construction: if (S,P) is a hyperbolic pair and $A \subset S^\circ$ is a closed submanifold of dimension 1 which represents a simplex of the curve complex, then for any A-permissible hyperbolic metric on $S^\circ - A$ we get a stable curve pointed curve (C_A,P). Thus the Harvey bordification $\mathcal{T}^+(S,P)$ parametrizes stable P-pointed curves of genus g.

Exercise 22. Prove that (C,P) is stable precisely when the group $\mathrm{Aut}(C,P)$ of automorphisms of C which preserve P pointwise is finite.

Stable pointed curves appear to have a deformation theory which is almost as good as the one for smooth pointed curves. If $\hat{P} \subset \hat{C}$ denotes the preimage of $P \cup C_{\mathrm{sg}}$, then one way to deform (C,P) is by deforming each connected component of the pair $(\hat{C},\hat{P})$ and then identifying the point pairs in $\hat{P}$. This will produce stable pointed curves homeomorphic to (C,P).

Exercise 23. Prove that the Teichmüller space $\mathcal{T}(\hat{C},\hat{P})$ (which is a product of Teichmüller spaces with one factor for every connected component of $\hat{C}$) has complex dimension $3g-3+n-|C_{\mathrm{sg}}|$.

However, we also want to include deformations which make the nodes disappear. If we look for a deformation space of the hoped for dimension

$3g-3+n$, then the above exercise suggests that the smoothing of each node needs one extra dimension.

Deformations of nodes. Let C be an abstract complex-analytic curve with a nodal singularity at $p \in C$. This means that there exists a neighborhood C_0 of p in C and an isomorphism of (C_0, p) onto the locus in $U \subset \mathbb{C}^2$ defined by $z_1 z_2 = 0$, where U is defined by $|z_i| < \varepsilon$, $i = 1, 2$. Thus is defined closed embedding ι of C_0 in U. If we let $f : U \to \mathbb{C}$ be defined by $f(z_1, z_2) = z_1 z_2$, then we may regard the pair (f, ι) as defining a *deformation* of C_0 in the sense that for small $|t|$ (e.g., $|t| < \varepsilon^2$), the fiber $U_t := f^{-1}(t)$ may be regarded as a 'deformed C_0'. Notice that in contrast to the deformation of a complex structure considered earlier, what is being deformed here is not just a complex structure, but also the underlying topological space: U_0 is a union of two disks with their center in common, whereas for $t \neq 0$ small, U_t is a nonsingular curve isomorphic to an annulus. On the other hand, this is very much a local affair: we can take C_0 and U as small as we like. Therefore the notion of a deformation here is really that one of a germ: what we are deforming is the curve germ (C, p) and the deformation is then the pair of analytic map-germs $(f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0), \iota : (C, o) \hookrightarrow (\mathbb{C}^2, 0))$.

We are here free to replace $(\mathbb{C}^2, 0)$ and $(\mathbb{C}, 0)$ by complex manifold germs $(\mathcal{C}, p)$ and (B, o) of dimension 2 and 1 respectively as long we can find coordinates (z_1, z_2) at $(\mathcal{C}, p)$ and a coordinate w at (B, o) such that wf takes the form $z_1 z_2$. There is in this case also a Kodaira-Spencer map (or rather its inverse) which grasps the first order content of the deformation: the derivative of $D_p f : T_p\mathcal{C} \to T_o B$ at p is zero (for p is a singular point), but then the Hessian of f is intrinsically defined as a *quadratic* map $H_p f : T_p \mathcal{C} \to T_o B$. If f is given in terms of coordinates as $z_1 z_2$, then $H_p f$ is just given as the product of the linear forms $dz_1|_0$ and $dz_2|_0$. So it sends $a_1 \frac{\partial}{\partial z_1}|_0 + a_2 \frac{\partial}{\partial z_2}|_0$ to $a_1 a_2$. We can also express this as follows: let $C^{\pm}$ be the two branches of C_0 (these are just names: there is no natural way of ordering them). Then the images of $T_p C^+$ and $T_p C^-$ in $T_p\mathcal{C}$ decompose the latter into two lines and the Hessian $H_p f$ defines a linear isomorphism $T_p C^+ \otimes T_p C^- \xrightarrow{\cong} T_o B$. Its inverse, $T_o B \xrightarrow{\cong} T_p C^+ \otimes T_p C^-$, is the analogue of the Kodaira-Spencer map: it goes from $T_o B$ to a space which only involves the object to be deformed (namely (C, p)).

A decent deformation theory should however not be restricted to bases of dimension one, nor should it require the total space germ $\mathcal{C}$ to be nonsingular. For instance, if (B, o) is an arbitrary smooth manifold germ, then the projection $(C, p) \times (B, o) \to (B, o)$ is perhaps uninteresting as a deformation (nothing is being deformed, after all), but should be allowed. These considerations lead to the following definition.

Definition 9.8. Let B be a complex manifold and $o \in B$. A *deformation* of the nodal curve germ (C, p) over (B, o) is a pair of analytic map-germs

$$(f : (\mathcal{C}, q) \to (B, o), \iota : (C, p) \hookrightarrow (\mathcal{C}, q)).$$

We demand that f factors through an isomorphism j of $(\mathcal{C}, q)$ onto an analytic hypersurface germ in $(\mathbb{C}^2, 0) \times (B, o)$ whose defining equation F has the form $z_1 z_2 = u$ for some holomorphic $u : (B, o) \to (\mathbb{C}, 0)$ and require that ι be an isomorphism of (C, q) onto the central fiber $(f^{-1}(o), q)$.

We call the deformation a *smoothing*, if the union of the singular fibers is nowhere dense in $\mathcal{C}$.

This covers the case considered at the beginning, where $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is given by $f(z_1, z_2) = z_1 z_2$, for then we may take $u(t) = t$ so that hypersurface in $(\mathbb{C}^2, 0) \times (\mathbb{C}, 0)$ is the graph of f, defined by $z_1 z_2 - t = 0$. Notice that this is in fact a smoothing. As we have already seen, the general fiber of a smoothing is homeomorphic a cylinder $(-1, 1) \times S^1$ and the central fiber is obtained from it by collapsing its midcircle $\{0\} \times S^1$ to a point.

Exercise 24. Prove that if in the above definition $\mathcal{C}$ is nonsingular, then f is a smoothing.

Remark 9.9. It is possible to set up a local deformation category in much the same way as $\mathfrak{Def}_{C,P}$. We shall not do that, but we observe that we have something like a universal object: for f, ι, j, u as in the definition, let $\iota' : (C, p) \hookrightarrow (\mathbb{C}^2, 0)$ be the restriction of j (so that $j(z) = (\iota'(z), b))$ and let $f' : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be given by $f'(z_1, z_2) = z_1 z_2$. Then (f', ι') defines a deformation of (C, p) and a morphism from (f, ι) to (f', ι') is then defined by the pair (Φ, ϕ), where $\Phi : (\mathcal{C}, q) \to (\mathbb{C}^2, 0)$ is the first component of j and and $\phi = u : (B, o) \to \mathbb{C}$. The morphism (Φ, ϕ) is not unique however, and this why (f', ι') is called a *semi-universal* deformation. It is so in first order though, for we may still define Kodaira-Spencer map $T_o B \to T_p C^+ \otimes T_p C^-$ as the composite of $D_o \phi = D_o u : T_o B \to T_0 \mathbb{C}$ with the inverse of the isomorphism $T_p C^+ \otimes T_p C^- \cong T_0 \mathbb{C}$ defined by the Hessian of f'. You may check that this definition is coordinate-independent (i.e., independent of j and u).

Deformations of nodal curves. We are going to combine the two types of deformations.

Definition 9.10. Let B be a complex manifold. A *family of stable* P*-pointed curves over* B consists a proper analytic map of complex-analytic varieties $f : \mathcal{C} \to B$ endowed with a set x_P of pairwise disjoint holomorphic sections $(x_p : B \to C)_{p \in P}$ such that every fiber $C_b := f^{-1}(b)$ forms with $x_P(b)$ a stable P-pointed curve of genus g. We require that if $z \in \mathcal{C}$, then the germ of f at z is either a submersion (this presupposes that $\mathcal{C}$ is nonsingular at z) or that z is a node on the fiber $C_{f(z)}$ and the germ of f at z defines a deformation of this node.

We also have a version relative to a fixed stable pointed curve:

Definition 9.11. Given a stable P-pointed curve (C, P), a complex manifold B and a point $o \in B$, then a *deformation* of C over (B, o) is a system of analytic map-germs

$$\big(f : (\mathcal{C}, C_o) \to (B, o), x_P = (x_p : (B, o) \to (\mathcal{C}, C_o))_{p \in P}, \iota : C \hookrightarrow \mathcal{C}\big)$$

with ι an isomorphism onto $C_o := f^{-1}(o)$, $x_p(o) = \iota(p)$ for all $p \in P$, and for which some representative of (f, x_P) is a family of P-pointed stable curves.

These definitions clearly extend the ones we gave for nonsingular curves (= compact Riemann surfaces). Since the section $x_p : B \to \mathcal{C}$ of f takes its values in the part where f is a submersion, we have defined a line bundle $\mathcal{L}_{f,p}$ on B whose fiber in b is the cotangent space $T^*_{x_p(b)}C_b$. More precisely, $\mathcal{L}_{f,p}$ is the pull-back under x_p of the relative cotangent bundle of $\mathcal{C}/B$.

The deformations of a stable pointed curve (C, P) are the objects of a deformation category $\mathfrak{Def}_{C,P}$ (the definition of a morphism being the obvious one) and so we have a notion of a universal deformation. As before, it is formal consequence of the universal property that the (finite) automorphism group of (C, P) acts naturally on its universal deformation.

Remark 9.12. Let (C, P) be a connected stable curve. There is a first order deformation space which serves as the generalization of the one in the smooth case (i.e., $H^1(C, \theta_C)$) and is for a deformation the target of a Kodaira-Spencer map. We briefly explain. Suppose (C, P) appears as the central fiber in a deformation $(f : (\mathcal{C}, C_o) \to (B, o), x_P, \iota : C \cong C_o)$ with $\mathcal{C}$ nonsingular. Then the derivative of f along C_o defines a map of $\mathcal{O}_C$-modules:

$$Df_{C,P} : \iota^*\theta_{\mathcal{C}}(\log x_P) \to \mathcal{O}_C \otimes T_oB,$$

where we recall that $\theta_{\mathcal{C}}(\log x_P)$ stands for the sheaf of holomorphic vector fields on $\mathcal{C}$ that are tangent to the sections x_p, $p \in P$. We will be concerned with the cohomology of the *mapping cone* of $Df_{C,P}$.

We digress for a moment to discuss that notion in its proper setting. We first mention that the notion of cohomology for sheaves extends to complexes of abelian sheaves that are zero for sufficiently small (negative) degree in such a manner that essentially all the fundamental properties subsist (it is then often referred to as *hypercohomology*). For instance, there is a long exact sequence attached to a short exact sequence $0 \to \mathcal{A}^\bullet \to \mathcal{B}^\bullet \to \mathcal{C}^\bullet \to 0$ of sheaf complexes on a space X. This is also true if for every k the row $0 \to \mathcal{A}_k \to \mathcal{B}_k \to \mathcal{C}_k \to 0$ is a sheaf complex (the composite is zero) and the sequence of cohomology sheaves $0 \to \mathcal{H}^k(\mathcal{A}^\bullet) \to \mathcal{H}^k(\mathcal{B}^\bullet) \to \mathcal{H}^k(\mathcal{C}^\bullet) \to 0$ is exact.

