Elliptic Homology and Topological Modular Forms

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Abstract

These are lecture notes on elliptic genera, elliptic homology and topological modular forms. They are based on a lecture course given in the summer term 2017 in Bonn. Please treat these informal notes with caution. If you find any mistakes (either typos or something more serious) or have other remarks, please contact the author. This (or any other) feedback is very welcome!

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1 Introduction

The main goal of this lecture course is to construct and understand certain homology theories constructed since the late 80s.

The most classical and best-known homology theories are the different variants of ordinary homology like singular homology or deRham cohomology.

Two further families of (co)homology theories have played a big role in topology and geometry since the 1950s. The first family are the bordism theories. These are based on the relation that two closed n-manifolds are cobordant if they form jointly the boundary of a compact $(n+1)$-dimensional manifold. While these theories have their roots in the work of Poincaré[1] and Pontryagin, they really came to prominence through the work of Thom, who calculated bordism rings and got the Fields medal for it, and Atiyah, who realized that one can actually define (generalized) homology theories by bordism. Bordism theories have played a big role in most attempts to understand manifolds eversince.

The other family consists of K-theory. It has its roots in Grothendieck's version of the Riemann–Roch theorem in algebraic geometry, but soon was transported by Atiyah, Bott and Hirzebruch to topology. Here, one considers the monoid of isomorphism classes of vector bundles on a compact Hausdorff space under direct sum and applies a group completion to it to define $K^0(X)$ – one can either take complex or real vector bundles, resulting in complex or real K-theory. Still in the 60s it was realized by Conner and Floyd that one can also construct complex K-theory from complex bordism [CF66].

[1] Actually, the first attempt of Poincaré to define homology in his Analysis Situs reads almost like the definition of bordism, but both definition and proofs have a certain vagueness, which could not be quite resolved with the differential topology at hands at that time (meaning virtually none). Thus, he switched to a more combinatorial definition in the spirit of cellular or simplicial homology.

We will study in this lecture course a fourth family of homology theories, consisting of elliptic homology theories and topological modular forms. The origins lie in the work of Ochanine and Witten who constructed ring homomorphisms from bordism rings to rings of modular forms (called elliptic genera or also the Witten genus) – this was partially motivated by an attempt to do index theory on free loop spaces. Landweber, Ravenel and Stong [LRS95] used these ideas to construct an *elliptic homology theory* from complex bordism whose coefficients are a ring of modular forms. Later it was realized that there are actually a lot of examples of such elliptic homology theories. These can be regarded as higher analogues of complex K-theory – though an equally close connection to geometry as the one of K-theory via vector bundles is still subject to research.[2]

The next step was to find an analogue of real K-theory in this context. This was much more demanding, but was constructed by Goerss, Hopkins and Miller (and later Lurie) under the name of *topological modular forms* [DFHH14], [Lur09]. This has seen applications to string bordism ([AHR10], [MH02] and [Hil09]) and the stable homotopy groups of spheres ([HM98], [BHHM08] and [BP04]).

Our basic plan for this lecture course is the following: First we will recall some basics of bordism theory and discuss some general results about genera and orientation, mostly in the context of complex bordism. A key concept here is that of a formal group law. Then there will be an interlude on the theory of elliptic curves. These will be used to construct elliptic genera. These in turn will allow to construct elliptic homology theories, once we have the Landweber exact functor theorem. We will first formulate it in elementary language, but then reformulate it in terms of stacks, which should make it more transparent and usable.

In the final section, we will talk about the spectrum of topological modular forms TMF. This are based on the moduli stack of elliptic curves. We will give the idea how to construct TMF and will sketch the computations of π_*TMF after inverting 2. At the end, we will give some outlook to applications.

Acknowledgments

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2 Bordism, genera and orientations

2.1 Homotopy colimits and limits

It is a well-known problem that taking (co)limits in spaces does not preserve (weak) homotopy equivalences. There is a notion of a *homotopy (co)limit* that solves this problem. We refer to [Dug08] and [Rie14] for the full theory. In this chapter, we will only need directed homotopy colimits and homotopy fiber products, which we will discuss now.

Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots$ be a directed diagram. We define its homotopy colimit $\operatorname{hocolim}_i X_i$ as the so-called *mapping telescope*. This is defined as $\coprod X_i \times [0,1]/\sim$, where $(x,1) \sim (f_i(x), 0)$ for $x \in X_i$. We leave it as an exercise to show that it preserves

[2] Which is an euphemism for "Things are difficult and unclear."

homotopy equivalences. There is also a pointed version: If all the spaces X_i are pointed by points x_i and the maps f_i are pointed, then we can define a pointed mapping telescope $\operatorname{hocolim} X_i/\sim$, where we additionally identify all $x_i \times [0,1]$ to the base point. If the X_i are well-pointed (which we will assume), then this is homotopy equivalent to the unpointed mapping telescope. In the well-pointed case, we have an isomorphism $\operatorname{colim}_i \pi_k X_i \to \pi_k \operatorname{hocolim}_i X_i$. Indeed, we can write $\operatorname{hocolim}_i X_i$ as the colimit over T_n, where $T_n = \coprod_{i \leq n} X_i \times [0,1]/\sim$; furthermore, $T_n \simeq X_n$ and $T_n \to T_{n+1}$ is a cofibration. In particular, we see that hocolim preserves weak homotopy equivalences (i.e. π_*-isomorphisms).

Given a diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

we define its *homotopy pullback* $X \times_Z^h Y$ as the subspace of those $X \times Z^{[0,1]} \times Y$ of (x, α, y) with $f(x) = \alpha(0)$ and $g(y) = \alpha(1)$. Again, we leave it as an exercise to show that this construction preserves homotopy equivalences. Note that we have a long exact sequence

$$\cdots \to \pi_k(X \times_Z^h Y) \to \pi_k X \times \pi_k Y \to \pi_k Z \to \pi_{k-1}(X \times_Z^h Y) \to \cdots$$

associated to the fibration sequence

$$X \times_Z^h Y \to (X \times_Z Z^{[0,1]}) \times Y \to Z.$$

This implies that homotopy pullbacks also preserve weak homotopy equivalences.

2.2 Remarks about spectra

For most of this lecture course, we will be pretty agnostic about which model of spectra we use. Most of the time, we will actually just work in the homotopy category of spectra, which we will denote by Ho(Sp). The most naive way to construct it is via sequential spectra; a (sequential) spectrum X consists of a sequence X_n of pointed spaces together with pointed maps $\Sigma X_n \to X_{n+1}$. This forms in an obvious way a category Sp. We define $\pi_k X$ as the colimit

$$\operatorname{colim}_i \pi_{k+i} X_i,$$

where the transition map is given as the composite $\pi_{k+i} X_i \to \pi_{k+i+1} \Sigma X_i \to \pi_{k+i+1} X_{i+1}$.

There is a way to define the *homotopy category of spectra* Ho(Sp) (often called the *stable homotopy category*). The functor $\mathrm{Sp} \to \mathrm{Ho}(\mathrm{Sp})$ can be characterized as the universal functor that sends π_*-isomorphisms to isomorphisms (i.e. every other functor $\mathrm{Sp} \to \mathcal{C}$ with this property factors uniquely over the functor $\mathrm{Sp} \to \mathrm{Ho}(\mathrm{Sp})$). There are different ways to construct it more explicitly. See e.g. [Ada74] and [BF78] for two classical approaches and [MMSS01] and [Mal11] for overviews of different approaches.

In spectra, we can also take negative suspensions of spectra by a shift construction. Thus, a model for $\Sigma^{-1} X$ has n-th space X_{n-1}. Another important construction is a homotopy colimit along a directed system. This can be constructed, e.g. as a levelwise mapping telescope. We have $\pi_* \operatorname{hocolim}_n X^n \cong \operatorname{colim}_n \pi_* X^n$ for a sequence $X^0 \to X^1 \to X^2 \to \cdots$ of spectra.

The latter property implies that

$$X \simeq \operatorname{hocolim}_n \Sigma^{-n} \Sigma^\infty X_n,$$

where we define $\Sigma^\infty Y$ for a pointed space Y as the spectrum with i-th space $\Sigma^i Y$ and the obvious structure maps.

There is a smash product on the homotopy (or ∞-)category of spectra. Indeed, it is characterized[3] as a symmetric monoidal product by the following two properties:

1. There is a natural equivalence: $\Sigma^\infty A \wedge \Sigma^\infty B \simeq \Sigma^\infty(A \wedge B)$
2. The smash product is compactible with homotopy colimits:

$$(\mathrm{hocolim}_i\, A^i) \wedge B \simeq \mathrm{hocolim}_i (A^i \wedge B).$$

For us, a *ring spectrum* is a monoid in Ho(Sp), i.e. we have maps $R \wedge R \to R$ (the multiplication) and $\mathbb{S} \to R$ (the unit) in Ho(Sp) such that the associativity and unitality diagrams commute in Ho(Sp). When we talk about more refined variants, we will use terms like A_∞- or E_∞-ring spectrum.

Every spectrum E represents both a reduced homology and a reduced cohomology theory. For a pointed space X, we define $E_k(X)$ as $\pi_k \Sigma^\infty X \wedge E$ and we define $E^k(X) = [\Sigma^\infty X, \Sigma^k E]$, where $[-,-]$ denotes morphisms in Ho(Sp). If E is a (homotopy commutative) ring spectrum, then E^* becomes a multiplicative cohomology theory. Note that if we want non-reduced homology theories, we just have to apply the reduced homology theory to X union a disjoint base point.

By Brown's representability theorem, every cohomology theory is represented by a spectrum. We denote the spectrum representing $H^*(-;A)$ (for an abelian group A) by HA.

Remark 2.1. These foundations are an extensive topic, of which barely anything will be relevant for us for most of the time, so I sweep it mostly under the rug. Let me comment though on the smash product. There are different ways to construct it. One can use ∞-categories as in [Lur12]. Or one can use symmetric or orthogonal spectra (see [Sch12] or [MMSS01]). If one is only interested in the homotopy category, one can also construct it directly on sequential spectra. First, we replace a sequential spectrum X by a cofibrant spectrum; for example, there is a CW-approximation theorem that we can replace X up to π_*-isomorphism (which are for us the relevant equivalences) by a CW-spectrum, i.e. one where X_0 is a CW-complex and the maps $\Sigma X_i \to X_{i+1}$ are relative CW-complexes; this is in particular cofibrant. Then we can define $X \wedge Y$ to have $2n$-th space $X_n \wedge Y_n$ and $(2n+1)$-st space $\Sigma X_n \wedge Y_n$, where the structure map $\Sigma\Sigma X_n \wedge Y_n \to X_{n+1} \wedge Y_{n+1}$ commutes one suspension past the X_n to let it act on Y_n (you can either introduce a sign while commuting or don't – both is possible). One can check that this defines a smash product on Ho(Sp). It is non-trivial to see that this is associative (as it is clearly *not* associative before passing to the homotopy category). See either [MMSS01, Section 11] or [Len12, Section 7] for comparisons of this approach to other approaches.

2.3 Bordism

We begin by recalling how to define bordism groups and Thom spectra with extra structure.

Recall first the stable normal bundle of a manifold. Given a closed k-manifold M, we can embed it via some map ι into some $\mathbb{R}^{n+k}$ with normal bundle ν_ι. If n is large enough any two such embeddings are homotopic through immersions and even this homotopy is unique up to homotopy [Hir59]. This implies that while $\nu_\iota \colon M \to BO(n)$ depends on the embedding, the composite $\nu \colon M \to BO(n) \to BO$ does not up to unique

[3] At least in ∞-categories. See [Lur12, Corollary 4.8.2.19] for a precise statement.

homotopy;[4] here, BO is the (homotopy) colimit of the $BO(n)$. We call this map ν the *stable normal bundle* and more generally a map into BO a *stable vector bundle*; an isomorphism between two stable vector bundles is a homotopy between the two maps into BO.

Extra structure on stable vector bundles can be encoded by a map $\xi\colon X \to BO$ as follows. Given a stable vector bundle $\nu\colon M \to BO$, an *X-structure* on it consists of an equivalence class of lifts $g\colon M \to X$ such that $\xi g = \nu$ or such that there is at least a chosen homotopy H between ξg and ν. Here, two such lifts g_1 and g_2 are called *equivalent* if they are homotopic over BO; more precisely, this means that the diagram

$$\begin{tikzcd} \xi g_1 \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & \xi g_2 \arrow[dl, Rightarrow] \\ & \nu & \end{tikzcd}$$

of homotopies commutes up to homotopy. If we denote the stable vector bundles classified by ξ and ν by the same names, we see in particular that an X-structure induces an isomorphism $g^*\xi \cong \nu$.

An X-structure on a closed manifold M is an X-structure on its stable normal bundle. (Strictly speaking, the notion of an X-structure depends on the precise map $\nu\colon M \to BO$; but a homotopy between two maps ν and ν' allows to transport X-structures on ν to X-structures on ν'. As the map ν is well-defined up to homotopy (which in turn is unique up to homotopy), there is no problem.)

Examples 2.2. 1. $X = BO$: An BO-manifold is just an unoriented manifold.

2. $X = BSO$: An BSO-manifold is equipped with an orientation of the stable normal bundle. Exercise: This is equivalent to an orientation of the tangent bundle.
3. $X = BSpin$: Recall that $Spin(n)$ is the unique connected 2-fold cover of $SO(n)$. Define $BSpin = \operatorname{colim} BSpin(n)$. Exercise: Show that a $BSpin$-structure on an n-manifold is equivalent to a $Spin(n)$-structure on the tangent bundle. (Hint: Show that if $E \oplus F$ and E are spin, then also F is spin.)
4. $X = BU$: This is a complex structure on the stable normal bundle of M.
5. $X = \mathrm{pt}$: This is a framing of the stable normal bundle of M, i.e. an isomorphism to the trivial stable bundle.

end of lecture 1

Remark 2.3. Passing between an X-structure on the (stable) tangent bundle and the (stable) normal bundle is in general subtle and not always possible. Let me give two examples, where this passage is possible.

First, consider a BSO-structure, which is equivalent to an orientation. As $TM \oplus \nu$ is a trivial bundle, an orientation on TM and on ν are equivalent data.

Now consider a BU-structure, i.e. a complex structure on the bundle after adding a trivial bundle of suitable dimension. If we have a complex structure on TM (e.g. if M is a complex manifold), then we obtain a complex structure on $TM \oplus \nu_{\mathbb{C}}$. As a real bundle, this is isomorphic to $(TM \oplus \nu) \oplus \nu$ and $TM \oplus \nu$ is trivial. Thus, we obtain a BU-structure on ν.

[4] We use here that the normal bundle of the composition $M \to \mathbb{R}^{n+k} \to \mathbb{R}^{n+k+1}$ is $\nu_\iota \oplus 1$, where 1 denotes the 1-dimensional trivial bundle.

Remark 2.4. Recall that the homotopy groups of BO are given by Bott periodicity. More precisely, they are 8-periodic and the first 8 groups are:

$$\begin{aligned}
&\pi_1 BO = \pi_0 O = \mathbb{Z}/2,\\
&\pi_2 BO = \pi_1 O = \pi_1 SO(3) = \mathbb{Z}/2,\\
&\pi_3 BO = 0,\\
&\pi_4 BO = \mathbb{Z}\\
&\pi_k BO = 0 \text{ for } 5 \leq k \leq 7\\
&\pi_8 BO = \mathbb{Z}.
\end{aligned}$$

(See [Mil63, Theorem 24.7].) As SO is connected, $BSO = BO\langle 2\rangle$ is the 1-connected cover of BO. As $Spin$ is simply-connected and $\pi_2 O = 0$ anyhow, we see that $BSpin = BO\langle 4\rangle$. The next interesting case is $BO\langle 8\rangle$, which is often denoted for fancy reasons by $BString$.

To define bordism groups, we also have to discuss stable normal bundles of manifolds with boundary. The story is the same, only that we have to require for such a manifold W that we embed it via a map i into $\mathbb{R}^{n+k} \times \mathbb{R}_{\geq 0}$ so that the boundary of W is nicely embedded into the boundary $\mathbb{R}^{n+k}$ of $\mathbb{R}^{n+k} \times \mathbb{R}_{\geq 0}$; here, nice means in particular that an open neighborhood of ∂W is embedded onto $i(\partial W) \times [0, t)$. Clearly, an X-structure on W induces an X-structure on ∂W.

Definition 2.5. We define Ω_k^X to be the cobordism classes of closed k-manifolds M with X-structure. More precisely, consider the monoid of closed k-manifolds with X-structure and define Ω_k^X to be the quotient monoid by the submonoid of manifolds of the form ∂W with W a compact $(k+1)$-dimensional manifold with X-structure. By the decomposition $\partial(M \times I) \cong M \coprod M'$ one sees that Ω_k^X is a group; note that M' is diffeomorphic to M, but has a different X-structure in general.

The Pontryagin–Thom construction identifies these groups with the homotopy groups of certain spectra, so-called Thom spectra. Recall that the Thom space $\mathrm{Th}(E)$ of a vector bundle $E \to B$ is set-theoretically defined as $E \coprod \mathrm{pt}$ such that neighborhoods of pt are complements of closed subsets of E whose intersection with every fiber is compact. If B is compact, this is just the 1-point compactification of E.

If $E_1 \to B_1$ and $E_2 \to B_2$ are vector bundles, there is a canonical isomorphism $\mathrm{Th}(E_1 \times E_2) \cong \mathrm{Th}(E_1) \wedge \mathrm{Th}(E_2)$. In particular, if $B_1 = \mathrm{pt}$ and $E_1 = \mathbb{R}$, we get $\mathrm{Th}(1 \oplus E) \cong \Sigma\,\mathrm{Th}(E)$, where we denote by 1 the trivial 1-dimensional bundle.

Definition 2.6. Let $X \to BO$ be a map. Define $X_n = X \times^h_{BO} BO(n)$ and denote the vector bundles classified by the projection $X_n \to BO(n)$ by E_n. Furthermore, we get maps $j_n \colon X_n \to X_{n+1}$ and we have $j_n^* E_{n+1} \cong 1 \oplus E_n$, where 1 denotes the trivial 1-dimensional vector bundle.

Define the Thom spectrum MX for X by $MX_n = \mathrm{Th}(E_n)$ and the structure maps are

$$\Sigma MX_n \cong \mathrm{Th}(1 \oplus E_n) \cong \mathrm{Th}(j_n^* E_{n+1}) \to \mathrm{Th}(E_{n+1}) = MX_{n+1}.$$

Theorem 2.7 (Pontryagin–Thom)**.** *There is an isomorphism* $\pi_* MX \cong \Omega_*^X$.

Examples 2.8.

1. If $X = BO$ (and hence $X_n \simeq BO(n)$), we obtain bordism of unoriented manifolds and we call the corresponding Thom spectrum MO. It is easy to see that $\pi_* MO$ consists of 2-torsion and this theory can be understood by Stiefel–Whitney classes ($H\mathbb{Z}/2$-characteristic classes).
2. If $X = BSO$ (and hence $X_n \simeq BSO(n)$), we obtain bordism of oriented manifolds and we call the corresponding Thom spectrum MSO. Thom and Wall showed

that oriented bordism is determined by Stiefel–Whitney and Pontryagin classes ($H\mathbb{Z}$-characteristic classes). The ring π_*MSO is known, but a little hard to write down. In contrast, $\pi_*MSO \otimes \mathbb{Q}$ is easy to understand: It is a polynomial ring generated by $[\mathbb{CP}^{2n}]$.

3. More generally, we can take $X = BO\langle n\rangle$. For $n = 4$, we obtain bordism of spin manifolds and we call the corresponding Thom spectrum $MSpin$. Anderson, Brown and Peterson have determined the structure of π_*MSpin, but it is complicated. It is detected by characteristic classes in $H\mathbb{Z}/2$ (i.e. Stiefel–Whitney classes) and characteristic classes in real K-theory KO. For $MO\langle 8\rangle = MString$, there is a connection to *topological modular forms*, but less is known.
4. If $X = BU$, we obtain bordism of stably almost complex manifolds and the corresponding Thom spectrum is called MU. Milnor computed that π_*MU is a polynomial ring in infinitely many generators. Rationally, it is generated by the $[\mathbb{CP}^n]$. This will be a major example for us.

References: Classic books on these topics are [Swi75] and [Sto68] and the latter also includes an extensive survey of the calculations known in 1968. If one wants to be all fancy and ∞-categorical, one can also look at the elegant treatment in [ABG+14]. The treatment in these notes is only partially following these approaches.

end of lecture 2

2.4 Properties of Thom spaces and spectra

Definition 2.9. Let h be a multiplicative cohomology theory and $E \to X$ an n-dimensional vector bundle. A *Thom class* is a class $\tau \in \tilde{h}^n(\mathrm{Th}(E)) \cong h^n(E, E_0)$ whose restriction to every compactified fiber $\overline{E}_x \cong S^n$ is a generator of $\tilde{h}^n(\overline{E}_x) \cong h^0(\mathrm{pt})$ as an $h^0(\mathrm{pt})$-module. We call a vector bundle with a Thom class *h-oriented*.

Theorem 2.10. *Let $E \to X$ be an h-oriented vector bundle with Thom class τ. Then there are natural* Thom isomorphisms

$$\cup\tau\colon h^m(X) \cong h^m(E) \to \tilde{h}^{m+n}(\mathrm{Th}(E)) \cong h^{m+n}(E, E_0)$$

and

$$\cap\tau\colon \tilde{h}^{m+n}(\mathrm{Th}(E)) \to h^m(X).$$

Remark 2.11. If $E \to X$ is a non-h-oriented vector bundle, there are twisted forms of the Thom isomorphism. This is easiest if $h = H\mathbb{Z}$, i.e. singular homology. Then there is an orientation-local system $\tilde{\mathbb{Z}}$ for E. This can, for example, be defined as $H^n(E_x, E_{0,x})$ over every point $x \in X$. Then there is a Thom isomorphism $\tilde{H}_{m+n}(\mathrm{Th}(E)) \cong H_m(X, \tilde{\mathbb{Z}})$.

Lemma 2.12. *Every Thom spectrum is connective, i.e. $\pi_k MX = 0$ for $k < 0$.*

Proof. This follows either from the Pontryagin–Thom isomorphism or from the observation that a Thom space of an n-dimensional bundle is always n-connective, i.e. its i-th homotopy group vanishes for every $i < n$ and every base-point. □

Lemma 2.13. *Let a factorization $E\colon X \xrightarrow{e} BO(n) \to BO$ be given. We denote the n-dimensional vector bundle classified by e by abuse of notation also by e. Then $MX \simeq \Sigma^{-n}\Sigma^\infty \mathrm{Th}(e)$.*

Proof. Then $(m+n)$-th space of $\Sigma^{-n}\Sigma^\infty \operatorname{Th}(e)$ is $\Sigma^m \operatorname{Th}(e) \cong \operatorname{Th}(e \oplus m)$. Now consider the diagram

$$\begin{tikzcd} X_{m+n} \arrow[rrr] \arrow[d, "f_m"] & & & BO(n+m) \arrow[d] \\ X \arrow[r] & BO(n) \arrow[urr] \arrow[rr] & & BO \end{tikzcd}$$

We see that the pullback bundle E_{m+n} on X_{m+n} agrees with $f_m^* e \oplus m$. Thus, the $(m+n)$-th space of MX is $\operatorname{Th}(f_m^* e \oplus m)$, which maps compatibly to $\operatorname{Th}(e \oplus m)$. Thus, we obtain a map $MX \to \Sigma^{-n}\Sigma^\infty \operatorname{Th}(e)$.

The (stable) vector bundle E defines compatible local orientation systems on X and all X_{m+n}. The Thom isomorphism induces isomorphisms

$$\tilde{H}_{*+m+n}(\operatorname{Th}(E_{m+n}); \mathbb{Z}) \cong H_*(X_{m+n}; \tilde{\mathbb{Z}})$$

and thus we obtain in the (co)limit that

$$\begin{aligned} H_* MX &\cong \operatorname{colim} \tilde{H}_{*+n+m}(\operatorname{Th}(E_{m+n}), \mathbb{Z}) \\ &\cong \operatorname{colim} H_*(X_{m+n}, \tilde{\mathbb{Z}}) \\ &\cong H_*(\operatorname{hocolim} X_{m+n}, \tilde{\mathbb{Z}}) \\ &\cong H_*(X, \tilde{\mathbb{Z}}) \\ &\cong \tilde{H}_{*+n}(\operatorname{Th}(e); \mathbb{Z}) \\ &\cong H_*(\Sigma^{-n}\Sigma^\infty \operatorname{Th}(e); \mathbb{Z}). \end{aligned}$$

One can see that the map $MX \to \Sigma^{-n}\Sigma^\infty \operatorname{Th}(e)$ induces an isomorphism between the homologies of these connective spectra and thus we can conclude by the Whitehead theorem that the map $MX \to \Sigma^{-n}\Sigma^\infty \operatorname{Th}(e)$ is an equivalence. $\square$

Example 2.14. For $X = \mathrm{pt} \to BO$, we obtain as the Thom spectrum the sphere spectrum that thus represents framed bordism.

Lemma 2.15. *Let $X^0 \to X^1 \to \cdots$ be a sequence of spaces with compatible maps $E_n \colon X^n \to BO$ and denote by E the map $X = \operatorname{hocolim}_n X^n \to BO$. Then $ME \simeq \operatorname{hocolim}_n ME_n$.*

Proof. We just have to use that both homotopy fiber products and the Thom space construction commute with homotopy colimits. $\square$

Lemma 2.16. *Let $X \xrightarrow{E} BO$ and $Y \xrightarrow{F} BO$ be maps and let $X \times Y \xrightarrow{E \times F} BO \times BO \to BO$ classify the external sum of the corresponding vector bundles. Then there is a natural equivalence $M(E \times F) \simeq ME \wedge MF$.*

Proof. Let X_n and E_n as above and consider analogously Y_n and F_n. Then $M(E_n \times F_n) \simeq ME_n \wedge MF_n$ because the analogous statement is true for Thom spaces. Thus,

$$\begin{aligned} ME \wedge MF &\simeq \operatorname{hocolim}_n ME_n \wedge \operatorname{hocolim}_m MF_m \\ &\simeq \operatorname{hocolim}_{n,m} ME_n \wedge MF_m \\ &\simeq \operatorname{hocolim}_n M(E_n \times F_n). \end{aligned}$$

As $\operatorname{hocolim}_n X_n \times Y_n \simeq X \times Y$, we see by the last lemma that this is equivalent to $M(E \times F)$. $\square$

Proposition 2.17. *Let $X \xrightarrow{E} BO$ be a map of H-spaces (where BO is equipped with the direct sum H-space structure). Then ME attains the structure of a ring spectrum. If X is homotopy commutative, then ME is homotopy commutative as well.*

Proof. This follows directly from the last lemma. □

Examples 2.18. Obviously, $BO \to BO$ and $BSO \to BO$ and $BU \to BO$ are H-space maps as these things are compatible with direct sums. It is also true (more generally) that if X is an H-space, then its connective covers $X\langle n\rangle$ are H-spaces and the map $X\langle n\rangle \to X$ is an H-space map. The reason is that the map $X\langle n\rangle \times X\langle n\rangle \to X \times X \to X$ factors over $X\langle n\rangle$ as the source is $2n$-connective.

Thus, we see that all the Thom spectra we have considered are actually homotopy commutative ring spectra. Actually, a more careful argument shows that they are all E_∞-ring spectra (i.e. they can be represented by commutative orthogonal ring spectra).

2.5 Orientations

We rephrase the theory of Thom classes from Thom spaces to Thom spectra. We will fix throughout a ring spectrum E:

Definition 2.19. Let $X \xrightarrow{\xi} BO$ be a map. Every point $x \in X$ determines a map $\mathbb{S} \cong Mx \to MX$ (where the isomorphism $\mathbb{S} \cong Mx$ in Ho(Sp) depends on the chosen trivialization of $\xi|_X$.) We say that $\tau \in E^0(MX)$ is a Thom class if $\tau|_x \in E^0(Mx)$ is a generator of this rank-1 free $E^0(\mathbb{S})$-module for every $x \in X$. If a Thom class exists, we say that ξ has an *E-orientation.*

If ξ actually comes from an n-dimensional vector bundle e, this reduces to the earlier notion of a Thom class. Here, we use that MX is in this case just $\Sigma^{-n}\Sigma^\infty \operatorname{Th}(e)$ so that $E^0MX \cong E^n \operatorname{Th}(e)$.

We observe that for a map $f\colon Y \to X$ and a stable vector bundle $X \xrightarrow{\xi} BO$ with a Thom class $\tau \in E^0(MX)$, the pulled back class $f^*\tau \in E^0(MY)$ is a Thom class as well. In particular, a Thom class for ξ is the same as a natural choice of Thom classes for all stable vector bundles with an X-structure.

We also obtain a Thom isomorphism in this world:

Theorem 2.20. *Let $X \xrightarrow{\xi} BO$ be a stable vector bundle with an E-orientation. Then $E_*MX \cong E_*X$ and $E^*MX \cong E^*X$.*

Proof. All the $X_k \to \xi_k BO(k)$ inherit E-orientations because they are pulled back from ξ. Thus, $\tilde{E}_{m+k}(\operatorname{Th}(\xi_k)) \cong E_m(X_k)$. Now the argument is as in the proof of Lemma 2.13. □

Remark 2.21. There is a theorem by Spanier and Milnor (see e.g. [Swi75, Theorem 14.43]) that for a closed n-manifold N with stable normal bundle ν, the Thom spectrum $\Sigma^n M\nu$ is equivalent to the Spanier–Whitehead dual DN_+. This can be constructed as the function spectrum $F(\Sigma^\infty_+ N, \mathbb{S})$. This implies that

$$\begin{aligned} E_*M\nu &\cong E_*DN_+ \\ &\cong \pi_*(F(\Sigma^\infty_+ N, \mathbb{S}) \wedge E) \\ &\cong \pi_*F(\Sigma^\infty_+ N, E) \\ &\cong E^{-*}(N) \end{aligned}$$

for every spectrum E. If ν is E-oriented, the source is isomorphic to $E_*\Sigma^{-n}N \cong E_{*+n}N$. Thus, N satisfies Poincaré duality for E if ν is E-oriented.

Definition 2.22. Let $X \to BO$ be a map of connected H-spaces and E be a ring spectrum. An *X-orientation* of E is a choice τ_ξ of E-Thom class for every stable vector bundle ξ with X-structure, where we demand

1. naturality, i.e. $\tau_{f^*\xi} = f^*\tau_\xi$, and
2. multiplicativity, i.e. if ξ and η are stable vector bundles on spaces Y and Z, then $\tau_{\xi\times\eta}$ corresponds to $\tau_\xi \times \tau_\eta$ under the identification $M(Y\times Z) \simeq MY \wedge MZ$. Here, we use the X-structure on $\xi \times \eta$ given by the composition $Y \times Z \to X \times X \to X$.

Example 2.23. Integral singular homology $H\mathbb{Z}$ is BSO-oriented: oriented vector bundles have Thom classes in singular cohomology and these are multiplicative.

Example 2.24. If $X \to BO$ is a map of connected H-spaces, then MX has a tautological X-orientation τ^{can}: If $Y \xrightarrow{g} X \to BO$ classifies a stable vector bundle with X-structure, we obtain an element $\tau_g^{can} = Mg \in [MY, MX] = MX^0(MY)$. This is clearly natural. For multiplicativity, we use the (homotopy) commutative diagram

$$\begin{tikzcd}
MY \wedge MZ \arrow[r] \arrow[d, "\simeq"] & MX \wedge MX \arrow[d, "\simeq"] \arrow[dr] & \\
 & & MX \\
M(Y\times Z) \arrow[r] & M(X\times X) \arrow[ur] &
\end{tikzcd}$$

We add a small observation: If $u\colon E \to F$ is a map of ring spectra and E has an X-orientation $\{\tau_\xi\}$, then $\{u_*\tau_\xi\}$ is an X-orientation for F.

Proposition 2.25. *Let $X \to BO$ be a map of connected H-spaces. Then*

$$\begin{aligned} \text{(Maps } MX \to E \text{ of ring spectra)} &\to (X\text{-orientations of } E) \\ u &\mapsto \{u_*\tau_\xi^{can}\} \end{aligned}$$

is a bijection. Here, maps are understood to be in the homotopy category $\mathrm{Ho}(\mathrm{Sp})$.

Proof. We can construct an inverse as follows: Let an X-orientation of E be given. In particular, we obtain a Thom class $E^0(MX) = [MX, E]$ of the universal stable bundle with X-structure. Multiplicativity of the Thom class shows that this is a map of ring spectra. And it is also easy to see that this is an inverse to the map described in the statement of the proposition. □

2.6 Complex orientations

Definition 2.26. We call a ring spectrum E *complex oriented* if it is equipped with a BU-orientation (or equivalently with a ring spectrum map $MU \to E$).

Example 2.27. Clearly, MU itself is complex orientable and also $H\mathbb{Z}$ (because every complex vector bundle is oriented). We will see later that complex K-theory KU is also complex oriented.

end of Lecture 3

Definition 2.28. Let ξ be a complex n-dimensional vector bundle on a space X and E be complex oriented. Then we define the *Euler class* of ξ to be the pull back $e(\xi) \in E^{2n}(X)$ of the Thom class $\tau_\xi \in E^{2n}(\mathrm{Th}(\xi))$ along the zero section $X \to \mathrm{Th}(\xi)$.

Proposition 2.29. *Let E be complex oriented and let $x_n \in E^2(\mathbb{CP}^n)$ (with n possibly ∞ and $x = x_\infty$) the Euler class of the tautological line bundle η_n. Then we have ring isomorphisms*

$$E^*(\mathbb{CP}^n) \cong E^*[x_n]/x_n^{n+1}$$
$$E^*(\mathbb{CP}^\infty) \cong E^*[\![x]\!].$$

Here, $E^* = E^*(\mathrm{pt})$.

Proof. We start with some general remarks. Clearly, x and x_{n+1} restrict to x_n on $\mathbb{CP}^n$. One can see that $\mathbb{CP}^{n+1}$ is the Thom space of η_n ([KT06, Lemma 3.8]) and the usual inclusion $\mathbb{CP}^n \to \mathbb{CP}^{n+1}$ corresponds to the zero section. Thus, x_{n+1} is a Thom class for η_n.

Now we argue by induction on (finite) n. The Thom isomorphism theorem implies that $\tilde{E}^*(\mathbb{CP}^{n+1})$ is a free $E^*(\mathbb{CP}^n) \cong E^*[x_n]/x_n^{n+1}$-module on x_{n+1}. As x_n is the restriction of x_{n+1}, we see that x_n acts on $\tilde{E}^*(\mathbb{CP}^{n+1})$ as x_{n+1}. It follows that $E^*(\mathbb{CP}^{n+1}) \cong E^*[x_{n+1}]/x_{n+1}^{n+2}$.

The statement for $n = \infty$ follows from the Milnor sequence, which we will state next. Indeed, it is easy to see that $\lim^1$ vanishes along a tower of surjective maps. □

Proposition 2.30 (Milnor sequence)**.** *Let E be a cohomology theory (satisfying the wedge axiom), $X_0 \to X_1 \to X_2 \to \cdots$ a diagram of spaces with homotopy colimit X (e.g. X_i might be the i-skeleton of a CW-complex X). Then there is a short exact sequence*

$$0 \to \lim\nolimits^1_i E^{*-1}(X_i) \to E^*X \to \lim_i E^*(X_i) \to 0.$$

Proof. One can write X as the homotopy coequalizer of two maps $\coprod_i X_i \to \coprod_i X_i$, namely the identity and the map induced by the $X_i \to X_{i+1}$. After taking Σ^∞_+ (i.e. adding a disjoint base point and doing the suspension spectrum), we obtain a cofiber sequence

$$\bigoplus_i \Sigma^\infty_+ X_i \xrightarrow{F} \bigoplus_i \Sigma^\infty_+ X_i \to \Sigma^\infty_+ X,$$

where F is the difference of the two induced maps. Taking E-cohomology we obtain a long exact sequence

$$\prod_i E^{*-1}X_i \xrightarrow{F^{*-1}} \prod_i E^{*-1}X_i \to E^*X \to \prod_i E^*X_i \xrightarrow{F^*} \prod_i E^*X_i.$$

From this, we obtain short exact sequences

$$0 \to \operatorname{coker} F^{*-1} \to E^*X \to \ker F^* \to 0.$$

This implies the result by the definition of limit and $\lim^1$. □

Remark 2.31. An alternative definition of complex orientability is the existence of a class $x \in \tilde{E}^2\mathbb{CP}^\infty$ whose restriction to $\tilde{E}^2(\mathbb{CP}^1) \cong \tilde{E}^2(S^2) \cong E^0(\mathrm{pt})$ is a generator. We have only proved that it is a necessary condition. For the equivalence of this approach with ours see either Adams [Ada74, Part II, Section 2 and 4] for an approach using the Atiyah–Hirzebruch spectral sequence or Kono–Tamaki [KT06], which is a rather nice read.

Remark 2.32. We can use this computation to show that *not* every spectrum is complex-orientable. The easiest counter example is the sphere spectrum $\mathbb{S}$ itself. Indeed, consider the cofiber sequence

$$S^3 \xrightarrow{\eta} \mathbb{CP}^1 \cong S^2 \to \mathbb{CP}^2$$

and the corresponding long exact sequence

$$\tilde{E}^2\mathbb{CP}^2 \to \tilde{E}^2\mathbb{CP}^1 \cong E^0 \xrightarrow{\eta} \tilde{E}^2(S^3) \cong E^1.$$

If E is complex-orientable, we can lift $1 \in \tilde{E}^2S^2 \cong E^0$ to $\tilde{E}^2\mathbb{CP}^2$. This is possible if and only if η operates as 0 on $1 \in E^0$, i.e. iff the Hurewicz image of $\eta \in \pi_1\mathbb{S}$ in $\pi_1 E = E^{-1}$ is zero. For $E = \mathbb{S}$, this is well-known to be non-zero and the same is true for $E = KO$.

There is a wealth of other computations that can be done purely formally for complex-oriented cohomology theories E. Let us list some of them.

1. $E^*((\mathbb{CP}^\infty)^{\times n}) \cong E^*[\![x_1, x_2, \ldots, x_n]\!]$, where the generators are the pullbacks of x along the projections $\mathrm{pr}_n\colon (\mathbb{CP}^\infty)^{\times n} \to \mathbb{CP}^\infty$.
2. $E^*BU(n) \cong E^*[\![c_1, \ldots c_n]\!]$ and $E^*BU \cong E^*[\![c_1, c_2, \ldots]\!]$. The classes c_i are called *Chern classes.*
3. $E_*(\mathbb{CP}^\infty) \cong E_*\{1, \beta_1, \ldots\}$ (this is an additive isomorphism of E_*-modules!)
4. $E_*(BU) \cong E_*[\beta_1, \beta_2, \ldots]$. Here, the isomorphism is multiplicative, using the H-space structure on BU given by direct sum of vector bundles. The map $E_*\mathbb{CP}^\infty \to E_*BU$ does the expected.

