Integer Representations of Convex Polygon Intersection Graphs

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\textbf{Abstract.} We study the grid size that is needed to represent intersection graphs of convex polygons. Here the polygons are similar to a base polygon $P$ whose corners have rational coordinates and each corner of each polygon in the representation must lie on a point of the integer grid. We provide constructions to show that for intersection graphs of
- translated copies of any fixed parallelogram a $\Omega(n^2) \times \Omega(n^2)$ grid is needed for some graphs;
- translated copies of any other fixed convex polygon a $2^{\Omega(n)} \times 2^{\Omega(n)}$ grid is needed for some graphs;
- homothetic copies of any fixed convex polygon a $2^{\Omega(n)} \times 2^{\Omega(n)}$ grid is needed for some graphs.

We complement these results by giving a matching upper bound in each case.

\section{Introduction}

If $A = \{A_i : i \in I\}$ is a collection of sets, then the \textit{intersection graph} of $A$ is a graph $G = (I, E)$ with vertex set $I,$ and an edge $ij \in E$ if and only if $A_i \cap A_j \neq \emptyset$. If $A$ has $G$ as its intersection graph then we say that $A$ realizes $G$. We consider \textit{geometric} intersection graphs in which the $A_i$ all represent bounded geometric domains in $\mathbb{R}^2$. We are especially interested in the case where the $A_i$ are all similar to a given \textit{convex polygon}.

Intersection graphs of geometric objects in the plane have a long and rich history, see for instance \cite{22, 23}. The graphs arise e.g. in the study of wireless communication networks, where the $A_i$ model transmission ranges of devices and one is interested in the structure of interference patterns. Intersection graphs

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have been studied e.g. for the following types of geometric objects: segments [15],
continuous curves or ‘strings’ [9, 13], unit disks [5, 3], disks [11], unit squares [7],
isosceles right triangles or ‘semi-squares’ [12] and arbitrary convex sets [18, 19].

The general study of intersection graphs of convex polygons was initiated
by Kratochvíl and Pergel [16]. They proved that the recognition problem for
these graphs is NP-hard in general and raised a series of interesting questions
for further research.

**Problem description.** We now define convex polygon intersection graphs more
precisely. We will consider the case where all elements of $A$ are either translates
of a fixed convex polygon $P$ in the plane, or homothets of $P$. (A homothet of $P$
is a scaled and translated copy of $P$.) We shall call the intersection graph of a
set of translates of $P$ a $P$-translate graph, and the intersection graph of a set of
homothets of $P$ a $P$-homothets graph.

**Definition 1.** Let $\text{trans}(P)$ denote the set of all $P$-translate graphs and let $\text{hom}(P)$ denote the set of all $P$-homothets graphs. Let $\text{trans}_n(P)$ resp. $\text{hom}_n(P)$ denote the set of all translate resp. homothets graphs on $n$ vertices.

In this work, we shall restrict attention to the case when all corner points of
the polygon $P$ have rational coordinates. In this case it can be shown that for
every $G \in \text{hom}(P)$ there is collection of $P$-homothets $A = \{A(v) : v \in V(G)\}$
such that $A$ realizes $G$ and moreover, every corner of every polygon of $A$ lies on
the integer grid $\mathbb{Z}^2$. (This follows for instance from [24] or the proof of Theorem 2
below.) We shall call such an $A$ an integer homothet realization of $G$.

Similarly, if $G \in \text{trans}(P)$ then there always is a collection $A = \{A(v) : v \in V(G)\}$ where each set is of the form $A(v) = p(v) + \lambda P$ for some $\lambda \geq 0$ (that is
each $A(v)$ is a translate of $\lambda P$) such that $A$ realizes $G$ and all corner points of
the $A(v)$ lie on $\mathbb{Z}^2$. (Again this will be proved formally in the proof of Theorem 2
below.) We shall call such an $A$ an integer translate realization of $G$.

If $A$ is an integer homothet/translate realization of $G$ then we will denote

$$m(A) := \min \{ K \in \mathbb{N} : A \subseteq [-K,K]^2 \text{ for all } A \in A \}.$$  

For $G \in \text{hom}(P)$ resp. $G \in \text{trans}(P)$ we set

$$h_P(G) := \min_{A} m(A), \quad t_P(G) := \min_{A} m(A),$$

where the minimum is taken over all integer homothet resp. translate realizations
$A$ of $G$. Finally, let us set

$$h_P(n) = \max_{G \in \text{hom}_n(P)} h_P(G), \quad t_P(n) = \max_{G \in \text{trans}_n(P)} t_P(G).$$

The key problem for the complexity of integer realizations of $P$-homothet resp.$P$-translate graphs can now be formulated as follows: *determine precise upper-
and lower bounds on $t_P(n)$ and $h_P(n)$.*
Results. First of all we prove that the quantity $t_P(n)$ displays the following remarkable behaviour:

**Theorem 1.** Let $P$ be a convex polygon with rational corner points. Then the following hold for $t_P(n)$:

(i) If $P$ is a parallelogram then $t_P(n) = \Theta(n^2)$;  
(ii) If $P$ is not a parallelogram then $t_P(n) = 2^{\Theta(n)}$.

For homothets, on the other hand, we will see that there is no qualitative difference between different (kinds of) convex polygons. We will prove:

**Theorem 2.** Let $P$ be any convex polygon with rational corner points. Then $h_P(n) = 2^{\Theta(n)}$.

Part (i) of Theorem 1 represents a drastic improvement over a result of Czyzowicz et al. [7], who showed that $t_U(n) \leq 2^{n-1}$ where $U$ denotes the unit square. Theorem 2 improves over a recent bound by Van Leeuwen and Van Leeuwen [24], who showed that $h_P(n) = 2^{O(n^4)}$ for $P$ any convex polygon with rational corner points.

Related work. By now there is a relatively long history of results on the smallest piece of the integer grid needed for various kinds of drawings of graphs, going back to a seminal work by De Fraysseix et al. [8] and independently Schnyder [20] who considered straight-line drawings of planar graphs. Not much later Bienstock [1] considered the smallest piece of the grid needed for straight-line drawings of a general graphs with as few crossings as possible. For geometric intersection graphs there are results of a similar nature to Theorems 1 and 2 for segment graphs [15, 17], string graphs [14], (unit) disk graphs [17] and general convex set intersection graphs [18, 19].

However, Theorem 1 and 2 show a major difference with the analogous results on these other types of geometric intersection graphs. For example, it is known that there are segment-, resp. (unit) disk graphs that require a doubly exponential grid if all endpoints resp. the centers and radii of the disks are to be integers, and that a doubly exponential grid is always large enough [15, 17]. Our results show that for convex polygon intersection graphs, singly exponential grids are necessary and sufficient.

Finally, observe that Theorem 1 and 2 show that it is possible to give an integer representation of a $n$-node $P$-translate resp. $P$-homothets graph that can always be stored using only $O(n^2)$ bits ($O(n)$ per vertex), when $P$ has rational corner points. The representation can be used to give very concise, $O(n^2)$-size certificates for the membership in NP of the recognition problem for trans($P$) and hom($P$). This improves on earlier results about the NP-recognition of these graphs [18, 24]. Amongst other things, Theorem 2 implies that the recognition problem for the well-studied class of max-tolerance graphs (see [10]) is NP-complete. Kaufmann et al. [12] showed that the class of intersection graphs of
homothets of a fixed triangle coin with the class of so-called max-tolerance graphs. Max-tolerance graphs are generalized interval intersection graphs, where two intervals $I_i$ and $I_j$ only induce an edge in the graph when they overlap by at least $\max\{t_i,t_j\}$ where $t_i$ is an individual ‘tolerance value’ associated with each interval $I_i$. Kaufmann et al. [12] also proved that max-tolerance graph recognition is NP-hard.

Organization of the paper. The upper bounds are proved in Section 2, the lower bounds for $P$-translates in Section 3 and Section 4, and for $P$-homothets in Section 5. In Section 6 we present some possible directions for further work.

2 Upper bounds

2.1 Upper bound for translates of a parallelogram

In this section we shall prove that, if $P$ is a parallelogram with rational corner points, then $t_P(n) = O(n^2)$.

Recall that an affine transformation is a non-singular linear map followed by a translation. We leave the straightforward proofs of the following two observations to the reader.

**Lemma 1.** Let $P,Q$ be two polygons. If there is an affine transformation $T$ such that $P = T[Q]$ then $\text{hom}(P) = \text{hom}(Q)$ and $\text{trans}(P) = \text{trans}(Q)$. ■

**Lemma 2.** Let $P,Q$ be two polygons whose corner points have rational coordinates. If there is an affine transformation $T$ such that $P = T[Q]$ then $t_P(n) = \Theta(t_Q(n))$ and $h_P(n) = \Theta(h_Q(n))$. ■

This last lemma has the following consequence:

**Corollary 1.** Let $P$ be a parallelogram with rational corner points, and let $U$ denote the unit square. Then $t_P(n) = \Theta(t_U(n))$.

**Theorem 3.** If $P$ is a parallelogram with rational corner points, then $t_P(n) = O(n^2)$.

**Proof.** By Corollary 1 we can restrict attention to the case when $P = U$. There are unit intervals $I_1,\ldots,I_n$ and $J_1,\ldots,J_n$ such that $ij \in E$ if and only if $I_i \cap I_j \neq \emptyset$ and $J_i \cap J_j \neq \emptyset$. In other words, $G = (V,E_1 \cap E_2)$ with $G_1 = (V,E_1)$ and $G_2 = (V,E_2)$ unit interval graphs.