A particular case of interest is when we only have a homomorphism $\phi : \mathcal{F} \to \mathcal{G}$ of abelian sheaves, for this then enables us to put the induced homomorphisms $H^k(X, \phi) : H^k(X, \mathcal{F}) \to H^k(X, \mathcal{G})$ into a long exact sequence defined as follows. First regard ϕ as defining a complex $\mathcal{C}^\bullet_\phi$ by putting $\mathcal{C}^\bullet_\phi = 0$ for $k \neq -1, 0$ and $\mathcal{C}^{-1}_\phi \to \mathcal{C}^0_\phi$ be given by ϕ. This complex is called the *mapping cone* of ϕ. We then have a short exact sequence of complexes of sheaves

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{C}^\bullet_\phi \longrightarrow \mathcal{F}[1] \longrightarrow 0,$$

where $\mathcal{G}$ and $\mathcal{F}$ are now regarded as one term complexes placed in degree 0 and -1 respectively. (The convention is that if $C^\bullet$ is a complex, then $C^\bullet[n]$ denotes the complex whose term in degree k is C^{k+n}.) This yields the exact sequence

$$\cdots \to H^k(X, \mathcal{F}) \xrightarrow{H^k(X,\phi)} H^k(X, \mathcal{G}) \to H^k(X, \mathcal{C}^\bullet_\phi) \to H^{k+1}(X, \mathcal{F}) \xrightarrow{H^{k+1}(X,\phi)} \cdots$$

We may however also form

$$0 \longrightarrow \operatorname{Ker}(\phi)[1] \longrightarrow \mathcal{C}^\bullet_\phi \longrightarrow \operatorname{Coker}(\phi) \longrightarrow 0.$$

This is exact on the cohomology sheaves, for $\mathcal{H}^k(\mathcal{C}^\bullet_\phi)$ equals $\operatorname{Ker}(\phi)$ for $k=-1$, $\operatorname{Coker}(\phi)$ for $k=0$, and is zero otherwise. So this gives the long exact sequence

$$\cdots \to H^{k-1}(X, \operatorname{Coker}(\phi)) \to H^{k+1}(X, \operatorname{Ker}(\phi)) \to H^k(X, \mathcal{C}^\bullet_\phi) \to H^k(X, \operatorname{Coker}(\phi)) \to \cdots$$

Returning to the case at hand, we take for ϕ the homomorphism $Df_{C,P}$ as defining a complex of $\mathcal{O}_C$-modules and put $\mathcal{T}^\bullet := \mathcal{C}^\bullet_{Df_{C,P}}[-1]$. Then the above discussion gives the exact sequence

$$\cdots \to H^0(C, \iota^*\theta_{\mathcal{C}}(\log x_P)) \to T_0 B \xrightarrow{KS} H^1(C, \mathcal{T}^\bullet_{C,P}) \to H^1(C, \iota^*\theta_{\mathcal{C}}(\log x_P)) \to \dots$$

We will see that the other long exact sequence helps us to understand $H^1(C, \mathcal{T}^\bullet_{C,P})$ as the space of first order deformations of the stable pointed curve (C, P) with KS playing the role of the Kodaira-Spencer map. For this we need to work out the kernel and cokernel of $Df_{C,P}$. The kernel of $Df_{C,P}$ can be identified with the sheaf of holomorphic fields tangent to C which vanish in P. At a node q this sheaf is in terms of local coordinates the $\mathcal{O}_{C,q}$-module generated by $z_1\frac{\partial}{\partial z_1}$ and $z_2\frac{\partial}{\partial z_2}$ and so that kernel is just the direct image of $\theta_{\hat{C},\hat{P}}$ under the normalization map $\pi : \hat{C} \to C$: $\operatorname{Ker}(Df_{C,P}) \cong \pi_*\theta_{\hat{C}}(-\hat{P})$. The cohomology of an abelian sheaf is the same as its direct image under a finite map and since π is such a map, we find that $H^k(C, \operatorname{Ker}(Df_{C,P})) \cong H^k(\hat{C}, \theta_{\hat{C}}(-\hat{P}))$. The cokernel is a skyscraper sheaf with support in the nodes: the stalk at a node y is the tensor product of the tangent spaces along the two branches $T_y C_y^+ \otimes T_y C_y^-$ and its cohomology is concentrated in degree zero. This yields the exact sequence

$$0 \to H^1(\hat{C}, \theta_{\hat{C}}(-\hat{P})) \to H^1(C, \mathcal{T}^\bullet_{C,P}) \to \oplus_{y\in C_{sg}} T_y C_y^+ \otimes T_y C_y^- \to 0.$$

The subspace $H^1(\hat{C}, \theta_{\hat{C}}(-\hat{P}))$ then corresponds to the space of first order deformations which preserve the nodes, and the quotient of these yields the first order deformations of the nodes.

The Deligne-Mumford compactification. We are now ready to state an extension to the central Theorem 8.11 in the stable case.

Theorem 9.13. *A deformation of the stable curve (C, P) is universal if and only if its Kodaira-Spencer map is an isomorphism. If these equivalent conditions are satisfied, then the deformation can be represented by a family $(f : \mathcal{C} \to B, x_P)$ endowed with an isomorphism $\iota : C \cong C_o$ which defines a universal deformation of each of its fibers and is such that*

(i) B *is contractible and admits a partial system of coordinates at $o \in B$ indexed by C_{sg}, $(t_q)_{q\in C_{sg}}$, such that $t_q = 0$ defines the locus where $q \in C_{sg}$ subsists as a singular point in the fiber,*
(ii) *the locus $B_o \subset B$ where all the t_q's vanish parametrizes the fibers homeomorphic to (C, P) and (simultaneous) normalization of these fibers yields a universal deformation of the normalization $(\hat{C}, P)$ of (C, P) over B_o as in Theorem 8.11,*
(iii) *the group $\operatorname{Aut}(C, P)$ of automorphisms of (C, P) extends its tautological action on (C, P) naturally to the family $(f : \mathcal{C} \to B, x_P)$.*

Notice that this theorem implies that the locus D_f parametrizing singular fibers is the normal crossing locus defined by $\prod_q t_q = 0$.

The theorem differs from Theorem 8.11 in that it does not involve Teichmüller space. Yet we may glue these together to form a family:

$$\big(\overline{\mathcal{C}}_{g,P} \to \overline{\mathcal{M}}_{g,P}, x_P = (x_p : \overline{\mathcal{M}}_{g,P} \to \overline{\mathcal{C}}_{g,P})_{p\in P}\big)$$

as an *orbifold* object in the holomorphic category: a family $(f : \mathcal{C} \to B, x_P)$ as in Theorem 9.13 defines a chart with $\mathrm{Aut}(C, P)$ as associated group. If we ignore the orbifold structures, then we get a map of complex-analytic spaces $\overline{C}_{g,P} \to \overline{M}_{g,P}$ with sections indexed by P for which the local model on $\overline{M}_{g,P}$ is given as by the $\mathrm{Aut}(C, P)$-quotient of a family as in Theorem 9.13: $\mathrm{Aut}(C, P)\backslash\mathcal{C} \to \mathrm{Aut}(C, P)\backslash B$.

The difference $D_{g,P} := \overline{\mathcal{M}}_{g,P} - \mathcal{M}_{g,P}$ is a normal crossing divisor (again in the orbifold sense). Such a normal crossing divisor is naturally partitioned into *strata*: these are the connected components of loci where this divisor has a fixed number of branches. We also see that the strata are effectively indexed by $\Gamma(S, P)$-orbits of simplices of the curve complex $\mathcal{C}(S, P)$.

We might say that we have a holomorphic line bundle $\mathcal{L}_p$ over $\overline{\mathcal{M}}_{g,P}$ ($p \in P$) in the orbifold sense. If we ignore the orbifold structure, then $\mathcal{L}_p$ is still defines a sheaf L_p on $\overline{M}_{g,P}$: in terms of a chart as above, the sections of L_p over $\mathrm{Aut}(C, P)\backslash B \hookrightarrow \overline{M}_{g,P}$ are the $\mathrm{Aut}(C, P)$-invariant sections of $\mathcal{L}_p$ on B.

Remark 9.14 (Smooth Galois covers of $\overline{\mathcal{M}}_{g,P}$). The technicalities that usually accompany an abstract orbifold setting can here be avoided, because one can show that there exist finite quotients $\phi : \Gamma(S, P) \twoheadrightarrow G$ of the mapping class group satisfying the following two properties: (i) $\mathrm{Ker}(\phi)$ acts freely on $\mathcal{T}(S, P)$ so that $\mathcal{M}[\phi] := \mathrm{Ker}(\phi)\backslash\mathcal{T}(S, P)$ is a complex manifold and (ii) the natural extension of the formation of the G-orbit variety $\mathcal{M}[\phi] \to \mathcal{M}_{g,P}$ to a G-covering over $\overline{\mathcal{M}}_{g,P}$, $\overline{\mathcal{M}}[\phi] \to \overline{\mathcal{M}}_{g,P}$, (in the language of algebraic geometry, this is the normalization of the map $\mathcal{M}[\phi] \to \overline{\mathcal{M}}_{g,P}$) has nonsingular total space. So $\overline{\mathcal{M}}_{g,P}$ then appears as the G-orbit space of a nonsingular variety. The orbifold line bundle $\mathcal{L}_p$ is obtained as a G-quotient of a genuine line bundle $\mathcal{L}_p[\phi]$ on $\overline{\mathcal{M}}[\phi]$.

Exercise 25. Prove that all but one of the irreducible components of $D_{g,P}$ are effectively indexed by unordered pairs $\{(g', P'), (g'', P'')\}$ with $g' + g'' = g$, $P = P' \sqcup P''$ and such that $g' + |P'| \geq 2$, $g'' + |P''| \geq 2$.

Let $\mathbf{a}$ be a simplex of the curve complex. If $A \subset S^\circ$ represents $\mathbf{a}$, then the choice of a permissible hyperbolic metric g on $S^\circ - A$ yields a stable pointed curve (C^A, P). We thus have defined a map $\tilde{\pi} : \mathcal{T}^+_{S,P} \to \overline{\mathcal{M}}_{g,P}$. This map is clearly constant on $\Gamma(S, P)$-orbits and so it factors through a map $\pi : \mathcal{M}^+_{g,P} \to \overline{\mathcal{M}}_{g,P}$.

Let us denote by $\tilde{b} \in \mathcal{T}^+_{S,P}$ the point defined by $(S, P; A, g)$, by $b \in \mathcal{M}^+_{g,P}$ its image and by $c \in \overline{\mathcal{M}}_{g,P}$ the point defined by (C^A, P). Suppose we have likewise $\mathbf{a}'$, A' and g' with the property that the resulting stable pointed curve $(C^{A'}, P)$ is isomorphic to (C^A, P). Then after some shearing, there is a element of $\mathrm{Diff}^+(S,P)$ which carries (A',g') onto (A,g). In other words, the element $\tilde b' \in \mathcal{T}^+_{S,P}$ defined by (A',g') lies in $\Gamma(S,P).\mathbb{R}^{\mathbf{a}}.\tilde b$. The intersection $\Gamma(S,P)\cap\mathbb{R}^{\mathbf{a}}$ consists of the Dehn twists around the vertices of **a** and so we see that on the fiber $\pi^{-1}(c)$ we have an principal action of the torus $(S^1)^{\mathbf{a}}$ and that the image of $\tilde b'$ in $\mathcal{M}^+_{g,P}$ is in the orbit of that torus. This shows that $\pi^{-1}(c)$ is an orbit of $(S^1)^{\mathbf{a}}$. We must be careful though, for it is possible that some mapping classes permute the elements of **a**. So this action only lives in an orbifold setting. We thus find:

Proposition 9.15. *We have a natural map of orbifolds $\pi:\mathcal{M}^+_{g,P}\to\overline{\mathcal{M}}_{g,P}$. This map is proper and over the stratum of $\overline{\mathcal{M}}_{g,P}$ defined by the $\Gamma(S,P)$-orbit of the simplex* **a** *of the curve complex it has the structure of a principal bundle of the torus $(S^1)^{C_{sg}}$ in the orbifold sense.*

The idea behind Geometric Invariant Theory. Before we discuss the projectivity of the moduli space of stable pointed curves, we need to review some basic facts from Geometric Invariant Theory. We assume here our base field K is algebraically closed of characteristic zero. If G is a reductive group (like $GL(n,K)$ or $SL(n,K)$) and W is a representation of G, then the algebra of G-invariant polynomial functions on W, $K[W]^G$, defines the *separated orbit space* of W: it is a finitely generated algebra and is able to separate any two orbits whose closures are disjoint. The geometric content of this statement is that $K[W]^G$ defines a variety (denoted $G\backslash\!\backslash W$) whose closed points are in bijection with the closed orbits in W. If we pass to the G-action on $\mathbb{P}(W)$, then we need to discard in W the closed orbit defined by the origin and hence with it also all orbits in W having the origin in their closure. The remaining open subset of W is called the *semistable locus* and is denoted W^{ss}. The closed orbits in $\mathbb{P}(W^{ss})\subset\mathbb{P}(W)$ are then in bijective correspondence with the closed points of the projective variety $\mathrm{Proj}(K[W]^G)$ for which $K[W]^G$ is the homogeneous coordinate ring (we therefore denote this variety also by $G\backslash\!\backslash\mathbb{P}(W^{ss})$). A point of W is called G-*stable* if its G-orbit is closed and its G-stabilizer is finite. Such a point clearly lies in W^{ss} and defines an open (possibly empty) subset $W^{st}\subset W^{ss}$. It defines subsets of $G\backslash\!\backslash W$ and $G\backslash\!\backslash\mathbb{P}(W^{ss})$ that can be identified with $G\backslash W^{st}$ and $G\backslash\mathbb{P}(W^{st})$ respectively.