To prove all of these, there are two strategies. The first is to do more elaborate versions of our arguments for the case $\mathbb{CP}^\infty$. The second is to assume the result for ordinary integral homology as known and deduce the general results by the Atiyah–Hirzebruch spectral sequence. The second approach appears to be much faster, but if one adds the work for deriving the formulae for ordinary homology, the work is about the same. We will skip this.

References: Good references are [Ada74], Part II, and [KT06].

2.7 Formal group laws

So far, we have already used the H-space structure on BU coming from adding vector bundles. But one can also tensor vector bundles and, in particular, line bundles. This induces a multiplication map $m\colon \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$. Commutativity and associativity of the tensor product show that this makes $\mathbb{CP}^\infty$ into a homotopy commutative H-space.

This induces a map

$$m^*\colon E^*(\mathbb{CP}^\infty) \cong E^*[\![x]\!] \to E^*[\![x_1, x_2]\!] \cong E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty).$$

This map is continuous for the usual topologies defined on rings of power series, i.e. for every k, l the preimage $(m^*)^{-1}(x_1^k, x_2^l)$ contains (x^n) for some n. This follows from the fact that the image of $m|_{\mathbb{CP}^{k-1}\times\mathbb{CP}^{l-1}}$ lies in some $\mathbb{CP}^{n-1}$ because $\mathbb{CP}^{k-1} \times \mathbb{CP}^{l-1}$ is compact.

As furthermore, m^* is an E^*-algebra morphism (as it is induced by a map of spaces), it follows that m^* is equivalent data to the power series $F = m^*(x)$. We record how the axioms for a homotopy commutative H-space translate into properties of F.

We know that the composition $\mathbb{CP}^\infty \xrightarrow{\mathrm{id}\times\mathrm{pt}} \mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{m} \mathbb{CP}^\infty$ is homotopic to the identity (right unitality). As the map $E^*[\![x]\!] \to E^*$ induced by $\mathrm{pt} \to \mathbb{CP}^\infty$ sets

$x = 0$, we see that this translates into $F(x_1, 0) = x_1$. Likewise, left unitality translates into $F(0, x_2) = x_2$. These two conditions are equivalent to

$$F(x_1, x_2) = x_1 + x_2 + \text{ higher terms.} \tag{2.33}$$

The twist map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty \times \mathbb{CP}^\infty$ just permutes x_1 and x_2. Thus, the homotopy commutativity of $\mathbb{CP}^\infty$ translates into

$$F(x_1, x_2) = F(x_2, x_1). \tag{2.34}$$

The homotopy associativity of m translates into

$$F(x_1, F(x_2, x_3)) = F(F(x_1, x_2), x_3). \tag{2.35}$$

Definition 2.36. Let R be a commutative ring. A power series $F \in R[\![x_1, x_2]\!]$ satisfying (2.33), (2.34) and (2.35) is called a *formal group law* over R.

If R has a grading, we say that F is a *graded formal group law* if the coefficient in front of $x_1^k x_2^l$ has degree $2k + 2l - 2$. This corresponds to $|x_1| = -2$, $|x_2| = -2$ and $|F| = -2$.

Example 2.37. If E is a complex oriented ring spectrum, we obtain a graded formal group law over $E_* = E^{-*}$.

end of lecture 4

Part of the strength of this observation is that formal group laws are well-studied objects in number theory and algebraic geometry (and since the 70s in algebraic topology as well!).

Remark 2.38. We may reinterpret the formal group law as a formula for the first Chern class of a line bundle. Let E be complex oriented and let x be the Euler class of the tautological bundle over $\mathbb{CP}^\infty$. Given a line bundle L on a space X, we obtain a classifying map $l \colon X \to \mathbb{CP}^\infty$. We define $c_1(L) \in E^2(X)$ as $l^* x$.

What is $c_1(L_1 \otimes L_2)$? We obtain it as the pullback of x along $X \xrightarrow{l_1 \times l_2} \mathbb{CP}^\infty \times \mathbb{CP}^\infty \xrightarrow{m} \mathbb{CP}^\infty$. We obtain

$$c_1(L_1 \otimes L_2) = (l_1 \times l_2)^* F(x_1, x_2) = F(c_1(L_1), c_1(L_2)).$$

You might worry what this actually means as the power series F can be infinite. This is easy to say if X has finite cup-length (for example if it is covered by finitely many contractibles, like a finite simplicial complex). In general you can topologize the abelian group $E^* X$ by declaring all kernels of maps $E^* X \to E^* Y$ for Y a finite complex with a map $Y \to X$ to be open. One can check that this is natural and in particular maps between spaces preserve convergent power series.

Example 2.39. Take $E = H\mathbb{Z}$, ordinary integral homology. What is $m^* \colon \mathbb{Z} \cong H^2(\mathbb{CP}^\infty) \to H^2(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \mathbb{Z} \oplus \mathbb{Z}$? Actually, grading and unitality directly imply that it has to send x to $x_1 + x_2$, i.e. $F(x_1, x_2) = x_1 + x_2$. This is called the *additive formal group law*.

Example 2.40. The next example is complex K-theory KU.

For a compact connected Hausdorff space X, we define $KU^0(X)$ to be the group completion of the monoid $\mathrm{Vect}_{\mathbb{C}}(X)$ of isomorphism classes of (finite-dim) complex vector bundles on X. Here, in general for an abelian monoid M, its group completion $\mathrm{Gp}(M)$ is the initial abelian group M is mapping into. This can be constructed as equivalence classes of pairs (m_1, m_2) with equivalence relation generated by $(m_1, m_2) \simeq (m_1 + m, m_2 + m)$. One usually writes the elements of the group completion as $m_1 - m_2$.

The reduced theory is slightly easier to define. By definition $\widetilde{KU}^0(X)$ is the quotient of $KU^0(X)$ by $KU^0(\mathrm{pt}) = \mathbb{Z}$. Actually, $\mathrm{Vect}_{\mathbb{C}}(X)$ modulo the trivial vector bundles is already a group as for every vector bundle ξ, there is a vector bundle η such that $\xi \oplus \eta$ is trivial. Thus, $\widetilde{KU}^0(X) \cong \mathrm{Vect}_{\mathbb{C}}(X)/\mathbb{N}$ and there is no need to group complete. On the other hand, choosing a base point $x \in X$, we can embed $\widetilde{KU}^0(X)$ into $KU^0(X)$ via $\xi \mapsto \xi - \dim_x \xi$.

For X connected compact Hausdorff, we can identify $\widetilde{KU}^0(X)$ with

$$\mathrm{colim}_n \, \mathrm{Vect}^n_{\mathbb{C}}(X) \cong \mathrm{colim}_n [X, BU(n)] \cong [X, \mathrm{hocolim}\, BU(n)] \cong [X, BU].$$

As $KU^0(X) \cong \widetilde{KU}^0(X) \oplus \mathbb{Z}$, we can deduce that $KU^0(X) \cong [X, BU \times \mathbb{Z}]$. Indeed, $X \to BU \times \mathbb{Z}$ factors over some $BU(n) \times \mathbb{Z}$, i.e. we obtain a vector bundle V and a number k. To this we associate the K-theory class $[V] - (n - k)$. Thus, we obtain for X still Hausdorff, compact and connected a map

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{KU}^0(X) \arrow[r] & KU^0(X) \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & {[X, BU]} \arrow[r] \arrow[u] & {[X, BU \times \mathbb{Z}]} \arrow[r] \arrow[u] & {[X, \mathbb{Z}]} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

of short exact sequences, where the outer two terms are isos.

In general, we define $KU^0(X)$ as $[X, BU \times \mathbb{Z}]$ and obtain $\widetilde{KU}^0(X) = [X, BU \times \mathbb{Z}]^\bullet$, where we assume for the latter that X is pointed and consider pointed maps.

We have $\Omega(BU \times \mathbb{Z}) \cong \Omega BU \simeq U$. (Indeed, for any topological group G, we have a principal G-fibration $EG \to BG$ with EG contractible. Thus, $G \simeq \Omega BG$.) By Bott periodicity, we also have $\Omega U \simeq BU \times \mathbb{Z}$. Thus, we obtain an Ω-spectrum made out of $BU \times \mathbb{Z}$ in degrees $2n$ and of U in degrees $2n+1$. We call this spectrum KU. As KU is an Ω-spectrum, we have that $\pi_k KU$ is simply $\pi_k BU \times \mathbb{Z}$ or more generally π_{k+l} of its l-th space.

Clearly, $\pi_* KU$ is 2-periodic. We easily compute $\pi_0 KU = \pi_0 BU \times \mathbb{Z} \cong \mathbb{Z}$ and $\pi_{-1} KU \cong \pi_0 U = 0$. One can check that KU is actually a ring spectrum (using the tensor product of vector bundles) and that $\pi_* KU \cong \mathbb{Z}[u^{\pm 1}]$ with $|u| = 2$.

Explicitly, one can show that $u \in \pi_2 KU \cong \widetilde{KU}^{-2}(S^0) \cong \widetilde{KU}^0(S^2)$ corresponds to the class of the tautological line bundle η_1 on $S^2 \cong \mathbb{CP}^1$. Thus, $[\eta_1]/u$ corresponds to 1. The class $x = [\eta]/u \in \widetilde{KU}^2(\mathbb{CP}^\infty)$ is a lift of this class, which is the Euler class, yielding $KU^*(\mathbb{CP}^\infty) \cong KU^*[\![x]\!]$. Thus, the corresponding Chern class of a line bundle in *reduced* K-theory is $\tilde{c}_1(L) = [L]/u$. We obtain $\tilde{c}_1(L_1 \otimes L_2) = [L_1][L_2]/u$ and hence $F(x_1, x_2) = x_1 + x_2 + u x_1 x_2$. This is called (a form of) the *multiplicative formal group law*.

Other examples are harder to write down, but we will see them later.

References: Accessible references for vector bundles and KU^0 are [Hat03] and [Ati67]; see also [Swi75] and [KT06] for K-theory as cohomology theories. For formal group laws see e.g. [Ada74], Part II, and [KT06] or [Haz12] for a comprehensive treatment.

end of lecture 5

2.8 $\pi_* MU$ and the universal formal group law

Given a morphism of (graded) rings $f \colon R \to S$, we can pushforward a (graded) formal group law $F \in R[\![x_1, x_2]\!]$ to S by just applying f to the coefficients of F. Given a map

$E \to F$ of ring spectra, we can also pushforward every complex orientation of E to F and it is easy to check that the corresponding formal group law over F_* is just the pushforward of that from E_*. As MU carries the universal complex orientation, we see that every formal group law associated to a complex oriented theory is actually pushed forward from MU_*. Amazingly, we can decouple this statement from topology.

Theorem 2.41 (Quillen). *The formal group law F_{MU} over MU_* is the universal formal group law, i.e. for every (graded) commutative ring R, pushforward defines a 1-1 correspondence between (graded) morphisms $MU_* \to R$ and (graded) formal group laws over R.*

Remark 2.42. It is easy to see that the ungraded version implies the graded version. The universal ring for formal group laws is often called the *Lazard ring* L as it was first computed by Lazard to be a polynomial ring $\mathbb{Z}[x_2, x_4, x_6, \ldots]$, where $|x_{2i}| = 2i$. The existence of L is actually easy; the determination that $L \cong \mathbb{Z}[x_2, x_4, x_6, \ldots]$ is harder, but was done before Quillen's theorem. An important point is that the generators x_i are not canonical – while there are some explicit choices, they are rather hard to work with.

There are two kinds of proofs of Quillen's theorem that the morphism $L \to \pi_* MU$ classifying the formal group law of MU is an isomorphism. One is working from the knowledge of L and $\pi_* MU$ (see e.g. [Lur10, Lecture 10]). Another proof of Quillen shows that $L \cong \pi_* MU$ differently (see [Qui71]).

Definition 2.43. A (graded) ring homomorphism $MX_* \to R_*$ is called a *genus*.

Thus, Quillen says that genera for MU are "classified" by formal group laws.

Remark 2.44. Clearly, every complex orientation $MU \to E$ induces a genus $MU_* \to E_*$. It is an interesting question whether we can lift a given genus to a transformation of ring spectra! This is the topic of Landweber's exact functor theorem, which we will deal with later.

2.9 Rational formal group laws

We will state most things for formal group laws, but they hold mutatis mutandis also for graded formal group laws.

Definition 2.45. Let F, G be two formal group laws over a ring R. A power series $f \in R[\![x]\!]$ with no constant term is called a *homomorphism* from $F \to G$ if

$$F(f(x_1), f(x_2)) = f(G(x_1, x_2)).$$

An invertible homomorphism is called an *isomorphism*.

Note that a homomorphism f is an isomorphism if and only if the linear coefficient is invertible.

Lemma 2.46. *Let R be a $\mathbb{Q}$-algebra. Then any two formal group laws over R are isomorphic.*

Proof. Left as an exercise. See e.g. [Rav86, Appendix]. □

Definition 2.47. Let R be a $\mathbb{Q}$-algebra and F be a formal group law over R. Then the unique isomorphism from the additive formal group law $\hat{\mathbb{G}}_a$ to F is called the *logarithm* $\log_F$ of F:

$$\log_F(F(x_1, x_2)) = \log_F(x_1) + \log_F(x_2).$$

Example 2.48. Consider the multiplicative formal group law $\hat{\mathbb{G}}_m(x, y) = x + y + xy$. It satistifes $1 + \hat{\mathbb{G}}_m(x, y) = (1 + x)(1 + y)$. Recall the classical logarithm series

$$f(x) = \log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots .$$

We know from calculus that $\log((1 + x)(1 + y)) = \log(1 + x) + \log(1 + y)$. That is $f(x) + f(y) = f(\hat{\mathbb{G}}_m(x, y))$. Thus, f is the logarithm for $\hat{\mathbb{G}}_m$.

There is also the variant for $F(x, y) = x + y + uxy$, where u is invertible (as for K-theory!). Then $(1 + ux)(1 + uy) = 1 + uF(x, y)$. Thus, the logarithm is

$$f(x) = \log(1 + ux) = ux - \frac{1}{2}u^2x^2 + \frac{1}{3}u^3x^2 - \cdots .$$

This already suggests that we need denominators in R if the additive formal group law should be isomorphic to a multiplicative one. This is indeed so. Before we prove this, we need some notation.

Definition 2.49. Let F be a formal group law over a ring R. Define inductively $[n]_F(x)$ by $[1]_F(x) = x$ and $[n]_F(x) = F([n-1]_F(x), x)$. I.e. we "multiply by n".

Examples 2.50. For $F(x, y) = x + y$, we obtain $[n]_F(x) = nx$.

For $F(x, y) = x + y + uxy$, we obtain $[n]_F(x) = \frac{(1+ux)^n-1}{u}$.

Lemma 2.51. *Fix a ring R and an invertible element $u \in R$. The additive formal group law $\hat{\mathbb{G}}_a(x, y) = x + y$ is only isomorphic to $F(x, y) = x + y + uxy$ over R if R is a $\mathbb{Q}$-algebra.*

Proof. Let f be such an isomorphism $\hat{\mathbb{G}}_a \to F$. Then

$$0 = f([n]_{\hat{\mathbb{G}}_a})(f^{-1}(x)) = [n]_F(x) = \frac{(1 + ux)^n}{u} - 1$$

over R/n. Thus $u^{n-1} = 0$ in R/n and thus $R/n = 0$ as u is invertible. As this is true for every n, the ring R must be a $\mathbb{Q}$-algebra. $\square$

Lemma 2.52. *Let $x, x' \in E^2(\mathbb{CP}^\infty)$ correspond to two different complex orientations of a ring spectrum E. Then the resulting formal group laws F and F' are isomorphic. More precisely, we can express x' as a power series $f(x)$ as $E^*(\mathbb{CP}^\infty) \cong E^*[\![x]\!]$. Then*

$$f(F(x_1, x_2)) = F'(x_1', x_2') = F'(f(x_1), f(x_2)).$$

Corollary 2.53. *The integral homology $H_*(KU; \mathbb{Z})$ is a rational vector space. In particular, $H_*(KU) \cong \mathbb{Q}[u^{\pm 1}]$ and $H_*(KU; \mathbb{F}_p) \cong H_*(KU/p; \mathbb{Z}) = 0$.*

Proof. By definition $H_*(KU; \mathbb{Z}) \cong \pi_* H\mathbb{Z} \wedge KU$. The spectrum $H\mathbb{Z} \wedge KU$ admits ring spectra maps from $H\mathbb{Z}$ and KU and thus carries both the additive and a multiplicative formal group law from the two different complex orientations. These are isomorphic by the last lemma. By Lemma 2.51, it follows that $\pi_* H\mathbb{Z} \wedge KU$ is a $\mathbb{Q}$-algebra.

The rational Hurewicz map

$$\pi_* KU \otimes \mathbb{Q} \to H_*(KU; \mathbb{Q}) \cong H_*(KU; \mathbb{Z})$$

is (as always) an isomorphism and $\pi_* KU \cong \mathbb{Z}[u^{\pm 1}]$.

The last part follows by the Bockstein sequence

$$\cdots \to H_*(KU; \mathbb{Z}) \xrightarrow{p} H_*(KU; \mathbb{Z}) \to H_*(KU; \mathbb{F}_p) \to \cdots$$ $\square$

2.10 Mischenko's theorem

In this section, we will give a proof of Mischenko's theorem modulo some parts from Adams's book [Ada74, Part II]. One reason that we give the proof here (though we did not treat it in detail in the lecture) is that Adams's proof of the crucial Lemma 9.1 is rather short.

Theorem 2.54 (Mischenko)**.** *The logarithm of the formal group law on MU_* equals*

$$\sum_{n\geq 0}^{\infty} \frac{[\mathbb{CP}^n]}{n+1} x^{n+1}.$$

Example 2.55. Let $\mathrm{Td}\colon MU_* \to \mathbb{Z}$ be the ring homomorphism classifying the formal group law $x+y+xy$. This is called the *Todd genus*. Mischenko's theorem and Example 2.48 imply that $\mathrm{Td}([\mathbb{CP}^n]) = 1$.

end of lecture 6

We know that $\pi_* MU \to H_* MU = \pi_* H\mathbb{Z} \wedge MU$ is a rational isomorphism. Thus, we can identify the logarithm on the easier side of $H_* MU$. As H_* is complex oriented, we have in particular $H_* MU \cong \mathbb{Z}[b_1, b_2, \dots]$ with $|b_i| = 2i$. The two complex orientations x (coming from H) and x^{MU} (coming from MU) can be translated into each other as follows [Ada74, II.6.3, 6.6]:

Lemma 2.56. *We have $x^{MU} = \sum_{i\geq 0} b_i x^{i+1}$.*

By Lemma 2.52, we see that this power series is the *exponential* of F^{MU}, i.e. the inverse of the logarithm. A purely algebraic manipulation shows the following [Ada74, II.7.5].

Lemma 2.57. *Setting $\log_F(x) = \sum_{i\geq 0} m_i x^{i+1}$, we have*

$$m_n = \frac{1}{n+1} \left[(\sum_{i\geq 0}^{\infty} b_i)^{-n-1} \right]_{2n},$$

where the lower n denotes the degree $2n$-part.

Now recall the Pontryagin–Thom construction to see what kind of element $[\mathbb{CP}^n]$ defines in $H_* MU$. Let ν be the (complex) normal bundle of some embedding $\mathbb{CP}^n \to S^{2n+2k}$. Then the element $\pi_n MU$ corresponding to $[\mathbb{CP}^n]$ is defined by the composition $S^{2n+2k} \to \mathrm{Th}(\nu) \to \mathrm{Th}(\xi_k^{univ}) = MU_{2k}$. On H_{2n+2k} this induces, we get:

$$\begin{tikzcd}
H_{2n+2k} S^{2n+2k} \arrow[r] \arrow[dr] & H_{2n+2k}(\mathrm{Th}(\nu)) \arrow[r] \arrow[d, "\cong"] & H_{2n+2k}(\mathrm{Th}(\xi_k^{univ})) \arrow[r] \arrow[d, "\cong"] & H_{2n} MU \arrow[d, "\cong"] \\
& H_{2n}\mathbb{CP}^n \arrow[r] & H_{2n} BU(k) \arrow[r] & H_{2n} BU
\end{tikzcd}$$

The diagonal map sends 1 to the fundamental class. All in all, we see that the image of $[\mathbb{CP}^n]$ in $H_* MU$ corresponds under the Thom isomorphism to $\nu_*[\mathbb{CP}^n] \in H_* BU \cong \mathbb{Z}[\beta_1, \beta_2, \dots]$ with β_i corresponding to b_i under the Thom isomorphism. We have thus have to show the following:

Proposition 2.58. *We have $\nu_*[\mathbb{CP}^n] = \left[(\sum_{i\geq 0}^{\infty} \beta_i)^{-n-1} \right]_{2n} \in H_* BU$.*

Milnor–Stasheff, Theorem 14.10, gives us the following:

Lemma 2.59. *The total Chern class of the tangent bundle of $\mathbb{CP}^n$ is $(1+x)^{n+1}$ in $H^*(\mathbb{CP}^n) \cong \mathbb{Z}[x]/x^{n+1}$. Thus, the total Chern class of the (stable) normal bundle $\nu_{\mathbb{CP}^n}$ is $(1+x)^{-n-1}$ (seen as a power series in x).*

This determines completely the map $H^*BU \xrightarrow{\nu^*} H^*\mathbb{CP}^n$. The challenge is now to describe the dual map $H_*\mathbb{CP}^n \to H_*BU$ in terms of the b_i. Our trick is to guess the correct map and then show that it dualizes to ν^*.

Denote the generator of $H_{2k}\mathbb{CP}^n$ by β_k. Thu coproduct on $H_*\mathbb{CP}^n$ is given by $\Psi(\beta_k) = \sum_{i+j=k} \beta_i \otimes \beta_j$. The same formula is true for the β_k (which are the images of the $\beta_k \in H_{2k}\mathbb{CP}^n$ under the standard map $\mathbb{CP}^\infty = BU(1) \to BU$). A map $H_*\mathbb{CP}^n \to H_*BU$ is compatible with coproducts iff the dual map is multiplicative.

Lemma 2.60. *The map*

$$\begin{aligned} H_*\mathbb{CP}^n &\xrightarrow{\Phi} H_*BU \\ \sum_k \beta_k &\mapsto [(\sum_k \beta_k)^{-n-1}]_{\leq 2n} \end{aligned}$$

is compatible with coproducts. Thus, Φ^* is multiplicative.

Proof. This is a simple check. □

Proof of proposition: To show that $\Phi^*\colon H^*BU \to H^*\mathbb{CP}^n$ equals ν^*, we just have to show that for $c \in H^*BU$ the total Chern class, we have $\Phi^*(c) = c(\nu_{\mathbb{CP}^n})$. Indeed,

$$\begin{aligned} \Phi^*(c) &= \sum_i \langle \Phi(\beta_i), c_i \rangle x^i \\ &= \sum_i \langle (\sum_k \beta_k)^{-n-1}, c_i \rangle x^i \\ &= \sum_i \langle (1+\beta_1)^{-n-1}, c_i \rangle x^i \\ &= (1+x)^{-n-1} \\ &= c(\nu_{\mathbb{CP}^n}). \end{aligned}$$

Here, we use that the only monomial in the β_k that pairs non-trivially with c_i is β_1^i and $\langle \beta_1^i, c_i \rangle = 1$. ([Ada74, Lemma 4.3])

Thus, $\Phi = \nu_*$. As $[\mathbb{CP}^n] = x^n$, the result follows. □

Corollary 2.61. *The $[\mathbb{CP}^n]$ are generators for $\pi_*MU \otimes \mathbb{Q}$.*

Proof. By Mischenko's theorem, it is enough to show that the m_n are rational generators. By the inverse formula to Lemma 2.57, the b_i are expressible in terms of the m_n. As the b_i generate H_*MU, which is rationally isomorphic to π_*MU, the corollary follows. □

2.11 Exercises

Exercise 2.62. Let M be a compact 2-dimensional complex manifold (like e.g. $\mathbb{CP}^2$). Show that the value of the Todd genus on M can be computed as $\frac{1}{12}\langle c_1(TM)^2 + c_2(TM), [M] \rangle$. Hint: You can use that this expression is invariant under bordism of manifolds with BU-structure.

Exercise 2.63. Let $x \in (H\mathbb{Z} \wedge MU)^2(\mathbb{CP}^\infty)$ be the image of the standard generator $x \in H^2(\mathbb{CP}^\infty)$. Consider the map

$$[\Sigma^k \mathbb{CP}^\infty, H\mathbb{Z} \wedge MU] \to \mathrm{Hom}(H_{*-k}\mathbb{CP}^\infty, H_* MU),$$

sending $f\colon \Sigma^k\mathbb{CP}^\infty \to H\mathbb{Z} \wedge MU$ to the composition $H_*\Sigma^k\mathbb{CP}^\infty \xrightarrow{H_* f} H_*(H\mathbb{Z} \wedge MU) \to H_*MU$ using the ring structure of $H\mathbb{Z}$. Show that it sends $\sum_i a_i x^{i+1}$ to the map sending β_i to a_i.

Does the argument change if I replace MU by any other spectrum E?

3 Algebraic geometry and elliptic genera

3.1 Schemes and group schemes

We start with a reminder on the functor of points approach to schemes. Denote the category of commutative rings by Alg.

Proposition 3.1. *The functor* $\mathrm{Sch} \to \mathrm{Fun}(\mathrm{Alg}, \mathrm{Set})$*, sending a scheme X and a commutative ring R to* $\mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}\, R, X)$ *is fully faithful.*

Proof. Every scheme can be covered by affine schemes. We directly see that the functor is faithful. For fullness: To describe a map $X \to Y$, it is enough to give a compatible map from all affine open subschemes of X to Y. □

By abuse of notation, we will often denote by $\mathrm{Spec}\, R$ just the functor $\mathrm{Alg} \to \mathrm{Set}$ represented by R.

We can also describe the image of the embedding $\mathrm{Sch} \to \mathrm{Fun}(\mathrm{Alg}, \mathrm{Set})$. We call a functor $\mathrm{Fun}(\mathrm{Alg}, \mathrm{Set})$ (or also a functor $\mathrm{Fun}(\mathrm{Sch}^{op}, \mathrm{Set})$) a *Zariski sheaf* if it is a Zariski sheaf restricted to the open subsets of any given $\mathrm{Spec}\, R$ (or X). We call a subfunctor $U \subset X\colon \mathrm{Fun}(\mathrm{Alg}, \mathrm{Set})$ *open* if $U \times_X \mathrm{Spec}\, R \subset \mathrm{Spec}\, R$ is representable by an open affine subscheme of $\mathrm{Spec}\, R$.

Proposition 3.2. *A functor* $X\colon \mathrm{Alg} \to \mathrm{Set}$ *is a scheme iff it is a Zariski sheaf and there is a collection of open subfunctors* $\mathrm{Spec}\, R_i \subset X$ *such that* $\coprod_i \mathrm{Spec}\, R_i(K) \to X(K)$ *is surjective for every field K.*

Definition 3.3. A *group scheme* is a group object in schemes, i.e. a lift of the functor $\mathrm{Sch}^{op} \to \mathrm{Set}$ to groups. (Equivalently, just lift $\mathrm{Alg} \to \mathrm{Set}$ to groups. Indeed, maps into a group object obtain a group structure.)

A group scheme over S is a lift of the functor $(\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ to groups.

Example 3.4. The scheme $\mathbb{A}^1_R$ represents the underlying set functor on R-algebras. We can lift this to abelian groups by addition. This gives the group schemes $\mathbb{G}_{a,R}$.

Example 3.5. The scheme $\mathrm{Spec}\, R[t^{\pm 1}]$ represents the set of invertible elements. This obtains a group structure by multiplication. This is called $\mathbb{G}_{m,R}$.

References: See e.g. the last chapter of [EH00] for this "functor of points"-approach.

3.2 Formal groups

On an affine scheme $\mathrm{Spec}\, A$, a group scheme structure is the same as a Hopf algebra structure on A, i.e. we need a map $A \to A \otimes_R A$ with certain properties. Can we use a formal group law to define the structure of a group scheme on $\mathrm{Spec}\, R[\![x]\!]$? Tricky. We need a map $R[\![x]\!] \to R[\![x_1]\!] \otimes_R R[\![x_2]\!]$ of R-algebras. Neither is such a map determined by the image of x nor is the image isomorphic to $R[\![x_1, x_2]\!]$. (We can look at the coefficients f_i of x_1^i. In the tensor product $R[\![x_1]\!] \otimes_R R[\![x_2]\!]$, these span a finitely generated R-submodule of $R[\![x_2]\!]$. This is not true for general power series in two variables.)

Solution: Use the topology on the power series ring. Let A be a ring with an ideal $I \subset A$. We can equip A with the minimal topology, where all I^n are open and which is closed under translation. We denote by $\operatorname{Spf} A$ the functor $\mathrm{Alg} \to \mathrm{Set}$, sending B to the continuous ring homomorphisms from $A \to B$. If A is an R-algebra, $\operatorname{Spf} A$ can also be seen as a functor

$$\mathrm{Alg}_R \to \mathrm{Set}, \quad B \mapsto \mathrm{Hom}_{\mathrm{cts}}(A, B),$$

where B carries the discrete topology.

For example, $\operatorname{Spf} R[\![t]\!]$ sends each R-algebra to its set of nilpotent elements. This functor is denoted by $\hat{\mathbb{A}}^1_R$. Its n-fold product with itself is denoted by $\hat{\mathbb{A}}^n_R$.

The functor Spf can indeed be extended to $(\mathrm{Sch}/R)^{op}$ by precomposing with the functor

$$(\mathrm{Sch}/R)^{op} \to \mathrm{Alg}_R, \quad X \mapsto H^0(X; \mathcal{O}_X).$$

Lemma 3.6. *Formal group laws over R are in one-to-one correspondence with lifts of $\hat{\mathbb{A}}^1_R$ to the category of groups.*

Proof. Exercise. □

A homomorphism of formal group laws corresponds to a homomorphism of the corresponding group valued functors on R-algebras. Note that this (obviously) is not the identity on $\hat{\mathbb{A}}^1_R$. So we might try to define a formal group to be a functor from Alg_R to groups whose underlying functor to sets is isomorphic to $\hat{\mathbb{A}}^1_R$. But this is an idea, which works as well as defining a vector bundle on M to be something that is isomorphic to $M \times \mathbb{R}^n$. You only want it locally!

——————————————————————————**end of lecture 7**

Definition 3.7. Let S be a scheme. An n-dimensional commutative *formal group* over S is a Zariski sheaf $F\colon \mathrm{Sch}_S^{op} \to \mathrm{Ab}$ such that there exists an open cover $\{U_i = \operatorname{Spec} R_i \subset S\}$ such that $F|_{U_i}$ is equivalent to $\hat{\mathbb{A}^n}_{R_i}$.

Let X be an S-scheme with an ideal sheaf $I \subset \mathcal{O}_X$ (corresponding to a closed subscheme). Then the *formal completion* $\hat{X}_I$ is defined by $\hat{X}_I(Y) = \{f\colon Y \to X : f^*I \text{ locally nilpotent }\}$. Here, a sheaf is locally nilpotent if there is an open cover and pulled back to every open in it, the sheaf is nilpotent. This agrees with the colimit in Zariski sheaves on Sch/S of the vanishing loci of the I^n. For example, we can take $X = \operatorname{Spec} R[t]$ and $I = (t)$. Then $\hat{X}_{(t)} = \hat{\mathbb{A}}^1_R$. More generally, it generalizes the Spf-construction above.

Note that if X is a separated S-scheme and $e\colon S \to X$ is a section of $p\colon X \to S$, then e is automatically a closed immersion and hence defines an ideal sheaf. Indeed, look at the pullback diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ X \arrow[r, "(\mathrm{id}, e)"] & X \times_S X. \end{tikzcd}$$

In particular, this is true for the identity section $e\colon S \to G$ of a separated group scheme G. We claim first that the group structure on G induces one on the formal completion $\hat{G}$. Indeed, let I be ideal sheaf corresponding to e. Let $f, g\colon X \to G$ be morphisms such that f^*I and g^*I are locally nilpotent. Equivalently, the compositions

$fi, gi\colon X^{red} \to X \to G$ from the underlying reduced scheme X factor over the vanishing locus of I, namely $\mathrm{im}(e)$. The same is true for $(fi) \cdot (gi) = (f \cdot g)i$ as e is the identity section. Thus, the pullback of I along $f \cdot g$ is locally nilpotent.

One can show that $\hat{G}$ does not only have a group structure, but is actually often a formal group:

Theorem 3.8. *Let G be a commutative group scheme that is smooth of relative dimension n over S. Then $\hat{G}$ is a formal group of dimension n over S.*

If S is a field, one can for example argue as follows: Every complete regular local ring containing a field k is of the form $k[\![x_1, \dots, x_n]\!]$.

We are only interested in one-dimensional formal groups. To produce interesting formal groups, we should look for one-dimensional smooth group schemes. Over $\mathbb{C}$, tori come to mind, so we might expect in addition to $\mathbb{G}_a$ and $\mathbb{G}_m$ also genus 1 curves as examples. Over algebraically closed fields this is all there is:

Theorem 3.9. *Every one-dimensional smooth connected group scheme over an algebraically closed field is isomorphic to $\mathbb{G}_a$ or to $\mathbb{G}_m$ or proper of genus* 1 *(and this case is called an* elliptic curve*).*

The argument goes roughly as follows: Let k be algebraically closed and G a smooth connected group scheme over k. Then it's easy to see that its cotangent bundle $\Omega^1_{G/k}$ is trivial. If G is proper, then $H^0(G; \Omega^1_{G/k})$ is the genus of G, which must thus be 1. The non-proper case is slightly more difficult. Embed G into a proper curve C (i.e. G is C without finitely many points). To do this note that every smooth connected curve over an algebraically closed field is quasi-projective (see e.g. [Oss]; a variety is for him a irreducible, separated scheme that is Zariski locally isomorphic to an affine variety over an algebraically closed field). The action of G on itself can be extended to an action of G on C via automorphisms. Curves of genus ≥ 2 have only finitely many automorphisms by Hurwitz. Hence, C has genus at most 1. The rest of the argument can be found in [You], who assumes that G is affine, but the quasi-projective case is similar. (See also [Con02] for another important result in the classification of group schemes.)

Summarizing: We care about elliptic curves.

References: There are several different approaches to formal groups, which makes the subject a bit confusing. See e.g. [Str99] or [Zin] for different approaches.

3.3 Elliptic curves over algebraically closed fields

We will begin with the theory over an algebraically closed field k. A variety will for us be a separated integral scheme of finite type over k and a curve is a variety of dimension 1.

Definition 3.10. An *elliptic curve* over k is a smooth proper curve C of genus[5] 1 with a chosen point $e \in C(k)$, i.e. a section of the structure morphism $C \to \operatorname{Spec} k$.

In the case $k = \mathbb{C}$, every elliptic curve is a torus, i.e. of the form $\mathbb{C}/L$ for a lattice $L \subset \mathbb{C}$. There are different ways to show this. The first uses uniformization: The universal cover of the elliptic curve C must be $\mathbb{C}$ or $\mathbb{H}$ or S^2 by uniformization. Thus, $C = \mathbb{C}/G$ or $\mathbb{H}/G$ or S^2/G, where G is a group of automorphisms (acting properly discontinuous), which is isomorphic to $\pi_1 C \cong \mathbb{Z}^2$. The last case is clearly impossible (S^2 cannot cover a genus 1-curve). If the universal cover of C is $\mathbb{H}$, then C gets a hyperbolic

[5]This is defined as $\dim_k H^0(C; \Omega^1_{C/k}) = \dim_k H^1(C; \mathcal{O}_C)$.

metric; this implies by Gauss–Bonnet that the Euler characteristic is negative. Thus, $C \cong \mathbb{C}/G$, where G is isomorphic to $\mathbb{Z}^2$. You can show that only translation subgroups can act properly discontinuously. Thus, G is a lattice inside of $\mathbb{C}$.

We can also sketch a different way: Let (C, e) be elliptic curve over $\mathbb{C}$. The vector space $H^0(C; \Omega^1)$ of differentials equals the genus of C, i.e. 1. For every path γ in C and every differential $\omega \in H^0(C; \Omega^1)$, we can consider the integral $\int_\gamma \omega$. This is homotopy invariant in γ if we leave the endpoints fixed. In particular, we obtan a map $H_1(C; \mathbb{Z}) \to H^0(C; \Omega^1)^\vee$ by evaluating on paths. Set $\mathrm{Jac}(C)$ to be the quotient. It is not too hard to see that $H_1(C; \mathbb{Z})$ embeds as a lattice so that $\mathrm{Jac}(C)$ is a complex torus. Note also that it has a group structure.

Define a map $D\colon C \to \mathrm{Jac}(C)$ as follows: Given a differential $\omega \in H^0(C; \Omega^1)$ and a point $x \in C$, we choose a path γ from e to x and integrate: $\int_\gamma \omega$. While this depends on the choice of γ any to choices of γ differ by an element of $H_1(C; \mathbb{Z})$ so that the image in $\mathrm{Jac}(C)$ is well-defined.

A holomorphic map is open if it is not constant. Clearly the map is not constant (locally, we can work in $\mathbb{C}$ and just write down examples). Thus D is open. As the image is also closed, D is surjective. The Abel–Jacobi theorem says that it is also injective and hence an isomorphism.

In the following, we will need some facts about divisors and their correspondence to line bundles. See e.g. [Har77, Sections II.6 and IV.1] or [Ful69, Chapter 8] for a more elementary account. We will use the notation $\mathcal{O}(D)$ to denote the line bundle associated with a divisor D. We will use the notation K to denote a divisor such that $\mathcal{O}(K) \cong \Omega^1_{C/k}$ on a curve C. This has degree $2g - 2$, where g is the genus of C. We denote by $l(D)$ the dimension of the global sections $H^0(C; \mathcal{O}(D))$.

______________________________ **end of lecture 8**

Theorem 3.11. *An elliptic curve has the (unique) structure of a group scheme over k with e as unity.*

Proof. We will just give an indication. We denote by $\mathrm{Jac}(C)(k)$ the set of all formal linear combinations $\sum_i a_i P_i$ of k-points of C (i.e. *divisors*) such that $\sum_i a_i = 0$ (degree 0) modulo the subgroup of *principal divisors*: Given a meromorphic function f on C, we give P_i the valuation a_i if f has a pole of order a_i at P_i or a zero of order $-a_i$; the formal combination $\sum_i a_i P_i$ is called a principal divisor. As the principal divisors form a subgroup of all divisors (just take $f \cdot g$), we see that $\mathrm{Jac}(C)(k)$ has an abelian group structure.