Kaufmann et al. actually prove that max-tolerance graphs are intersection graphs of semi-squares, where a semi-square is ‘a square with one half cut off along the bottom-right to top-left diagonal’. It follows from Lemma 1 below that the class of intersection graphs of scaled, translated copies of $P$ is the same for all triangles $P$. 

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By a result from Corneil et al. [6] we can assume that the endpoints of the $I_i$’s and $J_i$’s are multiples of $\frac{1}{n}$. Notice that for any component of $G_1$ of order $k$, the corresponding intervals $I_{1i}, \ldots, I_{ki}$ are contained in an interval of length at most $k$. By means of componentwise translations we can now ensure that all intervals $I_i$ are contained in $[0, n + \frac{n-1}{n}]$ (i.e. we order the components of $G_1$ arbitrarily, we simultaneously translate all intervals of first component so that $0$ is the leftmost endpoint, then we simultaneously translate all intervals of the second component so that its leftmost endpoint is $\frac{1}{n}$ to the right of the rightmost endpoint of the first component, and so on). Similarly we can assume that $J_1, \ldots, J_n \subseteq [0, n + \frac{n-1}{n}]$.

Finally, let us multiply all coordinates by $n$, to get a representation of $G$ by same-size squares in which all coordinates of the corner points are integers $\in \{0, \ldots, n^2 + n - 1\}$.

\section{Upper bound for translates and homothets}

We will now prove that, if $P$ is any convex polygon with rational corner points, then $t_P(n) = 2^{O(n)}$ and $h_P(n) = 2^{O(n)}$.

We will use the following observation.

\textbf{Lemma 3.} Let $P$ be a convex polygon with rational coordinates. Then there exists an $N = N(P) \in \mathbb{N}$ and $N$ vectors $C_i = C_i(P) = (a_i, b_i, c_i, d_i, e_i, f_i) \in \mathbb{Q}^6$ such that $((x_1, y_1) + \lambda_1 P) \cap ((x_2, y_2) + \lambda_2 P) \neq \emptyset$ if and only if $a_i x_1 + b_i y_1 + c_i \lambda_1 + d_i x_2 + e_i y_2 + f_i \lambda_2 \geq 0$ for all $i = 1, \ldots, N$.

\textbf{Proof.} We can write $P = \{x \in \mathbb{R}^2 : Ax \leq b\}$ for some matrix $A \in \mathbb{Q}^{k \times 2}$ and $b \in \mathbb{Q}^k$. Observe that $(p_1 + \lambda_1 P) \cap (p_2 + \lambda_2 P) \neq \emptyset$ if and only if there exists a $x \in \mathbb{R}^2$ such that

$$A \left( \frac{x - p_1}{\lambda_1} \right) \leq b, \text{ and } A \left( \frac{x - p_2}{\lambda_2} \right) \leq b.$$ 

Or, in other words, $(p_1 + \lambda_1 P) \cap (p_2 + \lambda_2 P) \neq \emptyset$ if and only if there exists an $x \in \mathbb{R}^2$ such that $A' x \leq b'$ where we set

$$A' := \begin{bmatrix} A \\ A \end{bmatrix}, \quad b' := \begin{bmatrix} \lambda_1 b + A p_1 \\ \lambda_2 b + A p_2 \end{bmatrix}.$$ 

By a variant of Farkas’ Lemma (Corollary 7.1e in [21]) the existence of an $x \in \mathbb{R}^2$ such that $A' x \leq b'$ is equivalent to $y' b' \geq 0$ for all $y \in W$ where $W := \{y \in \mathbb{R}^{2k} : y \geq 0, y^T A' = 0\}$. Observe that we can write

$$W = \{\mu_1 y_1 + \cdots + \mu_N y_N : \mu_1, \ldots, \mu_N \geq 0\},$$ 

where $y_1, \ldots, y_N \in \mathbb{Q}^{2k}$ are the vertices of the polytope $W' := \{y \in W : \sum y = 1\}$. (That the $y_i$s have rational coordinates follows from the fact that all entries
Theorem 4. Let $A$ be a convex polygon with rational corner points. Then $t_\mu(n) = 2^\Theta(n)$ and $h_\mu(n) = 2^\Theta(n)$.

**Proof.** By Lemma 2 we can assume without loss of generality that the corner points of $P$ lie on the integer grid. Let $C_1, \ldots, C_N \in \mathbb{Q}^d$ as be provided by Lemma 3. Let $G$ be a $P$-translate resp. $P$-homothet graph. For convenience we take $V(G) = \{1, \ldots, n\}$. We will say that $z = (x_1, y_1, \lambda_1, \ldots, x_n, y_n, \lambda_n) \in \mathbb{R}^{3n}$ realizes $G$ as a translate resp. homothet graph if, when we set $P_i := (x_i, y_i)^T + \lambda_i P$, the intersection graph of $P_1, \ldots, P_n$ is precisely $G$, where in the case of a translate representation we demand in addition that $\lambda_1 = \cdots = \lambda_n$. Observe that there are representations with all the $x_i$s, $y_i$s and $\lambda_i$s non-negative. Also observe that if $z$ realizes $G$ and $\mu > 0$ then $\mu z = (\mu x_1, \ldots, \mu \lambda_n)$ also realizes $G$. Thus, there is a realization $z_0$ such that for each $1 \leq i \leq n$ and for each $1 \leq j_1 \neq j_2 \leq n$ we have either $a_i x_{j_1} + b_i y_{j_1} + c_i \lambda_1 + d_i x_{j_2} + e_i y_{j_2} + f_i \lambda_2 \geq 0$ or $a_i x_{j_1} + b_i y_{j_1} + c_i \lambda_1 + d_i x_{j_2} + e_i y_{j_2} + f_i \lambda_2 \leq -1$.

Consider the set of $N \cdot \binom{n}{2}$ inequalities of this kind that $z_0$ satisfies. Let us write this set of inequalities as

$$Az \leq b,$$

where we also add the inequalities $x_i, y_i, \lambda_i \geq 0$ and in the translate graph case also the inequalities $\lambda_i = \lambda_{i+1}$ for $i = 1, \ldots, n-1$. This way we obtain a system with $A \in \mathbb{Q}^{m \times 3n}$, $b \in \{-1, 0\}^m$ where $m = N \binom{n}{2} + 4n - 1$ in the translate case and $m = N \binom{n}{2} + 3n$ in the homothet case.

The crucial observation for the proof is that if $z$ is another solution of (1), then $z$ also realizes $G$ as a homothets resp. translate graph. This follows from Lemma 3 and the fact that for each $1 \leq i \leq N, 1 \leq j_1 \neq j_2 \leq n$ the expression $a_i x_{j_1} + b_i y_{j_1} + c_i \lambda_1 + d_i x_{j_2} + e_i y_{j_2} + f_i \lambda_2$ is non-negative for $z$ if and only if it is non-negative for $z_0$.

The set $W := \{z \in \mathbb{R}^{3n} : Az \leq b\}$ is a polyhedron. Let $v$ be a vertex of $W$. Then $v = B^{-1} c$ is the unique solution to a system $Bz = c$ that is obtained by taking $3n$ of the inequalities of (1) and making them into equalities. By Cramer's rule

$$v_j = \det(B_j) / \det(B),$$

where $B_j$ is the matrix we get by replacing the $j$th column of $B$ by $c$. Observe that all entries of $B$ and $B_j$ are either 0 or are elements of a finite set of rationals.
$Q = \left\{ \frac{n_1}{d_1}, \ldots, \frac{n_K}{d_K} \right\}$ ($Q$ contains $\pm 1$ and $\pm$ the coefficients of $C_1, \ldots, C_N$). Let us also observe that every row of $A$ (and hence $B$) has at most $6$ nonzero elements, and hence every row of $B_j$ has at most $7$ nonzero elements.

Set

$$\mu := |(d_1 \cdots d_K)^{3n} \det(B)|, \quad w := \mu v.$$  

Then $w$ realizes $G$ by a previous remark. Recall that by the determinant formula

$$\det(B_j) = \sum_{\sigma} (-1)^{\sigma} \prod_{i=1}^{3n} (B_j)_{i\sigma(i)}, \quad (2)$$

where the sum is over all permutations of $\{1, \ldots, 3n\}$. Each nonzero summand of the right hand side of $(2)$ is a rational number whose denominator divides $(d_1 \cdots d_K)^{3n}$ (since all entries of $B_j$ are elements of $Q$). Thus, we see that $w \in \mathbb{Z}^{3n}$.

It remains to upper bound the entries of $w$. Recall that at most $7$ entries of each row of $B_j$ are non-zero. Hence in $(2)$ there are at most $7^{3n}$ summands that are not zero (corresponding to those permutations that map $i$ to a $\sigma(i)$ with $(B_j)_{i\sigma(i)}$ non-zero for all $i = 1, \ldots, 3n$). Hence

$$|w_j| = |(d_1 \cdots d_K)^{3n} \det(B_j)| \leq |(d_1 \cdots d_K)^{3n}| \cdot 7^{3n} \cdot \left( \max_{q \in Q} |q|^3 \right)^{3n} = C^n,$$

writing $C := |(d_1 \cdots d_K)^3| \cdot 7^3 \cdot \max_{q \in Q} |q|^3$. As remarked earlier, we can assume that $P$ has integer corner points. This implies that $w$ corresponds to a homothets resp. translates representation of $G$ with all corner points on integer points, and every coordinate of every corner point is $O(C^n) = 2^{O(n)}$.  

Theorem 4 considerably improves the bound given in [24]. We also note that Lemma 3 and Theorem 4 and their proofs generalize to higher dimensions. Thus, the proven bounds hold for the representation of $P$-translate and $P$-homothet graphs in any fixed, constant dimension greater than or equal to two and any convex polytope $P$ with rational corner points.

