Grothendieck constructions of model categories have recently received some attention as in [HP14]. We will show that the Grothendieck construction of (more general) relative categories is actually a homotopy colimit in the model category $\text{RelCat}$. This will be proven, after some definitions and lemmas, as Theorem [10]. At the end, we will give an example involving the classical Spanier–Whitehead category.

We begin by reviewing the definition of a relative category.

**Definition 1.** A relative category $\mathcal{M}$ is a category $\mathcal{M}$ together with a subcategory $\text{we}\mathcal{M}$ containing all objects of $\mathcal{M}$. The morphisms in $\text{we}\mathcal{M}$ are usually called weak equivalences. A relative functor between relative categories $\mathcal{M}$ and $\mathcal{M}'$ is a functor $F: \mathcal{M} \rightarrow \mathcal{M}'$ with $F(\text{we}\mathcal{M}) \subseteq \text{we}\mathcal{M}'$. We denote the category of small relative categories with relative functors between them by $\text{RelCat}$.

For a category $\mathcal{D}$, define a functor

$$\left(\cdot\right)^{\mathcal{D}}: \text{RelCat} \rightarrow \text{RelCat}, \quad \mathcal{C} \mapsto \mathcal{C}^{\mathcal{D}},$$

where $\mathcal{C}^{\mathcal{D}}$ has as morphisms those natural transformations that are objectwise weak equivalences.

For a relative category $\mathcal{C}$, denote by $N(\mathcal{C})$ the Rezk classifying diagram, a simplicial space whose $p$-th space $N(\mathcal{C})_p$ is given by $\text{nerve}(\text{we}(\mathcal{C}[^p]))$. Here, $[^p]$ the category of $p$ composable morphisms. In [BK12b], Barwick and Kan construct a model structure on $\text{RelCat}$ whose weak equivalences are detected by

$$N: \text{RelCat} \rightarrow s\mathcal{S},$$

where the category of simplicial spaces $s\mathcal{S}$ carries the Rezk model structure (see [Rez01]). This is a certain localization of the Reedy model structure. Barwick and Kan give in [BK12a] also an alternative characterization of the weak equivalences. They show that $f: \mathcal{M} \rightarrow \mathcal{N}$ is a weak equivalence in $\text{RelCat}$ if and only if it induces a Dwyer–Kan equivalence of the hammock localizations

$$L^H\mathcal{M} \rightarrow L^H\mathcal{N},$$

i.e. an equivalence of homotopy categories $\text{Ho}(\mathcal{M}) \rightarrow \text{Ho}(\mathcal{N})$ and weak equivalences of mapping spaces. Here, the homotopy category $\text{Ho}(\mathcal{M})$ is defined be the localization of $\mathcal{M}$ at the class of
Lemma 4. Let \( (\text{RelCat}) \) be a (not necessarily relative) category with objects pairs \((i \in \mathcal{D}, x \in F(i))\) and morphisms \((i, x) \to (j, y)\) given by a pair \((f: i \to j, g: (F(f))(x) \to y)\). We declare such a morphism to be a weak equivalence if \(f\) and \(g\) are.

The Grothendieck construction comes with a canonical map \(\alpha_F: \mathcal{D} \int F \to \text{colim}_\mathcal{D} F\) defined as follows: Given an object \(i \in \mathcal{D}, x \in F(i)\), we set \(\alpha_F(x)\) to be the image of \(x\) under the canonical map \(F(i) \to \text{colim}_\mathcal{D} F\). Given a morphism \((f, g): (i, x) \to (j, y)\) defined by a pair \((f: i \to j, g: (F(f))(x) \to y)\), define \(\alpha(f, g)\) to be the image of \(g\) under the canonical map \(F(i) \to \text{colim}_\mathcal{D} F\) where one uses that \(F((f))(x)\) and \(x\) become identified in the colimit.

If \(\mathcal{D}\) is an ordinary category, we will view it for this definition as a relative category with the maximal relative structure, where every morphism is a weak equivalence.