We have a map $C(k) \to \mathrm{Jac}(C)(k)$, sending P to $[P - e]$. We will show using Riemann–Roch that this is a bijection. Let D be a divisor of degree 0. Then Riemann–Roch implies that

$$l(D + e) - l(K - D - e) = \deg(D + e) + (1 - g(C)) = 1.$$

As K has degree 0, we see that $K - D - e$ has degree -1. Thus $l(K - D - e) = 0$. (Indeed, if there is a section of $\mathcal{O}(D)$ (without poles), then there is a divisor D' equivalent to D with all coefficients ≥ 0, i.e. an effective divisor.) We see that $l(D + e) = 1$. This means there is an effective divisor of degree 1 equivalent to $D + e$, which must be of the form P; thus D is equivalent to $P - e$. If Q is another point whose divisor class equals that of $D + e$, we see that there is a section of $\mathcal{O}(D + e)$ whose only zero is at Q. As the space of sections of $\mathcal{O}(D + e)$ is 1-dimensional, we see that $P = Q$.

One way to actually prove that C is a group scheme is to strengthen this theorem to see that C represents the functor $T \mapsto \mathrm{Pic}^0(C/T)$ from k-schemes (of finite type) to abelian groups. Here, $\mathrm{Pic}^0(C/T)$ is the group of line bundles on $C \times_k T$ that restrict to a degree 0-line bundle on each fiber of $\mathrm{pr}_2\colon C \times_k T \to T$ modulo those of the form $\mathrm{pr}_2^* \mathcal{L}$ for $\mathcal{L}$ a line bundle on T. (For details, see [Har77, Section IV.4].) □

Theorem 3.12. *Every elliptic curve (C, e) can be embedded into $\mathbb{P}^2_k$ and is cut out by an equation of the form*

$$Y^2 Z + a_1 XYZ + a_3 YZ^2 = X^3 + a_2 X^2 Z + a_4 XZ^2 + a_6 Z^3. \tag{3.13}$$

The point $[0 : 1 : 0]$ is the chosen point. Conversely, such an equation defines an elliptic curve iff a certain polynomial Δ in the a_i does not vanish.

Proof. Consider the divisor $n \cdot e$. By Riemann–Roch, we have $l(n \cdot e) = n$ for $n \geq 1$. We have obvious inclusions $\mathcal{O}(n \cdot e) \subset \mathcal{O}((n+1) \cdot e)$. We have $H^0(C; \mathcal{O}_C) = k \cdot 1$. As $H^0(C; \mathcal{O}(2e))$ is 2-dimensional, we have one basis vector 1 and call another x. In $H^0(C; \mathcal{O}(3e))$, we have one more basis vector and call it y. We can consider $1, x, x^2, x^3, xy, y, y^2$ in $\mathcal{O}(6e)$. As this is only 6-dimensional, there must be a relation

$$a_0 y^2 + a_1 xy + a_3 y = a_0' x^3 + a_2 x^2 + a_4 x + a_6. \tag{3.14}$$

If not both a_0 and a_0' are nonzero, the pole orders on the two sides do not agree. We can replace x by $a_0 a_0' x$ and y by $a_0 (a_0')^2 y$ and divide everything by $a_0^3 (a_0')^4$ and can thus set $a_0 = a_0' = 1$.

We get a map $\Phi\colon C \to \mathbb{P}^2_k$ that we can informally describe by $\Phi(p) = [x(p), y(p), 1]$. This makes sense unless $e = p$; in this case y has the highest pole order, so we set this to be $[0 : 1 : 0]$. General theory says that this is a morphism of schemes and the image is exactly given by the projective variety E cut out by (3.13), which is the homogenization of (3.14).

We sketch now, why $\Phi\colon C \to E$ is an isomorphism. The degree of a map $C_1 \to C_2$ can be computed as the number of preimages of a point in the target weighted by the order; this is multiplicative. The map $C \to E \to \mathbb{P}^1$ sending p to $[x(p), 1]$ is of degree 2 because ∞ has e as preimage with order 2. Thus, $E \to C$ has degree dividing 2. The map $C \to E \to \mathbb{P}^1$ sending p to $[y(p), 1]$ is of degree 3 because ∞ has e as preimage with order 3. Thus, $C \to E$ has degree 1. Next, we show that E is smooth. An explicit argument with the equation defining E shows that if it is singular, there is a rational map $E \to \mathbb{P}^1$ of degree 1. Thus, the composite $C \to E \to \mathbb{P}^1$ would be a rational map between smooth curves; such a map can be extended to a morphism, which is of degree 1 and hence an isomorphism, which is absurd. Thus, E is smooth and thus Φ is an isomorphism. [For details see [Sil09, Proposition 3.1] □

Remark 3.15. In class, we presented a different argument why Φ is an isomorphism. Essentially, we had to show that Φ is a closed immersion; in this case one says that $3e$ is a *very ample divisor*. Hartshorne [Har77, Prop 3.1] gives a criterion when a divisor is very ample. In our case, we could just use the Riemann–Roch formula that $l(D) = \dim |D| = \deg D$ if $\deg D > 0$. This is actually done in Example IV.3.3.3 in [Har77].

Remark 3.16. We will later see that one can simplify this equation tremendously if $\mathrm{char}(k)$ is neither 2 or 3, namely to $y^2 = x^3 + a_4 x + a_6$. The discriminant Δ is a constant multiple of $\Delta = 27a_6^2 + 4a_4^3$. If $\mathrm{char}(k)$ is indeed neither 2 or 3, the factor does not matter, of course. Let us actually prove in this case that the nonvanishing of Δ is equivalent to smoothness: Let $f(x, y) = x^3 + a_4 x + a_6 - y^2$ and C be the curve defined as its zero set. Then a point (x_0, y_0) on C is singular iff $\frac{\partial f}{\partial x}(x_0, y_0) = 3x_0^2 + a_4 = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) = 2y_0 = 0$. Assume that this is the case. Then $y_0 = 0$ and $x_0^3 + \frac{1}{3} a_4 x_0 = 0$. Substract the latter from $f(x_0, 0)$ and get $\frac{2}{3} a_4 x_0 + a_6 = 0$, i.e. $x_0 = -\frac{3a_6}{2a_4}$. Plugging this into the equation for $\frac{\partial f}{\partial x}$, wee see that $a_4 = -\frac{27a_6^2}{4a_4^2}$ or, equivalently, $27a_6^2 + 4a_4^3 = 0$. This only checked smoothness on the affine part. We still have to check smoothness at $[0:1:0]$, i.e. smoothness of $x^3 + a_4xz^2 + a_6z^3 - z$ at $(0,0)$. This is clear. (See [Sil09, III.1] for an extensive treatment of these equations.)

We can see what happens with the group law under this embedding. Consider a line $H \subset \mathbb{P}^2_k$ defined by an equation $s = ax + by + cz = 0$. Suppose that H intersects $\Phi(C)$ in three points P, Q and R; if counted with multiplicities, Bezout's theorem actually implies that it will always intersect C in three points. We can view s as a section of $\mathcal{O}(1)$ on $\mathbb{P}^2_k$; thus $\Phi^* s$ is a global section of $\Phi^*\mathcal{O}(1) = \mathcal{O}(3e)$, namely $ax + by + c$ (where $x, y, 1 \in H^0(C; \mathcal{O}(3e))$) as in the previous proof. The zero locus of $\Phi^* s$ consists exactly of P, Q and R. Thus, $P + Q + R - 3e$ is zero in the class group. Thus, $P + Q + R = 0$ in the group law. Upshout: To compute $P + Q$, draw a line connecting P and Q (tangent if $P = Q$); denote the third intersection with C by $-R$. Then draw a line connecting e and $-R$; the third point of intersection is $R = P + Q$.

One can use this description to show that the addition morphism of the group structure on points described above is actually given a a morphism of schemes. (Morphism of varieties are determined by what they do on points [GW10, Section 3.13].)

3.4 Elliptic curves over general base schemes

Definition 3.17. Let S be a scheme. An *elliptic curve* E over S is a smooth proper morphism $p\colon E \to S$ with a section $e\colon S \to E$ such that for every morphism $x\colon \operatorname{Spec} k \to S$ (with k algebraically closed), x^*E is an elliptic curve over k.

Theorem 3.18. *Every elliptic curve over S has the unique structure of a group scheme over S with unity e.*

We will not prove this statement. The basic idea is the same as for the algebraically closed case, but the details are quite a bit harder (and actually the general proof reduces at a crucial point to the case of a field). See [KM85, Theorem 2.1.2].

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Theorem 3.19. *For every elliptic curve E, we can find a Zariski covering $S = \bigcup_i U_i$ with $U_i = \operatorname{Spec} R_i$ so that $E \times_S U_i$ has a Weierstrass form, i.e. can be embedded into $\mathbb{P}^2_{R_i}$ and is cut out by an equation*

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

with Δ invertible.

For proofs see [Ols16, Section 13.1.6] or [KM85, Section 2.2]; a more detailed account appears in [MO17]. Note here the following lemma:

Lemma 3.20. *An element x of a ring R is invertible iff $f(x) \neq 0$ for every map $f\colon R \to k$ to a field k.*

Proof. The only if is clear. If x is not invertible, choose a maximal ideal $\mathfrak{m} \subset R$ not containing x. Then x is zero in the field $R/\mathfrak{m}$. $\square$

Remark 3.21. Under certain assumptions, the general Weierstrass equation of an elliptic curve C can be simplified. We will choose some ad hoc conventions. These will be different from e.g. [Sil09].

Assume first that $\frac{1}{2} \in R$. Then set $y' = y - \frac{1}{2}a_1x - \frac{1}{2}a_3$. The coefficient in front of xy and y become zero and thus Weierstrass equation gets the form $y^2 = x^3 + b_2x^2 + b_4x + b_6$.

Assume further that $\frac{1}{3} \in R$. Set $x' = x - \frac{1}{3}b_2$. Then the x^2-term vanishes and we get an equation of the form $y^2 = x^3 + c_4 x + c_6$.

Going back to the case where only $\frac{1}{2} \in R$ and we have simplified the Weierstrass equation to $0 = f(x, y) = x^3 + b_2x^2 + b_4x + b_6 - y^2$. Points of the form $(t, 0)$ are exactly those of exact order 2. Indeed: Exact order 2 means that the tangent line at that point goes through $[0 : 1 : 0]$ that is: of the form $ax + d = 0$. This happens iff $\frac{\partial f}{\partial y} = -2y$ is zero at that point, i.e. iff $y = 0$. Suppose that there is is a point on C of the form $(t, 0)$ with $t \in R$. By the coordinate change $x' = x - t$, we can move this point to $(0, 0)$. That $(0, 0)$ is on C means exactly that $b_6 = 0$. Upshot: If we have a point of exact order 2 on our elliptic curve (with $\frac{1}{2} \in R$, we can simplify the Weierstrass equation to $y^2 = x^3 + b_2x^2 + b_4x$.

This already shows that the Weierstrass form is not unique. But we understand isomorphisms between them. Let C be an elliptic curve over $\operatorname{Spec} R$ with a Weierstrass form given by chosen coordinates (x, y). Then $H^0(C; \mathcal{O}(2e)) \cong R \cdot 1 \oplus R \cdot x$ and $H^0(C; \mathcal{O}(3e)) \cong R \cdot 1 \oplus R \cdot x \oplus R \cdot y$. Other Weierstrass coordinates (x', y') must be of the form $x' = vx + r$ and $y' = wy + sx + t$ with $v, w \in R^\times$. If we plug this into $(y')^2 + \cdots = (x')^3 + \cdots$, we get $w^2y^2 + \cdots = v^3x^3 + \cdots$. This must be a invertible multiple of $y^2 + \cdots = x^3 + \cdots$ and thus $w^2 = v^3$. Putting $u = w/v$, we see that $u^3 = w^2$ and $u^2 = v^3$. Thus, the general form of coordinate change is $x' = u^2x + r$ and $y' = u^3 + sx + t$. (Note that e.g. [Sil09, III.1] has a slightly different convention.)

We don't want to compute what a general coordinate change does to the a_i (though Silverman does). We just want to see what coordinate changes preserve the simpler Weierstrass forms we have.

Example 3.22. Let $y^2 = x^3 + b_2x^2 + b_4x + b_6$. Then a coordinate change can only fix the form of the equation (i.e. $a_1 = a_3 = 0$) if $y' = u^3y$ and $x' = u^2x + r$.

Even simpler: Let $y^2 = x^3 + c_4x + c_6$ be a Weierstrass form. Set $x' = u^2x + r$ and $y' = u^3y + sx + t$. It is easy to see that this is of the form $(y')^2 = (x')^3 + c_4'x' + c_6'$ iff $r = s = t = 0$ and $c_4' = u^4c_4$ and $c_6' = u^6c_6$.

Is there any way, we can actually nail down c_4 and c_6 precisely? This we will see in the next section.

3.5 Invariant differentials

Proposition 3.23. *Let $p\colon G \to S$ be a group scheme with multiplication $m\colon G \times_S G \to G$ and unit section $e\colon S \to G$. Set $\omega_{G/S} = e^*\Omega^1_{G/S}$. Then there is a natural isomorphism $\Omega^1_{G/S} \cong p^*\omega_{G/S}$.*

Proof. View $G \times_S G$ as a G-scheme via pr_2. Consider now the automorphism $(m, \mathrm{pr}_2)\colon G \times_S G \to G \times_S G$. As this is an automorphism over G, we obtain $(m, \mathrm{pr}_2)^*\Omega^1_{G\times_S G/G} \cong \Omega^1_{G\times_S G/G}$. Combined with $\Omega^1_{G\times_S G/G} \cong \mathrm{pr}_1^*\Omega^1_{G/S}$, we obtain $m^*\Omega^1_{G/S} \cong \mathrm{pr}_1^*\Omega^1_{G/S}$. Pulling this back along $(ep, \mathrm{id})\colon G \to G \times_S G$, this gives $\Omega^1_{G/S} \cong (ep)^*\Omega^1_{G/S} = p^*\omega_{G/S}$. □

Let $C = G$ be an elliptic curve. Then C is smooth of relative dimension 1 over S so that $\Omega^1_{C/S}$ is a line bundle and thus also $\omega_{C/S}$. Zariski locally on S, this is trivial and hence also $\Omega^1_{C/S}$. A non-vanishing section (and hence a trivialization) of $\Omega^1_{C/S}$ is called an *invariant differential*. Actually, every elliptic curve in Weierstrass form has an invariant differential. The explicit formula is $\eta = \frac{dx}{2y + a_1x + a_3}$. (Idea: write the elliptic curve as zero locus of $f(x, y) = 0$. Then $(\frac{\partial f}{\partial y})^{-1}dx = -(\frac{\partial f}{\partial x})^{-1}dy$. Thus, if the differential $(\frac{\partial f}{\partial y})^{-1}dx$ has a pole somewhere, then $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x} = 0$ at that point; this point would be thus singular, which cannot hapen. Thus, η is a section of $\Omega^1_{C/S}$. If η has a zero somewhere, then it has a zero over a point of S; thus, we can show over an algebraically closed field that η is nowhere vanishing. In this case $\Omega^1_{C/S}$ is isomorphic to the structure sheaf; thus every section that vanishes somewhere, vanishes everywhere. But η is clearly not identically zero for any choice of a_1 and a_3.)

Remark 3.24. By adjunction, the proposition gives a map $\omega_{C/S} \to p_*\Omega^1_{C/S}$. If $S = \operatorname{Spec} k$ for k a field, this is an isomorphism. Indeed: Both sides have one-dimensional global sections (as the genus of C is one) and the map cannot be zero as else the adjoint would not be an isomorphism. By a technique called *cohomology and base change* (combined with Grothendieck duality) one can deduce that it is actually always true that $\omega_{C/S} \to p_*\Omega^1_{C/S}$ is an isomorphism. Thus, an invariant differential can be equivalently seen as a trivialization of $\omega_{C/S}$.

Now assume that C is given as $y^2 = x^3 + b_2x^2 + b_4x + b_6$ so that $\eta = \frac{dx}{2y}$. If we do the coordinate change $(x', y') = (u^2x, u^3y)$, the new differential $\frac{dx'}{2y'}$ is $u^{-1}\eta$. Thus, c_4 and c_6 (in the case $\frac{1}{6} \in R$) are uniquely determined, once we fix an invariant differential!

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This implies the following proposition:

Proposition 3.25. *The scheme* $\operatorname{Spec} \mathbb{Z}[\frac{1}{6}][c_4, c_6][\Delta^{-1}]$ *represents the moduli problem that associates to each scheme S over $\mathbb{Z}[\frac{1}{6}]$ the set of isomorphism classes of elliptic curves with a chosen invariant differential.*

The scheme $\operatorname{Spec} \mathbb{Z}[\frac{1}{2}][b_2, b_4][\Delta^{-1}]$ *represents the moduli problem that associates to each scheme S over $\mathbb{Z}[\frac{1}{2}]$ the set of isomorphism classes of elliptic curves with a chosen invariant differential and a chosen point of exact order* 2.

Proof. We only prove the first thing. We first claim that every elliptic curve over a scheme S over $\mathbb{Z}[\frac{1}{6}]$ with an invariant differential has a Weierstrass form. Indeed, it has one Zariski locally on S, namely one of the form $y^2 = x^3 + c_4x + c_6$ and c_4, c_6 are canonically determined (by using the chosen invariant differential); thus, they glue to functions $c_4, c_6 \in H^0(S; \mathcal{O}_S)$. Likewise, the functions x, y are canonically determined and they glue to a map $C \to \mathbb{P}^2_S$ that is locally a closed immersion with image cut out by $y^2 = x^3 + c_4x + c_6$ and thus also globally.

Thus, every elliptic curve over S with a chosen invariant differential is isomorphic to a unique elliptic curve of the form $y^2 = x^3 + c_4x + c_6$ with invariant differential $\frac{dx}{2y}$ such that $\Delta(c_4, c_6)$ is invertible. $\square$

This gives two elliptic curves over a localization of a polynomial ring in two variables. Both of these seem to have a natural grading ($|b_i| = i$ and $|c_i| = i$). How does this correspond to the functors they represent?

Proposition 3.26. *Let R be a ring. There is a natural one-to-one correspondence between $\mathbb{G}_m$-actions on the functor* $\operatorname{Spec} R$ *represented by R on* Alg *and $\mathbb{Z}$-gradings of R (with $\bigoplus_{i \in \mathbb{Z}} R_i \cong R$).*

Proof. We will only sketch the argument. Denote the functor represented by R on Alg by $\operatorname{Spec} R$. A $\mathbb{G}_m$-action on $\operatorname{Spec} R$ is a map

$$\operatorname{Spec}(R \otimes \mathbb{Z}[t^{\pm 1}]) \cong \mathbb{G}_m \times \operatorname{Spec} R \to \operatorname{Spec} R$$

satisfying the group action axioms. By Yoneda it corresponds to a map

$$\Phi \colon R \to R \otimes \mathbb{Z}[t^{\pm 1}].$$

Denote the t^i-component of Φ by Φ_i. The group action property corresponds to $r = \sum_{i\in\mathbb{Z}} \Phi_i(r)$ and $\Phi_j(\Phi_i(r)) = \delta_{ij}\Phi_i(r)$. If we set $R_i = \Phi_i(R)$, we see that Φ defines an isomorphism $R \to \bigoplus_{i\in\mathbb{Z}} R_i$ with the summing map as inverse. $\square$

The two functors in Proposition 3.25 have obvious $\mathbb{G}_m$-actions: We can just multiply the chosen invariant differentials with a unit. I leave it as an exercise to check that the gradings match.

Remark 3.27. One can generalize the last proposition to gradings by an arbitrary abelian monoid M if one replaces $\mathbb{G}_m$ by $\operatorname{Spec}\mathbb{Z}[M]$ (monoid ring). For example, $\mathbb{N}_{\geq 0}$-gradings.

3.6 The formal group

Every elliptic curve is smooth, so its completion is a formal group. Actually, a Weierstrass equation gives an explicit identification of the formal completion with $\hat{\mathbb{A}}^1_R$. For simplicity, we will only explain this in the case when the Weierstrass form is $Y^2Z = X^3 + c_4XZ^2 + c_6Z^3$, defining an elliptic curve C over a ring R. We want to have coordinates at the origin $[0 : 1 : 0]$. Thus, we see $y = 1$ and obtain $z = x^3 + c_4xz^2 + c_6z^3$. Thus, we can express z in terms of x and higher powers in z. Iterating, we obtain:

$$\begin{aligned} z &= x^3 + c_4xz^2 + c_6z^3 \\ &= x^3 + c_4x(x^3 + c_4xz^2 + c_6z^3)^2 + c_6(x^3 + c_4xz^2 + c_6z^3)^3 \quad = x^3 + c_4x^7 + c_6x^9 + \cdots =: f(x) \end{aligned}$$

There is a resulting morphism $\hat{\mathbb{A}}^1_R \to C$; on every R-algebra T, this sends a nilpotent element x to $[x : 1 : f(x)]$. One sees that this induces an isomorphism $\hat{\mathbb{A}}^1_R \to \hat{C}$: Indeed, on every R-algebra T, where x and z are nilpotent elements satisfying $z = x^3 + c_4xz^2 + c_6z^3$, we have $z = f(x)$ (with no additional constraints on x).

Via this isomorphism, the group structure on $\hat{C}$ induces a group structure on $\hat{\mathbb{A}}^1_R$, which is the same as a formal group law over R. Thus, every elliptic curve over a ring R in Weierstrass form produces a formal group law over R (which depends not only on the elliptic curve, but also on the chosen Weierstrass equation).

For the general case of a Weierstrass equation and also a procedure how to compute the formal group law explicitly, see [Sil09, Section IV.1]. We record a general fact:

Proposition 3.28. *Let C be an elliptic curve over a ring R in Weierstrass form. Assume that R is graded and $|a_i| = 2i$. Then the formal group law corresponding to the coordinates chosen above is graded.*

Proof. The idea is that choice $|x| = -2$ and $|z| = -6$ gives a consistent choice of grading everywhere. $\square$

Remark 3.29. You will have noticed a confusing issue. Before, we said that the natural algebraic grading of, say, c_i is i. Now we need it to be $2i$ to be compatible with our topological conventions. There's nothing that forbids us to double all the degrees of our generators, but we have to be careful, which convention we need at which place.

3.7 Weierstrass Elliptic genera

Definition 3.30. Let C be an elliptic curve over a ring R in Weierstrass form. Let $R = R_*$ be graded such that $|a_i| = 2i$ and let F be its associated graded formal group law as above. This defines a genus $MU_* \to R_*$.

We call such a genus a *complex Weierstrass elliptic genus.*

To really do something with it, we have to compute its logarithm, which gives the values on $\mathbb{CP}^n$ by Mischenko's theorem. We will do this later using complex-analytic methods. One thing we want to do already now though is to define a versions of Weierstrass elliptic genera that go from *oriented bordism*, which is geometrically maybe more important. Recall to that purpose the following theorem:

Theorem 3.31. *The ring $MU_* \otimes \mathbb{Q}$ is a polynomial ring generated by the $[\mathbb{CP}^i]$ and the ring $MSO_* \otimes \mathbb{Q}$ is a polynomial ring generated by the $[\mathbb{CP}^{2i}]$.*

If we denote the forgetful morphism $MU_* \to MSO_*$ by u, it follows that $u_{\mathbb{Q}}\colon MU_* \otimes \mathbb{Q} \to MSO_* \otimes \mathbb{Q}$ is onto with kernel generated by the $\mathbb{CP}^{2i+1}$. There is also the following refinement of this observation.

Theorem 3.32. *The morphism $\overline{u}\colon : MU_* \to MSO_*/\mathrm{tors}$ is onto.*

Proof. See [Sto68, p. 180]. □

Proposition 3.33. *Let R_* be a torsionfree graded ring and $\phi\colon MU_* \to R_*$ be a ring morphism. Then ϕ factors over $u\colon MU_* \to MSO_*$ if and only if $\phi([\mathbb{CP}^{2i+1}]) = 0$. This happens if and only if the logarithm of the formal group law classified by ϕ is of the form $\sum \lambda_{2i+1} x^{2i+1}$.*

Proof. As R is torsionfree, ϕ factors over u if and only if it factors over $\overline{u}$. The kernel of $\overline{u}\colon MU_* \to MSO_*/\mathrm{tors}$ equals the intersection of $\ker(u_{\mathbb{Q}})$ with MU_* as $MSO_*/\mathrm{tors} \to MSO_* \otimes \mathbb{Q}$ is injective. Thus, $\ker(\overline{u}) \subset \ker(\phi)$ if and only if $\ker(u_{\mathbb{Q}}) \subset \ker(\phi_{\mathbb{Q}})$. The latter happens iff $\phi([\mathbb{CP}^{2i+1}]) = 0$.

The last observation follows directly from Mischenko's theorem 2.54. Indeed, if we write the logarithm of the formal group law classified by ϕ as $\log_F = \sum \lambda_i x^i$, we have $\lambda_{n+1} = \frac{\phi([\mathbb{CP}^n])}{n+1}$; thus $\lambda_{2i} = 0$ iff $\phi([\mathbb{CP}^{2i-1}]) = 0$. □

In particular, this implies if R_* is concentrated in degrees divisible by 4. If we equip $\mathbb{Z}[\frac{1}{2}][b_2, b_4, \Delta^{-1}]$ with the gradings $|b_i| = 2i$, we obtain thus Weierstrass elliptic genus $MSO_* \to \mathbb{Z}[\frac{1}{2}][b_2, b_4, \Delta^{-1}]$. The same works with $\mathbb{Z}[\frac{1}{6}][c_4, c_6, \Delta^{-1}]$.

3.8 The Jacobi quartic

So far, we have considered elliptic curves defined by *cubic* equations. For applications of elliptic genera it is more convenient to define elliptic curves via quartic equations instead.

Let's consider the affine curve C defined by

$$y^2 = 1 - 2ux^2 + vx^4$$

over some $\mathbb{Z}[\frac{1}{2}]$-algebra R. This is nonsingular iff $u^2 - v$ is invertible. We assume this and further that v is invertible (so that we really have a quartic curve).

By homogenization, we obtain a curve in $\mathbb{P}^2_R$ defined via the equation:

$$Y^2 Z^2 = Z^4 - 2uX^2 Z^2 + vX^4.$$

If this were a smooth curve (over a field), it would have genus 3 (by the general genus formula $g = \frac{(d-1)(d-2)}{2}$), but we will see that it is singular. There is only one solution with $Z = 0$, namely $[0 : 1 : 0]$. We claim that this is singular. Dehomogenizing with $Y = 1$, indeed gives $z^2 = z^4 - 2ux^2 z^2 + vx^4$ and $(0, 0)$ is obviously singular. (Actually, this singularity is quite bad: it is not ordinary. Else, the genus-degree formula for ordinary singularities would give the wrong answer!)

We would like to produce a proper smooth curve that is birational to this curve. Consider to that purpose an affine curve C' defined by the equation

$$y^2 = x^4 - 2ux^2 + v.$$

Under our assumptions on u and v this is also smooth. Consider the part C° of C with $x \neq 0$ (i.e. if R is an integral domain, we are taking out two copies of $\operatorname{Spec} R$, namely $(0, \pm 1)$). This maps via the transformation

$$(x, y) \mapsto \left(\frac{1}{x}, \frac{y}{x^2}\right)$$

isomorphically onto the open part of C' also defined by $x \neq 0$. We define $\overline{C} = C \coprod_{C^\circ} C'$, where we embed C° into C' and once using the transformation above. As the gluing of two smooth R-curves, this is clearly a smooth R-curve again. The curve $\overline{C}$ has a section as C has one, namely $e = (0, 1)$.

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We have morphisms

$$\begin{aligned} C &\to \mathbb{P}^1_R, \qquad (x, y) \mapsto [x : 1] \\ C' &\to \mathbb{P}^1_R, \qquad (x, y) \mapsto [1 : x] \end{aligned}$$

that glue to a morphism $\Phi\colon \overline{C} \to \mathbb{P}^1_R$. The morphism Φ is finite as it is finite when restricted to the preimage of the two standard $\mathbb{A}^1_R$. [Indeed: Let's do it for the preimage of the first $\mathbb{A}^1_R$, which is C. Then we have to check that $R[x, y]/(y^2 - 1 + 2ux^2 - vx^4)$ is a finite $R[x]$-module. It is clearly generated by 1 and y.] Thus, $\overline{C}$ is a proper over R.

To show that $\overline{C}$ is an elliptic curve, it remains to compute the genus if $R = k$ is an algebraically closed field. The map Φ has degree 2 (as generically for every x, there are two possibilities for y). Moreover, there are four points where the map ramifies, namely the points in C and C' with $y = 0$ (the quartic polynomial has 4 zeros because $u^2 \neq v$ and $x = 0$ is no zero). By the Riemann-Hurwitz formula, we obtain that the Euler characteristic of $\overline{C}$ is $2(-2) + 4$, where -2 is the Euler characteristic of $\mathbb{P}^1_k$ and 4 is total ramification number (all 4 points are just ordinary double points as the degree of the map is 2). Thus, $\overline{C}$ is a genus 1 curve.

Thus, we have shown that $\overline{C}$ is an elliptic curve over $\operatorname{Spec} R$, called a *Jacobi quartic*. The universal case is $R = \mathbb{Z}[\frac{1}{2}][u, v][(u^2 - v)^{-1} v^{-1}]$.

Remark 3.34. It is indeed true that the universal Jacobi quartic is isomorphic to the universal elliptic curve with $\Gamma_1(2)$-structure and an invariant differential. We give a sketch. To produce a map, we have to give a 2-torsion point and a nowhere vanishing differential on the universal Jacobi quartic $\overline{C}$.

We claim that the point $P = (0, -1)$ is a 2-torsion point. It is enough to check this in the algebraically closed case (e.g. by reducing the universal case $R = \mathbb{Z}[\frac{1}{2}][u, v][(u^2 - v)^{-1} v^{-1}]$ to the algebraic closure of its quotient field). Then the 2-torsion property is equivalent to $(P - e) + (P - e)$ is equivalent to $(e - e) = 0$ as divisors, i.e. that there is a meromorphic function on $\overline{C}$ with double zero at P and double pole at e. We claim that the meromorphic function $f = \frac{1}{y - 1 + \sqrt{v} x} + \frac{1}{2}$ on C does the job.

By the same argument as for Weierstrass curves $\frac{dx}{2y}$ is a nowhere vanishing differential on C. On C', the differential takes the form $-\frac{dx}{2y}$ and is thus also nowhere vanishing.

All in all, we obtain a map $\mathbb{Z}[\frac{1}{2}][b_2, b_4][\Delta^{-1}] \to \mathbb{Z}[\frac{1}{2}][u, v][(u^2 - v)^{-1} v^{-1}]$. One can compute that the quantity Δ equals $b_4^2(b_2^2 - 4b_4)$ up to unit multiple. Thus, the two rings are isomorphic (i.e. with $b_2 \mapsto u$ and $b_4 \mapsto \frac{1}{2}v$); it is only to check whether the map we have is an isomorphism. Computing where the generators go can be done over an algebraically closed field, e.g. $\mathbb{C}$ (as $\mathbb{Z}[\frac{1}{2}][u, v][(u^2 - v)^{-1} v^{-1}]$ embeds into $\mathbb{C}$). We will come back to this point later (perhaps).

3.9 The Ochanine elliptic genus

Consider a Jacobi quartic $C : y^2 = 1 - 2ux^2 + vx^4$ over a $\mathbb{Z}[\frac{1}{2}]$-algebra R (or rather its compactification $\overline{C}$). We formally complete this elliptic curve at its neutral element $(0, 1)$ to obtain a formal group.

We can express y as a power series in x:

$$y = \sqrt{1 - 2ux^2 + vx^4} = 1 - \sum_{n=1}^{\infty} \binom{2n}{n} \frac{(2ux^2 - vx^4)^n}{(2n-1)4^n} =: f(x)$$

Note that the only denominators are powers of 2. As before, there is a resulting morphism $\hat{\mathbb{A}}^1_R \to C$; on every R-algebra T, this sends a nilpotent element x to $[x : 1 : f(x)]$. One sees that this induces an isomorphism $\hat{\mathbb{A}}^1_R \to \hat{C}$. This defines a formal group law F over R. This formal group law is graded if $|u| = 4$ and $|v| = 8$.

The universal case is $R = \mathbb{Z}[\frac{1}{2}][u, v][(u^2 - v)^{-1}v^{-1}]$ with $|u| = 4$ and $|v| = 8$. As the grading is divisible by 4, we see that the associated genus $MU_* \to R$ factors over MSO_*.

Definition 3.35. The genus $MSO_* \to \mathbb{Z}[\frac{1}{2}][u, v][(u^2 - v)^{-1}v^{-1}]$ just defined is called the *Ochanine genus*.

This is the most important elliptic genus. When we compute its logarithm (using analytic methods), we will actually see that it takes values in $\mathbb{Z}[\frac{1}{2}][u, v]$. A genus $MSO_* \to R$ that factors over the Ochanine genus is often just called *elliptic genera*.

As a teaser, we already mention a remarkable theorem of Euler:

Theorem 3.36 (Euler, 1761)**.** *Let $R(x) = 1 - 2ux^2 + vx^4$. Then*

$$F(x, y) = \frac{x\sqrt{R(y)} + y\sqrt{R(x)}}{1 - vx^2y^2}.$$

This is one of the few formal group laws with nice closed form. But why was Euler interested in formal groups, almost 200 years before their definition by Bochner in 1945? The answer is: elliptic integrals.

3.10 Elliptic integrals and logarithms

The goal of this section is to compute the logarithms of the formal group laws attached to elliptic curves we have defined and, at the same time, to give a short overview of elliptic functions and elliptic integrals.

Recall that the logarithm is an isomorphism between the given formal group law and the additive formal group law (over a $\mathbb{Q}$-algebra). The ideal situation is when your formal group law comes from a group scheme G and you give an isomorphism of G to some other group scheme that defines manifestly the additive formal group law. This works very well for elliptic curves over *the complex numbers*.

We already sketched in Section 3.3 a proof that every elliptic curve (C, e) over the complex numbers is isomorphic to $\mathbb{C}/L$ for a lattice L in $\mathbb{C}$. This went as follows: Choose an invariant differential η on C. Then we obtain a map

$$\Phi\colon C \to \mathbb{C}/L, \qquad P \mapsto \int_e^P \eta.$$

We have to understand this isomorphism explicitly near e. More precisely: Choose a (formal) coordinate x on C near e. Then the logarithm of the formal group law of C with respect to x is the (Taylor) power series that expresses Φ near e in terms of x.

Example 3.37. Let C be given by $Y^2Z = X^3 + c_4XZ^2 + c_6Z^3$ in $\mathbb{P}^2_{\mathbb{C}}$ (with $c_4, c_6 \in \mathbb{C}$). As we are interested in a neighborhood around $e = [0:1:0]$ set $y = 1$ and we obtain $z = x^3 + c_4xz^2 + c_6z^3$. This has an invariant differential $\eta = \frac{dx}{1-2c_4z^2-3c_6z^2}$. Then the logarithm of the formal group law of C (with respect to x) is $\int_0^t \frac{dx}{1-2c_4z(x)^2-3c_6z(x)^2}$, where $z(x)$ is the power series describing z in terms of x that we computed in Section 3.6. Sadly, this is not very explicit.

Example 3.38. The situation is a bit better for the Jacobi quartic. Let $\overline{C}$ be a Jacobi quartic with affine part $y^2 = 1 - 2ux^2 + vx^4$ and neutral element $e = (0,1)$. Consider the invariant differential $\eta = \frac{dx}{y}$. Away from the zeros of y (e.g. near e), we can write $y = \sqrt{R(x)}$ with $R(x) = 1 - 2ux^2 + vx^4$ as we did before. Thus, we obtain that the logarithm of the formal group F of $\overline{C}$ is $\int_0^t \frac{dx}{\sqrt{R(x)}}$.

We obtain $F(x,y) = \exp_F(\int_0^x \frac{dt}{\sqrt{R(t)}} + \int_0^y \frac{dt}{\sqrt{R(t)}})$. We cannot quite recover Euler's theorem from last section before we know more explicitly what $\exp_F$ does.

Remark 3.39. Recall from calculus that one can calculate integrals of rational functions of x (using arctan and stuff like this) and also integrals of the form $\int \frac{dx}{\sqrt{1-x^2}}$ by substituting $x = \sin(t)$. In contrast, integrals involving squareroots of polynomials of degree bigger than 2 cannot be integrated using only elementary function (including sin, arctan etc.). Integrals over rational functions of x and $\sqrt{R(x)}$ with $R(x)$ a polynomial of degree 3 or 4 are called *elliptic integrals*. An example is $T(k) = \int_0^1 \frac{1-k^2x^2}{\sqrt{(1-x^2)(1-k^2x^2)}}dx$. It turns out that the arc length of an ellipse $x^2/a + y^2/b = 1$ is given by $4aT(\sqrt{1-(b/a)^2})$. Thus, the name.

Another popular example is the integral $\int \frac{dx}{\sqrt{1-x^4}}$ calculating the arc length of the so-called lemniscate. This was studied by Fagnago. Euler studied more generally integrals of the form $\int \frac{dx}{\sqrt{R(x)}}$ with $R(x) = 1 - 2ux^2 + vx^4$, thus exactly the form we are interested in!

Let ϕ be the Ochanine genus and F be the associated formal group law. Recall that Mischenko's theorem states that $\log_F'(x) = \sum_n [\mathbb{CP}^n]x^n$ and we just computed that $\log_F'(x) = \frac{dx}{\sqrt{1-2ux^2+vx^4}}$. This is quite explicit, we can just Taylor expand $(1-t)^{-1/2}$ and plug $t = 2ux^2 - vx^4$ in; we could also use Mathematica to do this for us. Anyhow, the first few values are:

$$\begin{aligned}
\phi(\mathbb{CP}^2) &= u \\
\phi(\mathbb{CP}^4) &= \frac{1}{2}(3u^2 - v) \\
\phi(\mathbb{CP}^6) &= \frac{1}{2}(5u^3 - 3uv) \\
\phi(\mathbb{CP}^8) &= \frac{1}{8}(35u^4 - 30u^2v + 3v^2).
\end{aligned}$$

Note two things: First, $\phi(\mathbb{CP}^{2i+1})$ vanishes for grading reasons as we have already seen above. Second, $\phi(\mathbb{CP}^{2i})$ is always in $\mathbb{Z}[\frac{1}{2}][u,v]$ (as we see from the Taylor expansion) and thus the Ochanine genus actually takes values in $\mathbb{Z}[\frac{1}{2}][u,v]$.