3 Lower bound for $P$-translate graphs when $P$ is a parallelogram

In this section we will prove that, if $P$ is a parallelogram (with rational corner points), then $t_P(n) = \Omega(n^2)$.

The result is obtained by constructing an infinite family of graphs $G = \{G_n\}$ that are all $U$-translate graphs and that require grids of size $K = \Omega(n^2)$, where $U$ denotes the unit square. To construct the graphs $G_n$ we first define graphs $O_n$ and $L_n$. 

Let \( O_n \) be a graph on vertex set
\[
V(O_n) := \{b, t, v_1, \ldots, v_{2n}\},
\]
with edge-set
\[
E(O_n) := \{v_i t, v_i b : i = 1, \ldots, 2n\} \cup \{v_i v_j : |i - j| < n, i \neq j\}.
\]
Then \( O_n \) is a unit square translate graph for all \( n \) (see figure 1).

\[\text{Fig. 1. The graph } O_3, \text{ and a unit square representation of it.}\]

**Lemma 4.** Suppose that \( \{P(v) : v \in V(C_4)\} \) has the 4-cycle as its intersection graph, where \( P(v) = p(v) + \lambda U \) with \( \lambda \geq 0 \) and \( U \) the unit square. Let \( u, v \in V(C_4) \) with \( uv \notin E(C_4) \). Exactly one of \( |(p(u))_x, (p(u))_x + \lambda| \cap |(p(v))_x, (p(v))_x + \lambda| \) and \( |(p(u))_y, (p(u))_y + \lambda| \cap |(p(v))_y, (p(v))_y + \lambda| \) is non-empty.

**Proof.** Since \( P(u) \) and \( P(v) \) do not intersect, at least one of the intersections \( |(p(u))_x, (p(u))_x + \lambda| \cap |(p(v))_x, (p(v))_x + \lambda| \) and \( |(p(u))_y, (p(u))_y + \lambda| \cap |(p(v))_y, (p(v))_y + \lambda| \) must be empty.

Aiming for a contradiction, let us suppose that both intersections are empty. We can assume that \( (p(u))_x > (p(v))_x + \lambda \) and \( (p(u))_y > (p(v))_y + \lambda \) (the other possibilities can be dealt with in a similar way). Let \( w, s \) be the two vertices of \( C_4 \) different from \( u, v \). Observe that \( ws \notin V(C_4) \). Since \( w \) is adjacent to both \( u \) and \( v \), we must have
\[
(p(w))_x \in [(p(u))_x - \lambda, (p(u))_x + \lambda] \cap [(p(v))_x - \lambda, (p(v))_x + \lambda] \\
\subseteq [(p(u))_x - \lambda, (p(v))_x + \lambda] \\
\subseteq [(p(v))_x, (p(v))_x + \lambda],
\]
where we have used that \( (p(u))_x > (p(v))_x + \lambda \) in the last line. Completely analogously, we have \( (p(w))_y \in [(p(v))_y, (p(v))_y + \lambda] \), and by symmetry \( (p(s))_x \in [(p(v))_x, (p(v))_x + \lambda] \) and \( (p(s))_y \in [(p(v))_y, (p(v))_y + \lambda] \). But this implies that
\[
|(p(w))_x - (p(s))_x|, |(p(w))_y - (p(s))_y| \leq \lambda.
\]
In other words, \( P(w) \cap P(s) \neq \emptyset \), a contradiction. ■
Lemma 5. Suppose that \( \{ P(v) : v \in V(O_n) \} \) has \( O_n \) as its intersection graph, where we have \( P(v) = p(v) + \lambda U \) with \( p(v) \in \mathbb{Z}^2, \lambda \in \mathbb{N} \) and \( U \) the unit square. Then \( \lambda \geq n/2 \).

**Proof.** Since \( v_1, t, v_{2n}, b \) is an induced copy of \( C_4 \) in \( O_n \), appealing to Lemma 4, we can assume without loss of generality that

\[
(p(b))_x \leq (p(t))_x \leq (p(b))_x + \lambda, \\
(p(t))_y > (p(b))_y + \lambda
\]  

We must then have that \( (p(v_i))_y \in [(p(b))_y, (p(b))_y + \lambda] \) for all \( i = 1, \ldots, 2n \), since \( v_i \) is adjacent to both \( b \) and \( t \). This implies that \( |(p(v_j))_x - (p(v_j))_x| \leq \lambda \) if and only if \( v_i v_j \in E(O_n) \). Since the closed neighbourhoods \( N(v_1) = \{ v_1, \ldots, v_n, t, b \}, N(v_2) = \{ v_1, \ldots, v_{n+1}, t, b \}, \ldots, N(v_{n+1}) = \{ v_2, \ldots, v_{2n}, t, b \} \) are all distinct (recall that the closed neighbourhood of a vertex is \( N(v) := \{ v \} \cup \{ u : uv \in E \} \)), we must then also have that \( (p(v_i))_x, \ldots, (p(v_n))_x \) are all distinct. Since \( (p(v_1))_x, \ldots, (p(v_n))_x \subseteq [(p(b))_x - \lambda, (p(b))_x + \lambda] \) (because \( bv_i \) is an edge for all \( i \)), we must have that \( 2\lambda + 1 \geq n + 1 \). Thus \( \lambda \geq n/2 \) as required. \( \blacksquare \)

Let \( L_n \) denote the graph with vertex set

\[ V(L_n) := \{ v_1, \ldots, v_n, u_1, \ldots, u_{n-1}, w_1, \ldots, w_{n-1} \}, \]

and edge set

\[ E(L_n) := \{ v_i u_i, v_i w_i, u_i u_{i+1}, w_i w_{i+1} : i = 1, \ldots, n-1 \}. \]

Then \( L_n \) is clearly a \( P \)-translate graph for all \( n \) (see figure 2).

![Fig. 2. The graph \( L_5 \), and a unit square representation of it.](image)

Lemma 6. Suppose that \( \{ P(v) : v \in V(L_n) \} \) has as its intersection graph, where \( P(v) = p(v) + \lambda U \) with \( \lambda \geq 0 \) and \( U \) the unit square. Then \( \| p(v_1) - p(v_n) \| \geq \lambda(n-1) \).

**Proof.** Since \( v_1, u_1, v_2, w_1 \) is an induced copy of \( C_4 \) in \( O_n \), appealing to Lemma 4, we can assume without loss of generality that

\[
(p(u_1))_x \leq (p(w_1))_x \leq (p(u_1))_x + \lambda, \\
(p(u_1))_y > (p(w_1))_y + \lambda
\]
We must have \((p(v_1)), y, (p(v_2)), y \in \{(p(w_1))_y, (p(w_1))_y + \lambda\}\) and, appealing to Lemma 4, we can assume without loss of generality that \((p(v_1))_x + \lambda < (p(v_2))_x\). We shall use induction to prove that in fact

\[
\begin{align*}
(p(v_i))_x + \lambda < (p(v_{i+1}))_x, \\
(p(v_i))_x, (p(w_i))_x \in [(p(v_i))_x, (p(v_i))_x + \lambda], \\
|(p(u_i))_y - (p(w_i))_y| > \lambda,
\end{align*}
\]

for \(i = 1, \ldots, n - 1\). This clearly implies the claim.

Clearly (4) is true when \(i = 1\). Suppose that (4) holds for some \(i < n - 1\). Observe that two same-size squares intersect if and only if they both contain one of the other’s corner points. Hence \(P(u_i), P(w_i), P(u_{i+1}), P(w_{i+1})\) must each contain a different corner point of \(P(v_{i+1})\). Moreover, by the second line of (4), \(P(u_i), P(w_i)\) must contain the top-left and bottom-left corners of \(P(v_{i-1})\). Hence \(P(u_{i+1}), P(w_{i+1})\) contain the top-right and bottom-right corners of \(P(v_{i+1})\). This gives \((p(u_{i+1}))_x, (p(w_{i+1}))_x \in [(p(v_{i+1}))_x, (p(v_{i+1}))_x + \lambda]\). Since \(u_{i+1}w_{i+1} \notin E(L_n)\) must then have \(|(p(u_{i+1}))_y - (p(w_{i+1}))_y| > \lambda\). Let us assume that \((p(u_{i+1}))_y > (p(w_{i+1}))_y + \lambda\) (the other case is similar). Since \(u_{i+1}, w_{i+1}\) are both adjacent to \(u_{i+1}, w_{i+1}\) we must have \((p(v_{i+1}))_y, (p(v_{i+2}))_y \in [(p(u_{i+1}))_y, (p(v_{i+1}))_y + \lambda]\). Because \(v_{i+1}v_{i+2} \notin E(L_n)\), we must then have \(|(p(v_{i+1}))_x - (p(v_{i+2}))_x| > \lambda\). Moreover, since \(P(u_{i+1})\) contain either the top-right or bottom-right corners of \(P(v_{i+1})\), we must have \((p(v_{i+1}))_x \in [(p(u_{i+1}))_x - \lambda, (p(v_{i+1}))_x]\). On the other hand, since \(v_{i+1}w_{i+1} \in E(L_n)\) we must have \((p(v_{i+2}))_x \in [(p(u_{i+1}))_x - \lambda, (p(v_{i+1}))_x + \lambda]\). Because \(v_{i+1}v_{i+2} \notin E(L_n)\) we thus have \((p(v_{i+1}))_x + \lambda < (p(v_{i+2}))_x\), so that all three lines of (4) holds also for \(i + 1\).