Remark 3. Harpaz and Prasma consider in \([HP14]\) instead a functor \(F: \mathcal{M} \to \text{ModCat}\) from a model category to the category of model categories with left Quillen functors (with chosen adjoint) between them. They equip the Grothendieck construction \(\mathcal{M} \int F\) with notions of weak equivalences and (co)fibrations, which under certain conditions define a model structure. As a left Quillen functor is not necessarily a relative functor, this does not fall under the scope of our definition. But denote by \(\tilde{F}: \mathcal{M}^{\text{cof}} \to \text{RelCat}^b\) the postcomposition of the restriction of \(F\) to the full subcategory of cofibrant objects with the functor

\[
\text{Mod Cat} \to \text{RelCat}^b, \quad \mathcal{C} \mapsto \mathcal{C}^{\text{cof}},
\]

where \(\text{RelCat}^b\) denotes the (large) category of not-necessarily small relative categories. Then one can check that the relative category \((\mathcal{M} \int F)^{\text{cof}}\) of cofibrant objects agrees with \(\mathcal{M}^{\text{cof}} \int \tilde{F}\).

In the following, let \(\mathcal{D}\) always be a fixed (ordinary) category.

Lemma 4. Let \(F: \mathcal{D} \to \text{RelCat}\) be functor. Then there is an adjunction

\[
(\mathcal{D} \int F)[n] \rightleftarrows \mathcal{D} \int (([n] \circ F))
\]

that restricts to an adjunction between its categories of weak equivalences. Furthermore, the right adjoint is natural both in \(F \in \mathcal{M}^\mathcal{D}\) and in \([n] \in \Delta\).

Proof. An object of \((\mathcal{D} \int (([n] \circ F))\) consists of \(i \in \mathcal{D}\) and a sequence of composable morphisms \(x_0 \to \cdots \to x_n \in F(i)\). An object in \((\mathcal{D} \int F)[n]\) consists of a sequence of composable morphisms \(i_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{n-1}} i_n \in \mathcal{D}\), objects \(x_j \in F(i_j)\) and morphisms \((F(f_j))(x_j) \to x_{j+1} \in F(i_{j+1})\).

The left adjoint sends an object in \((\mathcal{D} \int F)[n]\) as above to \(i_n \in \mathcal{D}\) and to the sequence

\[
(F(f_{n-1} \circ \cdots \circ f_0))(x_0) \to (F(f_{n-1} \circ \cdots \circ f_1))(x_1) \to \cdots \to x_n
\]

in \(F(i_n)\).

The right adjoint sends an object in \((\mathcal{D} \int (([n] \circ F))\) as above to \(i \xrightarrow{id_i} \cdots \xrightarrow{id_i} i\) and to the sequence \(x_0 \to x_1 \to \cdots \to x_n \in F(i)\). \(\square\)
Lemma 5. Denote by we: RelCat → Cat the functor that sends \( \mathcal{M} \) to \( \text{we}\mathcal{M} \). Then there is a natural isomorphism \( \text{we}(\mathcal{D} \int F) \cong (\mathcal{D} \int (\text{we}\circ F)) \).

Proof. Clear. \( \square \)

We recall the definition of a homotopy colimit from [CS02]:

Definition 6. Given an indexing category \( \mathcal{D} \) and a relative category \( \mathcal{M} \) admitting \( \mathcal{D} \)-shaped colimits, a homotopy colimit is a terminal homotopical approximation of \( \text{colim} \), i.e. a functor \( H: \mathcal{M}^\mathcal{D} \to \text{Ho}(\mathcal{M}) \) together with a natural transformation \( \alpha: H \Rightarrow \text{colim} \), having the following two properties:

1. \( H \) is homotopical in the sense that it sends objectwise weak equivalences to isomorphisms.
2. If \( K: \mathcal{M}^\mathcal{D} \to \text{Ho}(\mathcal{M}) \) is another homotopical functor with a natural transformation \( \delta: K \Rightarrow \text{colim} \), then there is a unique natural transformation \( \gamma: K \Rightarrow H \) with \( \alpha\gamma = \delta \).