In practice one is only interested in a G-invariant subvariety $Z\subset\mathbb{P}(W)$. If $Z^{ss}:=Z\cap\mathbb{P}(W^{ss})$ is closed in $\mathbb{P}(W^{ss})$, then so is its image in $\mathrm{Proj}(K[W]^G)$, making it a projective variety. If it so happens that $Z^{ss}\subset\mathbb{P}(W^{st})$, then we see that the G-orbit space $G\backslash Z^{ss}$ of Z^{ss} is realized as a closed subset in $\mathrm{Proj}(K[W]^G)$. So $G\backslash Z^{ss}$ has then the structure of a projective variety.

Projectivity of the moduli space of stable pointed curves. We now return to the discussion *The quasi-projective structure on the universal family* of Section 9. We there constructed a vector space V, a representation W of $SL(V)$ and a $SL(V)$-invariant a subvariety $Z\subset\mathbb{P}(W)$ on which $SL(V)$ acts properly with finite stabilizers in such a manner that the orbit space is naturally identified with $M_{g,P}$. What was left unsaid is that if we take in *loc. cit.*

the parameters r and d large enough, then $\overline{Z} \cap \mathbb{P}(W^{ss})$ is closed in $\mathbb{P}(W^{ss})$ and contained in $\mathbb{P}(W^{st})$: any $z \in \overline{Z} \cap \mathbb{P}(W^{ss})$ defines a projective stable P-pointed curve (C_z, P) of genus g and the image of the natural action of the stabilizer $SL(V)_z$ on (C_z, P) yields all of $\mathrm{Aut}(C_z, P)$.

The above construction then leads to a projective compactification of $M_{g,P}$ which turns out to be just our $\overline{M}_{g,P}$. This implies in particular that if our base field is $\mathbb{C}$, then $M_{g,P}$ is compact in the Hausdorff topology. It then follows from Proposition 9.15 that the same is true for $M^+_{g,P}$.

Getting Harvey's bordification from the D-M compactification. We first need to discuss the notion of a real oriented blowup. If M is a C^∞-manifold and $N \subset M$ a closed submanifold, then the *real oriented blowup* of N in M is a C^∞-manifold $\tilde{M}_N$ with boundary which comes with a differentiable map $\pi : M_N \to M$ with the property that π is over M_N is a diffeomorphism and is over N is a the normal sphere bundle: a point over $p \in N$ is given by a ray in the normal space T_pM/T_pN. This sphere bundle is also the boundary of $\tilde{M}_N$. The differentiable structure on M_N can be given by local coordinates: if $(x', x'') = (x^1, \dots, , x^n, x^{n+1}, \dots, x^m)$ is a coordinate system on $U \subset M$ such that $N \cap U$ is given by $x'' = 0$, then on $\pi^{-1}U$ we have a parametrization by an open subset of $\mathbb{R}^n \times [0, \infty) \times S^{m-n-1}$ such that π is given over U by $(x', r, \xi) \mapsto (x', r\xi)$.

In case M is a complex manifold and N is a complex submanifold of codimension one, then the boundary of M_N is in a natural way a principal homogeneous bundle of the circle group $U(1)$ (for the rays in $\mathbb{C}$ emanating from $0 \in \mathbb{C}$ make up a principal homogeneous space for that group). If $N' \subset M$ is have another complex submanifold of codimension one which meets N transversally (so that $N \cap N'$ is a complex submanifold of codimension two), then the preimage of N' in M_N can be identified with manifold with boundary $N'_N \cap N'$ and the normal bundle of $N'_{N\cap N'}$ in M_N is the pull-back of the normal bundle of $N' \subset M$. So if we perform another real oriented blowup for the inclusion $N'_N \subset M_N$, then the resulting space $M_{N,N'}$ is a manifold with a boundary over $N \cup N'$ which has a corner over $N \cap N'$. Notice that a point in $M_{N,N'}$ over $p \in N \cap N'$ is a pair of rays, one in T_pM/T_pN' and another in T_pM/T_pN'. In fact, the obvious map $M_{N,N'} \to M$ is over $N \cap N'$ a principal $U(1) \times U(1)$-bundle and the order of blowing up is irrelevant: $M_{N,N'}$ may be identified with $M_{N',N}$. This generalizes in a straightforward manner to the situation of a normal crossing divisor: if D is a normal crossing divisor in M, then we have a real oriented blowup $M_D \to M$, which over the locus where D has precisely k local irreducible components has locally the structure of a principal $U(1)^k$-bundle; if $p \in M$, then a point over M_D is given by a ray in T_pM/T_pD_i for every local irreducible component (D_i, p) of (D, p).

We apply this in an orbifold setting: we take $M = \overline{\mathcal{M}}_{g,P}$ and we let D be the Deligne-Mumford boundary.

Proposition 9.16. *The Harvey compactification $\mathcal{M}^+_{g,P}$ is naturally identified with the real oriented blowup of the Deligne-Mumford boundary $D_{g,P}$ in $\overline{\mathcal{M}}_{g,P}$.*

Proof. We construct a map from this real oriented blowup to $\mathcal{M}^+_{g,P}$. A point $c \in M$ is given by a stable P-pointed curve (C, P). The local irreducible components of (D, c) correspond to the nodes of C and the normal space to the irreducible component indexed by the node q is naturally identified with $T_q C_q^+ \otimes T_q C_q^-$. So a point over c is specified by giving for every node $q \in C_{sg}$ a ray in $T_q C_q^+ \otimes T_q C_q^-$. We use these data to construct a surface with a permissible metric. Consider the normalization $\pi : \hat{C} \to C$ and denote by $N \subset \hat{C}$ the preimage of C_{sg}. This comes as a finite set of pairs: for each node $q \in C$ we get an unordered pair $\{q^+, q^-\}$ and we assume given a ray r_q in $T_q \hat{C}_{q^+} \otimes T_q \hat{C}_{q^-}$. Now consider the real-oriented blowup $\hat{C}_N \to C$. This is a surface with boundary. The boundary component defined by $q^\pm$ is the space of rays in $T_q \hat{C}_{q^\pm}$. We use the ray r_q to identify these boundary components: we identify the boundary point defined by the ray $r^+ \subset T_q \hat{C}_{q^+}$ the boundary point defined by $r^- \subset T_q \hat{C}_{q^-}$ if $r^+ \otimes r^- = r_q$. This produces in first instance a topological surface S over C that is with over each q an embedded circle A_q. But if we equip C°_{reg} with its hyperbolic metric, then S has a unique differentiable structure for which the hyperbolic metric is that is given on $S^\circ - \cup_{q \in C_{sg}} A_q \cong C^\circ_{reg}$ is permissible with respect to its embedding in S. The resulting the resulting object defines an element of the Harvey compactification $\mathcal{M}^+_{g,P}$.

This defines our map. It is not hard to check that it is continuous. It is also has a continuous two-sided inverse, which is somewhat easier to define and whose definition is clearly suggested by the above map: if $A \subset S^\circ$ defines a simplex of the curve complex, then a permissible metric on $S^\circ - A$ relative to its embedding in S defines a conformal structure on $S - A$ giving us a nodal curve C_A. We have (according to Exercise 21) at each connected component A_q of A a standard coordinate system that is unique up to a rotation and inversion. So an orientation of A_q determines on A_q the structure of a principal $U(1)$-space, (with opposite orientations defining opposite $U(1)$-actions. This 'opposition' of $U(1)$-structures then determines a ray in the deformation space of the node $q \in C_A$. □

This implies an important compactness property for the Harvey compactification.

Corollary 9.17. *The mapping class group $\Gamma(S, P)$ acts properly discontinuously on $\mathcal{T}(S, P)^+$ and the orbit space $\mathcal{M}^+_{g,P}$ and is a compact orbifold with corners and with interior $\mathcal{M}_{g,P}$.*

Proof. Since the Deligne-Mumford is compact, it follows from Proposition 9.16 that $\mathcal{M}^+_{g,P}$ is compact as well.

We show how the first assertion can also be deduced from a property of the Deligne-Mumford compactification. A point $z \in \overline{\mathcal{M}}_{g,P}$ is represented by a (C,P) be a stable P-pointed curve (C,P) and an orbifold atlas at that point is given by a universal deformation $(f:\mathcal{C}\to B, x_P)$ of that curve as in Theorem 9.13, the associated (finite) group being $\mathrm{Aut}(C,P)$. The discriminant of f is a normal crossing divisor $D\subset B$. The real oriented blowup $B_D\to B$ of D will have the homotopy type of its central fiber, a torus of dimension $|C_{sg}|$. Its universal cover $\tilde{B}_D\to B_D$ is then contractible and its covering group can be identified with $\mathbb{Z}^{C_{sg}}$, the generator indexed by $q\in C_{sg}$ being a Dehn twist over that point. Then we have an exact sequence of automorphisms of $\tilde{B}_D$:

$$0\to\mathbb{Z}^{C_{sg}}\to\widetilde{\mathrm{Aut}(C,P)}\to\mathrm{Aut}(C,P)\to 1.$$

This group clearly acts properly discontinuously on $\tilde{B}_D$. From our assumptions on the universal deformation it follows that $\tilde{B}_D$ parametrizes an open subset $U\subset\mathcal{T}(S,P)^+$ (more precisely, a $\Gamma(S,P)$-orbit of those) and that any mapping class $g\in\Gamma(S,P)$ with the property that $gU\cap U\neq\emptyset$ must come from $\widetilde{\mathrm{Aut}(C,P)}$. Since the open sets thus obtained cover $\mathcal{M}^+_{g,P}$, it follows that $\Gamma(S,P)$ acts properly discontinuously on $\mathcal{T}(S,P)^+$. □

Corollary 9.18. *For every $\varepsilon>0$, the locus $\mathcal{T}(S,P)^\varepsilon\subset\mathcal{T}(S,P)$ where all the geodesic length functions are $\geq\varepsilon$ is a $\Gamma(S,P)$-invariant subset on which $\Gamma(S,P)$ acts with compact fundamental domain. For small enough ε, $\mathcal{T}(S,P)^\varepsilon$ is a $\Gamma(S,P)$-equivariant deformation retract of $\mathcal{T}(S,P)^+$ (and hence of $\mathcal{T}(S,P)$).*

Proof. Clearly, $\mathcal{T}(S,P)^\varepsilon$ is closed in $\mathcal{T}(S,P)$ and $\Gamma(S,P)$-invariant. Hence its image $\mathcal{M}^\varepsilon_{g,P}$ in $\mathcal{M}^+_{g,P}$ is also closed. A closed subset of a compact set is compact and so $\mathcal{M}^\varepsilon_{g,P}$ is compact. With the help of Fenchel-Nielsen charts, we see the For $\varepsilon>0$ small enough, $\mathcal{M}^\varepsilon_{g,P}$ is a deformation retract of $\mathcal{M}^+_{g,P}$. Such a deformation retraction will lift to a $\Gamma(S,P)$-equivariant one of $\mathcal{T}^+_{g,P}$ onto $\mathcal{T}^\varepsilon_{g,P}$. □