**Theorem 5.** If \(P\) is a parallelogram with rational corner points, then \(t_P(n) = \Omega(n^2)\).

**Proof.** By Corollary 1, it suffices to prove the theorem for the special case in which \(P\) is the unit square. Let \(G\) be the disjoint union of \(O_n\) and \(L_n\). Then \(G\) is a unit square translate graph and it has \(2n + 3n - 1 = O(n)\) vertices. Suppose that \(K\) is such that \(G\) can be represented as a \(\lambda U\)-translate graph for some \(\lambda\) with all corner points on a \(K \times K\) subgrid of \(\mathbb{Z}^2\). By Lemmas 5 and 6, we must then have \(K = \Omega(n^2)\).

### 4 Lower bound for \(P\)-translate graphs when \(P\) is not a parallelogram

We now aim to prove the following result: if \(P\) is an arbitrary convex polygon (with rational corner points) that is not a parallelogram, then \(t_P(n) = 2\Omega(n)\). In Section 4.1 we first prove an auxiliary lemma, in Section 4.2 we use it to prove the lower bound for \(P\)-translate graphs when \(P\) is not a parallelogram.
4.1 Line arrangements and sign vectors

A line $\ell$ in the plane partitions $\mathbb{R}^2 \setminus \ell$ into two parts. In an oriented line we (arbitrarily) label one of these components $\ell^+$, the “positive side”, and the other $\ell^-$, the “negative side”. An oriented line arrangement $\mathcal{L} = (\ell_1, \ldots, \ell_n)$ is a collection of oriented lines in the plane. The sign vector $\sigma(p; \mathcal{L}) \in \{-1, 0, 1\}^{|\mathcal{L}|}$ of a point $p \in \mathbb{R}^2$ with respect to an oriented line arrangement $\mathcal{L}$ is given by:

$$
(\sigma(p; \mathcal{L}))_i := \begin{cases} -1 & \text{if } p \in \ell^-_i, \\ 0 & \text{if } p \in \ell^+_i, \\ +1 & \text{if } p \in \ell^+_i. 
\end{cases}
$$

If $W \subseteq \mathbb{R}^2$ is a set of points then we write

$$
\sigma(W; \mathcal{L}) := \{\sigma(p; \mathcal{L}) : p \in W\}.
$$

For $A \subseteq \{1, \ldots, n\}$ and $\sigma \in \{-1, 0, +1\}^n$ we write

$$
\sigma|A := (\sigma_i)_{i \in A},
$$

(that is, we drop all coordinates of $\sigma$ whose indices are not in $A$). Similarly we define, for $S \subseteq \{-1, 0, +1\}^n$, $S|A := \{\sigma|A : \sigma \in S\}$.

The slope of a line or line segment is an angle $s \in (-\pi, \pi]$ where $s = 0$ means “horizontal” and $s = \frac{\pi}{2}$ means “vertical”.

**Lemma 7.** For $k \geq 3$, let $s_1, \ldots, s_k$ be distinct slopes. There are constants $\alpha = \alpha(s_1, \ldots, s_k) > 1$ and $m_0 = m_0(s_1, \ldots, s_k) \in \mathbb{N}$ such that the following hold. For all $m \geq m_0$ there exists a set $S \subseteq \{-1, +1\}^{3m}$ with $|S| = 3m$ such that

(i) There exists an oriented line arrangement $\mathcal{L}$ and a set of points $W \subseteq \mathbb{R}^2$ with

(a) $\sigma(W; \mathcal{L}) = S$, and;

(b) Each line of $\mathcal{L}$ has slope $s_1, s_2$ or $s_3$;

(ii) For each $A \subseteq \{1, \ldots, 3m\}$ with $|A| \geq 2.99 \cdot m$ and every oriented line arrangement $\mathcal{L}$ and point set $W \subseteq \mathbb{R}^2$ with

(a) $\sigma(W; \mathcal{L})|A = S|A$, and;

(b) Each line of $\mathcal{L}$ has slope $s_1, \ldots, s_{k-1}$ or $s_k$, there exist distinct points $p, q, r, s \in W$ such that $\|p - q\| / \|r - s\| > \alpha^m$.

**Proof.** Let $T$ be a triangle whose sides have slopes $s_1, s_2, s_3$. Observe that with slopes $s_1, s_2, s_3$ two different kinds of triangles can be formed, those homothetic to $T$ and those homothetic to $-T$, the reflection of $T$ through the origin.

We start with a sequence of polygons $T_1 \supseteq T_2 \supseteq \cdots \supseteq T_m$, where $T_i$ is a homothet of $T$ when $i$ is odd and a homothet of $-T$ if $i$ is even. We also require that $T_{i+1} \subseteq \text{int}(T_i)$ for all $i = 1, \ldots, m - 1$.

We now place oriented lines $\ell_1, \ldots, \ell_{3m}$ in such a way that

$$
T_i = \ell_{3i-2} \cap \ell_{3i-1} \cap \ell_{3i},
$$

for $i = 1, \ldots, m$. Then we have

$$
\sigma(W; \mathcal{L}) = S,
$$

and every line of $\mathcal{L}$ has slope $s_1, s_2, s_3$. Moreover, for each set of indices $A \subseteq \{1, \ldots, 3m\}$ with $|A| \geq 2.99 \cdot m$, there exist distinct points $p, q, r, s \in W$ such that $\|p - q\| / \|r - s\| > \alpha^m$.
Fig. 3. With three slopes, two kinds of triangles can be formed up to translation and scaling.

and $\ell_{3i-2}$ has slope $s_1$, $\ell_{3i-1}$ has slope $s_2$ and $\ell_{3i}$ has slope $s_3$. Let us also place points $p_1, \ldots, p_{3m}$ such that:

\[
\begin{align*}
    p_{3i-2} &\in T_{i-1} \cap \ell_{3i-2}^- \cap \ell_{3i-1}^+ \cap \ell_{3i}^+; \\
    p_{3i-1} &\in T_{i-1} \cap \ell_{3i-2}^+ \cap \ell_{3i-1}^- \cap \ell_{3i}^+; \\
    p_{3i} &\in T_{i-1} \cap \ell_{3i-2}^- \cap \ell_{3i-1}^+ \cap \ell_{3i}^-; \\
\end{align*}
\]

where we set $T_0 := \mathbb{R}^2$. See figure 4 for a depiction of the construction.

Fig. 4. The construction of $S$ in Lemma 7.

Let us set $\mathcal{L} := (\ell_1, \ldots, \ell_{3m})$ and $S := \{\sigma(p_i) : i = 1, \ldots, 3m\}$. This finishes the construction of $S$. Let us observe that part (i) of the lemma holds by construction.

Now let $A \subseteq \{1, \ldots, 3m\}$, $\tilde{W} = \{\tilde{p}_1, \ldots, \tilde{p}_{3m}\} \subseteq \mathbb{R}^2$ and $\tilde{\mathcal{L}} = (\tilde{\ell}_1, \ldots, \tilde{\ell}_{3m})$ be as in part (ii) of the Lemma. Let us set

\[
J := \{i \in \mathbb{N} : i \text{ odd, and } \{3i-2, 3i+3\} \subseteq A\} \cup \{i \in \mathbb{N} : i \text{ even, and } \{3i-5, 3i\} \subseteq A\}.
\]
and let us relabel $J$ as $j_1 < j_2 < \cdots < j_m$. Then the following properties hold:

(i) $3j_i - 2, 3j_i - 1, 3j_i \in A$ for all $i = 1, \ldots, m$;
(ii) $j_i$ is even if $i$ is even and odd if $i$ is odd;
(iii) $m > 0.98m$.

(The first two properties follow directly from the definition of $J$ and the last follows because $|A| > 2.99m$ so that at most $0.01m$ pairs $1 \leq 2k - 1, 2k \leq m$ are missing from $J$.) Let us set $\hat{A} := \{3j - 2, 3j - 1, 3j : j \in J\}$. Observe that, up to a relabeling, $\hat{S}|\hat{A}$ is exactly set of sign vectors $\hat{S}$ we would have obtained if we had started the construction with $T_{j_1} \supseteq T_{j_2} \supseteq \cdots \supseteq T_{j_m}$ instead of $T_1 \supseteq \cdots \supseteq T_m$. Moreover, again up to a relabeling of the coordinates, this is exactly the set of sign vectors we would have ended up with if we had started with $T_1 \supseteq \cdots \supseteq T_m$.

For the remainder of the proof let us relabel the points of $\hat{W}$ and the lines of $\hat{L}$ in such a way that $j_i = i$, and discard all lines and points not corresponding to $J$.

For $i = 1, \ldots, m$ let us set

$$
\tilde{T}_i := \tilde{\ell}_{3i-2} \cap \tilde{\ell}_{3i-1} \cap \tilde{\ell}_{3i}.
$$

We have following observations:

Claim C-1. $\tilde{T}_i$ is a triangle.

Proof of Claim C-1. Since $\tilde{T}_i$ is the intersection of three half-planes, it suffices to show that $\tilde{T}_i \neq \emptyset$ and that no two of the lines $\tilde{\ell}_{3i-2}, \tilde{\ell}_{3i-1}, \tilde{\ell}_{3i}$ are parallel. That $\tilde{T}_i \neq \emptyset$ follows for instance from $\tilde{p}_{3i+1} \in \tilde{T}_i$ (recall that $\sigma(\tilde{p}_{3i+1}; \tilde{L})$ has $-1$ in positions $3i - 2, 3i - 1, 3i$ by construction of $\hat{S}$).

That $\tilde{\ell}_{3i-2}$ and $\tilde{\ell}_{3i-1}$ cross follows from the fact that $\tilde{p}_{3i-2}, \tilde{p}_{3i-1}, \tilde{p}_{3i}$ and $\tilde{p}_{3i+1}$ attain all possible signs with respect to $\tilde{\ell}_{3i-2}$ and $\tilde{\ell}_{3i-1}$. Similarly, $\tilde{\ell}_{3i-2}$ and $\tilde{\ell}_{3i-1}$ and $\tilde{\ell}_{3i}$ cross.