Example 3.40. If we set $u = v = 1$, we obtain the signature as then $\log_F' = \frac{1}{1-x^2}$ and thus $\phi(\mathbb{CP}^{2i}) = 1$ (this determines the genus uniquely).

Example 3.41. If we set $v = 0$ and $u = -\frac{1}{8}$, we obtain the $\hat{A}$-genus, which is geometrically extremely important. For example, it vanishes on a spin manifold if it has a non-trivial S^1-action or a metric of positive scalar curvature.

3.11 Elliptic functions and exponentials

To compute the exponential of Weierstrass and Jacobi formal group laws, we have to give an isomorphism from $\mathbb{C}/L$ to Weierstrass or Jacobi curve. This is done using the theory of elliptic functions, i.e. meromorphic functions on elliptic curves. So fix a lattice L in $\mathbb{C}$.

Given a meromorphic function f on $\mathbb{C}$, the section df of the sheaf Ω^1 on $\mathbb{C}$ is $f'dz$ with f' the classical meaning. The differential dz descends to a nowhere vanishing differential $\overline{dz}$ on $\mathbb{C}/L$. A meromorphic function f on $\mathbb{C}/L$ corresponds to a doubly-periodic function $\tilde{f}$. The differential df equals $f'\overline{dz}$, where f' corresponds to $\tilde{f}'$.

Proposition 3.42. *There is a meromorphic function $\wp$ on $\mathbb{C}/L$ satisfying*

$$(\wp')^2 = \wp^3 + c_4\wp + c_6$$

for $c_4, c_6 \in \mathbb{C}$ depending on L. If C is the curve in $\mathbb{P}^2_{\mathbb{C}}$ cut out by the equation $y^2 = x^3 + c_4x + c_6$, we obtain an isomorphism $\mathbb{C}/L \to C$ given by

$$z \mapsto [\wp(z), \wp'(z), 1] = [\frac{\wp(z)}{\wp'(z)} : 1 : \frac{1}{\wp'(z)}].$$

Proof. We follow the program laid out in Section 3.3 to find coordinates for elliptic curves. Choose Weierstrass coordinates x, y for $\mathbb{C}/L$ satisfying a Weierstrass equation

$$y^2 + a_1xy + a_3 = x^3 + a_2x^2 + a_4x + a_6.$$

Consider the meromorphic section dx of $\Omega^1_{(\mathbb{C}/L)/\mathbb{C}}$. The differential dz equals $\frac{udx}{y+\frac{1}{2}a_1x+\frac{1}{2}a_3}$ for some $u \in \mathbb{C}^\times$ as the space of invariant differentials is 1-dimensional. Thus, $dx = u^{-1}(y + \frac{1}{2}a_1x + \frac{1}{2}a_3)dz$, i.e. $x' = u^{-1}(y + \frac{1}{2}a_1x + \frac{1}{2}a_3)$. By a coordinate change, we can assume that $u = 1$. Then

$$(x')^2 = y^2 + a_1x + a_3y + \text{ polynomial in } x = x^3 + b_2x^2 + b_4x + b_6.$$

Setting $\wp = x + t$ for suitable t, produces then an equation of the form

$$(\wp')^2 = \wp^3 + c_4\wp + c_6.$$

The rest is as in Section 3.3. $\square$

Remark 3.43. The function $\wp$ is called *Weierstrass' p-function* and can be explicitly described by

$$\wp(z) = 4 \cdot \left(\frac{1}{z^2} + \sum_{\lambda \in L\setminus\{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) \right).$$

This implies that the logarithm of the Weierstrass curve associated with $\mathbb{C}/L$ can be computed as the Taylor expansion of $\frac{\wp(z)}{\wp'(z)}$ at $z = 0$.

Remark 3.44. We should really write c_4 and c_6 as $c_4(L)$ and $c_6(L)$. Thus, c_4 is a function from the set of all lattices in $\mathbb{C}$ to $\mathbb{C}$. It turns out that $c_4(uL) = u^{-4}c_4(L)$. Furthermore, it has a holomorphicity property: Restricting to lattices generated by 1 and $\tau \in \mathbb{H}$ (where $\mathbb{H}$ is the upper half plane), we obtain a function $c_4\colon \mathbb{H} \to \mathbb{C}$. This turns out to be holomorphic and bounded towards $\infty = \infty \cdot i$. These are exactly the defining properties of *modular forms*.

More precisely, a *modular form* of weight k is a function f from the set of all lattices in $\mathbb{C}$ to $\mathbb{C}$ such that

1. $f(uL) = u^{-k} f(L)$ for $u \in \mathbb{C}^\times$
2. $f\colon \mathbb{H} \to \mathbb{C}, \quad \tau \mapsto f([1,\tau])$ is holomorphic
3. f is bounded.

Actually, for every lattice $L \subset \mathbb{C}$, there is a $u \in \mathbb{C}^\times$ such that $uL = [1,\tau]$ for some $\tau \in \mathbb{H}$. The first transformation property becomes then: $f(\frac{a\tau+b}{c\tau+d}) = (cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and all z in the upper half plane.

We have seen before that for an elliptic curve $y^2 = x^3 + c_4x + c_6$ the points of exact order 2 are exactly those with $y = 0$. In terms of elliptic function, we see that $\wp'(z) = 0$ iff $z \in \frac{1}{2}L$, but not in L. If we fix a basis ω_1 and ω_2 of L, set $e_1 = \omega_1/2$, $e_2 = \omega_2/2$ and $e_3 = (\omega_1 + \omega_2)/2$. We obtain

$$\wp'(z)^2 = (\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

For the theory of Jacobi quartics there is equally both an analytic and an algebraic approach. For the analytic approach see [HBJ92, Section 2.2]. We will do an algebraic approach.

Proposition 3.45. *Let C/k be an elliptic curve over an algebraically closed field of characteristic not 2 with a chosen point P of exact order 2. Then there is a meromorphic function f on C satisfying*

$$(f')^2 = v - 2uf^2 + f^4$$

for some $u, v \in k$.

Proof. Let $D = e + P$. We have two involutions on $H^0(C; \mathcal{O}(nD))$. The involution S precomposes with $x \mapsto -x$ and the involution T precomposes with $x \mapsto x + P$. These involutions commute and so $H^0(C; \mathcal{O}(nD))$ decomposes into the eigenspaces $++$, $+-$, $-+$ and $--$. In particular, we can pick bases of eigenvectors. If we do this for $n = 1$, we obtain the constant function 1 ($++$) and further function f. As it has a simple pole at e, we see that it is odd. Assume that $Tf = f$. Then f factors as $C \to C/P \to \mathbb{P}^1$. The first map has degree 2 and f has also degree 2; but there is no degree 1-map $C/P \to \mathbb{P}^1$. Thus, f is $--$.

The space $H^0(C; \mathcal{O}(3D))$ is 6-dimensional and is spanned by $f, f', f^2, f'f$ and f^3; indeed, there can be no linear dependence because of the eigenspaces (as f^2 is non-constant). We have $f^4, (f')^2 \in H^0(C; \mathcal{O}(4D))$ and they are both of type $++$. But T cannot act as identity on $H^0(C; \mathcal{O}(4D))/H^0(C; \mathcal{O}(3D))$ since there is a function g with only pole at P and of order 4 and Tg has a pole of order 4 at e; their difference is clearly not in $H^0(C; \mathcal{O}(3D))$. Thus, there is a linear dependenc between $1, f^2, f^4$ and $(f')^2$. By scaling f, we can assume that it is of the form

$$(f')^2 = v - 2uf^2 + f^4.$$

□

This gives a map $\Phi\colon C \to C_2$, where $C_2 \subset \mathbb{P}^2_k$ is cut out by the projectivization of $y^2 = v - 2ux^2 + x^4$. The map is given away from ∞ as $[f, f', 1]$. If there are points Q, R with $f(Q) = f(R)$ and $f'(Q) = f'(R)$, we see that $\mathcal{O}(2D - R)$ and $\mathcal{O}(2D - R - Q)$ have the same global section, which cannot be. Thus, the map is injective away from the points e and P, where f has poles. Actually, one can check that it is a closed immersion away from e and P. By the universal property of normalization, the map Φ factors over the normalization of C_2, which corresponds to our construction $\overline{C}$. This way, we see that (at least over an algebraically closed field of characteristic not 2) every elliptic curve with a chosen point of order 2 can be written as a Jacobi quartic. We also can use f to compute the exponential of the formal group of a Jacobi quartic.

Note the f^2 factors over E/P and has there a pole of order 2 at the origin. Thus, it must be equal to $a + b\wp$. (Exercise: Determine a and b.)

Note also that f is essentially the same as the Jacobi sine function as it satisfies (essentially) the same differential equation. The Jacobi sine function is used to describe the motion of a frictionless pendulum.

end of lecture 12

3.12 Properties of the elliptic genus

Theorem 3.46. *A genus $\phi\colon MSO_* \to R$ (with R torsionfree) is elliptic if and only if we have $\phi(\mathbb{HP}^{2i+1}) = 0$ and $\phi(\mathbb{HP}^{2i}) = v^i$ for some $v \in R$.*

We will not prove this, but see [HBJ92, Section 1.7].

Next, we will talk about multiplicativity statements. Let $M \to B$ be a fiber bundle of closed smooth manifolds with fiber F closed smooth again. Assume that the fundamental group of B acts trivially on $H^*(F;\mathbb{Q})$. Let $\phi\colon MSO_* \to R$ (with R torsionfree) be a genus. Chern–Hirzebruch–Serre showed (by an application of the Serre spectral sequence) that the signature is multiplicative in such fiber bundles: $\mathrm{sgn}(M) = \mathrm{sgn}(F)\,\mathrm{sgn}(B)$. Borel and Hirzebruch showed that every $\phi\colon MSO_* \to R$ (with R torsionfree) that satisfies $\phi(M) = \phi(F)\phi(B)$ and $\phi(\mathbb{CP}^2) = 1$ agrees with the signature; if we drop the last condition, we have $\phi(M) = \phi(\mathbb{CP}^2)^{\dim(M)/4}\,\mathrm{sgn}(M)$, where the power is interpreted to be zero if the exponent is non-integral.

What if we ask multiplicativity only for certain classes of bundles, e.g. projectivizations of complex vector bundles with fiber $\mathbb{CP}^{2i-1}$? Recall that $\mathbb{CP}^{2i-1}$ is zero in MSO_*, so this would imply that the genus just vanishes on these bundles.

Theorem 3.47 (Ochanine)**.** *A genus $\phi\colon MSO_* \to R$ (with R torsionfree) is elliptic if and only if the genus ϕ vanishes on all projectivizations of even dimensional complex vector bundles.*

See [Och87] or again [HBJ92] for proofs.

There are more results of this form. A complex projective space $\mathbb{CP}^n$ has a spin structure iff n is odd. Indeed, they are always oriented so that $\mathbb{CP}^n$ has a spin structure iff $w_2(\mathbb{CP}^n) = 0$ (for w_2 the second Stiefel–Whitney class). The class $w_2(\mathbb{CP}^n)$ is the mod 2 reduction of the first Chern class $c_1(\mathbb{CP}^n)$. As already used earlier, the total Chern class of $\mathbb{CP}^n$ is $(1+x)^{n+1}$ and thus the first Chern class is $\binom{n+1}{n} = n+1$. Thus, one can ask whether the elliptic genus is actually multiplicative for bundles with spin fiber.

Theorem 3.48 (Bott-Taubes)**.** *Let ϕ be an elliptic genus. For every fiber bundle $M \to B$ with a compact, oriented spin manifold F as fiber and compact, connected Lie group as structure group, we have $\phi(M) = \phi(F)\phi(B)$.*

We will prove only a small part of these theorems. Namely, we will prove that if a genus is generally multiplicative it must be (essentially) the signature and if it vanishes on all projectivizations of even dimensional complex vector bundles it must be elliptic. The crucial tool are Milnor hypersurfaces, which are certain generators of the complex bordism ring. (Recall that the complex projective spaces were only rational generators.) This we will deal with in the next section.

3.13 Geometric cobordism, Milnor manifolds and multiplicativity

Recall that the (universal) formal group law for MU was the pullback of the complex orientation $x \in MU^2(\mathbb{CP}^\infty) \to MU^2(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$. We want to express this more geometrically.

Theorem 3.49. *For a closed smooth manifold X of dimension n, there is a natural isomorphism $MO^k(X)$ with bordism classes of maps $g\colon M \to X$ of closed smooth manifolds M of dimension $n-k$. The functoriality on the right hand side is given as follows: Let $f\colon Y \to X$ be a map; by a homotopy, we can assume that f is smooth and transverse to g. Then we can consider the pullback f^*M and the associated map $f^*M \to Y$.*

We will just give a natural transformation. The proof that this is an isomorphism is similar to the Pontryagin–Thom theorem.

Choose an embedding $i\colon M \hookrightarrow \mathbb{R}^m$ and let ν be the normal bundle of $(g,i)\colon M \hookrightarrow X \times \mathbb{R}^m$. By a Pontryagin–Thom collapse, we obtain a map

$$\Sigma^m X_+ \cong (X \times \mathbb{R}^m)_+ \to \mathrm{Th}(\nu) \to \mathrm{Th}(\xi^{univ}_{m+k}) = MO_{m+k}.$$

This gives the desired class in $MO^k(X)$.

To put a complex structure into the picture, we do the following: A BU-structure on $g\colon M \to X$ is an equivalence class factorizations $M \xrightarrow{i} X \times \mathbb{R}^m \xrightarrow{\mathrm{pr}_2} X$ of g, where i is an embedding with a chosen complex structure on its normal bundle. Two factorizations are equivalent if we can obtain them from each other by isotopy or enlargening m.

Theorem 3.50. *For a closed smooth manifold X of dimension n, there is a natural isomorphism $MU^k(X)$ with bordism classes of complex-oriented maps $g\colon M \to X$ of closed smooth manifolds M of dimension $n-k$.*

Again by a Pontryagin–Thom collapse, we obtain a map

$$\Sigma^m X \cong (X \times \mathbb{R}^m)^+ \to \mathrm{Th}(\nu_i) \to \mathrm{Th}(\xi^{\mathbb{C},univ}_{\frac{m+k}{2}}) = MU_{m+k}.$$

Example 3.51. The fundamental class in $MU^n(S^n) \cong \mathbb{Z}$ is given by the embedding of a point. More generally let $E \to X$ be an n-dimensional complex vector bundle over a closed manifold X. We claim that the Thom class in $MU^{2n}(\mathrm{Th}(E)) \cong MU^{2n}(E, E-X)$ is given by the embedding $X \to MU^{2n}(\mathrm{Th}(E))$. (The point that $\mathrm{Th}(E)$ might not be a manifold is not a problem because it is a manifold in a neighborhood of the image of X.) Indeed, restricting to each "fiber" S^{2n} gives the embedding of a point.

Question 3.52. *The multiplication $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ restricts to maps $\mathbb{CP}^i \times \mathbb{CP}^j \to \mathbb{CP}^N$. Can we describe the image of $x \in MU^2(\mathbb{CP}^N)$ in $MU^2(\mathbb{CP}^i \times \mathbb{CP}^j)$ geometrically?*

The canonical class $x \in MU^2(\mathbb{CP}^N)$ corresponds to the embedding of $\mathbb{CP}^{N-1}$ into $\mathbb{CP}^N$. Indeed, x is the restriction of the Thom class $t \in MU^2(\mathrm{Th}(\xi)) \cong MU^2(\mathbb{CP}^{N+1})$ of the canonical bundle ξ on $\mathbb{CP}^N$. The tranverse intersection of two $\mathbb{CP}^N$ in $\mathbb{CP}^{N+1}$ is a $\mathbb{CP}^{N-1}$.

Lemma 3.53. *The H-space structure on $\mathbb{CP}^\infty$ is unique. We can construct it as the colimit of the Segre embeddings*

$$\begin{aligned} \mathbb{CP}^i \times \mathbb{CP}^j &\xrightarrow{s} \mathbb{CP}^{(i+1)(j+1)-1} \\ ((z_0 : \cdots : z_i), (w_0 : \cdots : w_j)) &\mapsto (z_r w_s)_{r,s} \end{aligned}$$

Proof. Homotopy classes of maps $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ are in one-to-one correspondence with $H^2(\mathbb{CP}^\infty \times \mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Unitality implies that it must correspond to $(1,1)$.

That the colimit of the Segre embeddings defines an H-space structure follows because all injective linear maps $\mathbb{C}^\infty \to \mathbb{C}^\infty$ are homotopic. □

end of lecture 13

Definition 3.54. For $1 \leq i \leq j$, the Milnor manifold $H_{ij} \subset \mathbb{CP}^i \times \mathbb{CP}^j$ is the hypersurface cut out by the equation $x_0 y_0 + \cdots x_i y_i$. Set $H_{ji} = H_{ij}$.

Thus, H_{ij} is the preimage under the Segre embedding of the hyperplane $H \subset \mathbb{CP}^{(i+1)(j+1)-1}$ given by $x_{1,1} + \cdots + x_{i,i} = 0$. One can see that the Segre embedding is transverse to H. As all hypersurfaces in $\mathbb{CP}^N$ define the same class in $MU^2(\mathbb{CP}^N)$, we see that

$$[H_{ij} \hookrightarrow \mathbb{CP}^i \times \mathbb{CP}^j] = s^* x = F(x_1, x_2) = \sum_{r \leq i,\, s \leq j} a_{rs} x_1^r x_2^s \in MU^2(\mathbb{CP}^i \times \mathbb{CP}^j)$$

for x_1, x_2 the generators of $MU^2(\mathbb{CP}^i \times \mathbb{CP}^j)$. Note that the cup product in geometric $MU^*(X)$ is given by transverse intersection. Thus, $x_1^r x_2^s$ corresponds to the embedding $\mathbb{CP}^{i-r} \times \mathbb{CP}^{j-s} \hookrightarrow \mathbb{CP}^i \times \mathbb{CP}^j$. Note that we can also forget the embedding into $\mathbb{CP}^i \times \mathbb{CP}^j$ and get the following equality in MU_*:

$$[H_{ij}] = \sum_{r \leq i,\, s \leq j} a_{rs} [\mathbb{CP}^{i-r}][\mathbb{CP}^{j-s}].$$

Proposition 3.55. *The H_{ij} generate MU_*.*

Proof. From the above, we see that $H_{ij} \equiv a_{ij}$ modulo decomposables. Thus, we only have to see that the a_{ij} generate MU_*. Call the subring generated by them $i\colon R \subset MU_*$. Then $F^{univ} = i_* F$ for F the FGL $F(x,y) = \sum a_{ij} x^i y^j$ over R. But $F = j_* F^{univ}$ for some $j\colon MU_* \to R$. Furthermore, $(ij)_* F^{univ} = F^{univ}$ and thus $ij = \mathrm{id}$. Thus, i is surjective and hence $R = MU_*$. □

Corollary 3.56. *The H_{ij} generate $MSO_*/$-tors (and certainly $MSO_* \otimes \mathbb{Q}$).*

Proposition 3.57. *Set $H(x_1, x_2) = \sum_{i,j \geq 0} [H_{ij}] x_1^i x_2^j$. Let F be the universal FGL over MU_*. Then:*

$$H(x_1, x_2) = F(x_1, x_2) \log'_F(x_1) \log'_F(x_2).$$

Proof. We have

$$\begin{aligned}
H(x_1, x_2) &= \sum_{i,j} [H_{ij}] x_1^i x_2^j \\
&= \sum_{i,j \geq 0} \left(\sum_{r=0}^{i} \sum_{s=0}^{j} a_{rs} [\mathbb{CP}^{i-r}][\mathbb{CP}^{i-j}] \right) x_1^i x_2^j \\
&= \sum_{r,s \geq 0} a_{rs} x_1^r x_2^s \left(\sum_{i \geq r} [\mathbb{CP}^{i-r}] x_1^{i-r} \right) \left(\sum_{j \geq s} [\mathbb{CP}^{j-s}] x_2^{j-s} \right) \\
&= F(x_1, x_2) \log'_F(x_1) \log'_F(x_2).
\end{aligned}$$

□

Proposition 3.58. *We have* $[H_{ij}] = -\binom{i+j}{i} m_{i+j-1}$ *with* $m_n = \frac{[\mathbb{CP}^n]}{n+1}$ *modulo decomposables in* MU_* *and hence also in* MSO_*.

Proof. Let F be the universal formal group law. Recall that $\log_F = \sum_{i\geq 0} m_i x^{i+1}$ and set $\exp_F = \sum_{i\geq 0} b_i x^{i+1}$ to be its inverse, i.e. $\exp_F(\log_F(x)) = x$. Note that the inverse is *unique* for power series of the form $x + \cdots$ over *any* commutative ring R, in particular for $R = MU_*/$decomposables. Over R, the power series $x - m_1x^2 - m_2x^3 - \cdots$ is an inverse for $\log_F$. Thus, $b_i \equiv -m_i$ modulo decomposables.

Thus, modulo decomposables, we have:

$$\begin{aligned} F(x,y) &= \exp_F(\log_F(x) + \log_F(y)) \\ &= \sum_{i\geq 0} m_i x^{i+1} + \sum_{j\geq 0} m_j y^{j+1} - \sum_{n\geq 1} m_n (\sum_{i\geq 0} m_i x^{i+1} + \sum_{j\geq 0} m_j y^{j+1})^{n+1} \\ &= \sum_{i\geq 0} m_i x^{i+1} + \sum_{j\geq 0} m_j y^{j+1} - \sum_{n\geq 1} m_n \binom{n+1}{i+1} x^{i+1} y^{2n+1-(i+1)}. \end{aligned}$$

As $[H_{ij}]$ is modulo decomposable exactly the coefficient in front of x^iy^j in F, the result follows. □

Corollary 3.59. *Rationally,* $\mathbb{CP}^2$ *together with the* $H_{2,(2j+1)}$ *with* $j \geq 1$ *generate* MSO_*.

Another set of rational generators is $[\mathbb{CP}^2]$, $[H_{2,3}]$ *and* $[H_{3,2j}]$ *for* $j \geq 2$.

The crucial point for multiplicativity is the following: The map $H_{ij} \to \mathbb{CP}^i \times \mathbb{CP}^j \xrightarrow{\mathrm{pr}_1} \mathbb{CP}^i$ is a fiber bundle whose fibers are hyperplanes in $\mathbb{CP}^j$ and hence isomorphic to $\mathbb{CP}^{j-1}$.

Theorem 3.60. *Let* $\phi\colon MSO_* \to R$ *(with* R *torsionfree) be a multiplicative genus (for fiber bundles with fiber a complex projective space) with* $\phi(\mathbb{CP}^2) = 1$. *Then* ϕ *is the signature.*

Proof. We prove it by induction on the dimension. The base case is dimension ≤ 4, where ϕ visibly agrees with the signature as 1 and $[\mathbb{CP}^2]$ generate MSO_* in these degrees. Assume that $\phi(M) = \mathrm{sgn}(M)$ has been proven for all manifold of dimension $\leq 4j$. We know that

$$\phi(H_{2,(2j+1)}) = \phi(\mathbb{CP}^2)\phi(\mathbb{CP}^{2j}) = \phi(\mathbb{CP}^{2j}) = \mathrm{sgn}(\mathbb{CP}^{2j}) = \mathrm{sgn}(H_{2,(2j+1)}).$$

As the $H_{2,(2j+1)}$ together with $\mathbb{CP}^2$ generate MSO_* rationally, the result follows. □

Now we come to the elliptic genus. Let ϕ be a genus that vanishes on projectivizations of even dimensional complex vector bundles, e.g. on the $H_{3,2j}$. Then rationally, it has to factor over $MSO_*/(H_{3,2j}) \otimes \mathbb{Q} \cong \mathbb{Q}[[\mathbb{CP}^2], [H_{2,3}]] = R$. Ochanine shows in [Och87] that the quotient map $MSO_* \otimes \mathbb{Q} \to R$ agrees with the Ochanine genus (with $[\mathbb{CP}^2]$ corresponding to u and $[H_{2,3}]$ corresponding to v). That provides one direction of Theorem 3.47. Let's sketch a proof.

Set h_{ij} to be the image of $[H_{ij}]$ in R and $h(x,y)$, $f(x,y)$ and $g(x)$ the images of H, F and $\log_F$ in R. (At the beginning of the proof, it could be actually under any genus $\phi\colon MSO_* \to R$.) We are interested in the coefficients $h_{3,2j} = h_{2j,3}$ and will thus consider everything mod y^4.

Recall from [Rav86, Appendix] that we have the formula $g'(x)\frac{\partial f(x,0)}{\partial y} = 1$. Thus, it seems reasonable to write f as a Taylor series like follows:

$$f(x,y) = x + \frac{\partial f(x,0)}{\partial y} \cdot y + \frac{1}{2}\frac{\partial^2 f(x,0)}{\partial y^2} \cdot y^2 + \frac{1}{6}\frac{\partial^3 f(x,0)}{\partial y^3} \cdot y^3 \mod y^4$$

From Proposition 3.57, we obtain

$$h(x,y) = g'(y)\left(g'(x)x + y + \frac{1}{2}g'(x)\frac{\partial^2 f(x,0)}{\partial y^2}y^2\right) + \frac{1}{6}g'(x)\frac{\partial^3 f(x,0)}{\partial y^3}y^3 \mod y^4.$$

We are interested in the coefficients of $x^{2i}y^3$. Set $r(x) = \sum_{i=1}^{\infty} h_{3,2i}x^{2i} = h_{3,2}x^2$. Then we have $r(x) = \frac{1}{6}g'(x)\frac{\partial^3 f(x,0)}{\partial y^3}$. Indeed: $g'(y)$ has only even powers of y.

Set $b(x) = \frac{1}{g'(x)} = \frac{\partial f(x,0)}{\partial y}$. A computation shows that $\frac{\partial^3 f(x,y)}{\partial y^3} = b(x)(b''(x)b(x) + b'(x)^2 - b''(0))$. Thus,

$$6r(x) = b''(x)b(x) + b'(x)^2 - b''(0) = \frac{1}{2}(b(x)^2)'' - b''(0).$$

As $r(x)$ is a polynomial of degree 2, we see that $b(x)^2$ is a polynomial of degree ≤ 4. As does $g'(x)$, the power series $b(x)$ can only have even powers of x. Set $b(x)^2 = 1 - 2ux^2 + vx^4$. Then $g'(x) = \frac{1}{\sqrt{1-2ux^2+vx^4}}$. We have already seen above that the image of $[\mathbb{CP}^2]$ is u. Furthermore, we have

$$r(x) = \frac{1}{12}(b(x)^2)'' - \frac{1}{6}b''(0) = \frac{1}{12}(12vx^2 - 4u) - \frac{1}{6}b''(0)$$

and hence $h_{2,3} = v$.

end of lecture 14

3.14 Exercises

Exercise 3.61. Show that $X = \operatorname{Spec}\mathbb{R}[x,y]/(x^2+y^2-1)$ has the structure of a group scheme that is not isomorphic to $\mathbb{G}_{m,\mathbb{R}}$. (Also describe the functor that X represents on $\mathbb{R}$-algebras.) Show that in contrast the base change $X_{\mathbb{C}} = X \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$ is isomorphic to $\mathbb{G}_{m,\mathbb{C}}$ as group schemes.

Exercise 3.62. Formal group laws over R are in one-to-one correspondence with lifts of $\hat{\mathbb{A}}^1_R$ to the category of abelian groups. Moreover, homomorphisms between formal group laws are in one-to-one correspondence with natural transformations of the associated functors $\mathrm{Alg}_R \to \mathrm{AbGroups}$.

Exercise 3.63. Consider the ring $\mathbb{Z}$ with ideal (p) and the p-adics equipped also with ideal (p). Show that $\operatorname{Spf}\mathbb{Z} \cong \operatorname{Spf}\mathbb{Z}_p$ and describe this functor. Show that this is not (representable by) a scheme.

Also show that $\operatorname{Spf} R[x] \cong \operatorname{Spf} R[\![x]\!]$, where both $R[x]$ and $R[\![x]\!]$ are equipped with the ideal (x). Show that this is likewise not (representable by) a scheme.

Exercise 3.64. Let R be a ring equipped with an ideal I. Show that $\operatorname{Spf} R$ is a Zariski sheaf on Alg.

Exercise 3.65. Consider curves defined by the following three equations.

1. $y^2 = x^3 + 1$
2. $y^2 = x^3 + 2$
3. $y^2 + y = x^3 + \frac{3}{4}$

Over which subrings of $\mathbb{Q}$ are these elliptic curves? Over which rings are they isomorphic?

Exercise 3.66. Fill in the details in Proposition 3.24 (about gradings and $\mathbb{G}_m$-actions). Which gradings on the rings in Proposition 3.23 do the natural $\mathbb{G}_m$-actions induce?

Exercise 3.67. Let C/R be an elliptic curve embedded into $\mathbb{P}^2_R$ in Weierstrass form. Assume that $\frac{1}{3} \in R$. (You are allowed to assume that R is an algebraically closed field for the proofs in doubt.)

(a) Show that a point P on C is a point of exact order 3 iff the tangent line at P has a triple intersection point with C at P.

(b) Let P be a point of exact order 3 on C (i.e. a $\Gamma_1(3)$-structure). Show that there is a unique coordinate change of the form $x' = x + r$ and $y' = y + sx + t$ sending P to $(0,0)$ and its tangent line to the x-axis.

(c) Show that the functor from $\mathbb{Z}[\frac{1}{3}]$-algebras to sets, sending an $\mathbb{Z}[\frac{1}{3}]$-algebra R to the set of isomorphism classes of elliptic curves over R with a chosen $\Gamma_1(3)$-structure and a chosen invariant differential is representable by $\mathbb{Z}[\frac{1}{3}][a_1, a_3][\Delta^{-1}]$ with $\Delta = a_3^3(a_1^3 - 27a_3)$.

Exercise 3.68. Add some details to the discussion of Jacobi quartics, e.g. check that it is indeed smooth iff $u^2 - v$ is invertible.

4 Stacks and elliptic cohomology

4.1 Landweber's exact functor theorem

We have seen that complex orientations (i.e. maps $f\colon MU \to E$ of ring spectra) give rise to graded formal group laws. The map $f_*\colon MU_* \to E_*$ classifies this graded formal group law.

But what if we have an arbitrary graded formal group law F over a graded ring E_* (corresponding to $MU_* \to E_*$)? Is there a spectrum E with $\pi_*E = E_*$ with a complex orientation $MU \to E$ that gives rise to exactly this formal group? Consider the functor

$$h_F\colon X \mapsto MU_*(X) \otimes_{MU_*} E_*$$

from spectra to graded abelian groups. If this is a homology theory, we can represent it by a spectrum E by homological Brown representability:

Theorem 4.1 (Brown representability)**.** *The functor from the stable homotopy category to homology theories (on spaces or, equivalently, on spectra) is essentially surjective and full.*

By this theorem, the natural transformation $MU_*(-) \to h_F$ is realized by a map of spectra $MU \to E$. It is a priori not clear though that h_F is represented by a ring spectrum (the non-faithfulness of the functor above doesn't guarantee the unitality and associativity diagrams to commute) nor that the transformation $MU \to E$ is a map of ring spectra if E is a ring spectrum. Maps in the stable homotopy category that induce the zero map on homology theories are called *(strong) phantoms*; under rather mild conditions one can actually exclude the existence of these phantoms and live a happy phantom-free life. If this works, then we have realized our formal group law by a complex oriented ring spectrum.

Question 4.2. *When is h_F a homology theory?*

Equivalently, we can ask when does h_F send cofiber sequences to long exact sequences (as it is automatically homotopy invariant and additive). One obvious sufficient condition is that E_* is flat as a MU_*-module. But this is very restrictive, for example E_* cannot be finitely generated over $\mathbb{Z}$! The crucial insight by Landweber was that a much weaker condition suffices. This needs a bit of preparation, so we will state it a little later. This will apply to certain formal group laws coming from elliptic curves.

Later we will reinterpret Landweber's theorem in stack language, where it will become more transparent.

As a sequence is exact iff it is so at every prime p, we will work p-locally now for a fixed prime p. So let R be a $\mathbb{Z}_{(p)}$ and F be a FGL over R. Recall the definition of $[p]_F$ from Definition 2.49. We define $v_i \in R$ to be the coefficient in front of x^{p^i} in $[p]_F$.

Remark 4.3. For the experts: This is neither quite the same as Araki's nor as Hazewinkel's definition of the v_i. But they all agree mod $(p, v_1, \ldots, v_{i-1})$. Indeed: Let v_i^A be the Araki generators. Then $[p]_F(x) = \sum_{j\geq 0}^F v_j^A x^{p^j}$ (see [Rav86, A2.2.4]). You see that mod $(p, v_1, \ldots, v_{i-1})$ this equals $\sum_{j\geq i}^F v_j^A x^{p^j}$, whose first term is just $v_j^A x^{p^j}$. This shows that $v_i^A \equiv v_i \mod (p, v_1, \ldots, v_{i-1})$. And the Hazewinkel generators agree with the Araki generators mod p anyhow.

Theorem 4.4 (Landweber exact functor theorem)**.** *Let F be a graded formal group law over a graded ring E_* and let h_F be as above. Then h_F is a homology theory if for every p the sequence $p, v_1, v_2, \ldots$ is regular on E_*, i.e. that $v_i\cdot\colon E_*/(p, v_1, \ldots, v_{i-1}) \to E_*/(p, v_1, \ldots, v_{i-1})$ is injective.*

We will call (graded) formal group laws satisfying the assumptions of the theorem *Landweber exact*. If E is a complex oriented ring spectrum whose formal group law is Landweber exact, we call E *Landweber exact* as well. We will later see that these notions just depend on the underlying formal group.

Corollary 4.5. *Let E be a Landweber exact complex oriented ring spectrum. The map $MU_*(-) \to E_*(-)$ induces an isomorphism*

$$MU_*(X) \otimes_{MU_*} E_* \to E_*(X)$$

for all spectra X.

Proof. Both sides are homology theories (by Landweber's exact functor theorem). Thus, the transformation is an isomorphism iff it is one for X the sphere spectrum (or the point if our source would be spaces). In this case, it is clear. $\square$

Example 4.6. We claim that complex K-theory is Landweber exact. Recall that its formal group law is $x + y + uxy$ for $\pi_* KU = \mathbb{Z}[u^{\pm 1}]$. Recall further that

$$[p]_F(x) = \frac{(1+ux)^p - 1}{u}.$$

Thus, $v_0 = p$, $v_1 = u^{p-1}$ and $v_i = 0$ for $i \geq 2$. Thus, KU is Landweber exact. We recover the classic Conner–Floyd theorem:

$$KU_*(X) \cong MU_*(X) \otimes_{MU_*} KU_*.$$

4.2 Heights of formal groups

Definition 4.7. Let R be an $\mathbb{F}_p$-algebra. We say that a formal group law over R has *height* $\geq n$ if $v_1 = \cdots = v_{n-1} = 0$ and we say that it has *(exactly) height* n if additionally v_n is invertible.

Example 4.8. Every formal group law over an $\mathbb{F}_p$-algebra has height ≥ 1 as the coefficient of x in $[n]_F(x)$ is n. We have seen above that the multiplicative formal group law has height 1. The additive formal group law has height ∞ as $[p]_F = 0$.

Remark 4.9. One can show that $[p]_F$ is of the form $v_n x^{p^n} + \text{ higher terms }$ if F is of height n.

The height of a formal group law is unchanged under isomorphism. Indeed, the observation above implies that $k[\![x]\!]/[p]_F(x)$ has rank p^n over if F is a formal group law over a field k. It is easy to see that this quantity is unchanged under isomorphisms.

We want to show that the height of the formal group law of an elliptic curve is either 1 or 2. We need some preparation though. Recall that the degree of a finite map $f\colon \to Y$ over a k-valued point $y\colon \operatorname{Spec} k \to Y$ (for Y a field) is the rank of A, where $\operatorname{Spec} A = X \times_Y \operatorname{Spec} k$; this rank is finite as f is finite. If Y is connected and noetherian and f is *flat*, then the degree does not depend on the choice of y.

In general, it is easier to check that a map has finite fibers (i.e. the fiber over $y\colon \operatorname{Spec} k \to Y$ has only finitely many points for every y with k algebraically closed) than that it is finite. But if X and Y are both proper over a scheme S and f is over S, then f is automatically proper as well and every proper map with finite fibers (with locally noetherian target) is finite [Sta17, Tag 02OG]. [In general, quasi-finite morphisms do not need to be finite, even if they are surjective and flat. For example, $\operatorname{Spec} k[t^{\pm 1}] \coprod \operatorname{Spec} k[t, (t-1)^{-1}] \to \operatorname{Spec} k[t]$ is quasi-finite, but not finite.]

Lemma 4.10. *Let k a field. For a k-variety X, denote the cotangent space $H^0(X; \Omega^1_{X/k})$ by $\Omega^1(X)$. If C is an elliptic curve over k, then the multiplication by n-map $[n]$ induces multiplication by n on $\Omega^1(C)$.*

Proof. We have natural isomorphisms $\Omega^1(X \times_k Y) \cong \Omega^1(X) \times \Omega^1(Y)$. Thus, the group structure of C induces a group structure $\tilde{+}$ via linear maps on $\Omega^1(C)$ with 0 as neutral element (as $\Omega^1(\operatorname{Spec} k) = 0$). Two commuting monoid operations with the same neutral element agree:

$$x + y = (x \tilde{+} 0) + (0 \tilde{+} y) = (x+0) \tilde{+} (0+y) = x \tilde{+} y.$$

As id_C induces the identity on $\Omega^1(C)$, it follows that $[n]$ induces multiplication by n. ☐

Proposition 4.11. *Let C be an elliptic curve over a base scheme S. Then the multiplication by n-map $[n]$ is finite and flat of degree n^2.*

Proof. We first do the case $S = \operatorname{Spec} \mathbb{C}$. By writing $C = \mathbb{C}/L$, we see directly that every point has exactly n^2 preimages under $[n]$. Furthermore, $[n]$ is étale (i.e. topologically a covering map), thus there is no ramification. Thus, the degree of $[n]$ is n^2 in this case.

Let S be general. The statement is local on S, so we can assume that C has a Weierstrass form. Thus, we can reduce to the universal case for Weierstrass curves

$$S = \operatorname{Spec} \mathbb{Z}[a_1, \ldots, a_6, \Delta^{-1}].$$

The scheme C is regular as it is smooth over the regular scheme S. Thus, $[n]$ is automatically flat if it is finite (see e.g. [Gro65, Prop 6.1.5]; this is related to Hironaka's miracle flatness). As discussed above, thus we only need to show that $[n]$ has finite fibers and then it is automatically finite flat. Thus, we can reduce to the case that $S = \operatorname{Spec} k$ for k algebraically closed.