10. Cohomological properties of $\mathcal{M}_{g,P}$

Harer's stability theorem. Let (S,P) be a P-pointed connected closed surface of genus g such that (g,P) is stable. Let $o\in S-P$ and put $S^\diamond:=S-\{o\}$. Consider the group $\mathrm{Diff}_c(S^\diamond,P)$ of diffeomorphisms of S that fix P and are the identity on an unspecified neighborhood of o (the subscript stands for *compact support*). It is a subgroup of the group $\mathrm{Diff}_{\vec{o}}(S^\diamond,P)$ of diffeomorphisms of S that fix P and T_oC pointwise. It is not hard to verify that the inclusion $\mathrm{Diff}_c(S^\diamond,P)\subset\mathrm{Diff}_{\vec{o}}(S^\diamond,P)$ induces an isomorphism on connected components $\Gamma_c(S^\diamond,P)\cong\Gamma_{\vec{o}}(S,P)$ (it is in fact a weak homotopy equivalence). Let us fix an oriented isomorphism $\alpha:T_oS\cong\mathbb{C}$. Then we have a Teichmüller space $\mathcal{T}_{\vec{o}}(S,P)$ associated to $\Gamma_{\vec{o}}(S,P)$, namely the space of conformal structures S that extend α given up to isotopy in the identity component of $\mathrm{Diff}_{\vec{o}}(S^\diamond,P)$. If we divide out by $\Gamma_{\vec{o}}(S,P)$, then we get the moduli space $\mathcal{M}_{g,P,\vec{o}}$ of triples $(C,\tilde{P},v)$, where the first two items define a $\tilde{P}$-pointed curve and $v\in T_oC$ is a nonzero tangent vector (the vector which under α maps to 1). The structure becomes clearer if we consider the forgetful map $\mathcal{T}_{\vec{o}}(S,P) \to \mathcal{T}(S,\tilde{P})$. The tangent space T_oS defines the line bundle $\mathcal{L}_o^{-1}$ over $\mathcal{T}(S,\tilde{P})$ and $\mathcal{T}(S,P,\vec{o})$ may be identified with the universal cover of the complement of the zero section of this line bundle (this has the structure of an affine line bundle over $\mathcal{T}(S,\tilde{P})$). The group of covering transformations is generated by the class $D_o \in \Gamma_c(S^\diamond, P)$ of a Dehn twist along a small circle centered at o, so that dividing out by $D_o^{\mathbb{Z}}$ yields the complement of the zero section of $\mathcal{L}_o^{-1}$. We have a central extension

$$1 \to D_o^{\mathbb{Z}} \to \Gamma_c(S^\diamond, P) \to \Gamma(S,\tilde{P}) \to 1$$

that can be understood as an exact sequence of orbifold fundamental groups of the $\mathbb{C}^\times$ orbifold bundle $\mathcal{M}_{g,P,\vec{o}}$ over $\mathcal{M}_{g,\tilde{P}}$ defined by $\mathcal{L}_o^{-1}$.

Suppose we are given an embedding $i : S^\diamond \to S'$ of $S^\diamond$ in a connected surface S' (necessarily of genus $\geq g$; it could be $S' = S$ with i the inclusion, but it is also possible for $i(S^\diamond)$ to appear as the connected component of S' minus an embedded circle). Then 'extension by the identity' defines an embedding $\mathrm{Diff}_c(S^\diamond, P) \hookrightarrow \mathrm{Diff}_c^+(S', P)$. This induces a group homomorphism $\Gamma_c(S^\diamond, P) \to \Gamma_c(S', P)$. The stability theorem reads as follows.

Theorem 10.1 (Harer). *The homomorphism $\Gamma_c(S^\diamond, P) \to \Gamma_c(S', P)$ induces an isomorphism on group (co)homology in degrees $< \frac{2}{3}g$.*

So if we take $S' = S$, then we see that $\Gamma_c(S^\diamond, P) \to \Gamma_c(S,P) = \Gamma(S,P) \cong \Gamma_{g,P}$ has this property and if we take S' to be closed of genus $g+1$, then we see that $\Gamma_c(S^\diamond, P) \to \Gamma_c(S', P) = \Gamma(S', P) \cong \Gamma_{g+1,P}$ has that property, too. The maps $\Gamma_{g,P} \cong \Gamma(S,P) \leftarrow \Gamma_c(S^\diamond, P) \to \Gamma(S', P) \cong \Gamma_{g+1,P}$ involve choices. But the ambiguity is always a conjugacy in the target group and since interior automorphisms act trivially on the cohomology of a group, the induced maps on cohomology are independent of these choices. So for a fixed k, $H^k(\Gamma_{g,P})$ is independent of g if g is large enough. This so-called *stable cohomology* is in fact the cohomology of the group $\Gamma_{\infty,P} := \Gamma_c(S_\infty, P)$ where S_∞ is connected and of infinite genus.

Corollary 10.2. *For $g > \frac{3}{2}k$, $H^k(\mathcal{M}_{g,P};\mathbb{Q}) = H^k(\Gamma_{\infty,P};\mathbb{Q})$.*

Hopf algebra structure on stable cohomology. Let S_i be a closed connected surface of genus g_i and let $o_i \in S_i$, $i = 1, 2, \dots$ and let S be another closed connected surface of genus g and $o \in S$. Suppose that for $i = 1, 2$, we are given an embedding $j_i : S_i^\diamond \to S^\diamond$ whose images are disjoint. This induces a homomorphism of groups

$$\Gamma_c(S_1^\diamond) \times \Gamma_c(S_2^\diamond) \to \Gamma_c(S^\diamond).$$

If D_i is a small open disk on S_i centered at o_i, then $S_i - D_i$ is a compact surface with boundary ∂D_i and then the complement of the images of $S_1 - D_1$ and $S_2 - D_2$ in $S^\diamond$ is a punctured surface of genus $g - g_1 - g_2$ with two boundary components. Such a surface is unique up to diffeomorphism. This is easily seen to imply that all embeddings of $(S_1 - D_1) \sqcup (S_2 - D_2)$ in $S^\diamond$ lie in a single $\mathrm{Diff}_c(S^\diamond)$-orbit. So the choice of another pair (j_1, j_2) may alter the homomorphism $\Gamma_c(S_1^\diamond) \times \Gamma_c(S_2^\diamond) \to \Gamma_c(S^\diamond)$ by conjugation with an element of $\Gamma(S^\diamond)$ only. Since an inner automorphism of a group acts trivially on its cohomology, the induced ring homomorphism

$$\Delta_{1,2} : H^\bullet(\Gamma_c(S^\diamond)) \to H^\bullet(\Gamma_c(S_1^\diamond) \times \Gamma_c(S_2^\diamond)) \to H^\bullet(\Gamma_c(S_1^\diamond)) \otimes H^\bullet(\Gamma_c(S_2^\diamond))$$

is independent of choices. The inclusion of the trivial group $\{1\}$ in $\Gamma_c(S_2^\diamond)$ induces the *co-augmentation* $\varepsilon_2 : H^\bullet(\Gamma_c(S_2^\diamond)) \to H^\bullet(\{1\}) = \mathbb{Z}$ and so if we compose $\Delta_{1,2}$ with $1 \otimes \varepsilon_2$, then the resulting map $H^\bullet(\Gamma(S^\diamond)) \to H^\bullet(\Gamma(S_1^\diamond))$ is just the map that we used to state the stability theorem.

If it so happens that $(S_1, D_1, o_1) = (S_2, D_2, o_2)$, then precomposing the embedding of $(S_1 - D_1) \sqcup (S_2 - D_2)$ in $S^\diamond$ with the exchange map alters the embedding by an element of $\mathrm{Diff}_c(S^\diamond)$. This implies that $\Delta_{1,2}$ does not change if we compose it with the exchange map (the graded involution which is on $H^k(\Gamma_\infty) \otimes H^l(\Gamma_\infty)$ is given by $u \otimes v \mapsto (-1)^{kl} v \otimes u$).

A similar argument shows that if we are given pairwise disjoint embeddings $j_i : S_i^\diamond \to S^\diamond$ for $i = 1, 2, 3$, such that j_1 and j_2 factor through an embedding $S_{12}^\diamond \to S^\diamond$ and j_2 and j_3 factor through an embedding $S_{23}^\diamond \to S^\diamond$, then the diagram below commutes.

$$\begin{tikzcd}
H^\bullet(\Gamma_c(S^\diamond)) \arrow[r, "\Delta_{12,3}"] \arrow[d, "\Delta_{1,23}"] & H^\bullet(\Gamma_c(S_{12}^\diamond)) \otimes H^\bullet(\Gamma_c(S_3^\diamond)) \arrow[d, "\Delta_{1,2}\otimes 1"] \\
H^\bullet(\Gamma_c(S_1^\diamond)) \otimes H^\bullet(\Gamma_c(S_{23}^\diamond)) \arrow[r, "1\otimes\Delta_{2,3}"] & H^\bullet(\Gamma_c(S_1^\diamond)) \otimes H^\bullet(\Gamma_c(S_2^\diamond)) \otimes H^\bullet(\Gamma_c(S_3^\diamond)).
\end{tikzcd}$$

So if we invoke Harer's stability theorem, then we find a graded ring homomorphism $\Delta : H^\bullet(\Gamma_\infty) \to H^\bullet(\Gamma_\infty) \otimes H^\bullet(\Gamma_\infty)$ with the property that if $\varepsilon : H^\bullet(\Gamma_\infty) \to \mathbb{Z}$ denotes the co-augmentation map, then

co-unit: $(\varepsilon \otimes 1)\Delta$ and $(1 \otimes \varepsilon)\Delta$ are the identity,
co-associativity: $(\Delta \otimes 1) \otimes \Delta = (1 \otimes \Delta) \otimes \Delta$.
cocommutativity: $\Delta = T\Delta$, where T is the graded involution which on $H^k(\Gamma_\infty) \otimes H^l(\Gamma_\infty)$ is given by $u \otimes v \mapsto (-1)^{kl} v \otimes u$,

the last assertion being rather trivial. In other words, this makes $H^\bullet(\Gamma_\infty)$ a Hopf algebra that is both graded-commutative and graded-cocommutative. If we tensor with $\mathbb{Q}$, so that we are dealing with $H^\bullet(\Gamma_\infty; \mathbb{Q})$, then according to a theorem of Milnor-Moore [] such an algebra has a very simple structure. Let us abbreviate $H^\bullet(\Gamma_\infty; \mathbb{Q})$ by $H^\bullet$. The *primitive part* $H^\bullet_{pr}$ is defined as the space of $u \in H^\bullet$ with $\Delta(u) = u \otimes 1 + 1 \otimes u$. This is a graded subspace and the inclusion of $H^\bullet_{pr}$ in $H^\bullet$ extends of course to a $\mathbb{Q}$-algebra homomorphism

$$\mathrm{Sym}^\bullet(H_{pr}^{even}) \otimes_{\mathbb{Q}} \wedge^\bullet(H_{pr}^{odd}) \to H^\bullet.$$

The theorem in question asserts that this is in fact an isomorphism. Since Δ is a ring homomorphism, this description also determines the comultiplication on the left hand side. We may think of $\mathrm{Sym}^\bullet(H_{pr}^{even})$ as the coordinate ring of the the dual H^{pr}_{even} of H^{even}_{pr} (the *stable primitive homology*). The coproduct on $\mathrm{Sym}^\bullet(H^{even}_{pr})$ simply describes the additive structure on H^{pr}_{even}.

11. TAUTOLOGICAL ALGEBRAS

Duality on orbifolds. Suppose X is a space and G is a finite group acting on X. Then the quotient map $X \to G\backslash X$ maps the singular cochain complex of $G\backslash X$ to the G-invariant part of the cochain complex of X and under mild conditions (e.g., X admits the structure of a G-invariant CW complex) this identifies $H^\bullet(G\backslash X;\mathbb{Q})$ with $H^\bullet(X;\mathbb{Q})^G$.