Claim C-2. The sides of $\tilde{T}_i$ have slopes in $s_1, \ldots, s_k$.

Proof of Claim C-2. This is immediate from the fact that the sides of $\tilde{T}_i$ are the lines $\tilde{\ell}_{3i-2}, \tilde{\ell}_{3i-1}, \tilde{\ell}_{3i}$.

Claim C-3. $\tilde{T}_{i+1} \subseteq \tilde{T}_i$.

Proof of Claim C-3. It is clear that

$$
\tilde{T}_{i+1} \subseteq \text{conv}(\{\tilde{p}_{3i+1}, \tilde{p}_{3i+2}, \tilde{p}_{3i+3}\}),
$$

since the situation is as depicted in figure 5.

The claim nows follows from $\tilde{p}_{3i+1}, \tilde{p}_{3i+2}, \tilde{p}_{3i+3} \in \tilde{T}_i$. (This can again be seen from the sign vectors $\sigma(\tilde{p}_{3i+1}; \tilde{L}), \sigma(\tilde{p}_{3i+2}; \tilde{L}), \sigma(\tilde{p}_{3i+3}; \tilde{L})$.)

Claim C-4. $\tilde{T}_{i+1}$ is not a homothet of $\tilde{T}_i$. 


Proof of Claim C-4. Let the slopes of $\ell_{3i-2}', \ell_{3i-1}', \ell_{3i}'$ be $s_{i_1}, s_{i_2}, s_{i_3}$, and let the slopes of $\ell_{3i+1}', \ell_{3i+2}', \ell_{3i+3}'$ be $s_{j_1}, s_{j_2}, s_{j_3}$.

If $\{i_1, i_2, i_3\} \neq \{j_1, j_2, j_3\}$ then we are clearly done, so let us assume that $\{i_1, i_2, i_3\} = \{j_1, j_2, j_3\}$. Observe that, by construction

$$\tilde{p}_{3i-2}, \tilde{p}_{3i-1} \in \tilde{\ell}_{3i+1}, \quad \tilde{p}_{3i} \in \tilde{\ell}_{3i+1}^+, \quad \tilde{p}_{3i+1} \in \tilde{\ell}_{3i+1}, \quad \tilde{p}_{3i+2}, \tilde{p}_{3i+3} \in \tilde{\ell}_{3i+1}^+.$$  \hspace{1cm} (5)

(See figure 4.) For convenience, let us assume $s_{i_1}$ corresponds to “horizontal” and that $\tilde{\ell}_{3i-2}$ lies “above” $\tilde{\ell}_{3i-1}$. (Hence $\tilde{p}_{3i-2}$ and $\tilde{T}_i$ are also above $\tilde{\ell}_{3i-2}$.)

First suppose that $j_1 = i_2$. (In other words the slope of $\tilde{\ell}_{3i+1}$, is equal to the slope of $\tilde{\ell}_{3i-1}$.) Observe that $\tilde{\ell}_{3i+1}$ must pass through $\tilde{T}_i$ since it separates $\tilde{p}_{3i+1}$ from $\tilde{p}_{3i+2}, \tilde{p}_{3i+3}$. Since it is parallel to $\tilde{\ell}_{3i-1}$, it must therefore separate $\tilde{p}_{3i-1}$ from $\tilde{p}_{3i}$ (see figure 6).
But this contradicts (5). Hence we must have $j_1 \neq i_2$. Similarly, we cannot have that $j_1 = i_3$.

Hence we must have $i_1 = j_1$. In this case $\ell_{3i-2}, \ell_{3i+1}$ are both horizontal. The relations (5) give that $\tilde{p}_{3i+1}$ lies below $\ell_{3i+1}$ and $\tilde{p}_{3i+2}, \tilde{p}_{3i+3}$ lie above it. (see figure 7.)

**Fig. 7.** The triangle $\tilde{T}_i$ and the line $\tilde{\ell}_{3i+1}$ when $j_1 = i_1$.

Hence $\tilde{T}_{i+1}$ lies below $\tilde{\ell}_{3i+1}$. This means that $\tilde{T}_i, \tilde{T}_{i+1}$ are not homothets (since in one the horizontal segment is the lowest part of the triangle and in the other the horizontal segment is it the highest part of the triangle).

Now let $D_1, \ldots, D_K$ with $K = 2^k(3)$ be all possible triangles, up to translations and scalings, that have slopes $\in \{s_1, \ldots, s_k\}$. Let us set

$$\beta := \max_{i \neq j} \max_{D' \subseteq H(D_j)} \frac{\text{area}(D')}{{\text{area}}(D_j)};$$

where $H(A) := \{x + \lambda A : x \in \mathbb{R}^2, \lambda > 0\}$ denotes the set of all $A$-homothets. Clearly $\beta < 1$ and $\beta$ depends only on $s_1, \ldots, s_k$. By Claims C-1 to C-4 we now have

$$\text{area}(\tilde{T}_{i+1}) \leq \beta \cdot \text{area}(\tilde{T}_i) \quad \text{for all } i = 1, \ldots, \tilde{m} - 1.$$

There must be a set $J' \subseteq J$ with $|J'| \geq \tilde{m}/K$ such that $\tilde{T}_i, \tilde{T}_j$ are homothets for all $i, j \in J'$. So in particular there are indices $1 \leq i_1, i_2 \leq \tilde{m}$ such that $i_2 \geq i_1 + \tilde{m}/K - 1$ and $T_{i_1}, T_{i_2}$ are homothets. Then we have that

$$\text{area}(\tilde{T}_{i_2}) \leq \beta^{\tilde{m}/K - 1} \cdot \text{area}(\tilde{T}_{i_1}). \quad (6)$$

Since $\tilde{T}_{i_1}, \tilde{T}_{i_2}$ are homothets we must have

$$\left(\frac{\text{diam}(\tilde{T}_{i_1})}{\text{diam}(\tilde{T}_{i_2})}\right)^2 = \frac{\text{area}(\tilde{T}_{i_1})}{\text{area}(\tilde{T}_{i_2})}.$$
Together with (6) this gives:

\[ \text{diam}(T_{i2}) \leq \beta^{\hat{m}/K-1} \cdot \text{diam}(T_{i1}). \]

Since \( \tilde{T}_{i1} \subseteq \text{conv}(\{\tilde{p}_{3i_1-2}, \tilde{p}_{3i_1-1}, \tilde{p}_{3i_1}\}) \), we can assume (up to relabelling) that \( \|\tilde{p}_{3i_1-2} - \tilde{p}_{3i_1-1}\| \geq \text{diam}(T_{i1}). \) Since \( \tilde{p}_{3i_2+1}, \tilde{p}_{3i_2+2} \in T_{i2} \) are distinct (the line \( \ell_{3i_2+1} \) separates them), we have that \( \|\tilde{p}_{3i_2+1} - \tilde{p}_{3i_2+2}\| > 0. \) Setting \( \alpha := \beta^{-1/2}K \), we see that

\[ \frac{\|\tilde{p}_{3i_1-2} - \tilde{p}_{3i_1-1}\|}{\|\tilde{p}_{3i_2+1} - \tilde{p}_{3i_2+2}\|} \geq \beta^{-\hat{m}/K+1} \geq \alpha^m, \]

for \( m \) sufficiently large. This concludes the proof of the lemma. \( \blacksquare \)

4.2 A family of \( P \)-translate graphs

We are now ready to describe the construction of a \( P \)-translate graph that will need a large grid size. Let us set

\[ B := P + (-P). \]

(Here we use the notation \( A + B = \{p + q : p \in A, q \in B\} \). Observe that:

\[ (P + z_1) \cap (P + z_2) \neq \emptyset \text{ if and only if } z_1 \in z_2 + B. \] \hspace{1cm} (7)

Let us also observe that \( B \) is centrally symmetric with respect to the origin, that \( B \) will have two sides for each slope that occurs as the slope of some side of \( P \).

Let \( W = \{p_1, \ldots, p_{3m}\}, \mathcal{L} = (\ell_1, \ldots, \ell_{3m}) \) be as provided by part (i) of Lemma 7, where \( s_1, \ldots, s_k \) are the slopes occurring on the sides of \( P \) (and \( B \)).

Now let \( \varepsilon_1, \varepsilon_2 > 0 \) be small constants with \( \varepsilon_1 \ll \varepsilon_2 \), to be chosen later, and let \( N \in \mathbb{N} \) be large. We can assume without loss of generality that \( \text{diam}(W) < \varepsilon_1 \).
Let us pick vectors \( v_1, v_2, v_3 \) of length \( \|v_i\| = \varepsilon_2 \) where the slope of \( v_i \) is \( s_i \). For \( 1 \leq i \leq 3m \) and \( x = (x_1, x_2, x_3) \in \{0, \ldots, N\}^3 \) let us set
\[
\ell_i(x) := \ell_i + x_1 v_1 + x_2 v_2 + x_3 v_3.
\]
That is, we translate the point resp. oriented line by the vector \( x_1 v_1 + x_2 v_2 + x_3 v_3 \).

Here we "keep the orientation" in the obvious way, i.e. in the sense that \( \ell_i(x) = \ell_i + x_1 v_1 + x_2 v_2 + x_3 v_3 \). Let us also write \( W_x := \{p_1(x), \ldots, p_{3m}(x)\} \),
\[ L_x := (\ell_1(x), \ldots, \ell_{3m}(x)). \]
Clearly
\[ \sigma(W_x; L_x) = \sigma(W; L), \]
for all \( x \). Also observe that the same oriented line occurs in several \( L_x \), because if \( \ell_i \) has slope \( s_1 \) then
\[ \ell_i(0, x_2, x_3) = \ell_i(1, x_2, x_3) = \cdots = \ell_i(N, x_2, x_3), \]
for all \( x_2, x_3 \). Similar statements hold when \( \ell_i \) has slope \( s_2 \) or \( s_3 \).