In this case, every non-constant self map of C is automatically finite. Thus, we need to show that $[n]$ is non-constant. The map $[n]$ induces multiplication by n on (the global sections of) $\Omega^1(C)$ by the last lemma. Thus, it is non-constant if $\operatorname{char} k$ is prime to n. Actually, it is even étale as it induces an isomorphism on Ω^1 and is flat and thus it is a disjoint union of $\operatorname{Spec} k$. Thus, we already see that $[n]$ is flat if

$S = \operatorname{Spec} \mathbb{Z}[\frac{1}{n}][a_1, \ldots, a_6, \Delta^{-1}]$ and it is automatically of degree n^2 as it has so over every $\mathbb{C}$-valued point.

If $\operatorname{char} k$ divides n, we choose an $m \geq 2$ prime to n and $\operatorname{char} k$. Choose l such that $ln \equiv 1 \mod m$. Then $[n]^l$ acts as the identity on the kernel of $C[m]$ of $[m]$; the scheme $C[m]$ has m^2 points (by what we have seen in the last paragraph) and clearly a constant map cannot act as the identity on two distinct points. Thus, $[n]$ is also non-constant here and we deduce that $[n]$ is finite and flat in general (and hence automatically of degree n^2). □

This nice proof is taken from [KM85]; note that we used the characteristic zero result over $\mathbb{C}$ to deduce the degree of the map $[n]$ also in characteristic p by using non-field bases. Note also that this result directly implies that there are at most n^2 points of order n in any elliptic curve C over any (algebraically closed) field k. If $n = p$ and $\operatorname{char} k = p$, then there are less that p^2-points as we will see below.

Proposition 4.12. *Let k be a field of characteristic p and C an elliptic curve over it. Then the height of the formal group $\hat{C}$ associated with C is 1 or 2.*

Proof. We claim that the formal completion $\widehat{C[p]}$ agrees with the p-torsion in $\hat{C}$. Indeed both represent the functor

$$T \mapsto \{f \in C(T) : p \cdot f = e,\ f^*I \text{ nilpotent}\}$$

on k-schemes, where I is the ideal sheaf on C cutting out the neutral element e.

By the last proposition, we know that $C[p] = \operatorname{Spec} R$ for a finite k-algebra R of rank p^2. The (formal) group scheme $\widehat{C[p]}$ is represented by $\lim R/I^j$. As R is Artinian, we have $I^n = I^{n+1} = \cdots$ for some $n > 0$ and thus $\lim R/I^j \cong R/I^n$. Note that $\operatorname{Spec} R/I^n = \operatorname{Spf} R/I^n$ as I is already nilpotent.

If one chooses a coordinate on $\widehat{C}$, one obtains a formal group law F over k. As $k[\![x]\!]/[p]_F(x)$ represents the p-torsion $\hat{C}[p]$, we obtain $k[\![x]\!]/[p]_F(x) \cong R/I^n$. Thus,

$$p^{\text{height}}(F) = \dim_k k[\![x]\!]/[p]_F(x) = \dim_k R/I^n \leq \dim_k R = p^2.$$

□

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Use the notation of the last proof: As $C[p](k)$ is a p-torsion, it must have p^m elements for some m and $m \leq 2$ (by Proposition 4.11). The case $m = 2$ cannot occur. Indeed, if $C[p](k)$ has p^2 elements, these elements define a map $R \to \prod_{p^2} k$. This must be surjective (for example, we can decompose the Artin ring R is a product $\prod_i R_i$ of local Artin algebras; i runs over the set of points which are p^2 many and the map $R \to \prod_{p^2} k$ is just the product of the projections onto the residue fields) and thus an isomorphism. This would imply that $R/I^n = k$, but $\dim_k R/I^n \geq p$ as we saw in the last proof.

If the height of $\hat{C} = 2$, then $\dim_k R/I^n = \dim_k R$; as the scheme $\operatorname{Spec} R/I^n$ has only one point (its underlying reduced scheme is $\operatorname{Spec} R/I = \operatorname{Spec} k$), we see that $C[p] = \operatorname{Spec} R \cong \operatorname{Spec} R/I^n$ has only one point as well. Conversely, if $C[p](k)$ has only one point, then R has only one maximal ideal and this must coincide with I as $R/I \cong k$; furthermore, I must also be the only prime ideal (as every commutative finite-dimensional integral domain over k is a field). Every element in the intersection of all prime ideals is nilpotent and thus I is nilpotent; we see that $R = R/I^n$. Thus $C[p](k)$ has only one point iff $\hat{C}$ has height 2 and in this case, C is called *supersingular*. It follows that $C[p](k)$ has exactly p points iff $\hat{C}$ has height 1 and in this case, C is called *ordinary*.

Note that a supersingular elliptic curve is not a singular curve! The wording should rather indicate that supersingular elliptic curve are rarer than ordinary ones.

There is a useful criterion to determine whether an elliptic curve is ordinary:

Proposition 4.13. *Let C/k be an elliptic curve over a field of characteristic $p > 2$, given by a Weierstrass equation $y^2 = f(x)$. Then C is supersingular iff the coefficient of x^{p-1} in $f(x)^{(p-1)/2}$ is zero.*

For a proof see [Sil09, Theorem 4.1] and also [Har77, Proposition 4.21] for a generalization.

Example 4.14. Consider the elliptic curve $C : y^2 = x^3 + 1$ over any field k of characteristic bigger than 3 (else it is not an elliptic curve). The coefficient in front of x^{p-1} in $(x^3+1)^{(p-1)/2}$ is 0 if $p-1$ is not divisible by 3 and $\binom{(p-1)/2}{(p-1)/3}$ else; the latter is clearly non-zero in k. Thus C is supersingular iff $p \equiv 1 \mod 3$.

Given equation of the form $y^2 = f(x)$ with non-zero discriminant, the p for which this defines a supersingular elliptic curve over $\mathbb{F}_p$ are for most equation much rarer than the ones where it defines ordinary elliptic curves; note though that there are by a result of Elkies infinitely many primes where it is supersingular.

Proposition 4.15. *For every prime p, there is an ordinary elliptic curve over $\overline{\mathbb{F}}_p$.*

Proof. By [Sil09, Theorem 4.1] there are only finitely many supersingular elliptic curves over $\overline{\mathbb{F}}_p$ (less than $\frac{p}{12} + 2$). But there are infinitely many elliptic curves over the same field. (This can be seen e.g. by the j-invariant; this defines a bijection between isomorphism classes of elliptic curves over an algebraically closed field k and the elements of k. Another argument (at least for $p \geq 5$) looks at elliptic curves of the form $y^2 = x^3 + c_4 x + c_6$.) □

Theorem 4.16. *Let C be an elliptic curve over a ring R. Then the corresponding formal group is Landweber exact iff*

1. *R is torsionfree*
2. *v_1 is a non-zero divisor on R/p for every p.*

If R/p is an integral domain, then it is sufficient there being a morphism $f: R \to k$ for k a field of characteristic p such that the base change f^*C is an ordinary elliptic curve over k.

Proof. We can assume that R is p-local. The first part of the theorem says that $p, v_1, v_2, \ldots$ is a regular sequence on R iff p, v_1 is. One direction is clear. We claim that v_2 is invertible on $R/(p, v_1)$ so that $R/(p, v_1, v_2) = 0$. Indeed, suppose otherwise. Then there is a morphism $f: R/(p, v_1) \to k$ to a field such that $f(v_2)$ is zero. Thus, the height of the formal group law of f^*C on k is bigger than 2 in contradiction to the last proposition.

For the second part: If R/p is an integral domain, then we just have to show that v_1 is nonzero, which is certainly the case if it is nonzero after mapping to a field. □

Definition 4.17. Let C be an elliptic curve over a ring R. Let R be graded such that the resulting formal group law of C is graded as well and assume that this group law is Landweber exact. The resulting homology theory is called an *elliptic homology theory.*

The next example is the original example of an elliptic homology theory. We will see more later.

Example 4.18. Consider $R = \mathbb{Z}[\frac{1}{2}][b_2, b_4, \Delta^{-1}]$ and the elliptic curve $y^2 = x^3 + b_2x^2 + b_4x$. We want to check that the resulting formal group law is Landweber exact. Clearly, R and R/p are integral domains. We need to check that for every p, there is a morphism $f\colon R \to k$ such that $y^2 = x^3 + f(b_2)x^2 + f(b_4)x$ is ordinary. If C is an elliptic curve over an algebraically closed field k of characteristic not 2, then C has a 2-torsion point and hence is pushed forward from R (as we identified R as carrying the universal elliptic curve with a two-torsion point and an invariant differential over $\mathbb{Z}[\frac{1}{2}]$-algebras; see Proposition 3.25). Thus, we can use Proposition 4.15 to conclude.

There is an alternative proof, which does not use the Proposition 4.15. Consider the elliptic genus $MU_* \to \mathbb{Z}[\frac{1}{2}][u, v]$. We claim that v_1 is non-zero for the corresponding formal group law. This can be shown after postcomposing with $\mathbb{Z}[\frac{1}{2}][u, v] \to \mathbb{Z}[\frac{1}{2}]$ sending u and v to 1. Then the logarithm of the pushed forward formal group law becomes $\log = \int \frac{dx}{(1-x^2)}$. We claim that $v_1 = 1$ for this formal group law at any odd prime p. Indeed: The first denominator in log divisible by p is in front of x^p. Considering its inverse exp, a simple calculation shows that the coefficients in front of $x, \dots, x^{p-1}$ do not have a denominator divisible by p and the one in front of x^p is of the form $-\frac{1}{p}$ plus an element of $\mathbb{Z}_{(p)}$. Thus, the p-series $\exp(p\log(x))$ is congruent to $\exp(px + x^p) \equiv x^p$ modulo p and higher terms than x^p. Thus, $v_1 \equiv 1 \mod p$. (See also [Fra92], Section 2, for another version of this proof.)

This argument does also show Proposition 4.15, at least for $p \geq 3$. Indeed, as v_1 is nonzero in $\overline{\mathbb{F}}_p[u, v, \Delta^{-1}]$ (for $p \geq 3$), we see that there are points (u_0, v_0) in $\overline{\mathbb{F}}_p[u, v, \Delta^{-1}] \subset \mathbb{A}^2_{\overline{\mathbb{F}}_p}$, where v_1 does not vanish. There corresponding Jacobi elliptic curve defined by

$$y^2 = 1 - 2u_0x^2 + v_0x^4$$

is ordinary.

4.3 Stacks on topological spaces

This section is an introduction to stacks, a categorical concept similar to sheaves. To motivate this concept, we will first look at the topological situation of vector bundles.

Given a space X, we can look at the presheaf that sends every open set U to

$$\mathrm{Vect}_n(U) = \{\text{isomorphism classes of } n\text{-dimensional vector bundles on } U\}.$$

The restriction maps are just restriction (aka pullback) of vector bundles. This is in general not a sheaf. Indeed, take $X = S^k$ and cover it by two contractible subset U_1 and U_2. If Vect_n were a sheaf, then $\mathrm{Vect}_n(S^k)$ would be the equalizer of the two maps $\mathrm{Vect}_n(U_1) \times \mathrm{Vect}_n(U_2)$ to $\mathrm{Vect}_n(U_1 \cap U_2)$. But $\mathrm{Vect}_n(U_1) \times \mathrm{Vect}_n(U_2)$ consists just of one element – in contrast, $\mathrm{Vect}_n(S^k)$ can contain many elements.

The point is, of course, that we allowed to glue vector bundles in non-trivial manners. That is, we must not only remember isomorphism classes of vector bundles, but the groupoids of all vector bundles and their isomorphisms. We define a presheaf of groupoids $\mathcal{V}\mathrm{ect}_n$ on X by sending U to

$$\mathcal{V}\mathrm{ect}_n(U) = \{\text{groupoid of } n\text{-dimensional vector bundles on } U \text{ and isomorphism between them}\}.$$

This has two useful "sheafy" properties:

1. Let $E, F \in \mathcal{V}\mathrm{ect}(U)$ be objects. Then the presheaf $\mathrm{Isom}(E, F)$ defined by

$$\mathrm{Isom}(E, F)(V) = \{\text{isomorphisms between } E|_V \text{ and } F|_V\}$$

for $V \subset U$ open is a sheaf.

2. Let $\{U_i \subset X\}$ be an open cover, let E_i be vector bundles on U_i and

$$f_{ij}\colon (E_j)|_{U_i \cap U_j} \to (E_i)|_{U_i \cap U_j}$$

be isomorphisms such that $f_{ij}f_{jk} = f_{ik}$ on $U_i \cap U_j \cap U_k$. Then there is a vector bundle E on X with isomorphisms $\phi_i\colon E|_{U_i} \to E_i$ such that $f_{ij}\phi_j = \phi_i$ on $U_i \cap U_j$.

Definition 4.19. Let X be a topological space. A *stack* on X is a presheaf $\mathcal{F}$ of groupoids such that

1. for $E, F \in \mathcal{F}(U)$ the presheaf $\mathrm{Isom}(E, F)$ on U is a sheaf, and
2. for any open cover $\{U_i \subset X\}$, objects $E_i \in \mathcal{F}(U_i)$ and isomorphisms

$$f_{ij}\colon (E_j)|_{U_i \cap U_j} \to (E_i)|_{U_i \cap U_j}$$

such that $f_{ij}f_{jk} = f_{ik}$ on $U_i \cap U_j \cap U_k$, there is an $E \in \mathcal{F}(X)$ with isomorphisms $\phi_i\colon E|_{U_i} \to E_i$ such that $f_{ij}\phi_j = \phi_i$ on $U_i \cap U_j$.

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For any scheme X, we define groupoids $\mathcal{M}_{ell}(X)$ and $\mathcal{M}_{FG}(X)$ as follows: An object of $\mathcal{M}_{ell}(X)$ is an elliptic curve over X and a morphism is an isomorphism of elliptic curves over X. Similarly, an object of $\mathcal{M}_{FG}(X)$ is a formal group over X and a morphism is an isomorphism of formal groups over X.

Proposition 4.20. *Let X be a scheme. Then $\mathcal{M}_{ell}$ and $\mathcal{M}_{FG}$ define stacks on the Zariski topology of X.*

Proof. We will do the proof only for $\mathcal{M}_{ell}$. We pick an open cover $\{U_i \subset X\}$ and set $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_{ij} \cap U_k$.

Suppose $p\colon C \to X$ and $p'\colon C' \to X$ are elliptic curves over X. First suppose that we have two isomorphisms $f_1, f_2\colon C \to C'$ that are equal on all $C|_{U_i}$, i.e. equal on all $p^{-1}(U_i)$; as the $p^{-1}(U_i)$ form an open cover of C, the maps f_1 and f_2 are clearly equal. Likewise, if we have isomorphisms $f_i\colon p^{-1}(U_i) \to (p')^{-1}(U_i)$ that agree over U_{ij}, we can glue these maps to a morphism $C \to C'$ that is automatically an isomorphism.

If we have elliptic curves C_i over U_i with isos $f_{ij}\colon C_j|_{U_ij} \to C_i|_{U_{ij}}$ satisfying the cocycle conditions, we obtain a scheme C with a map $C \to X$. We also obtain a section by gluing the sections over the U_i. The scheme C is smooth and proper over X (as these properties can be tested locally) and if $\operatorname{Spec} k \to X$ is a point, it factors over some U_i, so the fiber is also an elliptic curve in the classical sense. ☐

Remark 4.21. For Ell the functor that takes every scheme X to the sheaf of isomorphism classes of elliptic curves on X, one sees that Ell is not a sheaf for the Zariski topology on a scheme X in general. E.g. let C be an elliptic curve over $\mathbb{C}$ and X be the union of for $\mathbb{A}^1_{\mathbb{C}}$ that built a quadrilateral. More precisely, you take the quotient of $\mathbb{Z}/4 \times \mathbb{A}^1_{\mathbb{C}}$ by $([k], 1) \simeq ([k+1], 0)$. Consider the two open subsets

$$U = \{([k], x) \in X \;:\; [k] \neq [0] \text{ and } x \neq 0 \text{ if } [k] = [1] \text{ and } x \neq 1 \text{ if } [k] = [-1]\}$$
$$V = \{([k], x) \in X \;:\; [k] \neq [2] \text{ and } x \neq 0 \text{ if } [k] = [-1] \text{ and } x \neq 1 \text{ if } [k] = [1]\}$$

The intersection $U \cap V$ decomposes as $(\mathbb{A}^1_{\mathbb{C}} - \{0,1\}) \times \{[-1],[1]\}$. We glue $U \times C$ with $V \times C$ by using the identity on $(\mathbb{A}^1_{\mathbb{C}} - \{0,1\}) \times \{[1]\}$ and the automorphism $[-1]$ on $(\mathbb{A}^1_{\mathbb{C}} - \{0,1\}) \times \{[-1]\}$. The result C' is clearly an elliptic curve (e.g. by the last proposition). We claim that it is not isomorphic to $X \times_{\mathbb{C}} C$ although both are isomorphic over U and V. Indeed, the 3-torsion of $X \times_{\mathbb{C}} C$ is $(\mathbb{Z}/3)^2$ (as is the one of C). Let P be a three-torsion section of $C' \to X$. The 3-torsion of both $C'|_U$ and of $C'|_V$ is isomorphic to $C[3] = (\mathbb{Z}/3)^2$; P must correspond in both to the same element p as we glued on $(\mathbb{A}^1_{\mathbb{C}} - \{0,1\}) \times \{[1]\}$ via the identity. But this implies that $p = -p$ by the glueing on $(\mathbb{A}^1_{\mathbb{C}} - \{0,1\}) \times \{[-1]\}$ and thus $p = 0$. Thus, $C'[3]$ has just the neutral element.

Recall from Section 3.1 that we can characterize (functors represented by) schemes as functors $\mathcal{F}$ from commutative rings to sets such that $\mathcal{F}$ restricted to the affine opens sets of each $\operatorname{Spec} A$ is a Zariski sheaf and such that $\mathcal{F}$ has an "open cover" by $\operatorname{Spec} A_i$.

If we want to define an analoug for stacks (an "algebraic stack") we could try to ask for contravariant functors $\mathcal{F}$ from schemes to groupoids such that $\mathcal{F}$ restricted to every scheme is a Zariski stack and which have an "open cover" by $\operatorname{Spec} A_i$. There are two problems with this, one merely technical the other one more serious.

First problem: If we want to define $\mathcal{M}_{ell}$ for all schemes, we run into the problem that if we have composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ and an elliptic curve C over Z, then $f^*g^*Z \cong (gf)^*Z$, but not $f^*g^*Z = (gf)^*Z$ as would befit a usual functor. Thus, we have to weaken the notion of a functor/presheaf. We will not work fully weakly by using the convention that $(\mathrm{id})^*C = C$.

Definition 4.22. Let $\mathcal{C}$ be a category. A *pseudo-presheaf* $\mathcal{F}$ of groupoids on $\mathcal{C}$ consists of the following data:

1. For each $c \in \mathcal{C}$ a groupoid $\mathcal{F}(c)$,
2. for each $f\colon c \to d$ a functor $f^*\colon \mathcal{F}(d) \to \mathcal{F}(c)$,
3. for each composable arrows $c \xrightarrow{f} d \xrightarrow{g} e$ a natural isomorphism $\phi_{f,g}\colon f^*g^* \xrightarrow{\cong} (gf)^*$.

These satisfy some axioms:

1. $(\mathrm{id})^* = \mathrm{id}$
2. $\phi_{e,g} = \mathrm{id}$
3. for composable arrows $b \xrightarrow{f} c \xrightarrow{g} d \xrightarrow{h} e$, we have an equality $\phi_{f,hg}(f^*\phi_{g,h}) = \phi_{gf,h}(\phi_{f,g}h^*)$.

We will usually just say *presheaf* when we mean pseudo-presheaf. (We will not have the opportunity to just work with "strict" presheaves of groupoids.)

Note that every presheaf of sets on Sch/S defines a presheaf of groupoids and in particular every scheme does. We will often identify a scheme with its associated presheaf.

Definition 4.23. A *Zariski stack* over a scheme S is a (pseudo-)presheaf of groupoids on (Sch/S) whose restriction to the opens of every scheme X over S is a stack in the sense of Definition 4.19.

Before we discuss the second problem, we have to make sense of what an open cover of a map of presheaves of groupoids is. For this, we have first to discuss fiber products.

Definition 4.24. Let

$$\begin{tikzcd} & \mathcal{G} \arrow[d, "g"] \\ \mathcal{F} \arrow[r, "f"] & \mathcal{H} \end{tikzcd}$$

be a diagram of presheaves of groupoids on Sch/S. We define the *fiber product* (or *2-pullback*) $(\mathcal{F} \times_{\mathcal{H}} \mathcal{G})(T)$ (for $T \in (\operatorname{Sch}/S)$) as the category of triples $(x \in \mathcal{F}(T), y \in \mathcal{G}(T), \alpha\colon f(x) \to g(y))$, where a morphism consists of a pair of $x \to x'$ in $\mathcal{F}(T)$ and

$y \to y'$ in $\mathcal{G}(T)$ such that the obvious diagram commutes. I leave the definition of functoriality and the natural isomorphisms for composition as an exercise. Note that the diagram

$$\begin{tikzcd}
\mathcal{F} \times_{\mathcal{H}} \mathcal{G} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & \mathcal{G} \arrow[d, "g"] \\
\mathcal{F} \arrow[r, "f"] & \mathcal{H}
\end{tikzcd}$$

does *not* commute. Indeed if we have $(x, y, \alpha) \in (\mathcal{F} \times_{\mathcal{H}} \mathcal{G})(T)$, $f(x) \neq g(y)$ – but the morphism α induces a natural isomorphism $\phi_{\mathcal{F},\mathcal{G},\mathcal{H}}$ between $f\,\mathrm{pr}_1$ and $g\,\mathrm{pr}_2$. This makes the diagram *2-commutative.*

Definition 4.25. A morphism $\mathcal{F} \to \mathcal{G}$ of presheaves of groupoids of Sch/S is called *representable* if for every morphism $X \to \mathcal{G}$ with X a scheme, the pullback $\mathcal{F} \times_{\mathcal{G}} X$ is equivalent to a scheme.[6]

Let P be a property of morphisms of schemes that is closed under pullback. We say that a morphism $\mathcal{F} \to \mathcal{G}$ is P if it is representable and $X \times_{\mathcal{G}} \mathcal{F} \to X$ satisfies P for every morphism $X \to \mathcal{G}$ with X a scheme.[7]

Now it is clear what an open cover should be. Define an open immersion of schemes to be a morphism that is isomorphic to the inclusion of an open subscheme. A collection of morphisms $\{U_i \to \mathcal{F}\}$ is a *Zariski open cover* if each $U_i \to \mathcal{F}$ is an open immersion and the corresponding morphism $\coprod_i U_i \to \mathcal{F}$ is surjective (both in the sense above).

Now we can formulate the second problem:

Proposition 4.26. *Let $\mathcal{F}$ be a stack on* (Sch/S) *with a Zariski open cover* $\{U_i \xrightarrow{j_i} \mathcal{F}\}$ *where each U_i is a scheme. Then $\mathcal{F}$ is equivalent to a set-valued presheaf and more precisely even to a scheme.*

Proof. Let X be a scheme over S and $x \in \mathcal{F}(X)$ be an object with an automorphism f. We have to show that it is trivial. Suppose the contrary. The object x corresponds by Yoneda to a morphism $X \to \mathcal{F}$. Let $V_i = U_i \times_{\mathcal{F}} X$, which is (equivalent to) a scheme. Then $\{V_i \to X\}$ is a Zariski open cover. Thus, there is an i such that $f|_{V_i}$ is a non-trivial automorphism of $x|_{V_i}$ (by the sheaf property of $\mathrm{Isom}(x, x)$). Construct a morphism $V_i \to V_i$, i.e. an element in $(U_i \times_{\mathcal{F}} X)(V_i)$ as the triple $(V_i \to X, V_i \to U_i, f\phi_{X,U_i,\mathcal{F}})$ (without this triple, this morphism would be the identity). But there can be no non-trivial automorphism of V_i over X as $V_i \to X$ is an open inclusion! Thus, f must be trivial and $\mathcal{F}$ is equivalent to a set-valued presheaf.

Now note that the stack condition is preserved under equivalences of presheaves and that a set-valued stack is automatically a set (exercise!). The open cover $\{U_i \to \mathcal{F}\}$ is exactly the atlas you need to show that $\mathcal{F}$ is a scheme. $\square$

This is a consequence we most certainly don't want! The problem was that an open immersion $U_i \to X$ could not have any automorphism over X. Thus, we should change the meaning of open.

4.4 Stacks on sites

We want to change the notion of an open cover in a way that it still makes good sense to talk about sheaves.

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[6]Some people use a weaker definition, where it suffices that $\mathcal{F} \times_{\mathcal{G}} X$ is an algebraic space.

[7]This is not always the best definition, e.g. one also wants to consider certain morphisms as proper that are not representable. See [LMB00, Definition 4.14].

Definition 4.27. Let $\mathcal{C}$ be category and $\{U_i \to U\}$ be a collection $\mathcal{T}$ of families of morphisms, called a *cover*. We demand that

1. If $V \to U$ is an isomorphism, then $\{V \to U\}$ is a cover.
2. If $V \to U$ is any arrow and $\{U_i \to U\}$ a cover, the pullbacks $U_i \times_U V$ exist and $\{U_i \times_U V \to V\}$ is still a cover.
3. If $\{U_i \to U\}$ is cover and $\{U_{ij} \to U_i\}$ are covers, then $\{U_{ij} \to U\}$ is also a cover.

Such a collection $\mathcal{T}$ is called a *Grothendieck topology* and the pair $(\mathcal{C}, \mathcal{T})$ is called a *site*.

Definition 4.28. Let $(\mathcal{C}, \mathcal{T})$ be a site and $\mathcal{F}$ be a presheaf (say, of sets) on $\mathcal{C}$. Then $\mathcal{F}$ is a *sheaf* if for every cover $\{U_i \to U\}$ the canonical morphism from $\mathcal{F}(U)$ to the equalizer of the two morphisms from $\prod_i \mathcal{F}(U_i)$ to $\prod_{i,j} \mathcal{F}(U_i \times_U U_j)$ is an isomorphism.

Likewise, we can define a *stack* on a site $(\mathcal{C}, \mathcal{T})$ to be a (pseudo-)presheaf of groupoids on $\mathcal{C}$ satisfying the stack conditions.

end of lecture 17

Which Grothendieck topologies $\mathcal{T}$ on Sch do we want to consider? We want at least two important properties of them:

1. Every presheaf on Sch that is represented by a scheme should be a sheaf. In this case $\mathcal{T}$ is called *subcanonical*.
2. Covers $\{U_i \to X\}$ can have automorphisms.

In topology, a thing satisfying the second (and actually both) conditions are covering spaces. The corresponding notion in algebraic geometry is that of an étale morphism, which we will define in two steps.

Definition 4.29. A morphism $f : X \to Y$ of schemes is called *flat* if for every $x \in X$ the induced map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ exhibits $\mathcal{O}_{Y,f(x)}$ as a flat $\mathcal{O}_{Y,f(x)}$-module.

Note that a morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ is flat iff A is flat as a B-module. Geometrically, you should have the intuition that it is a morphism, where the "fibers are varying concreteley".

Definition 4.30. A morphism $f : X \to Y$ is *étale* if it is locally of finite presentation, flat and satisfies $\Omega^1_{Y/X} = 0$.

The idea is: A flat morphism (locally of finite presentation) is smooth (something like a smooth fiber bundle in topology or an open part thereof) iff $\Omega^1_{Y/X}$ is locally free (i.e. has no jumps in its rank). For example, if X is the affine curve $y^2 = x^3$ over $\operatorname{Spec} k$, we see that the rank of Ω^1 is everywhere one but at the singularity, where it is two. If $\Omega^1_{Y/X} = 0$, then we have a "fiber bundle with fibers of dimension 0" which is much like covering space (or an open part of it).

Examples 4.31.

- Every open immersion is étale.
- A morphism between smooth complex varieties is étale iff it is a local homeomorphism in the complex topology.
- The map $\operatorname{Spec} R[\sqrt{t}] \to \operatorname{Spec} R$ for $t \in R$ is étale if 2 is invertible in R. Here, you have an automorphism $\sqrt{t} \mapsto (-\sqrt{t})$.
- Every Galois extension $K \to L$ defines an étale map $\operatorname{Spec} L \to \operatorname{Spec} K$. Recall that the Galois group are just the automorphism of L over K.

Thus, étale maps unite covering space theory with Galois theory. There is a huge theory of étale cohomology and étale fundamental group, which is very important in arithmetic geometry (see e.g. the Weil conjectures), but of which we will say almost nothing!

Definition 4.32. A family $\{U_i \to X\}$ is an *étale cover* if all $U_i \to X$ are étale and $\coprod_i U_i \to X$ is surjective.

Definition 4.33. A family $\{U_i \to X\}$ is an *fpqc cover* if all $\coprod_i U_i \to X$ is flat, surjective and every quasi-compact subset of X is a the image of a quasi-compact subset of $\coprod_i U_i$.

A morphism that is flat and surjective is also called *faithfully flat* because $\operatorname{Spec} A \to \operatorname{Spec} B$ is faithfully flat iff A is a faithfully flat B-module (faithful means that $\otimes_B A$ detects isomorphisms). In French, this is fidèlement plat, which explains the "fp" in fpqc. The quasi-compactness conditions is rather technical; note that it is automatically satisfied if f is of finite presentation or f is affine.

Note that every étale cover is also an fpqc cover! But there are many other examples, e.g. $\operatorname{Spec} A[x_1, x_2, \ldots] \to \operatorname{Spec} A$.

Theorem 4.34 (Grothendieck)**.** *Both fpqc covers and étale covers define subcanonical topologies.*

That these are subcanonical is not a formality at all![8] That both are topologies are easier though (e.g. it is easy to see that pullbacks of fpqc or étale covers are fpqc or étale covers again).

Theorem 4.35. *Both $\mathcal{M}_{ell}$ and $\mathcal{M}_{FG}$ are stacks for the fpqc (and hence for the étale) topology.*

While it is relatively easy to show that the first stack condition (that Isom is a sheaf) is satisfied, the "glueing" condition is less formal. See [MO17] and [Nau07, Section 4] for the respective details.

Remark 4.36. It is a little easier to show that the functor Ell from Remark 4.21 is no étale sheaf than showing that it is no a Zariski sheaf, as we did earlier. Consider the elliptic curves $E_1 : y^2 = x^3 - 1$ and $E_2 : y^2 = x^3 + 1$ over $\mathbb{R}$. These are not isomorphic. But after base change to $\mathbb{C}$ they are. Thus, there is an étale cover (namely $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$) such that E_1 and E_2 are isomorphic on this cover, but not isomorphic themselves.

References: [Vis05], [Góm01], [Ols16]

4.5 Algebraic stacks and Hopf algebroids

An algebraic stack should be a stack covered in a suitable sense by (affine) schemes. We will choose the following definition.

Definition 4.37. An *algebraic stack* is a stack $\mathcal{X}$ for the fpqc topology on Sch such that there is an affine fpqc map $\operatorname{Spec} A \to \mathcal{X}$ for some A. Here, affine means that $\operatorname{Spec} B \times_{\mathcal{X}} \operatorname{Spec} A$ is an affine scheme for every morphism $\operatorname{Spec} B \to \mathcal{X}$.

Remark 4.38. This definition is equivalent to the one in [Nau07] and [Goe08]. Indeed, they do not require the map $\operatorname{Spec} A \to \mathcal{X}$ to be affine, but the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ to be affine. The latter is equivalent to every morphism $\operatorname{Spec} B \to \mathcal{X}$ being representable and affine (analogously to, e.g., [Góm01, Proposition 2.19]). Thus, we have to show that if there exists an affine fpqc map $\operatorname{Spec} A \to \mathcal{X}$, every map $\operatorname{Spec} B \to \mathcal{X}$ is affine. Affineness can be checked fqpc-locally (see [GW10, Proposition 14.51]) and clearly $\operatorname{Spec} B \times_{\mathcal{X}} \operatorname{Spec} A \to \operatorname{Spec} A$ is affine as the source is affine.

[8] The fpqc topology is the biggest/finest subcanonical topology I have ever heard someone working with. There are though some topologies used that are even finer, e.g. the arc-topology of Bhatt and Mathew [BM18].

Algebraic geometers often use *algebraic stack* as a synonym for an Artin stack, which is something slightly different. This is a stack $\mathcal{X}$, where the diagonal is representable, quasi-compact and separated and there exists a surjective smooth morphism $X \to \mathcal{X}$ with X a scheme. Unfortunately, $\mathcal{M}_{FG}$ is not an Artin stack, which is why we will use our definition of an algebraic stack.

Proposition 4.39. *The stack $\mathcal{M}_{ell}$ is algebraic.*

Proof. Let $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, \Delta^{-1}]$. The Weierstrass equation defines a morphism $f\colon \operatorname{Spec} A \to \mathcal{M}_{ell}$ (because by Yoneda such a morphism corresponds to an elliptic curve over A). We need to show that f is fpqc and affine. This means that for every morphism $g\colon X \to \mathcal{M}_{ell}$ from a scheme X, the fiber product $F = X \times_{\mathcal{M}_{ell}} \operatorname{Spec} A$ is equivalent to a scheme and the projection to X is fpqc and affine.

Let C be the elliptic curve over X classified by g. Then F represents the functor, sending a $T = \operatorname{Spec} R$ to a triple of a morphism $h\colon T \to X$, a Weierstrass curve E on T and an isomorphism $h^*C \to E$. Assume now for a moment that $X = \operatorname{Spec} B$ and C admits a Weierstrass form. By Section 3.4, an isomorphism from h^*E to another Weierstrass curve is classified by $r, s, t, u \in R$ such that u is invertible. Thus, F is equivalent to $\operatorname{Spec} B[r, s, t, u^{\pm 1}]$ in this case (as the second Weierstrass curve is determined by the isomorphism). Note that in this case $F \to \operatorname{Spec} B$ is an affine, flat and surjective morphism (in particular, fpqc).

Back to the general case: The scheme X admits an open cover $\{U_i = \operatorname{Spec} B_i \to X\}$ such that $C|_{U_i}$ admit Weierstrass forms. Thus, F admits a Zariski open cover by the affine scheme and thus is equivalent to a scheme itself by Proposition 4.26. Moreover, the morphism $F \to X$ is locally on the target affine and fpqc; standard algebraic geometry implies that it is affine and fpqc itself. $\square$

The proof for $\mathcal{M}_{FG}$ is similar.

Proposition 4.40. *The stack $\mathcal{M}_{FG}$ is algebraic.*

Proof. Let $L = MU_*$ be the universal ring for formal group laws, carrying the unviersal formal group law F^{univ}. Analogously to the last proof, it follows that the obvious morphism $f\colon \operatorname{Spec} L \to \mathcal{M}_{FG}$ is affine and fpqc if the map $Y = \operatorname{Spec} B \times_{\mathcal{M}_{FG}} \operatorname{Spec} L \to \operatorname{Spec} B$ is affine and fpqc for every $\operatorname{Spec} B \to \mathcal{M}_{FG}$ coming from a formal group *law* F (indeed: every formal group is Zariski locally on the base isomorphic to a formal group coming from a formal group law). The fiber product F represents a triple of a morphism $h\colon \operatorname{Spec} C \to \operatorname{Spec} B$ and an isomorphism $h\colon h^*F \to G$, where G is any formal group law on C. This isomorphism is just any power series $a_0t + a_1t^2 + a_2t^3 + \cdots$ such that a_0 is invertible. Thus, $Y \simeq \operatorname{Spec} B[a_0^{\pm 1}, a_1, a_2, \dots]$. $\square$

Let $p\colon X \to \mathcal{X}$ be any morphism and let $Y = X \times_{\mathcal{X}} X$. The pair (X, Y) represents a groupoid valued functor on the category of schemes. Indeed, for every scheme T, we obtain a groupoid with objects maps $t\colon T \to X$ and morphisms the isomorphisms between (pt_1) and (pt_2) in $\mathcal{X}(T)$; such isomorphisms are exactly classified by the fiber product Y. Thus, (X, Y) is a groupoid object in the category of schemes. Concretely this means that we have maps

1. $s, t\colon Y \to X$ (called *source* and *target*, correspond to the two projections),
2. $e\colon X \to Y$ (the *unit*, corresponding to the diagonal)
3. $Y \times_X Y \to Y$ (the *composition*, corresponding to the map $Y \times_X Y \simeq X \times_{\mathcal{X}} X \times_{\mathcal{X}} X \to X \times_{\mathcal{X}} X \simeq Y$ that is projection onto the first and third coordinate)
4. $Y \to Y$ (the *inverse*, corresponding to the switch map of $X \times_{\mathcal{X}} X$)

satisfying some axioms.

end of lecture 18

Assume now that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} \Gamma$ are affine. Then (A, Γ) is a "cogroupoid object" in commutative rings, which means concretely that we obtain dual maps

1. $\eta_L, \eta_R \colon A \to \Gamma$ (called *left and right unit*),
2. $\epsilon \colon \Gamma \to A$ (called the *counit* or *augmentation*)
3. $\Psi \colon \Gamma \to \Gamma \otimes_A \Gamma$ (called the *diagonal*)
4. $c \colon \Gamma \to \Gamma$ (called the *conjugation*)

satisfying some axioms. Such a structure is called a *Hopf algebroid*. This generalizes a Hopf algebra, which is a cogroup object in A-algebras; this means simply that $\eta_L = \eta_R$ above.

Example 4.41. Consider $\operatorname{Spec} A \to \mathcal{M}_{ell}$ as above. Then $\Gamma = A[r, s, t, u^{\pm 1}]$. The map η_L is just the inclusion $A \to A[r, s, t, u^{\pm 1}]$. The map $\eta_R \colon A \to A[r, s, t, u^{\pm 1}]$ classifies the Weierstrass curve, we obtain from the universal Weierstrass curve after applying the coordinate change $x \mapsto u^2 x + r$ and $y \mapsto u^3 y + sx + t$. All these maps can be looked up in [Sil09, Table 1.2] or [Bau08, Section 3] (though check whether they have the same convention or the opposite ones!).