This is used to see that the space underlying an orbifold M of dimension m is a rational homology manifold of that same dimension. By the latter is meant that for every $p \in M$, $H_\bullet(M, M-\{p\};\mathbb{Q})$ is isomorphic to $H_\bullet(\mathbb{R}^m, \mathbb{R}^m-\{0\};\mathbb{Q})$ (in other words is of dimension one and concentrated in degree m). It is said to be *oriented* if for every $p \in M$ we have specified a generator μ_p for $H_m(M, M-\{p\};\mathbb{Q})$ which depends 'continuously' on p: if $U \subset M$ is the interior of an embedded disk, then for $p, q \in U$ the isomorphism $H_m(M, M-\{p\};\mathbb{Q}) \xrightarrow{\cong} H_m(M, M-U;\mathbb{Q}) \xleftarrow{\cong} H_m(M, M-\{q\};\mathbb{Q})$ takes μ_p to μ_q. In that case M satisfies $\mathbb{Q}$-Poincaré duality in dimension m: we have a natural isomorphism $H^k(M;\mathbb{Q}) \cong H^{lf}_{m-k}(M;\mathbb{Q})$, where $H^{lf}_\bullet$ stands for the homology of the complex of possibly infinite, but *locally finite* chains (if M is open in a compact set K, such that $M-K$ is not too wild, e.g., a compact manifold with boundary having M as its interior, then $H^{lf}_\bullet(M) \cong H_\bullet(K, M-K)$). Note that a cochain with compact support can be evaluated on a locally finite chain.

The preceding applies to the space underlying a complex-analytic orbifolds such as $\mathcal{M}_{g,P}$ and $\overline{\mathcal{M}}_{g,P}$ and so they satisfy $\mathbb{Q}$-Poincaré duality.

Review of the Gysin map. Let $f: N \to M$ be a proper map between manifolds of dimension n and m respectively which are endowed with orientations μ_M and μ_N. Then we can use duality on both N and M to define the *Gysin map*: if $d := m - n$ denotes the *formal codimension* of f, then put

$$f_! : H^k(N) \overset{\mu_N}{\cong} H^{lf}_{n-k}(N) \xrightarrow{f_*} H^{lf}_{n-k}(M) \overset{\mu_M}{\cong} H^{d+k}(M).$$

The ring homomorphism $f^* : H^\bullet(M) \to H^\bullet(N)$ turns $H^\bullet(N)$ into a $H^\bullet(M)$-module and with respect to that structure $f_!$ is a homorphism of $H^\bullet(M)$-modules: $f_!(f^*y \cup x) = y \cup f_!(x)$. We further note that if $g: M \to P$ is another proper map with P an oriented manifold, then gf is proper and $(gf)_! = g_! f_!$.

The following two cases are of special interest.

If f is a proper embedding of codimension d, then $f_! : H^\bullet(N) \to H^{d+\bullet}(M)$ can be obtained as the composite of the Thom isomorphism

$$H^\bullet(N) \cong H^{d+\bullet}(U, U-N)) \cong H^{d+\bullet}(M, M-N)$$

(where U is a tubular neighborhood of N) and the natural map $H^{d+\bullet}(M, M-N) \to H^{d+\bullet}(M)$ so that it fits in an exact sequence:

$$\cdots \to H^{k-1}(M-N) \to H^{k-d}(N) \xrightarrow{f_!} H^k(M) \to H^k(M-N) \to \cdots$$

The element $f_!(1) \in H^d(M)$ can be understood as assigning to a d-cycle on M its intersection number with N; we often refer to this as the *(cohomology) class defined by* N. Its image $f^*f_!(1) \in H^d(N)$ is the Euler class of the normal bundle of f.

If f is a proper submersion (so with with fiber dimension $-d \geq 0$), then the Gysin map $f_! : H^{\bullet-d}(N) \to H^\bullet(M)$ is 'integration along the fibers'. There is then a base change property: if $g : M' \to M$ is a smooth map of manifolds, then we have pull-back diagram

$$\begin{array}{ccc} N' & \xrightarrow{g'} & N \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ M' & \xrightarrow{g} & M \end{array}$$

with f' a proper submersion of manifolds and $g^*f_! = f'_!(g')^*$.

Remark 11.1. The Gysin map can be defined without requiring that f be proper, but then we need to restrict ourselves on N to cohomology with *proper* f*-support*, $H^\bullet_f(N)$: we only allow cochains such that the restriction of f to their support is proper. The aforementioned properties still hold and we then see that the general case is a combination of these two special spaces: if we factor f as the embedding $i_f : N \subset N \times M$ by its graph followed by the projection $\pi_M : N \times M \to M$: then $i_{f!}$ takes its values in $H^\bullet_{\pi_M}(M \times N)$ and we have $f_! = \pi_{M!} i_{f!}$.

If we are willing to take cohomology with $\mathbb{Q}$-coefficients, then the preceding is still true for rational homology manifolds. A *rational homology manifold of dimension* m is a Hausdorff space M that is locally the cone over a compact space and has the property that for every $p \in M$, $H_\bullet(M, M-\{p\}; \mathbb{Q})$ is isomorphic to $H_\bullet(\mathbb{R}^m, \mathbb{R}^m - \{0\}; \mathbb{Q})$ (in other words is of dimension one and concentrated in degree m). It is said be *oriented* if we have specified a generator $\mu_p \in H_m(M, M-\{p\}; \mathbb{Q})$ which depends 'continuously' on p in the sense that if U is an open ball in M containing p, then the isomorphism $H_m(M, M-\{p\}; \mathbb{Q}) \cong H_m(M, M-U; \mathbb{Q}) \cong H_m(M, M-\{q\}; \mathbb{Q})$ maps μ_p to μ_q. In that case the orientation defines a duality isomorphism $\mu : H^k(M; \mathbb{Q}) \cong H^{lf}_{m-k}(M; \mathbb{Q})$.

This applies for example to the space underlying a complex-analytic orbifold. This also shows that if $E \to M$ is an oriented vector bundle of rank r in the orbifold sense, then its rational Euler class $e(E/M) \in H^r(M; \mathbb{Q})$ can be defined as $o_!(1)$, where $o : M \to E$ is the zero section.

The moduli space of stable pointed curves as a category. Among moduli spaces $\overline{\mathcal{M}}_{g,P}$ and products of those there many interesting maps.

First, there is a rather trivial kind of map: if we have a bijection $Q \cong P$ of finite sets, then we have an evident identification $\overline{\mathcal{M}}_{g,Q} \cong \overline{\mathcal{M}}_{g,P}$. This already implies that the symmetric group $\mathrm{Aut}(P)$ acts on $\overline{\mathcal{M}}_{g,P}$. More generally, every injection $P \to Q$ determines a morphism of orbifolds $\overline{\mathcal{M}}_{g,Q} \to \overline{\mathcal{M}}_{g,P}$. Let us first see why this is so for the inclusion of P in a set that has one element more than P: $\tilde{P} = P \sqcup \{o\}$ (with part of the argument relegated to two exercises). If $(\tilde{C}, \tilde{P})$ is a stable pointed curve, then by forgetting o, the resulting $(\tilde{C}, P)$ need not be a stable, but nevertheless does give rise to a stable P-pointed genus g curve (we keep on assuming that $2g-2+|P|>0$, of course). The following two exercises make this precise.

Exercise 26. Show that $(\tilde{C}, P)$ is stable unless o lies on an irreducible component $\tilde{C}_o$ isomorphic to $\mathbb{P}^1$ which has exactly two points that are nodes or belong to P. Prove that in the last case, contraction of $\tilde{C}_o$ in $\tilde{C}$ yields a stable P-pointed curve (C, P).

Exercise 27. Next show a converse: let (C, P) be a stable pointed genus g curve. Prove that for every $q \in C$ there is naturally defined a stable $\tilde{P}$-pointed genus g curve $(\tilde{C}_q, \tilde{P})$ together with a surjective morphism $\tilde{C}_q \to C$ for which forgetting o yields the above contraction map. Show that the pair $(\tilde{C}_q, \tilde{P})$ is unique up to unique isomorphism and that the automorphism group of $(\tilde{C}_q, \tilde{P})$ can be identified with the group of automorphisms of (C, P) which fix q.

Exercise 27 can also be carried out 'with parameters': if $(\mathcal{C} \to B, x_P)$ is a family of stable P-pointed genus g curves, then $\mathcal{C}$ serves as a base for a naturally defined family of stable $\tilde{P}$-pointed genus g curves. This construction therefore defines a morphism of orbifolds $\overline{\mathcal{C}}_{g,P} \to \overline{\mathcal{M}}_{g,\tilde{P}}$. Exercise 26 provides an inverse and so we conclude:

Proposition 11.2. *We have a natural isomorphism $\overline{\mathcal{C}}_{g,P} \cong \overline{\mathcal{M}}_{g,\tilde{P}}$ of orbifolds. In particular, we have a morphism $f_{\tilde{P},P} : \overline{\mathcal{M}}_{g,\tilde{P}} \to \overline{\mathcal{M}}_{g,P}$ and every $p \in P$ defines a section $x_{P,\tilde{P}}$ of it.*

Iteration then implies our assertion above that any inclusion $P \hookrightarrow Q$ of finite sets determines a surjective morphism $\overline{\mathcal{M}}_{g,Q} \to \overline{\mathcal{M}}_{g,P}$ (the order of contraction is indeed immaterial).

These moduli spaces can be connected for different genera as follows. First assume we are given a *stable pair* $(g-1, Q)$ (by which we simply mean that $2(g-1)-2+|Q|>0$) and a 2-element subset $D \subset Q$. Out of any stable pointed curve $(\tilde{C}, Q)$ of genus g we construct a stable pointed curve $(\tilde{C}, Q)/D$ of genus g by simply identifying the two points of D so that this becomes a node. There is no difficulty in doing this in families and we thus get a finite morphism

$$i_D : \overline{\mathcal{M}}_{g-1,Q} \to \overline{\mathcal{M}}_{g,P},$$

where $P := Q - D$. The image is in fact an irreducible component of the Deligne-Mumford boundary of $\overline{\mathcal{M}}_{g,P}$ (that is often denoted by Δ_0).

Similarly, suppose we are given stable pairs (g', Q') and (g'', Q'') and elements $q \in Q'$ and $q'' \in Q''$. Put $g := g' + g''$, $D := \{q', q''\}$ and $P := Q' \sqcup Q'' - D$. If (C', Q') and (C'', Q'') are stable pointed curve of genus g' and g'' respectively, then joining these curves by identifying the members of D creates a P-pointed stable curve of genus g which at least one node. This defines a finite morphism

$$i_D : \overline{\mathcal{M}}_{g',Q'} \times \overline{\mathcal{M}}_{g'',Q''} \to \overline{\mathcal{M}}_{g,P}.$$

The image is an irreducible component of the Deligne-Mumford boundary (and the normal bundle of i_D may be identified with $\mathcal{L}_D$ as before. This irreducible component is sometimes denoted $\Delta_{(g',P')}$ $(= \Delta_{(g'',P'')})$.

Exercise 28. A special case of interest is when $g'' = 0$ and Q'' is a 3-element set $\{q'', q_0, o\}$. Then $\overline{\mathcal{M}}_{g'',Q''}$ is a singleton, $g' = g$ and P is obtained from Q' by replacing q' by the 2-element set $\{q_0, o\}$, so that $\overline{\mathcal{M}}_{g,P}$ can be identified with $\overline{\mathcal{C}}_{g,Q'}$. Show that the resulting map $i_D : \overline{\mathcal{M}}_{g,Q'} \to \overline{\mathcal{M}}_{g,P}$ can then be understood as the section $x_{q'}$, at least if we rename q_0 as q'.