For each \( i = 1, \ldots, 3m \) and each \( x \in \{0, \ldots, N\}^3 \), we now place points \( c_i^-(x), c_i^+(x) \) such that, writing \( B_i^-(x) := B + c_i^-(x), B_i^+(x) := B + c_i^+(x) \), we have:

(i) \( B_i^-(x) \cap B_i^+(x) = \emptyset \);
(ii) \( W_x \cap \ell_i^+(x) = W_x \cap B_i^+(x) \);
(iii) \( W_x \cap \ell_i^-(x) = W_x \cap B_i^-(x) \);
(iv) If \( \ell_i(x) = \ell_i(y) \) then \( c_i^+(x) = c_i^+(y) \) and \( c_i^-(x) = c_i^-(y) \).

It is not difficult to see that this is possible, provided we chose \( \varepsilon_1, \varepsilon_2 \) sufficiently small. (See figure 9.) Provided we chose \( \varepsilon_2 \) sufficiently larger than \( \varepsilon_1 \), we can assume that \( \text{conv}(W_x) \cap \text{conv}(W_y) = \emptyset \) whenever \( d_H(x, y) = 1 \) where \( d_H(\ldots) \) denotes Hamming distance (in other words, \( x, y \) differ in exactly one coordinate).

For each pair \( x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \) of Hamming distance one, let us place points \( a(x, y), b(x, y) \) in such a way that \( B + a(x, y) \) and \( B + b(x, y) \) are disjoint, and \( W_x \subseteq B + a(x, y) \) and \( W_y \subseteq B + b(x, y) \). (Provided \( \varepsilon_2 \) was chosen sufficiently larger than \( \varepsilon_1 \) this can be done). Now define
\[
W := \{p_i(x), c_i^+(x), c_i^-(x) : i = 1, \ldots, 3m, x \in \{0, \ldots, N\}^3\} \\
\cup\{a(x, y), b(x, y) : x, y \in \{0, \ldots, N\}^3 \text{ and } d_H(x, y) = 1\},
\]
(where \( d_H(\ldots) \) denotes the Hamming distance) and let \( \mathcal{A} \) define the corresponding family of \( P \)-translates:
\[ \mathcal{A} := \{p + P : p \in W\}, \]
and let \( G_{m, N} \) denote the intersection graph defined by \( \mathcal{A} \). Observe that \( G_{m, N} \) has \( \Theta(mN^3) \) vertices.
Theorem 6. Let $P$ be an arbitrary polygon with rational corner points that is not a parallelogram, then $t_P(n) = 2^\Omega(n)$.

Proof. Set $N := 10^{10} \cdot k$, and let $m \in \mathbb{N}$ be arbitrary. Let $G_{m,N}$ be as constructed above, and suppose that for some $\lambda \geq 0$ there is a realization $\tilde{A} := \{ p + \lambda P : p \in \tilde{W} \}$ of $G_{m,N}$, where each corner point of each translate $p + \lambda P$ lies on $\mathbb{Z}^2$.

For convenience let us write $\tilde{W}_x := \{ \tilde{p}_i(x) : i = 1, \ldots, 3m, x \in \{0, \ldots, N\} \} \cup \{ \tilde{a}(x,y), \tilde{b}(x,y) : x, y \in \{0, \ldots, N\} \} \} \cup \{ \tilde{a}(x,y), \tilde{b}(x,y) : x, y \in \{0, \ldots, N\} \} \} \} \cup \{ \tilde{a}(x,y), \tilde{b}(x,y) : x, y \in \{0, \ldots, N\} \} \}$ in the obvious way, and let us set $\tilde{W}_x := \{ \tilde{p}_i(x) : i = 1, \ldots, 3m \}$. We have:

Claim D-1. If $d_H(x,y) = 1$ then $\text{conv}(\tilde{W}_x) \cap \text{conv}(\tilde{W}_y) = \emptyset$.

Proof of Claim D-1. Set $\tilde{B}_a := \tilde{B} + \tilde{a}(x,y), \tilde{B}_b := \tilde{B} + \tilde{b}(x,y)$. By construction of $\tilde{W}$ and (8), we must have that

$$\tilde{W}_x \subseteq \tilde{B}_a \setminus \tilde{B}_b, \quad \tilde{W}_y \subseteq \tilde{B}_b \setminus \tilde{B}_a.$$
Thus, \(\tilde{W}_x\) and \(\tilde{W}_y\) lie in disjoint open half-spaces \(\ell^-, \ell^+\). This implies that \(\text{conv}(\tilde{W}_x)\) and \(\text{conv}(\tilde{W}_y)\) are disjoint.

For the remainder of the proof, let us thus suppose that \(\tilde{B}_a \cap \tilde{B}_b \neq \emptyset\) but \(\text{int}(\tilde{B}_a) \cap \text{int}(\tilde{B}_b) = \emptyset\). In this case we must have \(\tilde{B}_a \cap \tilde{B}_b = \partial \tilde{B}_a \cap \partial \tilde{B}_b\).

If \(|\tilde{B}_a \cap \tilde{B}_b| = 1\) then the intersection point must be a corner of either \(\tilde{B}_a\) or \(\tilde{B}_b\), say \(\tilde{B}_a\). Then \(\tilde{B}_a \setminus \tilde{B}_b\) is convex. By the hyperplane separation theorem for convex sets, there is a line \(\ell\) such that \(\tilde{B}_b \subseteq \ell^+ \cup \ell\) and \(\tilde{B}_a \setminus \tilde{B}_b \subseteq \ell^-\), which implies the claim.

Hence we suppose that \(|\tilde{B}_a \cap \tilde{B}_b| > 1\). Observe that \(\tilde{B}_a \cap \tilde{B}_b\) cannot contain points on two separate sides of the convex polygon \(\tilde{B}_a\), because then it would also contain the line segment between these two points which intersects \(\text{int}(\tilde{B}_a)\). Thus we see that \(\tilde{B}_a \cap \tilde{B}_b\) is a segment common to a side of each of them. Those two sides must clearly be contained in the the line \(\ell\) through the segment \(\tilde{B}_a \cap \tilde{B}_b\).

(See figure 11.)

Since \(\tilde{W}_x, \tilde{W}_y\) might intersect \(\ell\), but on different sides of the segment \(\tilde{B}_a \cap \tilde{B}_b\), we see that again \(\text{conv}(\tilde{W}_x) \cap \text{conv}(\tilde{W}_y) = \emptyset\). ■

**Claim D-2.** \(\text{conv}(\tilde{W}_x) = \text{conv}\{\hat{p}_1(x), \hat{p}_2(x), \hat{p}_3(x)\}\) for all \(x\).

**Proof of Claim D-2.** For \(i = 1, 2, 3\), let \(\ell_i\) denote an oriented line with \(\tilde{B}_i^+(x) \setminus \bar{B}_i^- (x) \subseteq \ell_i^+\) and \(\bar{B}_i^- (x) \setminus \tilde{B}_i^+ (x) \subseteq \ell_i^-\) (such lines exist as we saw in the proof of Claim D-1). From (8) and the construction of \(G_{m,N}\), it follows that

\[
\hat{p}_1(x) \in \ell_1^+ (x) \cap \ell_2^- (x) \cap \ell_3^- (x), \\
\hat{p}_2(x) \in \ell_1^+ (x) \cap \ell_2^- (x) \cap \ell_3^- (x), \\
\hat{p}_3(x) \in \ell_1^+ (x) \cap \ell_2^- (x) \cap \ell_3^- (x),
\]

which implies the claim (see figure 12). ■

Let us say that \(1 \leq i \leq 3m\) is bad for \(x\) if \(\text{conv}(\tilde{W}_x)\) contains a corner of \(\bar{B}_i^+ (x)\) or a corner of \(\tilde{B}_i^- (x)\). We will say that \(i\) is good for \(x\) if it is not bad for \(x\).
Claim D-3. If \( i \) is good for \( x \), then there is an oriented line \( \ell \) with slope \( s \in \{s_1, \ldots, s_k\} \) such that \( \tilde{W}_x \cap \ell^- = \tilde{W}_x \cap \tilde{B}_i^- (x) \) and \( \tilde{W}_x \cap \ell^+ = \tilde{W}_x \cap \tilde{B}_i^+ (x) \).

Proof of Claim D-3. By (8) the points \( \tilde{p}_j(x) \) lie in exactly one of \( \tilde{B}_i^- (x) \), \( \tilde{B}_i^+ (x) \) for \( j = 1, 2, 3 \). Assume that

\[
\tilde{B}_i^- (x) \cap \{\tilde{p}_1(x), \tilde{p}_2(x), \tilde{p}_3(x)\} = \{\tilde{p}_1(x)\}.
\]

(The other cases a similar.) Let \( t_1 \) denote point of \( [\tilde{p}_1(x), \tilde{p}_2(x)] \cap \tilde{B}_i^+ (x) \) closest to \( \tilde{p}_1(x) \) and let \( t_2 \) denote the point of \( [\tilde{p}_1(x), \tilde{p}_3(x)] \cap \tilde{B}_i^+ (x) \) closest to \( \tilde{p}_1(x) \). Observe that \( t_1, t_2 \in \partial \tilde{B}_i^+ (x) \). Let \( \ell \) denote the line through \( t_1 \) and \( t_2 \). (See figure 13.)