Example 4.42. For $\operatorname{Spec} L \to \mathcal{M}_{FG}$, we obtain the Hopf algebroid

$$(L, L[a_0^{\pm 1}, a_1, a_2, \dots] = W).$$

A crucial point is that Hopf algebroids can also arise from algebraic topology. Let E be a (homotopy) commutative ring spectrum and assume that $E_*E \cong \pi_* E \wedge E$ is flat over $\pi_* E$. (As $E^*E = [E, \Sigma^* E]$ are stable cohomology operations, one calls E_*E also the *homology cooperations*. We will see more about cooperations next section.) Then we have maps

1. $\pi_* E \to \pi_*(E \wedge E)$
2. $\pi_*(E \wedge E) \to \pi_* E$ (induced by multiplication)
3. $\pi_*(E \wedge E) \to \pi_*(E \wedge E \wedge E)$ (induced by unit in middle variable) and an isomorphism

   $$\pi_*(E \wedge E) \otimes_{\pi_* E} \pi_*(E \wedge E) \to \pi_*((E \wedge E) \wedge_E (E \wedge E)) \xrightarrow{1 \wedge \mu \wedge 1} \pi_* E \wedge E \wedge E.$$

   Indeed, the corresponding morphism also makes sense, if we replace one copy of E by an arbitrary spectrum X to get a map $\pi_*(X \wedge E) \otimes_{\pi_* E} \pi_* E \wedge E \to \pi_* X \wedge E \wedge E$. This is an isomorphism for X the sphere spectrum. Moreover, the class of spectra X where this is an isomorphism is closed under weak equivalences, direct homotopy colimits and cofibers of maps (as $\pi_* E \wedge E$ is a flat $\pi_* E$-module). Thus, it is an isomorphism for all spectra X, in particular for $X = E$.
4. $\pi_*(E \wedge E) \to \pi_*(E \wedge E)$ (induced by twist)

These satisfy the axioms of a (graded) Hopf algebroid.

Example 4.43. We have seen a long time ago that $E_* MU \cong E_*[b_1, b_2, \dots]$ for any complex oriented theory E, in particular for $E = MU$ itself, we obtain $MU_* MU = L[b_1, b_2, \dots]$. This is nearly the same as the W above! The difference is essentially one of gradings.

There is a functor U from the category of evenly graded Hopf algebroids to that of (ungraded) Hopf algebroids sending (A, Γ) to $(A, \Gamma[u^{\pm 1}])$. While η_L and c are essentially given by the same formula and ϵ sends u to 1, the new maps η_R and Ψ take the gradings into account. More precisely, we have

$$\eta_R^{U(A,\Gamma)}(a) = u^{|a|/2}\eta_R^{(A,\Gamma)}(a)$$

for homogeneous elements $a \in A$ and

$$\Psi^{U(A,\Gamma)}(x) = u^{|x|/2}\Psi^{(A,\Gamma)}(x)$$

for homogeneous elements $x \in \Gamma$. [Again, there are two possible conventions; we could have chosen negative powers of u as well.]

The statement (essentially proven by Quillen) is now that $U(MU_*, MU_*MU) \cong (L, W)$. This makes the connection between MU and formal groups even tighter!

References: [Rav86, Appendix],

4.6 Comodules and quasi-coherent sheaves

The importance of Hopf algebroids is that they encode structure the E-homology of any space has.

Definition 4.44. Let (A, Γ) be a Hopf algebroid. A (right) Γ-*comodule* is an A-module M together with a map $\Phi\colon M \to M \otimes_A \Gamma$ satisfying a counitality and a coassociativity axiom.

There is an obvious analogue for evenly graded Hopf algebroids. The category of evenly graded comodules over (A, Γ) is equivalent to that of comodules over $U(A, \Gamma)$.

Example 4.45. We can illustrate the last point by a simple example. Let R be an evenly graded ring, which we can view as an evenly graded Hopf algebroid (R, R). The associated ungraded Hopf algebroid is $(R, R[u^{\pm 1}])$. A comodule M consists of a map $M \to M[u^{\pm 1}]$ satisfying certain properties; we have seen in Proposition 3.26 how this corresponds to a grading on M.

If E is a homotopy commutative ring spectrum such that E_*E is flat over E_* and X is a space or spectrum, then E_*X has the structure of a graded E_*E-comodule:

$$E_*X \cong \pi_*E \wedge X \to \pi_*E \wedge E \wedge X \cong E_*E \otimes_{\pi_*E} E_*X.$$

Here, we use the unit in the middle factor again. More precisely, one can say that E defines a homology theory on spaces/spectra with values in graded E_*E-comodules.

In particular, MU_*X has always the structure of a graded (MU_*, MU_*MU)-comodule. Every such comodule decomposes uniquely into an evenly graded and an oddly graded part. As the evenly graded (MU_*, MU_*MU)-modules are equivalent to ungraded (L, W)-comodules, we see that the category of graded (MU_*, MU_*MU)-comodules is equivalent to $\mathbb{Z}/2$-graded (L, W)-comodules. A 2-fold shift corresponds to tensoring with $L[2]$, which is the L-module L together with the map $u \cdot \eta_R\colon L \to W$.

end of lecture 19

Can we express this in terms of the stack $\mathcal{M}_{FG}$? Indeed, in terms of quasi-coherent sheaves. First recall one possible definition in the scheme case.

Definition 4.46. Let X be a scheme. An $\mathcal{O}_X$-module[9] $\mathcal{F}$ is called *quasi-coherent* if for every affine open $U \subset X$ with $U \cong \operatorname{Spec} A$ and every $f \in A$, the canonical mapping

$$\mathcal{F}(U) \otimes_A A[\frac{1}{f}] \to \mathcal{F}(D(f))$$

is an isomorphism for $D(f) = \operatorname{Spec} A[\frac{1}{f}]$.

[9]This means a module over $\mathcal{O}_X$ in the category of (Zariski) sheaves on X.

We will do a similar definition in general.

Definition 4.47. Let $\mathcal{X}$ be an algebraic stack. Consider the categories $\mathrm{Aff}/\mathcal{X}$ and $\mathrm{Sch}/\mathcal{X}$. Their objects are morphisms from (affine) schemes to $\mathcal{X}$ and morphisms are 2-commutative diagrams. We equip them with the fpqc-topology, where a family $\{U_i \to X\}$ is an fpqc cover if it is an fpqc-cover of schemes – this defines a Grothendieck topology. We define a sheaf $\mathcal{O}_{\mathcal{X}}$ on this site by $\mathcal{O}_{\mathcal{X}}(X \to \mathcal{X}) = H^0(X; \mathcal{O}_X)$. [That this is indeed an fpqc-sheaf follows from fpqc descent as $H^0(X; \mathcal{O}_X)$ is naturally isomorphic to $\mathrm{Hom}_{\mathrm{Sch}}(X, \mathbb{A}^1)$.]

A presheaf[10] of $\mathcal{O}_X$-modules $\mathcal{F}$ on $\mathrm{Aff}/\mathcal{X}$ is called *quasi-coherent* if for every morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ in $\mathrm{Aff}/\mathcal{X}$, the canonical map

$$\mathcal{F}(\operatorname{Spec} B \to \mathcal{X}) \otimes_B A \to \mathcal{F}(\operatorname{Spec} A \to \mathcal{X})$$

is an isomorphism.

A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on $\mathrm{Sch}/\mathcal{X}$ is called *quasi-coherent* if its restriction to $\mathrm{Aff}/\mathcal{X}$ is quasi-coherent.

It turns out that the categories of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules on $\mathrm{Aff}/\mathcal{X}$ and $\mathrm{Sch}/\mathcal{X}$ are equivalent and we denote them by $\mathrm{QCoh}(\mathcal{X})$.

Theorem 4.48 (Faithfully flat descent, Grothendieck)**.** *Let $\mathcal{X}$ be an algebraic stack and $p\colon \operatorname{Spec} A \to \mathcal{X}$ be an affine fpqc morphism. Observe that $\operatorname{Spec} A \times_{\mathcal{X}} \operatorname{Spec} A$ is equivalent to an affine scheme $\operatorname{Spec} \Gamma$ and (A, Γ) is a Hopf algebroid.*

The functor

$$p^*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\operatorname{Spec} A) \simeq A\text{-mod}$$

can be lifted to a functor

$$\mathrm{QCoh}(\mathcal{X}) \to (A, \Gamma)\text{-comod}$$

that is an equivalence of categories. This takes $\mathcal{O}_X$ to the canonical comodule A.

The structure of an (A, Γ)-comodule on $\mathcal{F}(\operatorname{Spec} A)$ arises as follows: By the definition of a quasi-coherent sheaf, the map $\mathcal{F}(\operatorname{Spec} A) \otimes_A \Gamma \to \mathcal{F}(\operatorname{Spec} \Gamma)$ (arising from $\eta_L^*\colon \operatorname{Spec} \Gamma \to \operatorname{Spec} A$) is an isomorphism. But we have also the map $\eta_R^*\colon \mathcal{F}(\operatorname{Spec} A) \to \mathcal{F}(\operatorname{Spec} \Gamma) \cong \Gamma \otimes_A \mathcal{F}(\operatorname{Spec} A)$. This is the structure map of the comodule.

Upshot 4.49. *The spectrum MU defines a homology theory with values in quasi-coherent sheaves on $\mathcal{M}_{FG}$. More precisely, let $\mathcal{F}_i(X)$ be the quasi-coherent sheaf on $\mathcal{M}_{FG}$ corresponding to the even part of $MU_{*+i}(X)$. Then $\mathcal{F}_*$ is a homology theory with values in $\mathrm{QCoh}(\mathcal{M}_{FG})$. Likewise, we can define reduced homology sheaves $\tilde{\mathcal{F}}_*$.*

Remark 4.50. The sheaf $\mathcal{F}_2(\mathrm{pt})$ is a line bundle (as evaluated on $\operatorname{Spec} L$ it is isomorphic to L). We want to give a description of this sheaf in terms of formal groups. First, we record though another general viewpoint on comodules over (L, W). Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{M}_{FG}$, let F be a formal group law on a ring R (classified by a map $L \to R$) and let $f = a_0 t + a_1 t^2 + \cdots \in R[\![t]\!]$ be a power series with a_0 invertible. Note first that as $\operatorname{Spec} R \times_{\mathcal{M}_{FG}} \operatorname{Spec} R \simeq R[a_0^{\pm 1}, a_1, \ldots]$, we obtain maps

$$\mathcal{F}(\operatorname{Spec} R) \xrightarrow{\Psi} \mathcal{F}(\operatorname{Spec} L)[a_0^{\pm 1}, a_1, \ldots] \xrightarrow{F^*} \mathcal{F}(\operatorname{Spec} R)[a_0^{\pm 1}, a_1, \ldots] \xrightarrow{f} \mathcal{F}(\operatorname{Spec} R).$$

[10]While the reader might expect the word *sheaf* here, th sheaf condition in the faithfully flat topology is automatic from the quasi-coherence by faithfully flat descent.

Denote this composite by $\phi_{\mathcal{F},F,f}$. Another interpretation of the map $\phi_{\mathcal{F},F,f}$ is as follows: We have a 2-commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R \arrow[rr, "\mathrm{id}"] \arrow[dr, "F"'] & & \operatorname{Spec} R, \arrow[dl, "fF(f^{-1}(x), f^{-1}(y))"] \\ & \mathcal{M}_{FG} & \end{tikzcd} \tag{4.51}$$

where the isomorphism between the two formal groups is given by f. This induces a map $\mathcal{F}(\operatorname{Spec} R) \to \mathcal{F}(\operatorname{Spec} R)$, which should be the same as above. [There is something amiss here: One should not write $\mathcal{F}(\operatorname{Spec} R)$ as the value of $\mathcal{F}$ can depend on the concrete map chosen from $\operatorname{Spec} R \to \mathcal{M}_{FG}$. And then the map is $\mathcal{F}(\operatorname{Spec} R \xrightarrow{F} \mathcal{M}_{FG}) \to \mathcal{F}(\operatorname{Spec} R \xrightarrow{fF(f^{-1}(x),f^{-1}(y))} \mathcal{M}_{FG})$ – this should be sorted out...] Clearly, $\phi_{\mathcal{O}_{\mathcal{M}_{FG}},F,f} = \mathrm{id}$. We claim that $\phi_{\mathcal{F}_{2i},F,f} = a_0^{-i}\phi_{\mathcal{O}_{\mathcal{M}_{FG}},F,f} = a_0^{-1}$ as can be seen by chasing through the gradings. Furthermore, note that the morphisms $\phi_{\mathcal{F},F,f}$ determine the comodule associated with $\mathcal{F}$ by taking $F = F^{univ}$ on L and thus also the quasi-coherent sheaf $\mathcal{F}$ itself.

We want now to show that $\mathcal{F}_2(\mathrm{pt})$ is isomorphic to a line bundle ω whose inverse we define as follows: Let $\operatorname{Spec} R \to \mathcal{M}_{FG}$ be a morphism corresponding to a formal group $\mathcal{G}$. Then we define $\omega^{-1}(\operatorname{Spec} R \to \mathcal{M}_{FG})$ to be the kernel of $\mathcal{G}(R[t]/t^2) \to \mathcal{G}(R)$. Let's see how this looks like if $\mathcal{G}$ comes actually from a formal group law F: $\mathcal{G} = \operatorname{Spf} R[\![x]\!]$. Recall that $\mathcal{G}$ represents the set-valued functor sending every R-algebra to its nilpotent elements. Thus, the map $\mathcal{G}(R[t]/t^2) \to \mathcal{G}(R)$ is $R^{nil} \oplus tR[t]/t^2 \to R^{nil}$ and $\omega^{-1}(\operatorname{Spec} R \to \mathcal{M}_{FG})$ is isomorphic to $tR[t]/t^2$, which we can identify via the isomorphism $\tau\colon R \xrightarrow{\cong} tR[t]/t^2$ with R. This shows that the ω^{-1} we defined is actually a line bundle (aka invertible sheaf).

Given a diagram like (4.51), the induced map $\omega^{-1}(\operatorname{Spec} R) \to \omega^{-1}(\operatorname{Spec} R)$ is given by

$$f_*\colon tR[t]/t^2 \to tR[t]/t^2, \qquad t \mapsto f(t) = a_0 t.$$

We have $f_*\tau = a_0\tau$. Thus, identifying $\omega^{-1}(\operatorname{Spec} R)$ with R via τ, we obtain $\phi_{\omega^{-1},F,f} = a_0$. Thus, $\omega^{-1} \cong \mathcal{F}_{-2}(\mathrm{pt})$ and $\omega \cong \mathcal{F}_2$.

Remark 4.52. Let $\mathcal{F}$ be a quasi-coherent sheaf on an algebraic stack $\mathcal{X}$. We define its sheaf cohomology $H^i(\mathcal{X};\mathcal{F})$ as $\operatorname{Ext}^i_{\mathrm{QCoh}(\mathcal{X})}(\mathcal{O}_X, \mathcal{F})$.

In algebraic topology, one considers also often Ext of comodules. Indeed, the Adams–Novikov spectral sequence is a spectral sequence of the form

$$\operatorname{Ext}^p_{MU_*MU\text{-comod}}(MU_*, MU_{*+q}) \Rightarrow \pi_{q-p}\mathbb{S}.$$

These Ext-groups are zero if q is odd, so we can equally well just consider Ext-groups in the category of evenly graded MU_*MU-comodules, which is equivalent to the $\mathrm{QCoh}(\mathcal{M}_{FG})$. Thus, the E^2_{pq}-term is isomorphic to $H^p(\mathcal{X};\omega^{\otimes q/2})$ for q even as zero else.

4.7 Landweber's exact functor theorem revisited

Recall from the exercises that for a scheme X with a $\mathbb{G}_m$-action the stack $[X/\mathbb{G}_m]$ classifies étale $\mathbb{G}_m$-torsors $T \to S$ with a $\mathbb{G}_m$-equivariant map $T \to X$. Note that an étale $\mathbb{G}_m$-torsor is an equivalent datum to an $\mathcal{O}_S$-module that is étale locally isomorphic to $\mathcal{O}_S$; this is indeed equivalent to being Zariski locally trivial (i.e. to being a line bundle). Thus every étale $\mathbb{G}_m$-torsor is already Zariski locally trivial.

We claim that the canonical morphism $\operatorname{Spec} L \to \mathcal{M}_{FG}$ factors over $[\operatorname{Spec} L/\mathbb{G}_m]$. Let $p\colon T \to S$ be a $\mathbb{G}_m$-torsor with a $\mathbb{G}_m$-equivariant map $T \to \operatorname{Spec} L$. We can cover S by affine opens $U_i = \operatorname{Spec} A$, where the $\mathbb{G}_m$-torsor is trivial, i.e. $p^{-1}(U_i) \cong \operatorname{Spec} A[u^{\pm 1}]$. A $\mathbb{G}_m$-equivariant map $\operatorname{Spec} A[u^{\pm 1}] \to \operatorname{Spec} L$ is the same as a graded map $L \to A[u^{\pm 1}]$, where $|u| = 2$. This is equivalent to an ungraded map $L \to A$, producing a formal group law over A. It is easy to see that these formal group laws glue to a formal group on S, producing a morphism $q\colon [\operatorname{Spec} L/\mathbb{G}_m] \to \mathcal{M}_{FG}$.

There is also a different way to express this construction. Let R be an evenly graded ring and $F = \sum_{i,j} a_{ij}x^iy^j$ a graded formal group law on R (i.e. $|a_{ij}| = 2i + 2j - 2$). The canonical map $\operatorname{Spec} R \to [\operatorname{Spec} R/\mathbb{G}_m]$ is an fpqc cover and the pullback $\operatorname{Spec} R \times_{[\operatorname{Spec} R/\mathbb{G}_m]} \operatorname{Spec} R$ is equivalent to $R[u^{\pm 1}]$ with the two structure maps $\eta_L, \eta_R\colon R \to R[u^{\pm 1}]$ induced by the projection being the obvious inclusion and $a \mapsto u^i a$ if $|a| = 2i$. Because $\mathcal{M}_{FG}$ is an fpqc-stack, giving a morphism $[\operatorname{Spec} R/\mathbb{G}_m] \to \mathcal{M}_{FG}$ is equivalent to giving a formal group over $\operatorname{Spec} R$ and an isomorphism between the two pullbacks to $\operatorname{Spec} R[u^{\pm 1}]$. From F, we obtain a formal group on $\operatorname{Spec} R$ and the power series ux defines an isomorphism between $(\eta_L)_*F = \sum_{i,j} a_{ij}x^iy^j$ and

$$(\eta_R)_*F = \sum_{i,j} \eta_R(a_{ij})x^iy^j = \sum_{i,j} u^{i+j-1}a_{ij}x^iy^j = u^{-1}\sum_{i,j}(ux)^i(uy)^j.$$

As the universal formal group law on $\operatorname{Spec} L$ is graded (where L is graded compatibly with the isomorphism $L \cong MU_*$), we obtain a map $q\colon [\operatorname{Spec} L/\mathbb{G}_m] \to \mathcal{M}_{FG}$.

By the exercises $\operatorname{QCoh}([\operatorname{Spec} L/\mathbb{G}_m])$ is equivalent to graded L-modules. After forgetting the comodule-structure, q^* agrees with the equivalence from $\operatorname{QCoh}(\mathcal{M}_{FG})$ to evenly graded (MU_*, MU_*MU)-comodules. In particular, the degree 0-part of $q^*\mathcal{F}_i(X)$ is exactly MU_iX (with $\mathcal{F}_i$ as in the last section). Let R be an evenly graded ring and $F\colon L \to R$ be a graded ring homomorphism (corresponding to a graded formal group law F on R), which induces a map $f\colon [\operatorname{Spec} R/\mathbb{G}_m] \to [\operatorname{Spec} L/\mathbb{G}_m]$. Then the degree 0-part of $(qf)^*(\mathcal{F}_i(X)) \cong f^*q^*(\mathcal{F}_i(X))$ is the degree i-part of $MU_*(X) \otimes_{MU_*} R$.. We could write this as

$$\deg_0(qf)^*(\mathcal{F}_*(X)) \cong MU_*(X) \otimes_{MU_*} R.$$

The crucial observation is the following: Because $\mathcal{F}_*$ is a homology theories for spaces valued in $\operatorname{QCoh}(\mathcal{X})$ and $\deg_0$ is exact, the pullback $(qf)^*\mathcal{F}_*(X)$ is a homology theory *if* qf *is flat.* Thus, the Landweber exact functor theorem follows from the following purely algebraic theorem. (Note that $[\operatorname{Spec} R/\mathbb{G}_m] \to \mathcal{M}_{FG}$ is flat iff $\operatorname{Spec} R \to \mathcal{M}_{FG}$ is flat.)

Theorem 4.53 (Algebraic Landweber exact functor theorem)**.** *Let M be a module over L. Then M is flat over $\mathcal{M}_{FG}$ if and only if for every prime p, the sequence $p, v_1, v_2, \ldots$ on M is regular. Here, M is called flat over $\mathcal{M}_{FG}$ if for every morphism $\operatorname{Spec} R \to \mathcal{M}_{FG}$ the pullback of M to $\operatorname{Spec} R \times_{\mathcal{M}_{FG}} \operatorname{Spec} L \simeq \operatorname{Spec} S$ is a flat S-module.*

This is not exactly an easy theorem and the best exposition I know is in Lurie's notes on chromatic homotopy theory [Lur10], lectures up to 16. We will provide a rough sketch of his proof. First note that it is enough to prove the theorem p-locally for every prime p; thus, we fix a prime p in the following.

We sketch first the theory of p-typical formal group laws. A FGL F over a torsionfree ring is called *p-typical* if its logarithm is of the form $\sum_i l_i x^{p^i}$ with $l_0 = 0$. Facts:

1. The universal ring for p-typical formal group laws is $V = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, where these v_i agree with the previous v_i modulo $(p, v_1, \ldots, v_{i-1}) = I_{i-1}$.
2. Every FGL over a p-local ring is canonically isomorphic to a p-typical one, its p-typification.

3. We obtain morphisms $L_{(p)} \to V$ (classifying $F^{univ,p}$) and $V \to L$ (classifying the p-typification of F^{univ}). As $F^{univ,p}$ is already p-typical, the composition $V \to L_{(p)} \to V$ is the identity; thus V is a retract of $L_{(p)}$ and in particular flat as an $L_{(p)}$-module.
4. The morphism $\operatorname{Spec} V \to \mathcal{M}_{FG,(p)}$ is fpqc. Indeed, it is flat as V is flat as an L-module and surjectivity follows from part (2).

Consider the pullback diagram

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} L_{(p)} \arrow[d] \\ \operatorname{Spec} V \arrow[r] & \mathcal{M}_{FG,(p)} \end{tikzcd}$$

As $\operatorname{Spec} V \to \mathcal{M}_{FG,(p)}$ is fpqc, $M_{(p)}$ is flat over $\mathcal{M}_{FG,(p)}$ iff $M_B = M_{(p)} \otimes_{L_{(p)}} B$ over V. One can show that it is enough to show that M_B is flat over every $\mathbb{Z}_{(p)}[v_1, \dots, v_n]$. We will show indeed by downward induction that M/I_m is flat over $\mathbb{Z}_{(p)}[v_1, \dots, v_n]/I_m$, which is clear for $m = n+1$ as $\mathbb{Z}_{(p)}[v_1, \dots, v_n]/I_{n+1} = \mathbb{F}_p$.

We will use the following theorem from commutative algebra.

Proposition 4.54. *Let N be a module over a commutative ring A and $x \in A$ a non-zero divisor. Then N is flat over A if and only if the following three conditions are fulfilled:*

1. *The element x is a non-zero divisor on N,*
2. *N/x is flat over A/x, and*
3. *$N[x^{-1}]$ is flat over $A[x^{-1}]$*

Assume that we already know that M_B/I_{m+1} is flat over $\mathbb{Z}_{(p)}[v_1, \dots, v_n]/I_{m+1}$. Consider the non-zero divisor $v_m \in \mathbb{Z}_{(p)}[v_1, \dots, v_n]/I_m$. Then v_m is a non-zero divisor on M_B/I_m (by the assumption of the Landweber theorem) and $(M_B/I_m)/v_m \cong M/I_{m+1}$ is flat over $\mathbb{Z}_{(p)}[v_1, \dots, v_n]/(I_m, v_m) \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n]/I_{m+1}$. The only thing still to show is that $M_B/I_m[v_m^{-1}]$ is flat over $\mathbb{Z}_{(p)}[v_1, \dots, v_n]/I_m[v_m^{-1}]$.

Let $\mathcal{M}^m_{FG,(p)}$ be the moduli stack of formal groups of exact height m. This agrees with the fiber product $\operatorname{Spec} V/I_m[v_m^{-1}] \times_{\operatorname{Spec} V} \mathcal{M}_{FG,(p)}$ as strict height m is exactly determined by the vanishing of $p, v_1, \dots, v_{m-1}$ and v_m being invertible. In particular, we see that $\operatorname{Spec} B/I_m[v_m^{-1}]$ is the fiber product $\operatorname{Spec} L/I_m[v_m^{-1}] \times_{\mathcal{M}^m_{FG,(p)}} \operatorname{Spec} V/I_m[v_m^{-1}]$. Thus, $M_B/I_m[v_m^{-1}]$ is flat as an $V/I_m[v_m^{-1}]$-module if and only if $M_{(p)}/I_m[v_m^{-1}] \in L/I_m[v_m^{-1}]$-mod is flat over $\mathcal{M}^m_{FG,(p)}$.

Proposition 4.55. *Every quasi-coherent sheaf on $\mathcal{M}^m_{FG,(p)}$ is flat.*

Proof. We only give a very basic idea of it. The crucial fact is that there is an fpqc map $\operatorname{Spec} \mathbb{F}_p \to \mathcal{M}^m_{FG,(p)}$ (at least for $m \geq 1$). The surjectivity means that after suitable flat extension all formal group laws of height exactly m in characteristic p are isomorphic. (The statement is even slightly stronger, e.g. there is up to isomorphism a unique formal group law of height exactly m over $\overline{\mathbb{F}}_p$.) □

Remark 4.56. From this viewpoint, one sees that the Landweber exact homology theory associated to a formal group law only relies on the underlying formal group and not on the choice of coordinate.

end of lecture 20

4.8 Applications to elliptic cohomology and $TMF[\frac{1}{6}]$

Sending an elliptic curve to its associated formal group defines a morphism $\Phi\colon \mathcal{M}_{ell} \to \mathcal{M}_{FG}$ of stacks.

Theorem 4.57. *The morphism $\Phi\colon \mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is flat.*

Proof. Flatness can be tested fpqc-locally on the base. Thus, it suffices to show that the composite $\operatorname{Spec} A \to \mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is flat. As the formal group of the universal Weierstrass curve carries a coordinate, this factors over $\operatorname{Spec} L$, corresponding to a formal group law F. By Theorem 4.53, the morphism $\operatorname{Spec} A \to \operatorname{Spec} L \to \mathcal{M}_{FG}$ is flat if and only if F is Landweber exact. By Theorem 4.16, it suffices to show that there are ordinary elliptic curves over every $\mathbb{F}_p$ (as they have automatically Weierstrass forms). This is Proposition 4.15, which we have shown in Example 4.18 at least for $p > 2$. $\square$

Let $f\colon [\operatorname{Spec} R/\mathbb{G}_m] \to \mathcal{M}_{ell}$ be a flat morphism. Then the functor

$$f^*\Phi^* = (\Phi f)^*\colon \operatorname{QCoh}(\mathcal{M}_{FG}) \to \operatorname{QCoh}([\operatorname{Spec} R/\mathbb{G}_m]) \simeq (\text{evenly graded } R\text{-modules})$$

is exact (as Φ is also flat). Thus, $X \mapsto H^0([\operatorname{Spec} R/\mathbb{G}_m], (\Phi f)^*\mathcal{F}_i)$ defines a homology theory represented by an even spectrum E such that $\pi_{2*}E = R$.

The upshot is the following: We obtain a presheaf $\mathcal{O}^{hom}$ of homology theories on the category of flat morphisms $[\operatorname{Spec} R/\mathbb{G}_m] \to \mathcal{M}_{ell}$.

We want to do some examples. Let $\mathcal{M}_1(n)$ be the (pseudo-)presheaf of groupoids that associates with each scheme S over $\operatorname{Spec} \mathbb{Z}[\frac{1}{n}]$ the groupoid of elliptic curve C over S with a section $S \to C$ of exact order n; more precisely we demand that for every morphism $\operatorname{Spec} k \to S$ with k algebraically closed the pulled back section $\operatorname{Spec} k \to C \times_S \operatorname{Spec} k$ has exactly order n in $C(k)$. Isomorphisms have to respect this point. It is not too hard to check that $\mathcal{M}_1(n)$ is an fpqc-stack as well.

Proposition 4.58. *The map $\mathcal{M}_1(n) \to \mathcal{M}_{ell,\mathbb{Z}[\frac{1}{n}]}$ is étale and surjective.*

Proof. Let $S \to \mathcal{M}_{ell,\mathbb{Z}[\frac{1}{n}]}$ be a morphism classifying an elliptic curve C and C_n be the fiber product $S \times_{\mathcal{M}_{ell,\mathbb{Z}[\frac{1}{n}]}} \mathcal{M}_1(n)$. We have to show that $C_n \to S$ is étale and surjective.

The map $[n]\colon C \to C$ is étale by Proposition 4.11 as n is invertible on S. Thus, $C[n]$ is étale over S as the map $C[n] \to S$ is by definition the pullback of $[n]$ along the unit section $S \to C$. For every $m|n$, the map $C[m] \to C[n]$ is a closed immersion as $C[m] \to C$ is one (as it is the base change of the closed immersion $S \to C$). We claim that C_n is isomorphic to the complement $\tilde{C}_n$ of the images of the $C[m]$ for $m|n$ with $m \neq n$ in $C[n]$. Indeed: The scheme $C[n]$ represents the functor of morphisms $f\colon T \to S$ together with a choice of a point of order n in $C(T)$. Both C_n and $\tilde{C}_n$ can be identified with the subfunctor where the chosen point has exactly order n in the sense above. As $C[n]$ is étale over S and C_n is an open subscheme of $C[n]$, the morphism $C_n \to S$ is étale as well.

For surjectivity it is enough to show that for every $C\colon \operatorname{Spec} k \to \mathcal{M}_{ell,\mathbb{Z}[\frac{1}{n}]}$ with k algebraically closed, we can find a lift to $\mathcal{M}_1(n)$. But we have seen before (in the proof of Proposition 4.11) that the n-torsion of $C(k)$ is isomorphic to $(\mathbb{Z}/n)^2$; in particular, there is a point of exact order n. $\square$

Let $\mathcal{M}^1_{ell}(S)$ be the groupoid of elliptic curves with chosen invariant differential and let $\mathcal{M}^1_1(n) = \mathcal{M}_1(n) \times_{\mathcal{M}_{ell}} \mathcal{M}^1_{ell}$. We know (from Proposition 3.25 and the exercises) that

- $\mathcal{M}^1_{ell,\mathbb{Z}[\frac{1}{6}]} \simeq \operatorname{Spec} \mathbb{Z}[\frac{1}{6}][c_4, c_6, \Delta^{-1}]$

- $\mathcal{M}_1^1(2) \simeq \operatorname{Spec} \mathbb{Z}[\frac{1}{2}][b_2, b_4, \Delta^{-1}]$
- $\mathcal{M}_1^1(3) \simeq \operatorname{Spec} \mathbb{Z}[\frac{1}{3}][a_1, a_3, \Delta^{-1}]$

Clearly, $\mathcal{M}_1^1(n)$ is an étale $\mathbb{G}_m$-torsor over $\mathcal{M}_1(n)$, so we see by the exercises that $[\mathcal{M}_1(n)^1/\mathbb{G}_m] \simeq \mathcal{M}_1(n)$. [Strictly speaking, this identification was only shown in the scheme case, but follows also in general.] Thus, we can obtain homology theories from the examples above:

- $TMF[\frac{1}{6}] := \mathcal{O}^{hom}(\mathcal{M}_{ell,\mathbb{Z}[\frac{1}{6}]})$
- $TMF_1(2) := \mathcal{O}^{hom}(\mathcal{M}_1(2))$ (the same elliptic homology as considered before)
- $TMF_1(3) := \mathcal{O}^{hom}(\mathcal{M}_1(3))$

One can indeed show that the $\mathcal{M}_1^1(n)$ (and indeed the $\mathcal{M}_1(n)$) are also affine schemes for $n \geq 4$, resulting in homology theories $TMF_1(n)$.

Here, TMF stands for *topological modular forms*. We will explain how to construct TMF without 6 inverted in the next sections. What we want to comment on now is what this has to do with modular forms.

We will explain first what a modular function is.

Definition 4.59. An (algebraic) *modular function* with coefficients in a ring R is an element of $MF_{0,R} = H^0(\mathcal{M}_{ell,R}, \mathcal{O}_{\mathcal{M}_{ell,R}})$.

If you have seen the notion of a modular function before in the complex-analytic setting, this might not appear very similar. Let us sketch what happens if $R = \mathbb{C}$. Every elliptic curve over $\mathbb{C}$ is of the form $\mathbb{C}/L$ for some lattice. Thus, a function on $\mathcal{M}_{ell,\mathbb{C}}$ might be seen as a function that sends every lattice L to a complex number. This function must be invariant under isomorphism of elliptic curves and $\mathbb{C}/L$ and $\mathbb{C}/L'$ are isomorphic iff $L' = zL$ for some $z \in \mathbb{C}^\times$. Thus, we can normalize the lattice to have one generating vector equals 1 and one generating vector τ with $\operatorname{Im}(\tau) > 0$. The elliptic curves $\mathbb{C}/\langle 1, \tau\rangle$ and $\mathbb{C}/\langle 1, \tau'\rangle$ are isomorphic iff $\tau' = \frac{a\tau+b}{c\tau+d}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

Definition 4.60. A (complex-analytic) *modular function* is a holomorphic function f on $\mathbb{H} = \{\tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0\}$ such that $f(\frac{a\tau+b}{c\tau+d}) = f(\tau)$ and such that f has at most a pole at $\tau = i\infty$ (in a suitable sense).

One can show that these definitions agree (with $R = \mathbb{C}$). One can show that $MF_{0,R} \cong R[j]$ (where j is the so-called j-invariant). This is easy if 6 is invertible in R, but not so easy in general.

We go on to discuss modular *forms*. First recall that for every elliptic curve $p\colon C \to S$ with section $e\colon S \to C$, we obtain a line bundle $\omega_{C/S} = e^*\Omega^1_{C/S} \cong p_*\Omega^1_{C/S}$ on S. This defines a line bundle ω on $\mathcal{M}_{ell}$. Indeed: it suffices to show that for a morphism $f\colon T \to S$ we have a natural isomorphism $f^*e^*\Omega^1_{C/S} \cong e_{f^*C}\Omega^1_{f^*C/T}$, where $f^*C = C \times_S T$. But it is a general fact that $\Omega^1_{f^*C/T}$ is naturally isomorphic to $\tilde{f}^*\Omega^1_{C/S}$ for $\tilde{f}\colon f^*C \to C$.

Definition 4.61. An (algebraic, meromorphic) *modular form* with coefficients in R is an element of $MF_{k,R} = H^0(\mathcal{M}_{ell,R}, \omega^{\otimes k})$.

In the complex-analytic condition, we have to replace $f(\frac{a\tau+b}{c\tau+d}) = f(\tau)$ by $f(\frac{a\tau+b}{c\tau+d}) = (cz+d)^k f(\tau)$.

We want to claim that $TMF[1/6]_*(\mathrm{pt})$ is concentrated in even degrees and $\pi_{2k}TMF[\frac{1}{6}] \cong MF_{k,\mathbb{Z}[\frac{1}{6}]}$. By definition,

$$\pi_i TMF[\frac{1}{6}] = TMF[\frac{1}{6}]_i(pt) = \mathcal{F}_i(\mathrm{pt})(\mathcal{M}_{ell,\mathbb{Z}[\frac{1}{6}]}) \cong H^0(\mathcal{M}_{ell,\mathbb{Z}[\frac{1}{6}]}; \Phi^*\mathcal{F}_i(\mathrm{pt}))$$

for $\Phi\colon \mathcal{M}_{ell,\mathbb{Z}[\frac{1}{6}]} \to \mathcal{M}_{FG}$ as above. Thus, we need to show the following:

Proposition 4.62. *We have* $\Phi^*\mathcal{F}_{2i+1}(\mathrm{pt}) = 0$ *and* $\Phi^*\mathcal{F}_{2i}(\mathrm{pt}) \cong \omega^{\otimes i}$.

Before we prove this, we have to recall a property about differentials.

Lemma 4.63. *Denote by* Alg_R^{aug} *the category of augmented* R*-algebras, i.e. of commutative* R*-algebras* A *with a map* $p\colon A \to R$ *of* R*-algebras. Let* A *be such an augmented* R*-algebra and* M *an* R*-module. Denote by* $R \oplus M$ *the augmented* R*-algebra with* $(r, m) \cdot (r', m') = (rr', rm' + mr')$ *(i.e. a square-zero extension). Then there is a natural isomorphism*

$$\mathrm{Hom}_{A\text{-mod}}(\Omega^1_{A/R}, M) \cong \mathrm{Hom}_{\mathrm{Alg}_R^{aug}}(A, R \oplus M),$$

where A *acts on* M *via* p *In particular, we get for* $M = R$*:*

$$\mathrm{Hom}_{A\text{-mod}}(\Omega^1_{A/R}, R) \cong \mathrm{Hom}_{\mathrm{Alg}_R^{aug}}(A, R[t]/t^2).$$

Proof. By definition $\mathrm{Hom}_{R\text{-mod}}(\Omega^1_{A/R}, M)$ is in natural one-to-one correspondence with R-derivations $d\colon A \to M$, i.e. R-linear maps satisfying $d(ab) = d(a)b + ad(b)$, where A acts on M via p. Such a derivation defines a morphism

$$A \to R \oplus M, \qquad a \mapsto (p(a), d(a))$$

of augmented R-algebra. If $f\colon A \to R \oplus M$ is a morphism of augmented R-algebras, $\mathrm{pr}_2 f$ is a derivation □

Proof of proposition: First note that $\mathcal{F}_{2i+1}$ is obviously zero. Now denote the ω defined in Remark 4.50 by ω_{FG}. We have seen that $\mathcal{F}_2 \cong \omega_{FG}$ and it is easy to see that $\mathcal{F}_{2i} \cong \mathcal{F}_2^{\otimes i}$. Thus, it suffices to show that $\Phi^*\omega_{FG} \cong \omega$. As we know that the category of quasi-coherent sheaves on the site of affine schemes over $\mathcal{M}_{ell}$ is equivalent to them on the site of all schemes, it suffices to give a natural isomorphism $\Phi^*\omega_{FG}(\operatorname{Spec} R) \cong \omega(\operatorname{Spec} R)$ for all morphisms $\operatorname{Spec} R \to \mathcal{M}_{ell}$ (classifying an elliptic curve C).

Denote by $\hat{C}$ the formal group of C (i.e. the formal completion at the unity section). Recall that

$$\Phi^*\omega_{FG}^{-1}(\operatorname{Spec} R) = \omega_{FG}^{-1}(\operatorname{Spec} R) = \ker(\hat{C}(\operatorname{Spec} R[t]/t^2) \to \hat{C}(\operatorname{Spec} R)).$$

More concretly, this consists of all morphisms $f\colon \operatorname{Spec} R[t]/t^2 \to C$ such that the composition $\operatorname{Spec} R \to \operatorname{Spec} R[t]/t^2 \to C$ is the unit section s and the pullback $f^*\mathcal{I}$ is nilpotent, where $\mathcal{I}$ is the ideal sheaf cutting out the image of the unit section. But the latter condition is implied by the former as $s^*\mathcal{I} = 0$ by definition. Note also that the set-theoretic image of f is automatically exactly the set-theoretic image of the unit section; thus one can replace C by an affine neighborhood $U = \operatorname{Spec} A$ of the image of s (after possibly shrinking $\operatorname{Spec} R$ first).