Exercise 29. We may iterate these two preceding constructions: let be given a finite collection $\{(g_i, Q_i)\}_{i \in X_o}$ of stable pairs, put $Q := \sqcup_i Q_i$ and let $\{D_\alpha \subset \sqcup_i Q_i\}_{\alpha \in X_1}$ be a collection of pairwise disjoint 2-element subsets. We may then define in an evident manner a graph G whose vertex set is X_o and whose edge set is X_1. Put $P := \sqcup_i Q_i - \cup_\alpha D_\alpha$ and $g := b_1(G) + \sum_i g_i$. Prove that then is defined a finite map $\prod_{i \in X_o} \overline{\mathcal{M}}_{g_i,Q_i} \to \overline{\mathcal{M}}_{g,P}$. (In fact, the image of $\prod_{i \in X_o} \mathcal{M}_{g_i,Q_i}$ is a stratum of $\overline{\mathcal{M}}_{g,P}$ and every stratum of $\overline{\mathcal{M}}_{g,P}$ is so obtained.)

Remark 11.3. The system of maps defined above generate a categorical structure which contains what Getzler and Kapranov call a *modular operad*. This centers around the notion of a *stable weighted graph* which describes the complete isomorphism type of a simplex **a** of the curve complex of a pair (S, P) (with S allowed to be disconnected, but nonempty): if $A \subset S^\circ$ is a representative closed 1-manifold, then a vertex corresponds to a connected component of $S - A$ and we attach to that vertex the genus of this component, an internal edge corresponds to connected component of A and an loose end to an element of P. Formally, a stable weighted graph G is given by a quadruple $(X, \sim, \sigma_1, g)$ (where σ_1 is an involution which need not be fixed point free and $\pi_0 : X \to (X/\sim) =: X_0$ resp. $\pi_1 : X \to X/\langle\sigma\rangle =: X_1$ is the formation of the vertices resp. edges) and $g : X_0 \to \{0, 1, 2, \dots\}$ is such that for every $v \in X_0$, the pair $(g(v), \pi_1^{-1}(v))$ is stable. However, we want to allow that $X_1 = \emptyset$, in which case G is discrete.

More specifically, if $P \subset X$ denotes the fixed point set of σ and $g(G) := b_1(G) + \sum_{v \in X_1} g(v)$, then we call G a *stable* P*-pointed weighted graph of genus* $g(G)$. To such a graph G we associate the orbifold $\overline{\mathcal{M}}(G) := \prod_{v \in X_0} \overline{\mathcal{M}}_{g(v),\pi_1^{-1}(v)}$. This parametrizes the stable curves that are homeomorphic to (S_A, P) relative to P. Note that for the stable P-pointed graph $G_{g,P}$ which has single vertex of weight g we get $\overline{\mathcal{M}}(G_{g,P}) = \overline{\mathcal{M}}_{g,P}$.

The stable weighted graphs are the objects of a category $\mathfrak{G}^{st}$ whose morphisms are suggested by the operations we encountered and are such that a morphism $r : G \to G'$ gives rise to a morphism $\mathcal{M}(r) : \overline{\mathcal{M}}(G) \to \overline{\mathcal{M}}(G')$. There are four types:

(i) r collapses an internal edge, in other words, we remove from X a regular σ_1-orbit (in which case the new vertex has its weight prescribed accordingly). This corresponds to erasing a connected component of A. The associated morphism $\overline{\mathcal{M}}(G) \to \overline{\mathcal{M}}(G')$ takes a two element subset D of X_1 and applies i_D to the associated factor(s) in $\prod_{v\in X_0} \overline{\mathcal{M}}_{g(v),\pi_1^{-1}(v)}$.

(ii) r collapses a loose end, in other words, we remove from X a fixed point of σ_1 (since the result G' must be stable, this is not allowed if the vertex it is attached to has genus zero and has valency 3). If the loose end is indexed by $p \in P$, then this corresponds on S to replace P by $P-\{p\}$. The associated morphism $\overline{\mathcal{M}}(G) \to \overline{\mathcal{M}}(G')$ is given on a factor by a projection of the type $f_{P,P-\{p\}}$.

(iii) r erases a vertex of G, in other words, we remove from X an equivalence class of $\sim$. This amounts to the removal of a connected component of $S-A$. The associated morphism $\overline{\mathcal{M}}(G) \to \overline{\mathcal{M}}(G')$ is a projection along one of its factors. In case the vertex is of genus zero and has valency 3, then this morphism is an isomorphism and we then declare r to be an isomorphism as well.

(iv) r takes two loose ends of G and joins them up to produce an internal edge or does the inverse of cutting an internal edge thus producing two loose ends. In other words, we alter the definition of σ_1 on a σ_1-invariant 2-element subset of X. Either one determines an isomorphism $\overline{\mathcal{M}}(G) \cong \overline{\mathcal{M}}(G')$, but the genera of G and G' may differ.

We define a morphism of $\mathfrak{G}^{st}$ as being a composite of such maps. We have set things up in such a manner that $G \mapsto \overline{\mathcal{M}}(G)$ defines a covariant functor $\overline{\mathcal{M}}$ from $\mathfrak{G}^{st}$ to the category of projective orbifolds. The connected P-pointed graphs resp. the P-pointed graphs of genus g define a subcategory ${\mathfrak{G}^{st}}_P$ resp. ${\mathfrak{G}^{st}}_{g,P}$. The isomorphism classes of the latter are in bijective correspondence with the boundary strata closures of $\mathcal{M}_{g,P}$.

Exercise 30. Show that a section $x_p : \overline{\mathcal{M}}_{g,P} \to \overline{\mathcal{M}}_{g,\tilde{P}}$ is up to an isomorphism in the image of the functor $\overline{\mathcal{M}}$.

The notion of a tautological algebra. We begin our discussion with a definition.

Definition 11.4. A *tautological algebra functor* $\mathcal{T}^\bullet$ assigns to every moduli space $\overline{\mathcal{M}}_{g,P}$ a graded $\mathbb{Q}$-subalgebra $\mathcal{T}^\bullet(\overline{\mathcal{M}}_{g,P}) \subset H^\bullet(\overline{\mathcal{M}}_{g,P};\mathbb{Q})$ such that if for any stable weighted graph G we put

$$\mathcal{T}^\bullet(G) := \otimes_{v\in X_0}\mathcal{T}^\bullet(\overline{\mathcal{M}}_{g(v),\pi_0^{-1}(v)};\mathbb{Q}) \subset \otimes_{v\in X_0} H^\bullet(\overline{\mathcal{M}}_{g(v),\pi_0^{-1}(v)};\mathbb{Q}) = H^\bullet(\overline{\mathcal{M}}(G);\mathbb{Q}),$$

then for every morphism $r : G \to G'$ in $\mathfrak{G}^{st}$, the map $\overline{\mathcal{M}}(r) : \overline{\mathcal{M}}(G) \to \overline{\mathcal{M}}(G')$ has the property that $\overline{\mathcal{M}}(r)^*$ takes $\mathcal{T}^\bullet(G')$ to $\mathcal{T}^\bullet(G)$ and $\overline{\mathcal{M}}(r)_!$ takes $\mathcal{T}^\bullet(G)$ to $\mathcal{T}^\bullet(G')$ (equivalently, it must be stable under the maps on cohomology, ordinary and of Gysin type, induced by each of the morphisms (i)-(ii) listed in Remark 11.3).

We denote the image of $\mathcal{T}^\bullet(\overline{\mathcal{M}}_{g,P})$ in $H^\bullet(\mathcal{M}_{g,P};\mathbb{Q})$ by $\mathcal{T}^\bullet(\mathcal{M}_{g,P})$.

Note that since an automorphism of P defines an automorphism of the graph $G_{g,P}$, $\mathcal{T}^\bullet(\overline{\mathcal{M}}_{g,P})$ will be invariant under the permutation group.

The obvious maximal choice for $\mathcal{T}^\bullet$ is to let it be all of $H^\bullet(\overline{\mathcal{M}}(G);\mathbb{Q})$. Another is $H^{\mathrm{even}}(\overline{\mathcal{M}}(G);\mathbb{Q})$, but more interesting is the minimal one: an intersection of tautological functors is also one and so that there is a smallest such functor $\mathcal{R}$ with $\mathcal{R}(G) = \cap_{\mathcal{T}}\mathcal{T}^\bullet(G)$, where $\mathcal{T}^\bullet$ runs over all possible tautological algebra functors. We call $\mathcal{R}(G)$ the *tautological algebra* of $\overline{\mathcal{M}}(G)$. As $\mathcal{R}$ will have only even degrees, it will be commutative. We grade accordingly: $\mathcal{R}^k(G) \subset H^{2k}(\overline{\mathcal{M}}(G);\mathbb{Q})$.

Let us make this more concrete. If $(\mathcal{C} \to B, x_P)$ is a family of stable P-pointed genus g curves, then for every $p \in P$ we have defined the line bundle $\mathcal{L}_p$ on B: this is the pull-back along the section x_p of the cotangent bundle along the fibers. It has a first Chern class in $H^2(B)$ whose image in $H^2(B;\mathbb{Q})$ we denote by $\Psi_p(f)$. We can do this universally and find a line bundle on $\overline{\mathcal{M}}_{g,P}$ in an orbifold sense that we shall also denote by $\mathcal{L}_p$ and whose rational first Chern class we denote by $\Psi_p(\overline{\mathcal{M}}_{g,P}) \in H^2(\overline{\mathcal{M}}_{g,P};\mathbb{Q})$.

From the definition it is immediate that under a map i_D as defined before, $i_D^*\mathcal{L}_p$ can be identified with its namesake on $\overline{\mathcal{M}}_{g-1,Q}$ (assuming $p \in P = Q - D$) or its pull-back via $\overline{\mathcal{M}}_{g',Q'} \times \overline{\mathcal{M}}_{g'',Q''} \to \overline{\mathcal{M}}_{g',Q'}$ (assuming $p \in Q' - \{q'\}$) and so a similar property holds for Ψ_p: $i_D^*\Psi_p(\overline{\mathcal{M}}_{g,P})$ is equal to $\Psi_p(\overline{\mathcal{M}}_{g-1,Q})$ resp. $\Psi_p(\overline{\mathcal{M}}_{g',Q'}) \otimes 1$.

Lemma 11.5. *Write P' for $P - \{p\}$ so that in case $\mathcal{M}_{g,P'}$ is defined, we have the sections $(x'_q : \overline{\mathcal{M}}_{g,P'} \to \mathcal{M}_{g,P})_{q \in P'}$. If $x_p : \mathcal{M}_{g,P} \to \mathcal{M}_{g,\tilde{P}}$, then we have*

$$\Psi_p(\overline{\mathcal{M}}_{g,P}) = -x_p^* x_{p!}(1) + \sum_{q \in P'} x'_{q!}(1).$$

In particular, $c_1(\mathcal{L}_p) \in \mathcal{R}^\bullet(\overline{\mathcal{M}}_{g,P})$ and the images of $c_1(\mathcal{L}_p)$ and $-x_p^* x_{p!}(1)$ in $\mathcal{R}^\bullet(\mathcal{M}_{g,P})$ are equal.