There can be no point of \( \tilde{B}_i^+ (x) \cap \text{conv}([\tilde{p}_1(x), \tilde{p}_2(x), \tilde{p}_3(x)]) \) on the same side of \( \ell \) as \( \tilde{p}_1(x) \), because otherwise a corner of \( \tilde{B}_i^+ (x) \) would be contained in \( \text{conv}(\tilde{W}_x) \). Thus, the segment \([t_1, t_2]\) is contained in \( \partial \tilde{B}_i^+ (x) \) and \( \ell \) supports a side of \( \tilde{B}_i^+ (x) \). In particular \( \ell \) has slope \( s \in \{s_1, \ldots, s_k\} \). We can orient \( \ell \) such that...
Claim D-3. \( \bar{B}^+_i(x) \subseteq \ell^+ \cup \ell \). Translating \( \ell \) very slightly in case some points of \( \bar{W}_x \) lie on it, we get an oriented line that is as required.

Claim D-4. There exists an \( x \in \{1, \ldots, N\}^3 \) such that at least \( 2.99 \cdot m \) indices \( i \) are good for \( x \).

Proof of Claim D-4. We shall first give an upper bound on the number of pairs \((i, x)\) such that \( i \) is bad for \( x \). We will show that

\[
|\{ (i, x) : i \text{ bad for } x \} | \leq 3m \cdot 4k \cdot (N+1)^2. \tag{9}
\]

In order to prove (9), let us pick an arbitrary \( 1 \leq i \leq 3m \). We can suppose that in the original line arrangement of part (i) of Lemma 7 the line \( \ell_i \) has slope \( s_1 \). (The other cases are analogous.)

For each \((a, b) \in \{0, \ldots, N\}^2\), let us set \( X(a, b) := \{ x \in \{0, \ldots, N\}^3 : x_2 = a, x_3 = b \} \). Observe that the sets \( X(a, b) \) partition \( \{0, \ldots, N\}^3 \). Recall that \( \tilde{c}^+_i(x) = \tilde{c}^+_i(y) \) and \( \tilde{c}^-_i(x) = \tilde{c}^-_i(y) \) for all \( x, y \in X(a, b) \). Since \( \tilde{B}^-_i(x) \) and \( \tilde{B}^+_i(x) \) both have \( 2k \) corners, and the sets \( \text{conv}(\bar{W}_x) : x \in X(a,b) \) are disjoint by Claim D-1, we see that \( i \) is bad for at most \( 4k \) elements of \( X(a,b) \). This gives that \( i \) is bad for at most \( 4k \cdot (N+1)^2 \) of all \( x \in \{0, \ldots, N+1\}^3 \). Since \( i \) was arbitrary, we get (9).

Because there are a total of \( (N+1)^3 \) choices of \( x \), there is some \( x \) for which at most

\[
\frac{3m \cdot 4k \cdot (N+1)^2}{(N+1)^3} = \frac{3m \cdot 4k}{N+1} < 0.01m,
\]

indices \( i \) are bad (recall that \( N = 10^{10} \cdot k \)). This proves the claim.

Now let \( x \) be a provided by Claim D-4. Taking the lines provided by Claim D-3 for the good indices, and adding arbitrary lines (with slopes \( \in \{s_1, \ldots, s_k\} \)) for the bad indices we get an oriented line arrangement \( \bar{L} \) such that \( \bar{L}, \bar{W}_x \) satisfy the conditions of part (ii) of Lemma 7. Thus, there are distinct \( p, q, r, s \in \bar{W}_x \) such that

\[
\|p - q\| \geq \alpha^m \|r - s\| \geq \alpha^m,
\]
using that $r - s$ is exactly the difference vector between a corner of $r + \tilde{P}$ and a corner of $s + \tilde{P}$, so that $r - s \in \mathbb{Z}^2$ and hence $\|r - s\| \geq 1$. Hence if all corner points are contained in $[-K,K]^2$ then we must have $K \geq \frac{\alpha^m}{2\sqrt{2}} = 2^{\Omega(m)}$. Since $|V(G_m,N)| = O(m)$ this concludes the proof.

\section{Lower bound for $P$-homothet graphs}

In this section, we will prove the following result: let $P$ be an arbitrary polygon with rational corner points, then $h_P(n) = 2^{\Omega(n)}$.

For $m \in \mathbb{N}$ let $T_m$ denote the triangular prism of height $m$. That is, $T_m$ is the graph we get if we take $m$ vertex-disjoint triangles $C_1, \ldots, C_m$ and we add matchings between $V(C_i)$ and $V(C_{i+1})$ for all $i = 1, \ldots, m-1$. For $k, m \in \mathbb{N}$ let $T_{k,m}$ denote the graph that is obtained from $T_m$ by subdividing the edges of the triangles once, and subdividing the edges of the matchings $k$ times. (see figure 14).

![Fig. 14. The triangular prism $T_4$ of height 4 and the graph $T_{2,4}$.](image)

We will rely on the following observation. (See figure 15.)

**Lemma 8.** Let $P$ be an arbitrary polygon. Then there is a $k = k(P)$ such that $T_{k,m}$ is a $P$-homothets graph for all $m \geq 1$.

We leave the straightforward proof of Lemma 8 to the reader. To prove that $T_{k,m}$ needs a large grid size, we will first need to prove a number of intermediate results.

Recall that a planar embedding of a planar graph $G$ is a set of distinct points $\{p(v) : v \in V(G)\}$ in the plane, together with simple (i.e. non self-intersecting) curves $\{c(e) : e \in E(G)\}$ such that $c(uv)$ has endpoints $p(u), p(v)$ and two of the curves $c(e), c(f)$ do not intersect except possibly at a common endpoint.

**Lemma 9.** Suppose that $A = \{A(v) : v \in T_{k,m}\}$ is a family of closed convex sets in the plane, each with nonempty interior, that realize $T_{k,m}$ as their intersection.
We shall first define sets $B(v)$ such that $B(v) = A(v)$ if $d(v) = 3$ and $B(v) \subseteq A(v)$ is a line segment when $d(v) = 2$, and moreover

$$|B(u) \cap B(v)| = \begin{cases} 1 & \text{if } uv \in E(T_{k,m}), \\ 0 & \text{otherwise}. \end{cases}$$

(10)

Order the vertices as $v_1, \ldots, v_n$ arbitrarily. We shall define the sets $B(v) = B^{(i)}(v)$ by means of the following procedure. We set $B^{(0)}(v) = A(v)$ for all $v \in V(T_{k,m})$. At step $1 \leq i \leq n$, we consider the vertex $v_i$. If $d(v_i) = 3$ then we set $B^{(i)}(v) = B^{(i-1)}(v)$ for all $v$. If $d(v_i) = 2$ then we do the following. Write $N(v_i) = \{u, w\}$. We pick points $a \in B^{(i-1)}(v_i) \cap B^{(i-1)}(u), b \in B^{(i-1)}(v_i) \cap B^{(i-1)}(w)$ of smallest possible distance, and we put $B^{(i)}(v_i) := [a, b]$ and $B^{(i)}(v) = B^{(i-1)}(v)$ for all other $v$. A straightforward inductive argument shows that the sets $B(v)$ satisfy (10).

For $uv \in E(T_{k,m})$, let $q(uv)$ be the unique point of $B(u) \cap B(v)$. For $v \in V(T_{k,m})$ with degree $d(v) = 2$ we define a point $p(v)$ to be the midpoint of the segment $[q(uv), q(uw)] \subseteq B(v)$ where $u, w$ are the two neighbours of $v$. For $d(v) = 3$ then we do the following. Write $N(v) = \{u, w, s\}$, and $B(v) \cap B(u) = \{a\}, B(v) \cap B(w) = \{b\}, B(v) \cap B(s) = \{c\}$. Observe that $a, b, c \in \partial A(v)$ (recall $B(v) = A(v)$), since $B(u), B(w), B(s)$ are segments that intersect $A(v)$ in exactly one point. Let $\ell_1, \ell_2, \ell_3$ denote the lines through $a$ resp. $a, c$ resp. $a, c$. We now pick an arbitrary $p(v) \in \text{int}(A(v)) \setminus (\ell_1 \cup \ell_2 \cup \ell_3)$.

By construction, for every two edges $uv, st \in E(T_{k,m})$:

![Graph](image-url)
\[ [p(v), q(uv)] \cap [p(s), q(st)] = \begin{cases} [p(v), q(uv)] & \text{if } v = s, u = t, \\ \{p(v)\} & \text{if } v = s, u \neq t \\ \{q(uv)\} & \text{if } v = t, u = s, \\ \emptyset & \text{otherwise.} \end{cases} \tag{11} \]

To see this recall that \([p(v), q(uv)] \subseteq B(v), [p(s), q(st)] \subseteq B(s)\), so that \([p(v), q(uv)] \cap [p(s), q(st)] \neq \emptyset\) implies that either \(v = s\) or \([p(v), q(uv)] \cap [p(s), q(st)] = \{q(vs)\}\).

The statement (11) now follows from the fact that the segments \([p(v), q(uv)]\) with \(w \in N(v)\) intersect only in \(p(v)\) by construction.

Finally, let us observe that the required embedding can be defined by setting \(c(uv) := [p(v), q(uv)] \cup [q(uv), p(u)]\) for every edge \(uv \in E(T_{k,m})\). That this is a proper embedding follows immediately from (11).

A simple closed curve \(c\) divides \(\mathbb{R}^2 \setminus c\) into two components. The bounded one is called the inside of \(c\), and we shall denote it by inside\((c)\).

If \(c_1, \ldots, c_k\) are simple closed curves in the plane then we say that they are nested if either \(c_i \subseteq \text{inside}(c_{i+1})\) for all \(i = 1, \ldots, k - 1\), or \(c_{i+1} \subseteq \text{inside}(c_i)\) for all \(i = 1, \ldots, k - 1\). We need the following straightforward observation, whose proof we leave to the reader.