Here and in the following, we view \( \text{colim} \) also as a functor \( \mathcal{M}^\mathcal{D} \to \text{Ho}(\mathcal{M}) \).

Remark 7. It is clear that all homotopy colimit functors are unique up to unique isomorphism. Note also that this definition agrees with the definition of a total left derived functor of \( \text{colim} \) in the sense of [Rie14, Definition 2.1.16].

Lemma 8. Let \( \mathcal{M} \) and \( \mathcal{N} \) be relative categories admitting \( \mathcal{D} \)-shaped colimits and homotopy categories and let

\[
G: \mathcal{N} \to \mathcal{M}
\]

be a homotopy equivalence between them, which means that there is a relative functor \( F: \mathcal{M} \to \mathcal{N} \) and zig-zags of natural weak equivalences between \( FG \) and \( \text{id}_\mathcal{N} \) and between \( GF \) and \( \text{id}_\mathcal{M} \). Then \( G \) detects homotopy colimit functors in the following sense:

Denote by \( \mathbb{F} \) and \( \mathbb{G} \) the induced equivalences between \( \text{Ho}(\mathcal{M}) \) and \( \text{Ho}(\mathcal{N}) \). Let furthermore

\[
(H: \mathcal{M}^\mathcal{D} \to \text{Ho}(\mathcal{M}), \alpha: H \Rightarrow \text{colim}^\mathcal{M})
\]

be a homotopy colimit functor and

\[
(J: \mathcal{N}^\mathcal{D} \to \mathcal{N}, \beta: J \Rightarrow \text{colim}^\mathcal{N})
\]

be another pair of a functor and a natural transformation. Assume there is an isomorphism \( f: HG \cong \mathbb{G}J \) such that

\[
\begin{array}{ccc}
HG & \xrightarrow{\alpha G} & \text{colim}^\mathcal{M} G \\
\downarrow f & & \downarrow \\
\mathbb{G}J & \xrightarrow{\mathbb{G}\beta} & \mathbb{G}\text{colim}^\mathcal{N}
\end{array}
\]

commutes. Then \( J \) is a homotopy colimit functor as well.

Proof. First note that there are natural isomorphisms \( \mathbb{F}G \cong \text{id}_{\text{Ho}(\mathcal{N})} \) and \( GF \cong \text{id}_{\text{Ho}(\mathcal{M})} \). These can be chosen such that they define an adjunction between \( \mathbb{F} \) and \( \mathbb{G} \) and we will fix such a choice.

The assumptions imply that \( \beta: J \Rightarrow \text{colim}^\mathcal{N} \) is isomorphic to

\[
\epsilon: \mathbb{F}HG \cong \mathbb{F}\text{colim}^\mathcal{M} G \Rightarrow \mathbb{F}\text{colim}^\mathcal{N} \Rightarrow \text{colim}^\mathcal{N}.
\]
Clearly, $FHG$ is homotopical. Let $\delta : K \Rightarrow \text{colim}^N$ be a natural transformation. Then the natural transformation

$$GKF \Rightarrow G\text{colim}^N F \Rightarrow GF\text{colim}^M \Rightarrow \text{colim}^M$$

induces a unique natural transformation $GKF \Rightarrow H$ and hence $\gamma : K \cong FGFH \Rightarrow FHG$. Here we use that $K$ factors over $\text{Ho}(N^D)$ so that $KFG$ is naturally isomorphic to $K$. It is a tedious, but routine check that we have $\epsilon \gamma = \delta$ and that $\gamma$ is unique with this property. □

**Remark 9.** The derived functors of every Quillen equivalence between model categories with functorial factorization define a homotopy equivalence of the underlying relative categories.