Proof. The expression $x_p^* x_{p!}(1)$ is the Euler class of the normal bundle ν_p of x_p. Let $b \in \overline{\mathcal{M}}_{g,P}$ and let (C, P) a representative of b. Then the fiber $\mathcal{L}|_p$ is T_p^*C by definition. The fiber of the normal bundle of x_p at b is T_pC, unless p lies on an irreducible component C_o which destabilizes after removal of p, i.e., C_o is a smooth genus zero curve which meets P' in precisely one point q and meets the rest C' of C in a single node p', in other words, if b lies in the image of x'_q for some $q \in P'$. There is in fact a natural section of $\mathcal{L} \otimes \nu_p$ (the pairing) whose zero divisor is the sum of the images of the x'_q's. The lemma follows from this. □

We now consider the universal family $f : \overline{\mathcal{C}}_{g,P} = \overline{\mathcal{M}}_{g,\tilde{P}} \to \overline{\mathcal{M}}_{g,P}$. The preceding lemma (or rather its proof) shows that the line bundle $\mathcal{L}_o$ on $\overline{\mathcal{M}}_{g,\tilde{P}}$ is away from the images of the sections x_p like the cotangent bundle along the fibers. We define for $r = 0, 1, 2, \ldots,$

$$\kappa_r(\overline{\mathcal{M}}_{g,P}) := f_!(\Psi_o(\overline{\mathcal{M}}_{g,P})^{r+1}) \in H^{2r}(\overline{\mathcal{M}}_{g,P};\mathbb{Q}).$$

Its restriction to $\mathcal{M}_{g,P}$ is denoted by $\kappa_r \in H^{2r}(\mathcal{M}_{g,P};\mathbb{Q})$ and is known as the rth *Miller-Morita-Mumford class*; geometrically this class is represented by the cycle defined by the locus where r generic topological perturbations $x_o^{(1)}, \dots x_o^{(r)} : \mathcal{M}_{g,P} \to \mathcal{M}_{g,\tilde{P}}$ of x_o take the same value as x_o. If there is a risk of ambiguity, we denote this class by $\kappa_r(\mathcal{M}_{g,P})$, but the simpler notation κ_r is justified by the fact that if $P \subset Q$, then the forgetful morphism $\mathcal{M}_{g,Q} \to \mathcal{M}_{g,P}$ has the property that it takes $\kappa_r(\mathcal{M}_{g,P})$ to $\kappa_r(\mathcal{M}_{g,Q})$ (this is in general not true on the Deligne-Mumford compactifications).

It follows from Lemma 11.5 that these classes all lie in $\mathcal{R}^\bullet(\overline{\mathcal{M}}_{g,P})$ resp. $\mathcal{R}^\bullet(\mathcal{M}_{g,P})$. The following proposition shows that these kappa classes behave nicely with respect to the morphisms considered above.

Proposition 11.6. *For* $i_D : \overline{\mathcal{M}}_{g-1,Q} \to \overline{\mathcal{M}}_{g,P}$ *resp.* $i_D : \overline{\mathcal{M}}_{g',Q'} \times \overline{\mathcal{M}}_{g'',Q''} \to \overline{\mathcal{M}}_{g,P}$*, we have that* $i_D^*\kappa_r(\overline{\mathcal{M}}_{g,P})$ *equals* $\kappa_r(\overline{\mathcal{M}}_{g-1,Q})$ *resp.* $\kappa_r(\overline{\mathcal{M}}_{g',Q'})\otimes 1 + 1\otimes \kappa_r(\overline{\mathcal{M}}_{g'',Q''})$.

Proof. We need to go through the definitions. In the first case we have the commutative diagram of universal families

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g-1,\tilde{Q}} & \xrightarrow{\tilde{i}_D} & \overline{\mathcal{M}}_{g,\tilde{P}} \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \overline{\mathcal{M}}_{g-1,Q} & \xrightarrow{i_D} & \overline{\mathcal{M}}_{g,P}, \end{array}$$

where $\tilde{P} = P \sqcup \{o\}$ and $\tilde{Q} = Q \sqcup \{o\}$. This is in fact a pull-back diagram. Moreover, the pull-back of $\mathcal{L}_p$ under $\tilde{i}_D$ is its namesake on $\overline{\mathcal{M}}_{g-1,\tilde{Q}}$ and so $\tilde{i}_D^*$ takes $\Psi_o(\overline{\mathcal{M}}_{g,\tilde{P}})$ to $\Psi_o(\overline{\mathcal{M}}_{g-1,\tilde{Q}})$. By the base change property, i_D^* then maps $\kappa_r(\overline{\mathcal{M}}_{g,P}) = f_!(\Psi_o(\overline{\mathcal{M}}_{g,\tilde{P}})^{r+1})$ to $f_!(\Psi_o(\overline{\mathcal{M}}_{g-1,\tilde{P}})^{r+1}) = \kappa_r(\overline{\mathcal{M}}_{g-1,Q})$.

The second case is somewhat more complicated. We have the universal families $\overline{\mathcal{M}}_{g',\tilde{Q}'} \to \overline{\mathcal{M}}_{g',Q'}$ and $\overline{\mathcal{M}}_{g'',\tilde{Q}''} \to \overline{\mathcal{M}}_{g'',Q''}$. Each of these can be pulled back over the product $\overline{\mathcal{M}}_{g',Q'} \times \overline{\mathcal{M}}_{g'',Q''}$ and then we can form their disjoint union $\hat{\hat{\mathcal{C}}} \to \overline{\mathcal{M}}_{g',Q'} \times \overline{\mathcal{M}}_{g'',Q''}$. But we can also take their union in the product $\overline{\mathcal{M}}_{g',\tilde{Q}'} \times \overline{\mathcal{M}}_{g'',\tilde{Q}''}$ and then the two total spaces meet along the natural section of each of the factors. In fact, we thus obtain the pull-back along i_D of the universal family over $\overline{\mathcal{M}}_{g,P}$. So we have a commutative diagram

$$\begin{array}{ccc} (\pi'')^*\overline{\mathcal{M}}_{g',\tilde{Q}'} \sqcup (\pi')^*\overline{\mathcal{M}}_{g'',\tilde{Q}''} & \xrightarrow{\tilde{i}_D} & \overline{\mathcal{M}}_{g,\tilde{P}} \\ {\scriptstyle (f'\times 1)\sqcup(1\times f'')}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \overline{\mathcal{M}}_{g',Q'} \times \overline{\mathcal{M}}_{g'',Q''} & \xrightarrow{i_D} & \overline{\mathcal{M}}_{g,P}, \end{array}$$

which is almost Cartesian. The pull-back of $\Psi_o(\overline{\mathcal{M}}_{g,\tilde{P}})^{r+1}$ is on one component equal to $\Psi_o(\overline{\mathcal{M}}_{g',\tilde{Q}'})^{r+1} \otimes 1$ and $1 \otimes \Psi_o(\overline{\mathcal{M}}_{g'',\tilde{Q}''})^{r+1}$ on the other. This implies that

$$f'_!\tilde{i}_D^*\Psi_o(\overline{\mathcal{M}}_{g,\tilde{P}})^{r+1} = \kappa_r(\overline{\mathcal{M}}_{g',Q'}) \otimes 1 + 1 \otimes \kappa_r(\overline{\mathcal{M}}_{g'',Q''}).$$

Hence it remains to show that the left hand side can be identified with $i_D^* f_! \Psi_o(\overline{\mathcal{M}}_{g,\tilde{P}})^{r+1}$, in other words, that we can pretend that base change applies here. This can be justified by the fact $\mathcal{L}_o$ is canonically trivial along the common section of the two components so that its Chern class has its support away from that locus. We then need to invoke Remark 11.1. □

Remark 11.7. We may restate this proposition as follows. Define for every stable weighted graph G, $\kappa_r(G) \in H^\bullet(\overline{\mathcal{M}}(G);\mathbb{Q})$ by

$$\kappa_r(G) := \sum_{v\in X_o(G)} \pi_v^*\kappa_r(\overline{\mathcal{M}}_{g(v),\pi_0^{-1}v}),$$

where $\pi_v : \overline{\mathcal{M}}(G) \to \overline{\mathcal{M}}_{g(v),\pi_0^{-1}v}$ is the projection. Then for any morphism $r : G \to G'$ in $\mathfrak{G}^{st}{}_{g,P}$, we have $\overline{\mathcal{M}}(r)^*(\kappa_r(G')) = \kappa_r(G)$.

Proposition 11.8. *The tautological algebra $\mathcal{R}^\bullet(\overline{\mathcal{M}}_{g,P})$ is as a $\mathbb{Q}$-algebra generated by the classes $\{\kappa_r(\overline{\mathcal{M}}_{g,P})\}_{r\geq 1}$, $\{\Psi_p(\overline{\mathcal{M}}_{g,P})\}_{p\in P}$, and the classes of the boundary strata closures (i.e., the images of the element of 1 under their Gysin maps). In particular, $\mathcal{R}^\bullet(\mathcal{M}_{g,P})$ is as a $\mathbb{Q}$-algebra generated by the kappa classes and the psi classes.*

Proof. Denote by $\mathcal{R}_o^\bullet(\overline{\mathcal{M}}_{g,P}) \subset H^\bullet(\overline{\mathcal{M}}_{g,P};\mathbb{Q})$ the $\mathbb{Q}$-algebra generated by the classes $\tilde{\kappa}_r(\overline{\mathcal{M}}_{g,P})$. It is clear that $\mathcal{R}_o^\bullet(G) \subset \mathcal{R}^\bullet(G)$ and so it suffices to show that $G \mapsto \mathcal{R}_o^\bullet(G)$ is a tautological algebra functor. In this case that comes down to checking invariance under the operations (i) and (ii). That is relatively straightforward. □

The tautological algebra $\mathcal{R}^\bullet_{0,P}$ is quite nice. First notice that the boundary divisors of $\overline{\mathcal{M}}_{0,P}$ are indexed by the splittings of P into two sets with least two elements (so this assumes $|P| \geq 4$): for every such splitting $S = \{S'|S''\}$ we have an embedding $i_S : \overline{\mathcal{M}}_{0,\tilde{S}'} \times \overline{\mathcal{M}}_{0,\tilde{S}''} \hookrightarrow \overline{\mathcal{M}}_{0,P}$ whose image is a boundary divisor. The following lemma gives in $\mathcal{R}^\bullet_{0,P}$ multiplicative relations in degree 4 and linear relations in degree 2.

Lemma 11.9. *Assume that $|P| \geq 4$ so that $\overline{\mathcal{M}}_{0,P}$ is defined.*

(mult) *If S and T are splittings of P into two sets with least two elements with no common member, then $i_{S!}(1).i_{T!}(1) = 0$.*

(add) *Let $p \in P$, $Q \subset P$ a 2-element subset and denote by $\mathcal{S}(p|Q)$ the collection of splittings of P into two sets with at least two elements which separate p from Q. Then $\Psi_p = \sum_{S\in\mathcal{S}(p|Q)} i_{S!}(1)$.*

Proof. The first identity follows from the geometry: if the codimension one strata parametrized by i_S and i_T meet, then they do so along a codimension two stratum. But a codimension two stratum defines a partition of P into three parts which refines both S and T. This implies that S and T have a common member.

The proof of the second identity is more subtle and it is only sketched. We define a section s of $\mathcal{L}_p$ with the help of Q as follows. Denote the two elements of Q by q_0 and q_∞. If $c \in \overline{\mathcal{M}}_{0,P}$, then represent it by a stable P-pointed curve (C, P) of genus zero. You may verify that there is a unique morphism $f_c : C \to \mathbb{P}^1$ with $f_c(p) = 1$, $f_c(q_0) = 0$, $f_c(q_\infty) = \infty$. We may then view $s(c) := f_c^*(\frac{dz}{z})|T_p^*C$ as an element of the fiber of $\mathcal{L}_p$ over c. It turns out that the naturality of this construction leads to the conclusion that s indeed defines a section of $\mathcal{L}_p$. So its zero divisor defines the class Ψ_p. But $s(c) = 0$ if and only if p lies on a component of C that is contracted by f_c and this happens precisely when c lies on the image of an i_S which separates p from q, r. The multiplicity of vanishing is 1 and so the the identity follows. □

Remark 11.10. Keel has shown that $\mathcal{R}^\bullet(\overline{\mathcal{M}}_{0,P}) = H^\bullet(\overline{\mathcal{M}}_{0,P}; \mathbb{Q})$ and that as an algebra, it is generated by the classes of the boundary divisors $i_{S!}(1)$ and the Ψ_p's subject to the relations above (in particular, we may eleminate the psi classes).

It is not known whether it is true in general ($g > 0$) that $\mathcal{R}^\bullet(\overline{\mathcal{M}}_{g,P})$ makes up a nondegenerate subspace of $H^\bullet(\overline{\mathcal{M}}_{g,P}; \mathbb{Q})$ with respect to the intersection pairing.

The lambda classes. Many of the classes that are naturally defined on a moduli space of (stable) pointed curves turn out to be tautological. Here is an example.