We see that $\Phi^*\omega_{FG}^{-1}(\operatorname{Spec} R)$ consists of augmented R-algebra homomorphisms $A \to R[t]/t^2$. By the last lemma, we see that

$$\begin{aligned}
\Phi^*\omega_{FG}^{-1}(\operatorname{Spec} R) &\cong \mathrm{Hom}_{A\text{-mod}}(\Omega^1_{A/R}, R) \\
&\cong \mathrm{Hom}_{R\text{-mod}}(\Omega^1_{A/R} \otimes_A R, R) \\
&\cong \mathrm{Hom}_{R\text{-mod}}((s^*\Omega^1_{\operatorname{Spec} A/R})(\operatorname{Spec} R), R) \\
&\cong \mathrm{Hom}_{R\text{-mod}}((s^*\Omega^1_{C/R})(\operatorname{Spec} R), R) \\
&\cong \mathrm{Hom}_{R\text{-mod}}(\omega(\operatorname{Spec} R), R)
\end{aligned}$$

This is exactly what we wanted to show. □

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4.9 Exercises

Exercise 4.64. Consider the elliptic curves $y^2 = x^3 \pm x$ over $\mathbb{Z}[\frac{1}{2}]$. Show that these become isomorphic after base change to an étale extension of $\mathbb{Z}[\frac{1}{2}]$.

Exercise 4.65. Let $\mathcal{F}$ be a stack on a site $(\mathcal{C}, \mathcal{T})$ that takes values in discrete groupoids (aka sets). Show that the stack axioms reduce to the sheaf axioms for presheaves of sets.

Exercise 4.66. Let G be an affine group scheme. The analogue of a G-principal bundle in algebraic geometry is that of an étale G-torsor: An *étale G-torsor* consists of a scheme T with a (left) G-action and a G-equivariant morphism $T \to S$, where S is equipped with the trivial action, such that there exists an étale cover $S' \to S$ so that there is a G-equivariant isomorphism of $T \times_S S'$ with $G \times S'$ over S'. Let BG be the pseudo-presheaf of groupoids that associates with each scheme S the groupoid of étale G-torsors over S.

(a) Show that BG is an fpqc stack. (Use Corollary A.17 of `https://www.math.uzh.ch/index.php?file&key1=5171`)

(b) Show that BG is an algebraic stack if G is flat over $\operatorname{Spec}\mathbb{Z}$.

Exercise 4.67. Let G be again a flat affine group scheme. Let X be a scheme with a (left) G-action. Define $[X/G]$ to be the pseudo-presheaf of groupoids that associates with each S the groupoid of étale G-torsors $T \to S$ with G-equivariant maps $T \to X$.

(a) Show that $[X/G]$ is an fpqc-stack.

(b) Let $X = \operatorname{Spec} A$. Show that $[X/G]$ is an algebraic stack.

(c) Specialize to $G = \mathbb{G}_m = \operatorname{Spec}\mathbb{Z}[u^{\pm 1}]$. Show that the Hopf algebroid associated with the cover $X \to [X/\mathbb{G}_m]$ is $(A, A[u^{\pm 1}])$.

(d) Recall that a $\mathbb{G}_m$-action on $\operatorname{Spec} A$ corresponds to a grading on A; we choose here the convention that this grading is automatically even (corresponding to the choice $|u| = 2$). Deduce that the category of quasi-coherent sheaves on $[\operatorname{Spec} A/\mathbb{G}_m]$ is equivalent to evenly graded modules over A. Taking global sections coresponds to taking the degree 0-part of the module.

Exercise 4.68. Let $T \to S$ be an étale G-torsor. Then $[T/G] \simeq S$. Bonus: As the notion of an étale G-torsor is closed under pullback, it makes also sense if T and S are stacks. Is the last assertion still true if S, T are stacks?

5 Topological modular forms

The goal of this section is to construct the spectrum TMF and discuss its basic properties.

5.1 The construction of TMF and the sheaf $\mathcal{O}^{top}$

We defined a (meromorphic) modular form of weight k to be a global section of $\omega^{\otimes k}$ on $\mathcal{M}_{ell}$. Note that ω was a priori only defined on the site $\mathrm{Sch}/\mathcal{M}_{ell}$ and in particular not on $\mathcal{M}_{ell}$ itself. But if we choose an fpqc (or even étale) cover $X \to \mathcal{M}_{ell}$, we can define $H^0(\mathcal{M}_{ell}, \omega^{\otimes k})$ simply as the equalizer of the two natural maps $\omega^{\otimes k}(X) \to \omega^{\otimes k}(X \times_{\mathcal{M}_{ell}} X)$. A more functorial definition is to define it as $\lim_{(X,f) \in \mathrm{Sch}/\mathcal{M}_{ell}} \omega^{\otimes k}(X, f)$. These definitions are easily seen to coincide if we use that ω is an fpqc-sheaf on $\mathrm{Sch}/\mathcal{M}_{ell}$ (which is automatic from the quasi-coherence on affine schemes and the Zariski sheaf condition).

Let $\mathcal{O}^{hom}$ be as in the last section. We would like to define TMF to be the global sections of $\mathcal{O}^{hom}$. The problem: $\mathcal{O}^{hom}$ is only defined on affine schemes (or more generally on stacks of the form $[\operatorname{Spec} A/\mathbb{G}_m]$). To define its global sections, we have to form a kind of limit. But the category of homology theories does not have all limits! So we should lift it to a setting where limits exist. For this purpose, let $\operatorname{Aff}^{ét}/\mathcal{M}_{ell}$ be the full subcategory of $\operatorname{Aff}/\mathcal{M}_{ell}$ of *étale* morphisms $\operatorname{Spec} A \to \mathcal{M}_{ell}$.

Theorem 5.1 (Goerss–Hopkins–Miller)**.** *There is a presheaf of spectra*[11] $\mathcal{O}^{top}$ *on* $\operatorname{Aff}^{ét}/\mathcal{M}_{ell}$ *whose underlying presheaf of homology theories is isomorphic to* $\mathcal{O}^{hom}$*. Actually,* $\mathcal{O}^{top}$ *can be chosen to be even a presheaf of* E_∞*-ring spectra.*

Here, E_∞-ring spectra can mean different things and the theorem is true for every interpretation. It is a refinement of the notion of a commutative monoid in Ho(Sp) to something stricter/more structured. One possible model: Commutative monoids in the category Sp^O of orthogonal spectra. [One can also treat it via ∞-categories.]

To define global sections, it makes more sense in this homotopical setting to use a homotopy limit than a usual limit. (We will talk later about how to construct/define homotopy limits.)

Definition 5.2. Define TMF as the "global sections" of $\mathcal{O}^{top}$; more precisely, we define

$$TMF := \operatorname{holim}_{(\operatorname{Spec} A \xrightarrow{f} \mathcal{M}_{ell}) \in \operatorname{Aff}^{ét}/\mathcal{M}_{ell}} \mathcal{O}^{top}(\operatorname{Spec} A \xrightarrow{f} \mathcal{M}_{ell})$$

More generally, if $\mathcal{X} \to \mathcal{M}_{ell}$ is an étale map from an algebraic stack, we define

$$\mathcal{O}^{top}(\mathcal{X}) = \operatorname{holim}_{(\operatorname{Spec} A \xrightarrow{f} \mathcal{X}) \in \operatorname{Aff}^{ét}/\mathcal{X}} \mathcal{O}^{top}(\operatorname{Spec} A \xrightarrow{f} \mathcal{X} \to \mathcal{M}_{ell}).$$

To access the homotopy groups of these spectra, we will use the descent spectral sequence. This requires some preparation to state.

Definition 5.3. Let $\mathcal{X}$ be an algebraic stack. We denote by H^q the q-th derived functor of the global sections functor $\mathrm{QCoh}(\mathcal{X}) \to \mathrm{AbGrps}$.

Remark 5.4. More commonly, sheaf cohomology is defined not as the derived functor of global sections from quasi-coherent sheaves on some site, but rather as the derived functor of global sections from sheaves of abelian groups on some suitable site. One can show that these notions agree if we use the fqpc topology for algebraic stacks. This is Proposition B.8 in [TT90] in the case of a quasi-compact and semi-separated scheme and the case of algebraic stacks is similar.

If $\mathcal{F}$ is an presheaf of spectra on a site, we denote by $\pi_k\mathcal{F}$ the *sheafification* of the presheaf $X \mapsto \pi_k(\mathcal{F}(X))$. On $\operatorname{Aff}^{ét}/\mathcal{M}_{ell}$ the presheaf $X \mapsto \pi_k(\mathcal{O}^{top}(X))$ is already a sheaf, namely 0 if k is odd and $\omega^{\otimes k/2}$ if k is even. In particular, $\pi_k\mathcal{O}^{top}$ is 0 for k odd isomorphic to $\omega^{\otimes k/2}$ if k is even.

Theorem 5.5. *Let* $\mathcal{X} \to \mathcal{M}_{ell}$ *be an étale map from an algebraic stack. There is a spectral sequence*

$$H^q(\mathcal{X} : \pi_p\mathcal{O}^{top}) \Rightarrow \pi_{p-q}\mathcal{O}^{top}(\mathcal{X}).$$

In particular, we have a spectral sequence

$$H^q(\mathcal{M}_{ell} : \pi_p\mathcal{O}^{top}) \Rightarrow \pi_{p-q}TMF.$$

The edge homomorphism is

$$MF_{k,\mathbb{Z}} = H^0(\mathcal{M}_{ell}; \omega^{\otimes k}) \to \pi_{2k}TMF.$$

[11]It is of importance here that this is really a presheaf in the category of spectra and *not* in the homotopy category of spectra.

Lemma 5.6. *If $\mathcal{X} = \operatorname{Spec} A$ or $[\operatorname{Spec} A/\mathbb{G}_m]$ and $\mathcal{F}$ is a quasi-coherent sheaf on $\mathcal{X}$, then $H^q(\mathcal{X}; \mathcal{F}) = 0$ for $q > 0$.*

Proof. The former case is well-known. For the latter: We observed before that $\mathrm{QCoh}(\mathcal{X})$ is equivalent to evenly graded A-modules and that the global sections functor corresponds to taking degree-0. This is obviously exact and so all higher derived functors vanish. □

Proposition 5.7. *Let A_* be an evenly graded ring and $\mathcal{X} = [\operatorname{Spec} A_*/\mathbb{G}_m]$ with an étale map to $\mathcal{M}_{ell}$. Then $\pi_*\mathcal{O}^{top}(\mathcal{X}) \cong A_*$. Thus, $\mathcal{O}^{top}$ also refines $\mathcal{O}^{hom}$ on stacks of the form $[\operatorname{Spec} A/\mathbb{G}_m]$ étale over $\mathcal{M}_{ell}$.*

This discussion also confirms that $\pi_{2*}TMF[\frac{1}{6}]$ is just the ring of modular forms $MF_{*,\mathbb{Z}[\frac{1}{6}]} \cong \mathbb{Z}[\frac{1}{6}][c_4, c_6, \Delta^{-1}]$. The homotopy groups are considerably more difficult to calculate without 6 inverted.

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5.2 The spectral sequence for a tower of fibrations

Lemma 5.8. *Let $Y^\bullet = (\cdots \to Y^1 \to Y^0)$ be a tower of Serre fibrations. Then we obtain a natural exact sequence*

$$0 \to \lim{}^1_n \pi_{*+1} Y^n \to \pi_* \lim_n Y^n \to \lim_n \pi_* Y^n \to 0.$$

In particular we get the following: Let $Y^\bullet = (\cdots \to Y^1 \to Y^0)$ and $Z^\bullet = (\cdot \to Z^1 \to Z^0)$ be two towers of Serre fibrations. Let $Y^\bullet \to Z^\bullet$ be a map of towers that is levelwise a weak homotopy equivalence. Then we obtain a weak equivalence on the inverse limits.

We obtain an analogous statement for a tower of levelwise Serre fibrations of Ω-spectra $Y^\bullet$. We obtain an exact couple

$$\begin{tikzcd} D = \bigoplus_n \pi_* Y^n \arrow[rr] & & D = \bigoplus_n \pi_* Y^n \arrow[dl] \\ & E = \bigoplus_n \pi_*(\mathrm{fib}(Y^n \to Y^{n-1}) \arrow[ul] & \end{tikzcd}$$

This produces a spectral sequence with E_1-term $\bigoplus_n \pi_*(\mathrm{fib}(Y^n \to Y^{n-1})$ and converging to $\lim_n \pi_* Y^n$, which is under good circumstances the same as $\pi_* \lim_n Y^n$.

5.3 The spectral sequence for a cosimplicial spectrum

Before we discuss general homotopy limits, we discuss as a first example a homotopy equalizer and let's start in the category of (compactly generated, weak Hausdorff) topological spaces. Recall that the equalizer of two maps $f, g \colon X \to Y$ consists of all $x \in X$ such that $g(x) = g(x)$. In the homotopy world, we replace equality by paths. Thus, the *homotopy equalizer* of f and g consists of the space of $x \in X$ together with a path $\gamma \colon I \to Y$ such that $\gamma(0) = f(x)$ and $\gamma(1) = g(x)$. We generalize this to the totalization of a semi-cosimplicial diagram.

Let Δ_{inj} be the category with objects $\underline{n} = \{0, 1, \ldots, n\}$ (for $n \in \mathbb{Z}_{\geq 0}$) and (not necessarily strictly) monotonic injective maps as morphisms. We denote the injection $\underline{n-1} \to \underline{n}$ that has i not in its image by d^i. All morphisms are composites of d^i. For a category $\mathcal{C}$, we will call a functor $\Delta_{\mathrm{inj}} \to \mathcal{C}$ a *semi-cosimplicial object*. By the remark above, an equivalent description is a sequence of objects $X^n \in \mathcal{C}$ with morphisms $d^i \colon X^{n-1} \to X^n$ satisfying the compatibility $d^j d^i = d^i d^{j-1}$ if $i < j$.

Definition 5.9. Let $\Delta^\bullet$ be the semi-cosimplicial space whose n-th space is the n-simplex (with d^i the inclusion of the face opposite to the i-th vertex). Let $X^\bullet$ be a semi-cosimplicial space or spectrum.

We define its *totalization* $\mathrm{Tot}(X^\bullet)$ as the mapping space/spectrum $\mathrm{Map}_{\Delta_{\mathrm{inj}}}(\Delta^\bullet, X^\bullet)$. (i.e. in the space case the subspace of $\prod_n \mathrm{Map}(\Delta^n, X^n)$ compatible with cofaces and codegeneracies; in the spectrum case we do this construction levelwise).

Let $X^{\bullet \leq n}$ be the restriction of the diagram to the full subcategory $(\Delta_0^n)_{\mathrm{inj}}$ of Δ on $[0], \ldots, [n]$. We define the *n-th partial totalization* $\mathrm{Tot}_n(X^\bullet)$of $X^\bullet$ to be the mapping spaces (or spectrum) $\mathrm{Map}_{(\Delta_0^n)_{\mathrm{inj}}}(\Delta^{\leq n}, X^{\leq n})$.

Example 5.10. Let $X^\bullet$ be a semi-cosimplicial space with $X^n = \mathrm{pt}$ for $n > k$. Then the map $\mathrm{Tot}(X^\bullet) \to \mathrm{Tot}_k(X^\bullet)$ is a homeomorphism. If $k = 1$, this totalization recovers the homotopy equalizer.

Lemma 5.11. *Let $X^\bullet$ be a semi-cosimplicial space. The map $\mathrm{Tot}_n(X^\bullet) \to \mathrm{Tot}_{n-1}(X^\bullet)$ is a Hurewicz fibration.*

Proof. We have to show that every commutative diagram

$$\begin{tikzcd} Y \times 0 \arrow[r, "f"] \arrow[d] & \mathrm{Tot}_n(X^\bullet) \arrow[d] \\ Y \times I \arrow[r, "\alpha"] \arrow[ur, dashed] & \mathrm{Tot}_{n-1}(X^\bullet) \end{tikzcd}$$

has a diagonal lift. It suffices to show this for $Y = \mathrm{pt}$ since the general case reduces to this by considering $(X^\bullet)^Y$. From f and the $d^i\alpha$, we obtain a map $\Delta^n \times 0 \cup \partial\Delta^n \times I \to X^n$. We precompose with the standard retraction $\Delta^n \times I \to \Delta^n \times 0 \cup \partial\Delta^n \times I$ to obtain a map $\Delta^n \times I \to X^n$ or, by adjunction, $I \to \mathrm{Map}(\Delta^n, X^n)$. This map defines together with α the required lift $I \to \mathrm{Tot}_n(X^\bullet)$. ☐

Lemma 5.12. *Let $X^\bullet$ be a pointed semi-cosimplicial space (i.e. a semi-cosimplicial object in pointed spaces with base point $*$). Then the fiber of $\mathrm{Tot}_n(X^\bullet) \to \mathrm{Tot}_{n-1}(X^\bullet)$ is homeomorphic to $\mathrm{Map}(\Delta^n/\partial\Delta^n, X^n) \cong \Omega^n X^n$. The induced map $\Omega^n X^{n-1} \to \Omega\,\mathrm{Tot}_{n-1}(X^\bullet) \to \Omega^n X^n$ is homotopic to $d^n - d^{n-1} + \cdots \pm d^0$ if $n \geq 2$ and to $(d_0)^{-1}d^1$ if $n = 1$.*

Proof. By definition, the fiber is the space of all maps $\Delta^n \to X^n$, which maps all faces to the base point. This shows the first part.

For the second: The map $\partial\colon \Omega\,\mathrm{Tot}_{n-1}(X^\bullet) \to \mathrm{fib}(\mathrm{Tot}_n(X^\bullet) \to \mathrm{Tot}_{n-1}(X^\bullet)) \cong \Omega^n X^n$ is defined as follows: Given $\alpha\colon I \to \mathrm{Tot}_{n-1}(X^\bullet)$ with $\alpha(0) = \alpha(1) = *$, lift it in the manner of the last lemma to $\tilde{\alpha}\colon I \to \mathrm{Tot}_n(X^\bullet)$ (using the map $*\colon 0 \to \mathrm{Tot}_n(X^\bullet)$ as a start) and $\partial(\alpha) = \tilde{\alpha}(1)$. Restriction ∂ to $\Omega\Omega^{n-1}X^{n-1}$ means that the map $I \times \Delta^n \to X^n$ is the base point on the $(n-2)$-skeleton. If we identify $\Delta^n \times 0 \cup (\partial\Delta^n \times I)$ with Δ^n (as the retraction defining $\tilde{\alpha}$ does), we see that the resulting map $\Delta^n/\partial\Delta^n$ is the alternating sum of the maps on the faces, which are exactly the $d^i\alpha$ (as the bottom face is constantly $*$). [This should be worked out more carefully with the orientations/signs!] ☐

Proposition 5.13. *Let $X^\bullet \to Y^\bullet$ be a levelwise weak homotopy equivalence of semi-cosimplicial spaces. Then the induced map on totalizations is a weak homotopy equivalence as well.*

The same is true if $X^\bullet \to Y^\bullet$ is a π_*-isomorphism between Ω-spectra.

Proof. By the five lemma and the Lemmas 5.11 and 5.12, we see that the induces map $\mathrm{Tot}_n(X^\bullet) \to \mathrm{Tot}_n(Y^\bullet)$ is a weak equivalence. [One has to be a little careful with π_0 and π_1 as these are not abelian groups; we will not be so careful with this because we will switch to spectra in a moment.] By the Lemma 5.8, we obtain the same result for Tot itself. The case of Ω-spectra follows because a π_*-isomorphism between Ω-spectra is a levelwise weak homotopy equivalence and every levelwise weak homotopy equivalence is a π_*-isomorphism. □

Because of this homotopy invariance property, we call $\mathrm{Tot}(X^\bullet)$ also the *homotopy limit* of $X^\bullet$ (if the X^n are spaces or Ω-spectra). If $X^\bullet$ is any semi-cosimplicial diagram of spectra, we can first functorially replace the spectra by π_*-isomorphic Ω-spectra and then take Tot to obtain the homotopy limit.

Construction 5.14. Let $X^\bullet$ be a semi-cosimplicial spectrum.[12] We replace all X^n by Ω-spectra without changing their π_*. Then the tower

$$\cdots \mathrm{Tot}_n(X^\bullet) \to \mathrm{Tot}_{n-1}(X^\bullet) \to \cdots \to \mathrm{Tot}_0(X^\bullet) = X^0$$

defines a spectral sequence in the manner of the last section. It takes the form

$$E_1^{pq} = \pi_p \Omega^q X^q \cong \pi_{p-q} X^q \Rightarrow \lim_q \pi_p \mathrm{Tot}_q X^\bullet.$$

In this indexing, this is an upper half-plane spectral sequence. If only finitely many differential exit each spot, then the spectral sequence converges strongly and the target can be identified with

$$\pi_p \lim_q \mathrm{Tot}_q X^\bullet \cong \pi_p \mathrm{Tot}\, X^\bullet \cong \pi_p \operatorname{holim}_{\Delta_{\mathrm{inj}}} X^\bullet.$$

(See [Boa99, Theorem 7.4] for the last point) The d^1-differential on E_1 is induced by the alternating sums of the d^i in the cosimplicial spectrum.

The spectral sequence is called the *Bousfield–Kan spectral sequence* associated with the semi-cosimplicial spectrum $X^\bullet$.

References: [GJ99, Chapter VIII], [BK72], [Dou07]

5.4 Descent spectral sequence again

As noted above, the maps $\mathcal{M}_1(n) \to \mathcal{M}_{ell} \times \mathrm{Spec}\, \mathbb{Z}[\frac{1}{n}]$ are étale and surjective. Moreover $\mathcal{M}_1(n)$ is equivalent to an affine scheme for $n \geq 4$. Thus, there exists an étale cover of $\mathcal{M}_{ell}$ by an affine scheme. The same is true for every algebraic stack $\mathcal{X}$ that is étale over $\mathcal{M}_{ell}$. Indeed, just pull the cover $\mathrm{Spec}\, A \to \mathcal{M}_{ell}$ back to $\mathrm{Spec}\, A \times_{\mathcal{M}_{ell}} \mathcal{X} \to \mathcal{X}$; the source is equivalent to a scheme as $\mathcal{X} \to \mathcal{M}_{ell}$ is representable and the map is étale and surjective (because these properties are closed under pullbacks). Now we can just Zariski cover the source by affine opens.

Let $U \to \mathcal{X}$ be an étale cover by an affine scheme U. We obtain the so-called *descent spectral sequence*

$$E_1^{pq} = \pi_p \mathcal{O}^{top}(U^{\times_{\mathcal{X}} n+1}) \Rightarrow \pi_{p-q} \operatorname{holim}_\Delta \mathcal{O}^{top}(U^{\times_{\mathcal{X}} n+1}).$$

We want to identify source and target with more familiar quantities. Let's begin with the source. The map $U \to \mathcal{X}$ is affine (as $\mathcal{X}$ is algebraic) so that the $U^{\times_{\mathcal{X}} n+1}$ are affine

[12] There is an analogous version for pointed spaces; the difficulty is however that π_0 and π_1 are not abelian groups so that one gets a spectral sequence where some things are just non-abelian groups or pointed sets, which complicates everything. The version for non-pointed spaces is even more difficult, partially because $\mathrm{Tot}(X^\bullet)$ might be a priori empty.

schemes. Then the cochain complex $E_1^{p\bullet}$-term can be identified with the Cech complex for the $\omega^{\otimes *}$ for the cover $U \to \mathcal{X}$. Using that all intersections are affine (and hence acyclic for cohomology), one sees that the cohomology of this cochain complex (i.e. E_2^{pq}) is isomorphic to $H^q(\mathcal{X}; \pi_p \mathcal{O}^{top})$. Now to the target:

Definition 5.15. A presheaf $\mathcal{F}$ of spectra on a site $\mathcal{C}$ is called a *sheaf* if for every cover $U \to X$, the map

$$\mathcal{F}(X) \to \operatorname{holim}_\Delta \mathcal{O}^{top}(U^{\times_X n+1})$$

is an equivalence.

Lemma 5.16. *$\mathcal{O}^{top}$ is a sheaf on $\mathrm{Aff}^{\acute{e}t} / \mathcal{M}_{ell}$ (with the étale topology).*

Proof. Consider the descent spectral sequence for a cover $U \to V$ of affine schemes. The E_2-term is concentrated in line zero as affine schemes have no higher cohomology. This also implies (by a degenerate form of the Mittag–Leffler criterion) that the $\lim^1$-term vanishes. We obtain that $\pi_k \operatorname{holim}_\Delta \mathcal{O}^{top}(U^{\times_V n+1})$ is $\pi_k \mathcal{O}^{top}(V)$, as was to be shown. □

One can show that extending a sheaf on $\mathrm{Aff}^{\acute{e}t} / \mathcal{M}_{ell}$ to all algebraic stacks étale over $\mathcal{M}_{ell}$ preserves the sheaf property. Thus, we have $\operatorname{holim}_\Delta \mathcal{O}^{top}(U^{\times_{\mathcal{X}} n+1}) \simeq \mathcal{O}^{top}(\mathcal{X})$. This gives the final form of the descent spectral sequence:

Proposition 5.17. *For any algebraic stack $\mathcal{X}$ étale over $\mathcal{M}_{ell}$, there is a spectral sequence*

$$E_2^{pq} \cong H^q(\mathcal{X}; \pi_p \mathcal{O}^{top}) \Rightarrow \pi_{p-q} \mathcal{O}^{top}(\mathcal{X}).$$

This is conditionally convergent in the sense of [Boa99].

Recall that we showed that $\pi_p \mathcal{O}^{top} = 0$ vanishes if p odd and $\pi_{2p} \mathcal{O}^{top} \cong \omega^{\otimes p}$ so that the E_2-term is indeed completely algebraic.

Remark 5.18. We do not have to take an affine scheme U for this to work; we can any étale cover $\mathcal{Y} \to \mathcal{X}$ instead on which the $\omega^{\otimes i}$ have no higher cohomology, e.g. $\mathcal{Y}$ could be of the form $[\operatorname{Spec} R / \mathbb{G}_m]$.

end of lecture 23

5.5 A tale of two spectral sequences

There is another, even more important spectral sequence: The Adams–Novikov spectral sequence. We will consider it only in the special case, where it computes (potentially) the stable homotopy groups of the sphere spectrum. The importance for us is that there is a comparison map to a descent spectral sequence for TMF, which allows to transfer information both ways.

We want to use in this section a smash product of spectra that is not only defined on the homotopy category. For definiteness we will use orthogonal spectra for this (see [MMSS01] for the original source or [Sch12] for a comprehensive treatment of the similar theory of symmetric spectra, which are almost equally as good for our purposes; it also contains a brief treatment of orthogonal spectra in Section I.7 – one can also have a look at [Mal11] that is an introduction to stable homotopy theory including orthogonal spectra). The basic idea is that the category of orthogonal spectra is an enhancement of the category of usual spectra (essentially the n-th level is equipped with a nice $O(n)$-action). This enhancement allows us to define a smash product on the category of orthogonal spectra. Furthermore, the forgetful map from orthogonal spectra to usual spectra induces an "equivalence of homotopy theories". In the background, we will use the positive convenient model structure of [Sto11, Section 1.3].

Construction 5.19. Let E be a spectrum with a map $e\colon \mathbb{S} \to E$. We obtain a semi-cosimplicial spectrum $X^\bullet$ with $X^n = E^{\wedge(n+1)}$ with the d^i induced by e. The resulting Bousfield–Kan spectral sequence is called the *E-based Adams spectral sequence (for $\mathbb{S}$)*. In this generality, it is rather useless though. (In particular, it should be only applied if E is cofibrant as else the smash product will not b homotopically correct.)

Remark 5.20. The construction is obviously functorial in maps of spectra $f\colon E \to E'$, where we have $e' = fe$ (where $e\colon \mathbb{S} \to E$ and $e'\colon \mathbb{S} \to E'$ are the two "unit maps"). We claim that it is even functorial if we only have $e' \simeq fe$ with specified homotopy $H\colon fe \Rightarrow e'$. It is enough to consider for this the case $f = \mathrm{id}$. We denote the n-th partial totalization associated with e by Tot_n and the one associated with e' by Tot'_n. It is enough to construct maps $f_n\colon \mathrm{Tot}_n \to \mathrm{Tot}'_n$ such that the squares

$$\begin{tikzcd}
\mathrm{Tot}_n \arrow[r] \arrow[d] & \mathrm{Tot}'_n \arrow[d] \\
\mathrm{Tot}_{n-1} \arrow[r] & \mathrm{Tot}'_{n-1}
\end{tikzcd}$$

are commutative. Assume we have already constructed f_i for $i \leq n$. Recall that a point in the k-th level of Tot_n consists of a point $(g_0, \dots, g_{n-1}) \in (\mathrm{Tot}_{n-1})_k$ together with a map $g_n\colon \Delta^n \to (E^{\wedge\, n+1})_k$ satisfying $d^i_e g_{n-1} = g_n d^i$; here $d^i_e\colon E^{\wedge\, n} \to E^{\wedge(n+1)}$ uses e at the i-th factor. We obtain maps $I \times \Delta^{n-1} \to (E^{\wedge\, n+1})_k$ by $(t, x) \mapsto (d^i_{H(t)} g_{n-1})_k(x)$. We can glue them together with g_n to obtain a map $(\Delta^n \times 0) \cup (\partial\Delta^n \times I) \to (E^{\wedge\, n+1})_k$. Using the usual retraction, we obtain a map $\Delta^n \times I \to (E^{\wedge\, n+1})_k$, whose restriction to $\Delta \times 1$ we use as the image of $(g_0, \dots, g_n)$ in $(E^{\wedge\, n+1})_k$. [There is probably an easier way to do this by replacing E' by E'^{Δ^1} to make the equality $e' = fe$ strict.]

What if e and e' just define the same map $\mathbb{S} \to E$ in the stable homotopy category? Using model category language (see [MMSS01] for the stable model structure we are using), we find a π_*-isomorphisms $r\colon E \to E^f$ to a fibrant orthogonal spectrum. Clearly, r induces isomorphisms of spectral sequence the ones for (E, e) and (E^f, re) and the ones for (E, e') and (E^f, re'). Furthermore, re and re' are really homotopic as $\mathbb{S}$ is cofibrant. Thus, we can use the previous arguments.

The two most important examples are the following:

Example 5.21. Take $E = H\mathbb{F}_p$. Then the spectral sequence converges (strongly) to the p-completion of $\pi_*\mathbb{S}$. The E_2-term can be identified with $E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{A}}(\mathbb{F}_p, \mathbb{F}_p[t])$, where $\mathcal{A}$ denotes the mod-p-Steenrod algebra and $\mathbb{F}_p[t]$ indicates a grading shift. This is one of the most powerful tools to compute the homotopy groups of spheres (especially at $p = 2$). This was the original example of an Adams spectral sequence, with which Adams solved Hopf invariant 1 problem.

Example 5.22. Let $E = MU$.[13] Then the spectral sequence converges (strongly) to $\pi_*\mathbb{S}$. The E_2-term can be identified with $E^2_{s,t} = \mathrm{Ext}^s_{MU_*MU}(MU_*, MU_{*+t})$. Here, the Ext is taken in the category of (MU_*, MU_*MU)-comodules. This is as well an extremely powerful tool to compute the homotopy groups of spheres, especially at odd primes. This spectral sequence is also called the *Adams–Novikov spectral sequence*, which we will just abbreviate to ANSS.

Let us make the identification of the E_2-term more explicit. From the general identification of the E_1-term of the Bousfield–Kan spectral sequence we know that the E_1-term is a cochain complex with n-th term $\pi_*(MU^{\wedge\, n+1})$. The differential is the alternating sum of the maps induced by taking the unit in the i-th factor.

[13] We choose a nice model for MU as a commutative monoid in orthogonal spectra that is cofibrant (as a commutative monoid and hence as an underlying orthogonal spectrum).

Recall that $MU_*MU = \pi_*MU \wedge MU$. Thus, we obtain a map

$$f_n\colon (MU_*MU)^{\otimes_{MU_*} n} \to \pi_*(MU^{(n+1)})$$

as follows: First, we have a map

$$(MU_*MU)^{\otimes n} \to \pi_*(MU \wedge MU)^{\wedge n} \cong \pi_*(MU^{\wedge 2n}).$$

Then we use the map $MU^{\wedge 2n} \to MU^{\wedge n+1}$ that multiplies the second and third entry, the fourth and fifth entry etc. This descends to a map f_n as above. The argument above Example 4.43 shows that f_n is an isomorphism. One checks that the differential on the E_1-term becomes under f_n the following:

$$d^0(a) = \eta_R(a) - \eta_L(a) \tag{5.23}$$

$$d^n(\gamma_1 \otimes \cdots \otimes \gamma_n) = (1 \otimes \gamma_1 \otimes \cdots \otimes \gamma_n) + \sum_{i=1}^{n} (-1)^i \gamma_1 \otimes \cdots \otimes \Psi(\gamma_i) \otimes \cdots \gamma_n + (-1)^{n+1}(\gamma_1 \otimes \cdot \otimes \gamma_n \otimes 1). \tag{5.24}$$

This is called the *cobar complex* for the graded Hopf algebroid (MU_*, MU_*MU).

We can interpret this also as a Cech complex. We have maps

$$g_n\colon (MU_{2*}MU)^{\otimes_{MU_{2*}} n} \to \omega^{\otimes *}([\operatorname{Spec} L/\mathbb{G}_m]^{\times_{\mathcal{M}_{FG}} n+1})$$

defined in a fashion analogous to the maps f_n and these are also isomorphisms. Under g_n the cobar differential corresponds exactly to the Cech differential. The cohomology H^i of quasi-coherent sheaves vanishes for $i > 0$ on the

$$[\operatorname{Spec} L/\mathbb{G}_m]^{\times_{\mathcal{M}_{FG}} n+1} \simeq [\operatorname{Spec} MU_{2*}MU^{\otimes_{MU_{2*}} n}/\mathbb{G}_m]$$

(for $n \geq 1$) and thus the cohomology of the Cech complex is actually $H^*(\mathcal{M}_{FG}; \omega^{\otimes *})$. As we have discussed before,

$$H^s(\mathcal{M}_{FG}; \omega^{\otimes k}) \cong \operatorname{Ext}^s_{MU_{2*}MU}(MU_{2*}, MU_{2*+2k}).$$

One easily sees that $E^2_{s,t} = 0$ for t odd.

Construction 5.25. Let $\mathcal{X}$ be an algebraic stack with an étale map $f\colon \mathcal{X} \to \mathcal{M}_{ell}$ and $U = [\operatorname{Spec} R/\mathbb{G}_m] \to \mathcal{X}$ an étale cover (for R an evenly graded ring). We want to construct a comparison map between the ANSS and the descent spectral sequence (DSS) computing $\pi_*\mathcal{O}^{top}(\mathcal{X})$. Set $X := \mathcal{O}^{top}(U)$.[14] Note that $\pi_*X = R$. As X is even, we can choose a complex orientation $MU \to X$, which commutes up to homotopy with the unit maps. This induces a map from the ANSS to the X-based Adams spectral sequence.

Now recall that the smash product is the coproduct in commutative monoids in orthogonal spectra. We have the $n+1$ maps $\mathrm{pr}_i^*\colon X = \mathcal{O}^{top}(U) \to \mathcal{O}^{top}(U^{\times_{\mathcal{M}_{ell}} n+1})$, which induces a map $X^{\wedge n+1} \to \mathcal{O}^{top}(U^{\times_{\mathcal{M}_{ell}} n+1})$, which actually defines a map of cosimplicial objects. Thus, we obtain a map from the X-based Adams spectral sequence to the DSS computing $\pi_*\mathcal{O}^{top}(\mathcal{X})$.

We have identified the E_1-term of the ANSS with the Cech complex for $\omega^{\otimes ?}$ for $[\operatorname{Spec} L/\mathbb{G}_m] \to \mathcal{M}_{FG}$ and the induced map on E_1-terms is just that to the Cech complex for $\omega^{\otimes ?}$ for $U \to \mathcal{M}_{ell}$ using the commutative square

$$\begin{tikzcd} {[\operatorname{Spec} R/\mathbb{G}_m]} \arrow[r] \arrow[d] & {[\operatorname{Spec} L/\mathbb{G}_m]} \arrow[d] \\ \mathcal{X} \arrow[r, "\Phi f"] & \mathcal{M}_{FG} \end{tikzcd}$$

[14]We should replace X cofibrantly (up to π_*-isomorphism) for all things to be good.

Here, the lower map uses the map $\Phi\colon \mathcal{M}_{ell} \to \mathcal{M}_{FG}$ considered before (sending an elliptic curve to its formal group) and the upper map comes from the graded formal group law on $R = \pi_{2*}X$.

In particular, the map on E_2-terms is just the map $H^s(\mathcal{M}_{FG}; \omega^{\otimes t}) \to H^s(\mathcal{X}; f^*\omega^{\otimes t})$ induced by $f\Phi$.

As we have already computed $\pi_*TMF[\frac{1}{6}]$, we are interested in computing $\pi_*TMF_{(2)}$ and $\pi_*TMF_{(3)}$. Here, we can take the covers $\mathcal{M}_1(3)_{(2)} \to \mathcal{M}_{ell,(2)}$ and $\mathcal{M}_1(2) \to \mathcal{M}_{ell,(3)}$, which are very explicit. We will concentrate mostly on the prime 3 as computations here are easier.

5.6 Computations in cobar complex: Moduli of elliptic curves

We will invert everywhere implicitly 2 in this section. Recall that

$$\mathcal{M}_1(2) \simeq [\operatorname{Spec}\mathbb{Z}[b_2, b_4, \Delta^{-1}]/\mathbb{G}_m].$$

Set $A = \mathbb{Z}[b_2, b_4, \Delta^{-1}]$. The pullback $\operatorname{Spec} A \times_{\mathcal{M}_{ell}} \mathcal{M}_1(2)$ classifies 2-torsion points on elliptic curves of the form $y^2 = x^3 + b_2x^2 + b_4$; these 2-torsion points are exactly points of the form $(r, 0)$, where r is a zero of the right hand side. Thus, $\mathcal{M}_1(2) \times_{\mathcal{M}_{ell}} \mathcal{M}_1(2) \simeq [\operatorname{Spec}\Gamma/\mathbb{G}_m]$ with $\Gamma = A[r]/r^3 + b_2r^2 + b_4r$. By the same arguments as for MU_*, we can compute now the cohomology of $\mathcal{M}_{ell}$ as the cohomology of the *cobar complex* of (A, Γ). This has n-th term $\Gamma^{\otimes_A n}$ and differential as in (5.23).