**Theorem 10.** The Grothendieck construction

$$\int : \text{RelCat}^D \rightarrow \text{RelCat}, \quad F \mapsto D \int F$$

together with the canonical natural transformation $\alpha : \int \Rightarrow \text{colim}$ is a homotopy colimit functor.

**Proof.** As $sS$ is a simplicial model category in which every object is cofibrant, by [Rie14, Theorems 5.1.1 and 2.2.8] the Bousfield–Kan homotopy colimit $\text{hocolim}_{BK}$ defines a homotopy colimit functor $sS^D \rightarrow \text{Ho}(sS)$. We use the convention that for $K$ a simplicial set and $X \in sS$, we have $(K \otimes X)_p = K \times X_p$.

The functor $N : \text{RelCat} \rightarrow sS$ is a homotopy equivalence. This follows from the natural equivalence $N \simeq N_\xi$ to a left Quillen functor shown in [BK12b], but actually Barwick and Kan present in [BK13] an easier proof (that also works for $n$-relative categories). It follows from Lemma 8 that it suffices now to show that $N \int \simeq \text{hocolim}_{ BK} N$.

Our main ingredient is that by [Tho79], there is for every functor $G : D \rightarrow \text{Cat}$ a natural weak equivalence $\text{hocolim}_{BK} \text{nerve} G \Rightarrow \text{nerve}(D \int G)$. Fix now a functor $F : D \rightarrow \text{RelCat}$. We have

$$(\text{hocolim}_{BK} NF)_p \cong \text{hocolim}_{BK} (NF)_p = \text{hocolim}_{BK} \text{nerve}(\text{we}(\cdot[p] \circ F))$$

and compose this isomorphism with the natural weak equivalence

$$\text{hocolim}_{BK} \text{nerve}(\text{we}(\cdot[p] \circ F)) \cong \text{nerve}(D \int (\cdot[p] \circ F))$$

by Thomason. The latter is by Lemma 5, isomorphic to $\text{nerve} \text{we}(D \int (\cdot[p] \circ F))$ and we have a weak equivalence

$$\text{nerve} \text{we}(D \int (\cdot[p] \circ F)) \cong \text{nerve} \text{we}(D \int F)[p]$$

induced by Lemma 4, both natural in $p$ and $F$. The target is equal to $N(D \int F)_p$.

Composing gives a zig zag of weak equivalences between $(\text{hocolim}_{BK} NF)_p$ and $N(D \int F)_p$ that is natural both in $p$ and in $F$. This induces a zig zag of weak equivalence between $\text{hocolim}_{BK} NF$ and $N(D \int F)$. It is easy to check that this is compatible with the natural transformation to the colimit. □

**Example 11.** Denote by $CW^{fin}$ the relative category of pointed spaces that are (pointed) homotopy equivalent to a finite CW-complex with homotopy equivalences as weak equivalences. Then the homotopy colimit of

$$CW^{fin} \xrightarrow{\Sigma} CW^{fin} \xrightarrow{\Sigma} \cdots$$
as constructed above is a relative category \( SW \), whose objects are given by pairs \((X, n)\) with \( X \in CW^{fin} \) and \( n \) a natural number. If \( m \geq n \), a morphism from \((X, n) \to (Y, m)\) is given by a morphism \( \Sigma^{m-n}X \to Y \) in \( \text{Top} \). Its homotopy category agrees with the classical Spanier–Whitehead category \( \overline{SW} \). Recall that \( \overline{SW} \) has the same objects and

\[
\overline{SW}((X, n), (Y, m)) = \text{colim}_d [\Sigma^{d-n}X, \Sigma^{d-m}Y]^{\bullet}.
\]

The key idea for showing that \( SW \to \overline{SW} \) is a localization at \( we_{\overline{SW}} \) is to associate with \([f: \Sigma^{d-n}X \to \Sigma^{d-m}Y]\) the zig zag \((X, n) \to (\Sigma^{d-m}Y, d) \leftarrow (Y, m)\).

References