If C is a projective nodal curve of genus g, then its *dualizing sheaf* ω_C is on the smooth part of C of the sheaf of of differentials and at a node it is a differential on each branch with at most a simple pole at that node such that the two residues add up to 0. Notice that ω_C is locally free on C

Lemma 11.11. *Let* $\pi : \hat{C} \to C$ *be a partial normalization of* C *(with the branches at some nodes being separated), and denote by* G *the graph attached to* π *(the points in* $\hat{C}$ *lying over a node define oriented edges, connected components of* $\hat{C}$ *define vertices, normalized nodes define edges). Then we have a natural exact sequence*

$$0 \to H^0(\hat{C}, \omega_{\hat{C}}) \to H^0(C, \omega_C) \to H^1(G; \mathbb{C}) \to 0,$$

so that in particular, $\dim H^0(C, \omega_C) = g$.

Proof. We first do the case when π is the full normalization so that $\hat{C}$ is smooth. An element of $H^0(C, \omega_C)$ defines a meromorphic differential on $\hat{C}$. Its residues on every connected component sum up to zero and so do the two residues that over a node any given node. So if we think of the set $X(G)$ of oriented edges of G as a set of simplicial 1-chains, then these residue data define an element of $H^1(G; \mathbb{C}) \subset \mathbb{C}^{X(G)}$. On the other hand, on a closed connected Riemann surface we can arbitrarily prescribe the polar parts of a meromorphic differential as along as the sum of their residues is zero. This means that every element of $H^1(G;\mathbb{C})$ is so obtained. The genus g of C is indeed the sum of the genera of the connected components of $\hat{C}$ and the first Betti number of G.

The general case (of a partial blowup) is proven in the same way. □

The formation of ω_C behaves well in families: if we have a family of projective nodal curves $f : \mathcal{C} \to B$ of genus g, then we have a relatively dualizing sheaf $\omega_{\mathcal{C}/B}$ and its direct image on B, $f_*\omega_{\mathcal{C}/B}$, is a rank g vector bundle on M. The kth rational Chern class of this vector bundle is denoted by $\lambda_k(f) \in H^{2k}(B;\mathbb{Q})$. We have of course that $\lambda_k(f) = 0$ for $k > g$. If the family admits a partial normalization $\pi : \hat{\mathcal{C}} \to \mathcal{C}$ with constant graph G, and we put $\hat{f} := f\pi$, then we have an exact sequence

$$0 \to \hat{f}^*\omega_{\hat{\mathcal{C}}/B} \to f_*\omega_{\mathcal{C}/B} \to \mathcal{O}_B \otimes_{\mathbb{C}} H^1(G;\mathbb{C}) \to 0.$$

Since the quotient is trivial, we see that $\lambda_k(\hat{f}) = \lambda_k(f)$. So then $\lambda_k(f) = 0$ for $k > g - b_1(G)$.

If we do this universally for stable nodal curves of genus $g \geq 2$, then we have thus defined $\lambda_k(\overline{\mathcal{M}}_{g,P}) \in H^{2k}(\overline{\mathcal{M}}_{g,P};\mathbb{Q})$ with $\lambda_k = 0$ for $k > g$. Since the pull-back of $\lambda_k(\overline{\mathcal{M}}_{g,P})$ over a projection $\mathcal{M}_{g,Q} \to \mathcal{M}_{g,P}$ is $\lambda_k(\overline{\mathcal{M}}_{g,Q})$, we will usually simply write λ_k instead. Mumford has shown that in fact $\lambda_k \in \mathcal{R}^\bullet(\overline{\mathcal{M}}_g)$. The following proposition follows in a straightforward manner from Lemma 11.11.

Proposition 11.12. *For $i_D : \overline{\mathcal{M}}_{g-1,Q} \to \overline{\mathcal{M}}_{g,P}$ we have that $i_D^*\lambda_r = \lambda_r$ and for $i_D : \overline{\mathcal{M}}_{g',Q'} \times \overline{\mathcal{M}}_{g'',Q''} \to \overline{\mathcal{M}}_{g,P}$ we have $i_D^*\lambda_r = \sum_{k+l=r} \lambda_k \otimes \lambda_l$.*

Corollary 11.13. *The restriction of $\lambda_g\lambda_{g-1} \in H^{4g-2}(\overline{\mathcal{M}}_g;\mathbb{Q})$ to every boundary divisor of $\overline{\mathcal{M}}_g$ is zero and so is the restriction of $\lambda_g \in H^{2g}(\overline{\mathcal{M}}_g;\mathbb{Q})$ to the boundary divisor parametrized by $\overline{\mathcal{M}}_{g-1,2}$.*

Proof. This is indeed immediate from the previous proposition. □

So if we write $\mathcal{M}_g^c \subset \overline{\mathcal{M}}_g$ for the complement of the divisor parametrized by $\overline{\mathcal{M}}_{g-1,2}$, then the first part of Corollary 11.13 implies that that λ_g is in the image of $H_c^{2g}(\mathcal{M}_g^c;\mathbb{Q})$. By means of a degeneration argument using Hodge theory, Faber [] shows that similarly $\lambda_g\lambda_{g-1}$ restricted to the Deligne-Mumford boundary is zero and hence lies in the image of $H_c^{2(2g-1)}(\mathcal{M}_g;\mathbb{Q})$. This implies that the multiplication by $\lambda_g\lambda_{g-1}$ in $\mathcal{R}^\bullet(\overline{\mathcal{M}}_g)$ factors through a map $\mathcal{R}^\bullet(\mathcal{M}_g) \to \mathcal{R}^{2g-1+\bullet}(\overline{\mathcal{M}}_g)$. That observation plays an important role in a set of conjectures that he stated on the structure of $\mathcal{R}^\bullet(\mathcal{M}_g)$ (which is generated by the kappa classes).

Faber's conjectures. Rather than recall these conjectures in their original form, let us first state what is essentially the proven part. The *hyperelliptic locus* $\mathcal{H}_g$ is the locus in $\mathcal{M}_g$ parameterizing the isomorphism classes of hyperelliptic curves of genus g. Since a hyperelliptic curve of genus g is given up to isomorphism as a double cover of $\mathbb{P}^1$ ramified in $2g+2$ points, we have an identification of $\mathcal{H}_g$ with the $2g-1$-dimensional variety $\mathcal{M}_{0,2g+2}$ (though not as orbifolds, because of the hyperelliptic involution). The hyperelliptic locus is closed and of codimension $g-2$ in $\mathcal{M}_g$.

(i) (Morita) The $\mathbb{Q}$-algebra $\mathcal{R}^\bullet(\mathcal{M}_g)$ is generated by the κ_r, with $r \leq \lfloor \frac{1}{3}g \rfloor$.
(ii) (Looijenga) The $\mathbb{Q}$-algebra $\mathcal{R}^\bullet(\mathcal{M}_g)$ is zero in degree $> g-2$ and $\mathcal{R}^{g-2}(\mathcal{M}_g)$ is spanned by class of the *hyperelliptic locus* $\mathcal{H}_g$.
(iii) (Faber) Multiplication by $\lambda_g\lambda_{g-1}$ yields isomorphisms $\mathcal{R}^{g-2}(\mathcal{M}_g) \cong \mathcal{R}^{3g-3}(\overline{\mathcal{M}}_g) \cong \mathbb{Q}$.

Still open is:

Conjecture 11.14 (Faber). *The $\mathbb{Q}$-algebra $\mathcal{R}^\bullet(\mathcal{M}_g)$ is Gorenstein with socle $\mathcal{R}^{g-2}(\mathcal{M}_g)$. Equivalently, the pairing*

$$(r, r') \in \mathcal{R}^k(\mathcal{M}_g) \times \mathcal{R}^{g-2-k}(\mathcal{M}_g) \mapsto \lambda_g\lambda_{g-1}rr' \in \mathcal{R}^{3g-3}(\overline{\mathcal{M}}_g) \cong \mathbb{Q}$$

is perfect.

Furthermore, Faber has a candidate formula for $\lambda_g\lambda_{g-1}\kappa_1^{r_1}\kappa_2^{r_2}\cdots \in \mathbb{Q}$ (where $\sum_i i\kappa_i = g-2$).

There are a number of variants of this conjecture, one of which involves $\mathcal{R}^\bullet(\mathcal{M}_g^c)$ (supposedly to be Gorenstein with socle in degree $2g-3$, the role of $\lambda_g\lambda_{g-1}$ taken by λ_g).

Primitivity of the kappa classes. The maps appearing in the Harer stability theorem and in the above discussion can be made rather concrete in terms of the Harvey bordification. Recall that $\mathcal{M}^+_{g,P}$ is a compact manifold with corners whose interior is $\mathcal{M}^+_{g,P}$. So $\mathcal{M}_{g,P} \subset \mathcal{M}_{g,P}$ induces an isomorphism on cohomology. For $g' > 0$ we have the boundary stratum of $\overline{\mathcal{M}}_{g+k,P}$ parameterized by $\mathcal{M}_{g,\tilde{P}} \times \mathcal{M}_{g',\{o\}} \to \overline{\mathcal{M}}_{g+g',P}$. The preimage of this parameterization over $\mathcal{M}^+_{g+g',P}$ is given by a map

$$\mathcal{M}_{g,P,\vec{o}} \times \mathcal{M}_{g',\vec{o}} \to \mathcal{M}^+_{g+g',P}$$

(which actually factors through the orbit space of $\mathcal{M}_{g,P,\vec{o}} \times \mathcal{M}_{g',\vec{o}}$ for the $\mathbb{C}^\times$-action defined by $\lambda \in \mathbb{C}^\times : (w, w') \mapsto (\lambda w, \lambda^{-1}w')$). Then the $\mathbb{Q}$-stability theorem amounts to the assertion that the projection map $\mathcal{M}_{g,P,\vec{o}} \to \mathcal{M}_{g,P}$ and the maps $\mathcal{M}_{g,P,\vec{o}} \times \{w'\} \to \mathcal{M}^+_{g+g',P}$, with $w' \in \mathcal{M}_{g',\vec{o}}$, induce isomorphisms on rational cohomology in degree $< \frac{2}{3}g$. With the help of Proposition 11.6 we observe that each of these maps respects the kappa class κ_r and so κ_r defines in fact a stable cohomology class $H^{2r}(\Gamma_\infty)$. For $P = \emptyset$, the map $\mathcal{M}_{g,\vec{o}} \times \mathcal{M}_{g',\vec{o}} \to \mathcal{M}^+_{g+g'}$ yields in the stable range the coproduct for the Hopf algebra structure and again Proposition 11.6 shows that $\Delta(\kappa_r) = \kappa_r \otimes 1 + 1 \otimes \kappa_r$. We conclude that κ_r gives rise to a primitive element of $H^{2r}(\Gamma_\infty, \mathbb{Q})$: for $g > \frac{3}{2}r$, the class $\kappa_r \in H^{2r}(\mathcal{M}_g; \mathbb{Q}) \cong H^{2r}(\Gamma_g; \mathbb{Q}) \cong H^{2r}(\Gamma_\infty; \mathbb{Q})$ is independent of g and is primitive.

Proposition 11.15 (E.Y. Miller). *Every κ_r is nonzero and so the natural map of graded algebras $\mathbb{Q}[K_1, K_2, \dots] \to H^\bullet(\Gamma_\infty; \mathbb{Q})$ which sends K_r to κ_r (so that* $\deg K_r = 2r$*) is an embedding.*

We will not prove this. Mumford made in [] the conjecture that this is in fact an isomorphism. This has been proved by Madsen and Weiss [] , who in fact obtained a considerably stronger statement in terms of homotopy theory. From the preceding it is not hard to deduce that for a finite set P, the psi classes stabilize to $(\psi_p \in H^2(\Gamma_{\infty,P}))_{p \in P}$ (ψ_p is in fact the Euler class of the infinite central extension of $\Gamma_{\infty,P}$ defined by connected group of the compactly supported diffeomorphisms that fix P pointwise and fix a nonzero tangent vector at p) and that $H^\bullet(\Gamma_{\infty,P}; \mathbb{Q})$ is the free polynomial algebra generated by the kappa classes and the psi classes.