Let us make the structure maps η_L, η_R and Ψ explicit. The map $\eta_L\colon A \to \Gamma$ is the obvious inclusion; indeed, the projection $\mathrm{pr}_1\colon \mathcal{M}_1(2) \times_{\mathcal{M}_{ell}} \mathcal{M}_1(2) \to \mathcal{M}_1(2)$ forgets in our identification just the 3-torsion point. The map η_R corresponds however to pr_2. If we want to bring the elliptic curve $y^2 = x^3 + b_2x^2 + b_4x$ with 2-torsion point $(r, 0)$ into the standard form we have to move $(r, 0)$ to $(0, 0)$. The coordinate change $x \mapsto x + r$ sends exactly the $(0, 0)$-point to the point $(r, 0)$. We compute

$$(x+r)^3 + b_2(x+r)^2 + b_4(x+r) = x^3 + (b_2+3r)x^2 + (b_4+2b_2r+3r^2)x + (r^3+b_2r^2+b_4r).$$

Note that the constant term is zero. We obtain

$$\begin{aligned} \eta_R(b_2) &= b_2 + 3r \\ \eta_R(b_4) &= b_4 + 2b_2r + 3r^2 \end{aligned}$$

To compute Ψ, we observe that composing $x \mapsto x + r$ with $x \mapsto x + r'$, we obtain $x \mapsto x + (r + r')$. This is represented by the map

$$\Psi\colon \Gamma \to \Gamma \otimes_A \Gamma, \qquad r \mapsto r \otimes 1 + 1 \otimes r.$$

Note that this is both a map of A-bimodules and of algebras, thus the image of r determines the whole map.

This makes the cobar complex completely algebraic. Its cohomology has been computed in [Bau08, Section 5] (without Δ inverted). We will do just a few sample computations and cite then the result of Bauer.

Proposition 5.26. *The ring of modular forms $MF_{*,\mathbb{Z}[\frac{1}{2}]}$ is the zeroth cohomology of the cobar complex above and it is $\mathbb{Z}[\frac{1}{2}][c_4, c_6, \Delta]/(27\Delta = 4c_4^3 - c_6^2)$.*[15]

Proof. The 0-th cohomology of the cobar complex is the equalizer of η_L and η_R. By the definition of a sheaf (and the matching of gradings and the $\omega^{\otimes *}$), this agrees with the global sections of the sheaves $\omega^{\otimes *}$.

[15] Usually one chooses a different convention for Δ, c_4 and c_6, but these differ just by powers of 2.

It is easy to see that the elements

$$c_4 = b_2^2 - 3b_4$$
$$c_6 = 2b_2^3 - 9b_2b_4$$

lie in the equalizer of η_L and η_R. Thus, also

$$\Delta = \frac{1}{27}(4c_4^3 - c_6^2) = b_4^2(b_2^2 - 4b_4)$$

is in the equalizer. We obtain a map

$$\mathbb{Z}[\frac{1}{2}][c_4, c_6, \Delta]/(27\Delta = 4c_4^3 - c_6^2) \to MF_{*,\mathbb{Z}[\frac{1}{2}]},$$

which is easily seen to be injective. As we know the right-hand side after inverting 6, we see that this map is a rational isomorphism. As all (monic) monomials in c_4, c_6 and Δ are indivisible by any natural number, we see that the map is also integrally surjective. [This might be slightly more subtle than I write.] $\square$

end of lecture 23

We identify two cocycles in the cobar complex:

$$\begin{aligned} d(r) &= r \otimes 1 - \Psi(r) + 1 \otimes r = 0 \\ d(r^2 \otimes r - r \otimes r^2) &= 1 \otimes r^2 \otimes r - 1 \otimes r \otimes r^2 - \Psi(r^2) \otimes r + \Psi(r) \otimes r^2 \\ &+ r^2 \otimes \Psi(r) - r \otimes \Psi(r^2) + r^2 \otimes r \otimes 1 - r \otimes r^2 \otimes 1 \\ &= 0, \end{aligned}$$

where we use $\Psi(r^2) = r^2 \otimes 1 + 2r \otimes r + 1 \otimes r^2$. We set $\alpha = [r] \in H^1(\mathcal{M}_{ell}; \omega^{\otimes 2})$ and $\beta = [r^2 \otimes r - r \otimes r^2] \in H^2(\mathcal{M}_{ell}; \omega^{\otimes 6})$.

As $d(r^2) = 2r \otimes r$, we see that $\alpha^2 = [r \otimes r]$ is zero. Furthermore, $d(b_2) = 3r$ (hence $3\alpha = 0$) and ?? (hence $3\beta = 0$) We obtain a map $\Lambda(\alpha) \otimes \mathbb{F}_3[\beta] \otimes \mathbb{F}_3[\Delta^{\pm 1}] \to H^*(\mathcal{M}_{ell}; \omega^{\otimes *})$. By [Bau08], this is an isomorphism in positive cohomological degree. (See also [Mat12] for a more detailed treatment.)

Remark 5.27. There is a more conceptual way to see that all cohomology classes in positive degree have to be 3-torsion. Consider the map $f\colon \mathcal{M}_1(2) \to \mathcal{M}_{ell}$. As

$$H^0(\mathcal{M}_1(2); \mathcal{F}) \cong H^0(\mathcal{M}_{ell}; f_*\mathcal{F})$$

and f_* is exact (as f is finite and in particular affine), we have

$$H^i(\mathcal{M}_{ell}; f_*\mathcal{F}) \cong H^i(\mathcal{M}_1(2); \mathcal{F})$$

for all $i \geq 0$. If $\mathcal{F}$ is quasi-coherent, this means in particular that these groups vanish for $i > 0$. Note furthermore that if $\mathcal{F}$ is a line bundle (e.g. $f^*\omega^{\otimes j}$) that $f_*\mathcal{F}$ is a vector bundle because f is finite and flat and finite flat modules are projective (hence locally free); the rank of $f_*\mathcal{F}$ is 3 as the degree of f is 3.

For quasi-coherent sheaves $\mathcal{G}$ and $\mathcal{G}'$ on an algebraic stack $\mathcal{X}$, we denote by $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{G}, \mathcal{G}')$ we denote the sheaf that is on every $g\colon U \to \mathcal{X}$ the abelian group $\mathrm{Hom}_{\mathcal{O}_U}(g^*\mathcal{G}, g^*\mathcal{G}')$. If $\mathcal{G}$ is a vector bundle of rank n, then the map $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{G}, \mathcal{O}_\mathcal{X}) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G} \to \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{G}, \mathcal{G})$ is an isomorphism because it is locally so. Consider now the composite

$$\mathcal{O}_\mathcal{X} \to \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{G}, \mathcal{G}) \cong \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{G}, \mathcal{O}_\mathcal{X}) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G} \to \mathcal{O}_\mathcal{X}.$$

Locally one can check that the first morphism corresponds to the unit matrix and the second morphism to the trace. Thus, the composite is just multiplication by n. If $\mathcal{G}$ is a sheaf of algebras, the map $\mathcal{O}_{\mathcal{X}} \to \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{G}, \mathcal{G})$ factors through $\mathcal{G}$ (via left multiplication). The resulting map $\mathcal{G} \to \mathcal{O}_{\mathcal{X}}$ is also called the *transfer*.

In particular, we see that the multiplication by 3 map on $\mathcal{O}_{\mathcal{M}_{ell}}$ factors over $f_*\mathcal{O}_{\mathcal{M}_1(2)}$ and hence the multiplication by 3 map on $\omega^{\otimes i}$ factors over $f_*\mathcal{O}_{\mathcal{M}_1(2)} \otimes \omega^{\otimes i} \cong f_* f^* \omega^{\otimes i}$. (In the last step, we use the projection formula, see e.g. [Har77, Ch III, Exercise 8.3].) Thus, the multiplication by 3 map on $H^j(\mathcal{M}_{ell}; \omega^{\otimes i})$ factors over $H^j(\mathcal{M}_1(2); f^*\omega^{\otimes i})$, which is zero for $j > 0$. Thus, all cohomology in positive degrees of $\omega^{\otimes i}$ on $\mathcal{M}_{ell}$ is 3-torsion. (Recall that we are still inverting 2 here! In general, it is only 24-torsion by using a similar argument with the cover $\mathcal{M}_1(3) \to \mathcal{M}_{ell,\mathbb{Z}[\frac{1}{3}]}$ of degree 8.)

We can actually compute the transfer $\mathrm{tr}\colon f_*\mathcal{O}_{\mathcal{M}_1(2)} \to \mathcal{O}_{\mathcal{M}_1(2)}$ (or the same map after tensoring with $\omega^{\otimes *}$) concretely via the equivalence between quasi-coherent sheaves on $\mathcal{M}_{ell}$ and (A, Γ)-comodules. Evaluating tr on Spec A gives a map $\mathrm{Tr}\colon \Gamma = f_*\omega^{\otimes *}(\mathcal{M}_1(2)) \to A$. This transfer is the composite of $M\colon \Gamma \to \mathrm{Hom}_A(\Gamma, \Gamma)$ (given by (left) multiplication) and the trace $\mathrm{Trace}\colon \mathrm{Hom}_A(\Gamma, \Gamma) \to A$. The latter trace is just the usual trace of a matrix if we identify $\Gamma = A\{1, r, r^2\}$ so that $\mathrm{Hom}_A(\Gamma, \Gamma)$ becomes the algebra of A-valued 3×3-matrices. Both maps are A-linear. We have

$$M(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$M(r) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -b_4 \\ 0 & 1 & -b_2 \end{pmatrix}$$
$$M(r^2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -b_4 & b_2b_4 \\ 1 & -b_2 & b_2^2 - b_4 \end{pmatrix}$$

Here, we use the equality $r^3 = -b_2r^2 - b_4r$. Thus, we obtain

$$\begin{aligned} \mathrm{Tr}(1) &= 3 \\ \mathrm{Tr}(r) &= -b_2 \\ \mathrm{Tr}(r^2) &= b_2^2 - 2b_4. \end{aligned}$$

We can compute tr on (graded) global sections (i.e. as a map $\mathrm{tr}\colon A \to MF_{*,\mathbb{Z}[\frac{1}{2}]}$) as $\mathrm{Tr}\,\eta_R$. This map is $MF_{*,\mathbb{Z}[\frac{1}{2}]}$-linear. We have

$$\begin{aligned} \mathrm{tr}(1) &= \mathrm{Tr}(1) = 3 \\ \mathrm{tr}(b_2) &= \mathrm{Tr}(b_2 + 3r) = 3b_2 + 3\,\mathrm{Tr}(r) = 0 \\ \mathrm{tr}(b_4) &= \mathrm{Tr}(b_4 + 2b_2r + 3r^2) = 3b_4 + 2b_2(-b_2) + 3(b_2^2 - 2b_4) \\ &= b_2^2 - 3b_4 = c_4 \\ \mathrm{tr}(b_2b_4) &= \mathrm{Tr}((b_2 + 3r)(b_4 + 2b_2 + 3r^2)) \\ &= 9b_2b_4 - 2b_2^3 = -c_6 \end{aligned}$$

We see that the ideal $(3, c_4, c_6) \subset MF_{*,\mathbb{Z}[\frac{1}{2}]}$ lies in im(tr); one can actually show that this containment is equality and so $MF_{*,\mathbb{Z}_{(3)}}/im(\mathrm{tr}) = \mathbb{F}_3[\Delta^{\pm 1}]$. Indeed: The ideal $(3, c_4, c_6)$ is a maximal homogeneous ideal (as the quotient is the graded field $\mathbb{F}_3[\Delta^{\pm 1}]$) and thus it suffices to show that 1 is not in im(tr). One can show that in general for $x \in H^*(\mathcal{M}_{ell}; \omega^{\otimes *})$ we have $x\,\mathrm{tr}(y) = \mathrm{tr}(xy)$, where we view tr as a map

$H^*(\mathcal{M}_1(2); f^*\omega^{\otimes *}) \to H^*(\mathcal{M}_{ell}; \omega^{\otimes *})$; but for x in positive cohomological degree $\mathrm{tr}(xy)$ must be zero, but if $\mathrm{tr}(y) = 1$ it must also be equal to x. This is a contradiction as there are classes in $H^*(\mathcal{M}_{ell}; \omega^{\otimes *})$ of positive cohomological degree.

5.7 Computations in cobar complex: Moduli of formal groups

It is much harder to make the Hopf algebroid (MU_*, MU_*MU) explicit, but we will do it in very low degrees. Recall that $MU_* \cong \mathbb{Z}[u_1, u_2, \dots]$ and

$$F(x, y) = x + y + u_1xy + \cdots .$$

Furthermore, $MU_*MU \cong MU_*[m_1, m_2, \dots]$ and the m_i correspond to power series $x + m_1x^2 + m_2x^3 + \cdots$ that defines an isomorphism. The morphism η_L is under these identifications just the obvious inclusion.

Lemma 5.28. $\eta_R(u_1) = u_1 - 2m_1$

Proof. Let $f(x) = x + m_1x^2 + m_2x^3 + \cdots$. We have to compute $f^{-1}F(f(x), f(y))$ in low degrees (i.e. modulo terms of degree higher than 2). For these low degrees we can pretend that $f(x) = x + m_1x^2$ and $f^{-1}(x) = x - m_1x^2$ and $F(x, y) = x + y + u_1xy$. With this identification, we have (modulo terms of degree higher than 2):

$$\begin{aligned} f^{-1}F(f(x), f(y)) &\equiv (x + m_1x^2) + (y + m_1y^2) + u_1xy - m_1(x + y)^2 \\ &\equiv x + y + (u_1 - 2m_1)xy. \end{aligned}$$

□

Lemma 5.29. *We have*

$$\begin{aligned} \Psi(m_1) &= m_1 \otimes 1 + 1 \otimes m_1 \\ \Psi(m_2) &= m_2 \otimes 1 + 2m_1 \otimes m_1 + 1 \otimes m_2. \end{aligned}$$

Proof. Let $f(x) = x + m_1x^2 + m_2x^3 + \cdots$ and $g(x) = x + m_1'x^2 + m_2'x^3 + \cdots$. We have (modulo terms of degree higher than 3):

$$f(g(x)) \equiv x + m_1'x^2 + m_2'x^3 + m_1(x + m_1'x^2)^2 + m_2x^3 \equiv x + (m_1' + m_1)x^2 + (m_2' + 2m_1m_1' + m_2)x^3.$$

This implies the result. □

It is easy to see that m_1 and $m_2 - m_1^2$ are cocycles. We set $\eta = [m_1]$ and $\nu = [m_2 - m_1^2]$. As $d(u_1) = -2m_1$, we see that η is 2-torsion. It is a little harder to see that ν is 24-torsion.

Let Γ be as in the last section. We want to make the map $MU_*MU \to \Gamma$ explicit. Consider the coordinate change $(x, y) \mapsto (x + r, y)$. In homogeneous coordinates, this becomes $[x, y, z] \mapsto [x + rz, y, z]$. Recall from Section 3.6 that in a neighborhood of $[0 : 1 : 0]$, we have $z = x^3 + \cdots$. Thus, the coordinate change induces $x \mapsto x + rx^3 + \cdots$ on the level of formal groups. This shows that $MU_*MU \to \Gamma$ sends m_1 to 0 and m_2 to r.

In particular, we see that the map $H^*(\mathcal{M}_{FG}; \omega^{\otimes *}) \to H^*(\mathcal{M}_{ell,\mathbb{Z}[\frac{1}{2}]}; \omega^{\otimes *})$ maps ν to α. We remark that there is a further class β_1 in the E_2-term of the ANSS that is mapped to β.

Remark 5.30. It becomes clear how hard it is to do these computations via the cobar complex based on MU_*MU. They become easier when using BP instead of MU, but even for BP the cobar complex should be replaced by better methods. See [Rav86].

5.8 Homotopy groups of TMF at the prime 3

We will implicitly 3-localize everything in this section. Recall from Section 5.6 that the E_2-term of the descent spectral sequence for computing $\pi_* TMF$ is

$$\mathbb{Z}[c_4, c_6, \Delta^{\pm 1}, \alpha, \beta]/(27\Delta - (4c_4^3 - c_6^2), 3\alpha, 3\beta, \alpha^2, c_4\alpha, c_6\alpha, c_4\beta, c_6\beta).$$

We have four major weapons to compute the differentials, namely the comparison map from the ANSS, the transfer, the Toda differential and Toda brackets. We begin with the transfer.

Construction 5.31. Consider again the map $f\colon \mathcal{M}_1(2) \to \mathcal{M}_{ell}$ and the $\mathcal{O}^{top}$-module $f_*f^*\mathcal{O}^{top}$; this is defined by $f_*f^*\mathcal{O}^{top}(U) = \mathcal{O}^{top}(U \times_{\mathcal{M}_{ell}} \mathcal{M}_1(2))$. It is easy to see that $\pi_* f_* f^* \mathcal{O}^{top} \cong f_* f^* \pi_* \mathcal{O}^{top}$, i.e. the odd homotopy groups are zero and the evens are $\pi_{2k} f_* f^* \mathcal{O}^{top} \cong f_* \mathcal{O}_{\mathcal{M}_1(2)} \otimes \omega^{\otimes k}$.

Note that $f_*f^*\mathcal{O}^{top}$ is a sheaf of E_∞-rings. This allows us to define the transfer as in the algebraic setting as the composite

$$\mathrm{tr}^{top}\colon f_*f^*\mathcal{O}^{top} \to \mathcal{H}om_{\mathcal{O}^{top}}(f_*f^*\mathcal{O}^{top}, f_*f^*\mathcal{O}^{top}) \xleftarrow{\simeq} \mathcal{H}om_{\mathcal{O}^{top}}(f_*f^*\mathcal{O}^{top}, \mathcal{O}^{top}) \wedge_{\mathcal{O}^{top}} f_*f^*\mathcal{O}^{top} \to \mathcal{O}^{top}.$$

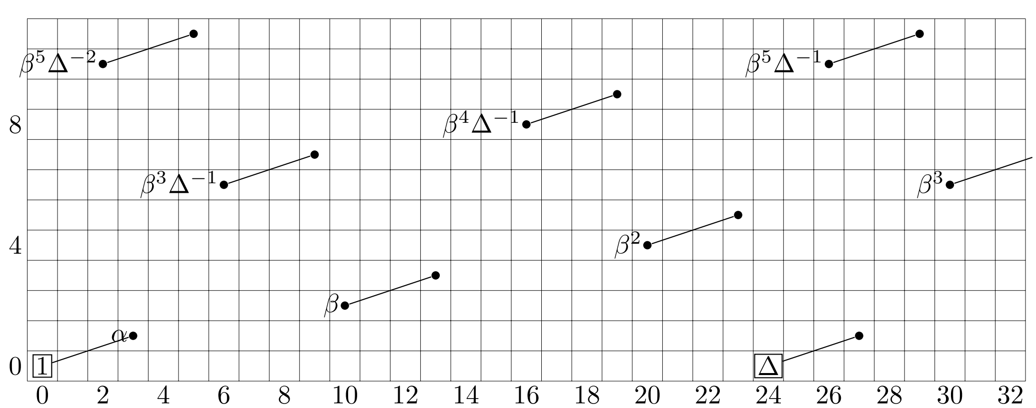
The equivalence in the middle can be checked locally and follows there because $f_*f^*\mathcal{O}^{top}$ is (étale) locally free as an $\mathcal{O}^{top}$-module. It follows easily that tr^{top} induces on homotopy groups just the transfer considered above.

Above, we set up the descent spectral sequence only for the sheaf $\mathcal{O}^{top}$, but one can do it equally well for $f_*f^*\mathcal{O}^{top}$ and we obtain a map of descent spectral sequences

$$\mathrm{tr}^{SS}\colon \mathrm{DSS}(f_*f^*\mathcal{O}^{top}) \to \mathrm{DSS}(\mathcal{O}^{top}).$$

We have seen in Remark 5.27 that the $H^*(\mathcal{M}_{ell}; f_*f^*\omega^{\otimes k})$ vanishes for $* > 0$. Thus, the E_2-term of $\mathrm{DSS}(f_*f^*\mathcal{O}^{top})$ is concentrated in the 0-line and there can be no differentials. Thus, the image of tr^{SS} consists of permanent cycles. On the zero line, tr^{SS} is just the algebraic transfer discussed in 5.27 and thereafter. Thus, the ideal $(3, c_4, c_6)$ consists of permanent cycles.

Let us draw the descent spectral sequence for TMF modulo the image of the transfer (where we know anyhow that it consists of permanent cycles; as it is completely in the line 0 it can also not be hit by any differentials). Note that the E_2-term is 24-periodic (with periodicity element Δ).



By the discussion in the last subsection, we know that α is a permanent cycle, namely the image of $\nu \in \pi_3\mathbb{S}$ under the unit map $\mathbb{S} \to TMF$. To proceed, we have to talk about Massey products and Toda brackets.

Definition 5.32. Let $C^\bullet$ be a differential graded algebra, i.e. a (graded) commutative monoid in cochain complexes. Let $\alpha, \beta, \gamma \in H^*(C^\bullet)$ with $\alpha\beta = 0 = \beta\gamma$. We obtain a new cohomology class as follows: Choose cocycles x, y, z representing α, β, γ. Choose u with $d(u) = xy$ and v with $d(v) = yz$. Then $xv - (-1)^{|x|}uz$ is again a cocycle. The (non-empty) set of the classes of all such cocycles is called the Massey product $\langle \alpha, \beta, \gamma \rangle \subset H^{|\alpha|+|\beta|+|\gamma|-1}(C^\bullet)$. If $\delta \in \langle \alpha, \beta, \gamma \rangle$, then

$$\langle \alpha, \beta, \gamma \rangle = \delta + \alpha H^{|\beta|+|\gamma|-1}(C^\bullet) + H^{|\alpha|+|\beta|-1}(C^\bullet)\gamma.$$

Example 5.33. We have $\langle \alpha, \alpha, \alpha \rangle = \{\beta\}$ in the cobar complex (A, Γ) from Section 5.6.

Theorem 5.34. [16] *Let R be an E_∞-ring spectrum (actually A_∞ would be enough) and $\alpha, \beta, \gamma \in \pi_* R$ with $\alpha\beta = 0 = \beta\gamma$. Then there is a naturally defined set $\langle \alpha, \beta, \gamma \rangle \subset \pi_{|\alpha|+|\beta|+|\gamma|-1} R$, called the* Toda bracket *of α, β and γ. The formula for the indeterminancy is analogous to the one for Massey products.*

What makes this really useful is the interplay between Massey products and Toda brackets In most spectral sequences of interest the following is true under usual circumstances[17]: Consider a spectral sequence converging to $\pi_* R$. Let $\alpha, \beta, \gamma \in \pi_* R$ be classes with $\alpha\beta = 0 = \beta\gamma$ and let $x \in \langle \alpha, \beta, \gamma \rangle$. The spectral sequence comes with a filtration on $\pi_* R$. If the filtrations of α, β and γ are p, q, r, then the filtration of x is at least $p + q + r - 1$. If moreover α, β, γ reduce to $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ in the spectral sequence of the E^t-page, then x reduces to an element in the Massey product $\langle \overline{\alpha}, \overline{\beta}, \overline{\gamma} \rangle$.

This is proven in the Adams–Novikov spectral sequence in the book [Koc96]. I do not know a published reference for this fact for the descent spectral sequence. In the case of TMF one can actually (via a non-trivial theorem) identify the descent spectral sequence with the Adams–Novikov spectral sequence for TMF, but this is not the way it should be done. Anyhow, we will use it.

Corollary 5.35. *The Toda bracket $\langle \alpha, \alpha, \alpha \rangle$ consists of exactly one element and this reduces in the spectral sequence to β; by abuse of notation, we will also call it β.*

Now we use the following theorem of Toda:[18]

Theorem 5.36 (Toda)**.** *Let R be any E_∞-ring spectrum and $x \in \pi_* R$ a 3-torsion element. Then $\nu x^3 = 0 \in \pi_* R$.*

To apply this, we need $3\beta = 0$. The easiest way to see it is that β is the image of $\langle \nu, \nu, \nu \rangle$. Furthermore, $\pi_{10}\mathbb{S} = \mathbb{Z}/3$ (as follows directly from the computation of the E_2-term of the Adams–Novikov spectral sequence). Thus, we must have $3\langle \nu, \nu, \nu \rangle = 0$ and hence $3\beta = 0$. Thus, Todas's theorem yields $\alpha\beta^3 = 0 \in \pi_* TMF$.

The only possible differential causing this is $d_5(\beta\Delta) = \pm\alpha\beta^3$. By multiplicativity (which we also did not show for the descent spectral sequence! But this should follow from the methods of [Dug03]), this implies $\beta d_5(\Delta) = \pm\alpha\beta^3$ and hence $d_5(\Delta) = \pm\alpha\beta^2$. Multiplicativity also implies some other differentials, in particular:

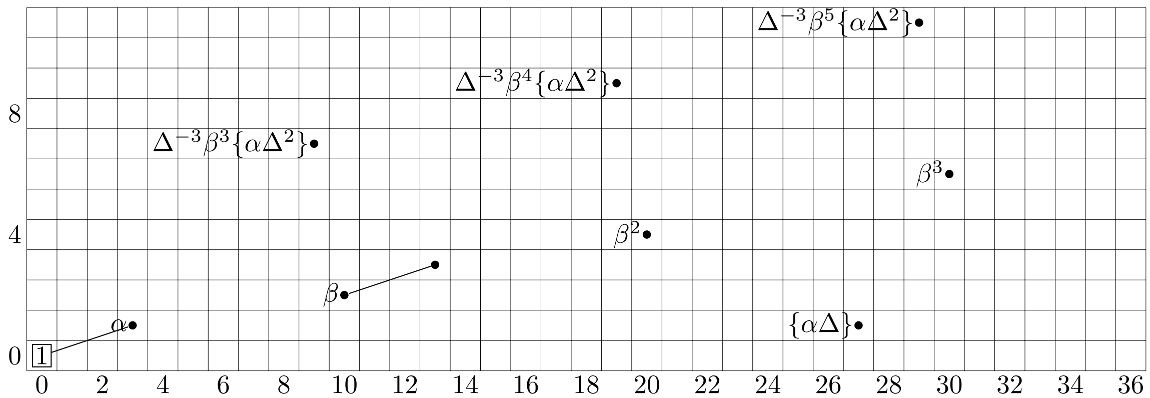
$$\begin{aligned} d_5(\Delta^n) &= \pm n\alpha\beta^2\Delta^{n-1} \\ d_5(\alpha\beta^k\Delta^n) &= 0 \\ d_5(\beta^k\Delta^n) &= \pm n\alpha\beta^{k+2}\Delta^{n-1} \end{aligned}$$

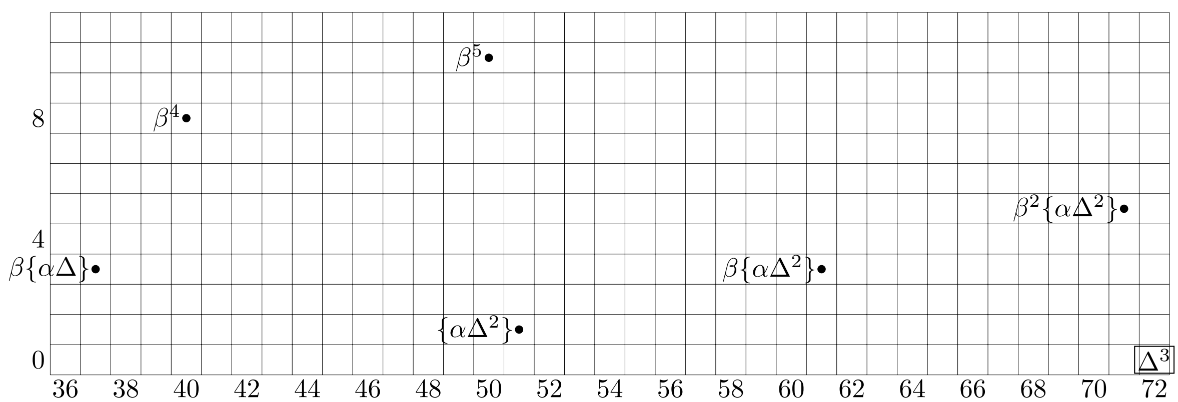
[16] The state of the literature on Toda brackets is less than ideal. But see for example [Koc96] or [Mei12, Section 4.6].

[17] These are: If you draw the spectral sequence, no two differentials cross each other.

[18] The original source is [Tod68, Theorem 3]. It states it only for the sphere spectrum, but the proof should generalize to the result stated here. In any case, our applications would also follow from the spherical case.

One sees that there is no room for further d_5 or d_7-differentials (and d_{2n}-differentials cannot occur for degreee reasons anyhow). The E_9-page is 72-periodic (with periodicity element Δ^3) and looks as follows:

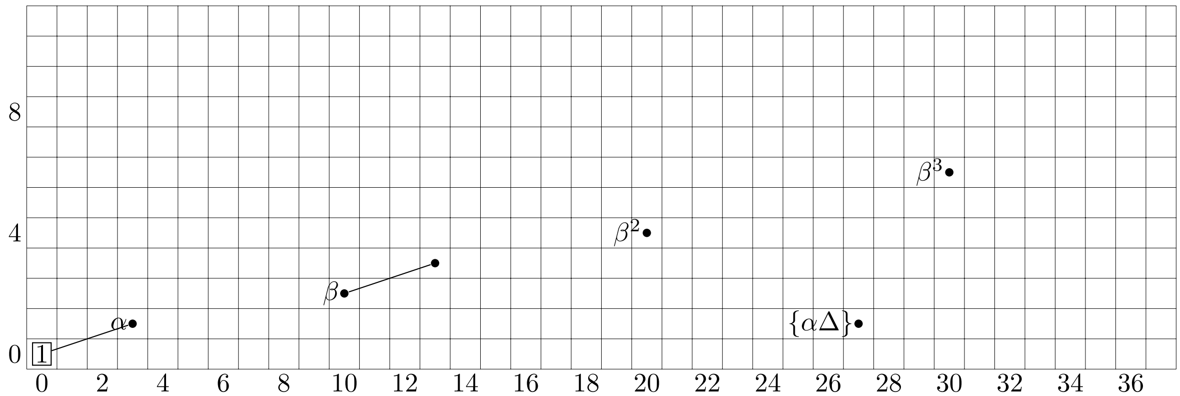


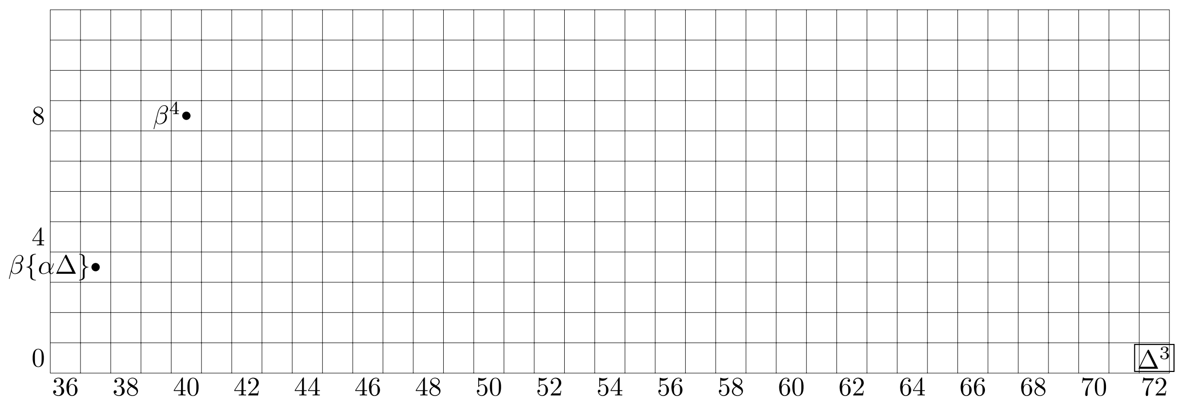


Recall that $\beta = \langle \alpha, \alpha, \alpha \rangle$ (as Toda bracket). The rules of Toda brackets imply that

$$\beta^3 = \beta^2 = \langle \beta^2\alpha, \alpha, \alpha \rangle = \langle 0, \alpha, \alpha \rangle.$$

The latter Toda brackets contains obviously 0 and thus equals $\alpha\pi_{27}TMF$. Thus, the non-zero class β^3 must be $\pm\alpha\{\alpha\Delta\}$ (as every other class than $\{\alpha\Delta\}$ is in a filtration to high for β^3). It follows that $\beta^5 = \pm\beta^2\alpha\{\alpha\Delta\} = 0$. The only differential that can possibly kill it is $d_9(\pm\{\alpha\Delta^2\}) = \beta^5$. One can see that there is no room for further differentials or extension issues on $E_{10} = E_\infty$:





Thus, we see the homotopy groups of TMF (localized at 3 modulo transfer) here with the non-visible relation $\alpha\{\alpha\Delta\} = \beta^3$. For degree reasons that is the only "multiplicative extension".

5.9 Outlook

There is a number of applications of TMF to problems in homotopy theory and geometry (although at this point certainly not as many as of K-theory!). We want to sketch some of these.

The Adams–Novikov spectral sequence comes with a filtration on the stable homotopy groups of spheres, corresponding to the lines in the Adams–Novikov spectral sequence. The 0-line consists just of one copy of $\mathbb{Z}$ corresponding to $\pi_0\mathbb{S} = \mathbb{Z}$.

The 1-line has been completely computed and it has been shown that all elements here are permanent cycles for $p > 2$. The differentials in the case $p = 2$ are also known. It turns out that at least for odd primes all these permanent cycles lie in the *image of* J. One way to describe the image of J is via framed manifolds. Recall that $\pi_*\mathbb{S} \cong \Omega_*^{fr}$ the bordism of manifolds with a (stable) framing on their stable normal bundle (or equivalently stable tangent bundle). We know that the spheres S^n have stable framings (their normal bundle is trivial in the standard embedding). But how many stable framings does the trivial bundle on a sphere have? Every element in $\pi_n O = [S^n, O]$ defines a reframing of the standard framing. [Here, $O = \operatorname{colim} O(n)$.] One can check that this induces a homomorphism

$$J\colon \pi_n O \to \Omega_n^{fr} \cong \pi_n \mathbb{S},$$

whose image consists just of all stable framing of spheres.

Let X be an oriented smooth manifold that is homeomorphic to S^n, i.e. a (possibly) exotic sphere. One can show that its stable normal bundle is trivial as well [KM63]. Choosing framings defines a map from Θ_n (the diffeomorphism classes of X as above) to Ω_n^{fr}, which is well-defined after quotiening out the image of J. Actually, Θ_n is a group (via connected sum) and we obtaina homomorphism $\Phi\colon \Theta_n \to \mathrm{Im}(J)$.

Question 5.37. *When does Θ_n have only one element, i.e. when is every manifold homeomorphic to S^n also (orientedly) diffeomorphic to S^n?*

This is true classically for $n = 1, 2, 3$ and very much open for $n = 4$. The methods for [KM63] say a lot about the case $n \geq 5$ using the homomorphism Φ. In particular, for n odd they show that Φ is surjective. Moreover, for $n = 4k-1$, the kernel of Φ is big and Θ_n has many elements. For $n = 4k+1$ the size of $\ker(\Phi)$ is intimately related to the so-called Kervaire invariant. In monumental work by people like Browder, Mahowald, Xu and Hill–Hopkins–Ravenel, it was shown that $\ker(\Phi)$ has order 2 unless $n = 2^l - 3$ for $l \leq 6$ and possibly for $l = 7$; if it has not order 2 it is trivial. Thus, the only possibilities for odd n in the question above are $n = 2^l - 2$ for $l \leq 7$. Recently, Wang and Xu [WX16] have deduced the following:

Theorem 5.38 (Wang–Xu)**.** *The only odd n for which Θ_n is trivial are $n = 1, 3, 5$ and 61.*

This was known for $n \leq 60$. The main achievement of Wang and Xu was to show that $\pi_{61}\mathbb{S} = 0$ to get the result in dimension 61, but they also needed to show that $\pi_{125}\mathbb{S}/\mathrm{Im}(J) \neq 0$ and they used TMF for that purpose. Let us briefly sketch what they did: The homotopy groups $\pi_n\mathbb{S}$ are known for $n \leq 61$ and one also knows what their image is under the map $u\colon \mathbb{S} \to TMF$. Beyond $\pi_{61}\mathbb{S}$ we know much less and we know very little about $\pi_*\mathbb{S}$ for, say, $* \geq 90$ at the prime 2. In contrast π_*TMF is completely known. What Wang and Xu do is the following: There are classes $\overline{\kappa} \in \pi_{20}\mathbb{S}$ and $w \in \pi_{45}\mathbb{S}$ whose image is non-trivial in π_*TMF. More precisely, we know even that $u(\overline{\kappa}^4 w) = u(\overline{\kappa})^4 u(w)$ is non-trivial and thus, $\overline{\kappa}^4 w \in \pi_{125}\mathbb{S}$ must be non-trivial as well! It is not too hard to show that it is not in the image of J (as this image is completely known). Thus, $\pi_{125}\mathbb{S}/\mathrm{Im}(J) \neq 0$ and thus $\Theta_{125} \neq 0$. I know of no other way to show this. More applications in the same direction can be found in [BHHM17].

Let us shortly sketch some other directions of applications. For this, it is useful to know that there is a connective variant of tmf with $\pi_*tmf[\frac{1}{6}] \cong \mathbb{Z}[\frac{1}{6}][c_4, c_6]$ (without inverting Δ).

1. As Ando, Hopkins and Rezk [AHR10] have shown, the map $\mathbb{S} \to TMF$ factors over $MString \to TMF$, making TMF string-oriented. Actually, this factors over a map $MString \to tmf$. As shown in [Hil09], the map $MString \to tmf$ is 15-connected and thus provides a pretty good approximation to study string bordism.

2. A theorem by Hopkins and Mahowald shows that $H^*(tmf; \mathbb{F}_2) \cong \mathcal{A}/\mathcal{A}(2)$, where this is the Hopf algebra quotient of the Steenrod algebra by the sub algebra generated by $\mathrm{Sq}^1, \mathrm{Sq}^2$ and Sq^4. [Mat16] It was not known before that a spectrum with this cohomology exists; more precisely, Davis and Mahowald had shown that such a spectrum does not exist, but their proof had a mistake.

3. Much of our knowledge which elements on the 2-line of the Adams–Novikov spectral sequence are permanent cycles (at the primes 2 and 3) stems from tmf (and more precisely from the construction of certain Smith–Toda complexes accomplished by tmf). See [HM98], [BP04] and [BHHM08].

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