**Lemma 10.** Let \(p(v) : v \in V(T_{k,m})\) and \(c(e) : e \in E(T_{k,m})\) be a planar embedding of \(T_{k,m}\). Let \(\gamma_1, \ldots, \gamma_m\) be the closed curves corresponding to the \(m\) six-cycles of \(T_{k,m}\). Then either \(\gamma_1, \ldots, \gamma_{\lfloor m/2 \rfloor}\) are nested, or \(\gamma_{\lfloor m/2 \rfloor + 1}, \ldots, \gamma_m\) are nested.

**Lemma 11.** For all \(k \in \mathbb{N}\) and \(c > 0\) there exists an \(\alpha = \alpha(k,c) > 1\) and an \(m_0 = m_0(k,c) \in \mathbb{N}\) such that the following holds. If \(m \geq m_0\) and \(\mathcal{A} = \{A(v) : v \in V(T_{k,m})\}\) is a collection of subsets of the plane with the properties:

(i) The intersection graph of \(\mathcal{A}\) is isomorphic to \(T_{k,m}\);
(ii) Every \(A \in \mathcal{A}\) is closed, convex and has non-empty interior;
(iii) area\((A) \geq c \cdot \text{diam}^2(A)\) for all \(A \in \mathcal{A}\),

then there exist \(A, A' \in \mathcal{A}\) such that \(\text{diam}(A) \geq \alpha^m \cdot \text{diam}(A')\).

**Proof.** Let \(C_1, \ldots, C_m\) denote the \(m\) six-cycles of \(T_{k,m}\). Let \(c_1, \ldots, c_m\) be the corresponding closed curves in the embedding provided by Lemma 9, and let \(v_{1,i}, \ldots, v_{6,i}\) denote the vertices of \(C_i\). We can assume without loss of generality that the numbering is such that \(\text{diam}(A(v_{1,i})) \geq \text{diam}(A(v_{2,i})), \ldots, \text{diam}(A(v_{6,i}))\) for all \(i = 1, \ldots, m\). Appealing to Lemma 10, we can assume without loss of generality that \(c_{i+1} \subseteq \text{inside}(c_i)\) for \(i = 1, \ldots, \lfloor m/2 \rfloor - 1\).

Since \(c_i \subseteq A(v_{1,i}) \cup \cdots \cup A(v_{6,i})\) we have:

\[
\text{diam}(c_i) \leq 6 \cdot \text{diam}(A(v_{1,i})).
\tag{12}
\]

By the isodiametric inequality, we must have

\[
\text{area(inside}(c_i)) \leq \frac{\pi}{4} (6 \text{diam}(A(v_{1,i})))^2 = 9\pi \text{diam}^2(A(v_{1,i})).
\tag{13}
\]
Now let $K$ be defined as

$$K := \lceil 1000/c \rceil.$$ 

We claim that there must be a sequence $i_0 := 1 < i_1 < \cdots < i_{\lfloor m/K \rfloor} \leq m$ such that $\text{diam}(A(v_{i_{j-1}+1})) \leq \frac{1}{2} \text{diam}(A(v_{i_{j}}))$ for $j = 0, \ldots, \lfloor m/K \rfloor$. To see this, suppose that for some $j \geq 1$ we have already defined $i_0, \ldots, i_{j-1}$. Suppose that $\text{diam}(A(v_{i_{j-1}+1})), \ldots, \text{diam}(A(v_{i_{j-1}+K})) > \frac{1}{2} \text{diam}(A(v_{i_{j-1}}))$. Then we have

$$\text{area}\left(\bigcup_{k=1}^{K} A(v_{i_{j-1}+z})\right) \geq K \cdot \frac{1}{2} \cdot c \cdot \text{diam}^2(A(v_{i_{j-1}}))$$

$$\geq 500 \text{diam}^2(A(v_{i_{j-1}}))$$

$$> 9\pi \text{diam}^2(A(v_{i_{j-1}}))$$

$$\geq \text{area}(\text{inside}(c_{i_{j-1}})).$$

using that the sets $A(v_{i,j}) : i = 1, \ldots, m$ are disjoint (as the corresponding vertices are non-adjacent in $T_{k,m}$) in the first line.

On the other hand, since $c_{i_{j-1}+1}, \ldots, c_{i_{j-1}+K} \subseteq \text{inside}(c_{i_{j-1}})$ and $A(v_{i_{j-1}+z})$ does not intersect $A(v_{i_{j-1}}) \cup \cdots \cup A(v_{i_{j-1}+z})$ for all $z \geq 1$, we must have

$$A(v_{i_{j-1}+1}), \ldots, A(v_{i_{j-1}+K}) \subseteq \text{inside}(c_{i_{j-1}}).$$

This implies that

$$\text{area}\left(\bigcup_{i=1}^{m} A(v_{i_{j-1}+1}) \cup \cdots \cup A(v_{i_{j-1}+K})\right) \leq \text{area}(\text{inside}(c_{i_{j-1}})),$$

which contradicts (14).

It follows that one of $\text{diam}(A(v_{i_{j-1}+1})), \ldots, \text{diam}(A(v_{i_{j-1}+K}))$ is at most $\frac{1}{2} \text{diam}(A(v_{i_{j-1}}))$. Thus, we can pick $i_{j-1} < i_j \leq i_{j-1}+K$ such that $\text{diam}(A(v_{i_j})) \leq \frac{1}{2} \text{diam}(A(v_{i_{j-1}}))$. This proves the claim.

To finish the proof of the Theorem, note that

$$\frac{\text{diam}(A(v_{i_1}))/\text{diam}(A(v_{i_{\lceil m/K \rfloor}})) \geq 2^{m/K} \geq \alpha^m,}$$

for $m$ sufficiently large, where we can for instance take $\alpha := 2^{\sqrt{2}}$. ■

**Theorem 7.** Let $P$ be an arbitrary convex polygon with rational corner points. Then $h_P(n) = 2^{2\Omega(n)}$.

**Proof.** By Lemma 8, there is a $k = k(P)$ such that $T_{k,m}$ is a $P$-homothets graph for all $m \in \mathbb{N}$.

Now let $m \in \mathbb{N}$ be arbitrary and let $\mathcal{A} = \{A(v) : v \in V(T_{k,m})\}$ be a collection of homothets of $P$, whose intersection graph is exactly $T_{k,m}$, and all of whose corner points lie on $\mathbb{Z}^2$. Let us write $n := |V(T_{k,m})|$. Observe that $n = \Theta(m)$.

Since each $A \in \mathcal{A}$ is a $P$-homothet, we have

$$\text{area}(A) \geq c \cdot \text{diam}^2(A) \quad \text{for all } A \in \mathcal{A},$$

(15)
with \( c := \text{area}(P) / \text{diam}^2(P) \). Hence we can apply Lemma 11. Thus, there are \( A, A' \in A \) with \( \text{diam}(A) \geq \alpha^m \text{diam}(A') \) with \( \alpha = \alpha(P) > 1 \) a constant that depends only on \( P \). Since the corner points of \( A' \) lie on the grid, we must have \( \text{diam}(A') \geq 1 \). Let \( p, q \) be the corner points of \( A \) that realize its diameter. Then

\[
\|p - q\| \geq \alpha^m.
\]

This shows that if \( p, q \) both lie inside a \([-K, K]^2\) then we must have \( K \geq \alpha^m / 2\sqrt{2} = 2^{\Omega(n)} \).

\[\blacksquare\]

6 Further work

The recognition problem for \( \text{hom}(P) \) was shown to be NP-hard \([18, 16]\) for all convex polygons \( P \), but we are not aware of any result for the recognition problem for \( \text{trans}(P) \) other than for the case when \( P \) is the unit square. In this case Breu \([2]\) proved the recognition problem to be NP-hard.

**Problem 1.** Is \( \text{trans}(P) \)-recognition NP-hard for all convex polygons \( P \)?

As in \([24]\) our results imply that, provided the base polygon has rational corner points, the recognition problem for \( \text{trans}(P) \) and \( \text{hom}(P) \) both belong to NP. Thus the recognition problems for \( \text{hom}(P) \) is NP-complete if \( P \) is rational, with certificates of NP-ness of bit size \( O(n^2) \). This however does not settle the problem for polygons \( P \) that do not have rational corner points.

**Problem 2.** Is \( \text{trans}(P) \)-recognition in NP for all convex polygons \( P \)?

**Problem 3.** Is \( \text{hom}(P) \)-recognition in NP for all convex polygons \( P \)?

(Here we think of \( P \) as implicit, i.e. not part of the input.) Observe that by Lemma 1 our results also apply if \( P \) can be transformed into a rational polygon by means of an affine transformation (this is for instance the case for all triangles). A naive approach to try to settle problems 2 and 3 might be to find for every polygon \( P \) a rational polygon \( Q \) such that \( \text{trans}(P) = \text{trans}(Q) \) resp. \( \text{hom}(P) = \text{hom}(Q) \). We however strongly suspect this is not possible. In particular we suspect that Lemma 1 can be turned into an if and only if statement:

**Problem 4.** Is \( \text{trans}(P) = \text{trans}(Q) \) if and only if \( P = T[Q] \) for some affine transformation \( T \)?

**Problem 5.** Is \( \text{hom}(P) = \text{hom}(Q) \) if and only if \( P = T[Q] \) for some affine transformation \( T \)?

A result of Černý et al. \([4]\), Theorem 1) suggests that the answer may be “yes” in both cases.

Another possible direction for further work is the generalization to higher dimensions. As mentioned in section 2 the proofs of our upper bounds extend trivially to higher dimensions. It is however not clear how much of the lower bound constructions can be salvaged in dimension \( d \geq 3 \).
References


