Supporting document to the paper "Logical limit laws for minor-closed classes of graphs"

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Here we provide a hand-checkable proof for Lemma 4.9 in the paper [2].

Definition 1 **(B_0, B_2; cf. [1, p. 327])** *We have to work with the following functions:*

(1) $B_0 = \frac{(3t-1)^2(t+1)^6\log(t+1)}{512t^6} - \frac{(3t^4-16t^3+6t^2-1)\log(3t+1)}{32t^3} - \frac{(3t+1)^2(-t+1)^6\log(2t+1)}{1024t^6} + \frac{1}{4}\log(t+3) - \frac{1}{2}\log(t) - \frac{3}{8}\log(16) - \frac{(217t^6+920t^5+972t^4+1436t^3+205t^2-172t+6)(-t+1)^2}{2048t^4(3t+1)(t+3)}$,

(2) $B_2 = \frac{(-t+1)^3(3t-1)(3t+1)(t+1)^3\log(t+1)}{256t^6} - \frac{(-t+1)^3(3t+1)\log(3t+1)}{32t^3} + \frac{(3t+1)^2(-t+1)^6\log(2t+1)}{512t^6} + \frac{(t-1)^4(185t^4+698t^3-217t^2-160t+6)}{1024t^4(3t+1)(t+3)}$.

Definition 2 **($h_1(t)$, $h_2(t)$)** *For every $t \in (0,1)$ we define*

(1) $h_1(t) := \frac{2t+1}{(3t+1)(-t+1)}$,

(2) $h_2(t) := -\frac{t^2(-t+1)\left(5t^2+36t+18\right)}{2(t+3)(2t+1)(3t+1)^2}$.

Definition 3 **($Y(t)$; cf. [1, p. 310])** *For every $t \in (0,1)$, and with h_1 and h_2 as in Definition 2, we define $Y(t) := -1 + h_1(t)\ \exp(h_2(t))$.*

Lemma 4 *The function $t \mapsto Y(t)$ is strictly monotone increasing in the open interval $(0,1)$.*

Proof: The derivative of Y is

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t) = \frac{3t^2(144+736t+1256t^2+799t^3+141t^4+t^5-5t^6)}{(2t+1)(3t+1)^4(t^2+2t-3)^2}\ \exp\left(-\frac{t^2(-t+1)\left(5t^2+36t+18\right)}{2(t+3)(2t+1)(3t+1)^2}\right) \quad . \qquad (1)$$

The exponential function being a strictly positive real number for any real argument, (1) implies

$$t > 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}Y(t) > 0 \qquad \Leftrightarrow \qquad 5t^6 < t^5 + 141t^4 + 799t^3 + 1256t^2 + 736t + 144 \quad , \qquad (2)$$

the latter of which is obviously true since already $5t^6 < 144$ for every $0 < t < 1$. ■

Lemma 5 *With h_1 and h_2 as in Definition 2 we have*

(1) $2.0941746325 - 10^{-10} < h_1(0.6263716633 - 10^{-10}) < 2.0941746325 + 10^{-10}$,

(2) $2.0941746335 - 10^{-10} < h_1(0.6263716633 + 10^{-10}) < 2.0941746335 + 10^{-10}$,

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(3) $-0.0460123254 - 10^{-10} < h_2(0.6263716633 - 10^{-10}) < -0.0460123254 + 10^{-10}$,

(4) $-0.0460123253 - 10^{-10} < h_2(0.6263716633 + 10^{-10}) < -0.0460123253 + 10^{-10}$.

Proof: Checking these statements is left to the reader, who is advised to entrust *this* entirely routine task to an electronic computer. The functions h_1 and h_2 being rational, the statements can be checked via exact computations with arbitrary long integers, a standard functionality of several computer algebra systems (note that to check (3) and (4) one of course does not have to compute fractions, but one can rewrite (3) and (4) as a statement about adding, subtracting and multiplying integers).

Let us add that for reaching certainty about the equalities (3) and (4), the closest non-commercial automated alternatives to hand-evaluation seem to be some `C` libraries for arbitrary precision arithmetic, like `GMP` or `iRRAM`. According to [3], the code in the `iRRAM` package itself is currently in the process of being formally verfied. ∎

We now derive Taylor polynomials taylormade for our purposes (the approximation in (II) is designed to be used twice: both for the evaluations of exp within Y, and later on for evaluations $\exp(-\tilde{\nu})$ with $\tilde{\nu}$ an approximation of ν):

Lemma 6 (some Taylor approximations to $\exp$**)**

(I) *for every* $x \in (0.48, 0.49)$,

(1) $\left|\exp(x) - \sum_{0\leqslant i\leqslant 11} \frac{x^i}{i!}\right| < 0.11998784433 \cdot 10^{-11}$

(2) $0.39995948109 \cdot 10^{-12} + \sum_{0\leqslant i\leqslant 11} \frac{x^i}{i!} < \exp(x) < 0.11998784433 \cdot 10^{-11} + \sum_{0\leqslant i\leqslant 11} \frac{x^i}{i!}$

(II) *for every* $x \in (-0.05, 0)$,

(1) $\left|\exp(x) - \sum_{0\leqslant i\leqslant 5} \frac{x^i}{i!}\right| < 2.1701388889 \cdot 10^{-11}$

(2) $1.0850694444 \cdot 10^{-11} + \sum_{0\leqslant i\leqslant 5} \frac{x^i}{i!} < \exp(x) < 2.1701388889 \cdot 10^{-11} + \sum_{0\leqslant i\leqslant 5} \frac{x^i}{i!}$

Proof: As to (I), we develop exp around[1] 0 and use Lagrange's error term for Taylor's theorem: for every k and every $x \in (0, 0.49)$ there exists $\xi_x \in (0, 0.49)$ such that $\exp(x) = \sum_{0\leqslant i\leqslant k-1} \frac{x^i}{i!} + \frac{\exp(\xi_x)}{k!}x^k$. Because of $1 = \exp(0) < \exp(\xi_x) < \exp(0.49) < \exp(1) < 3$, we therefore know

$$\frac{1}{k!}x^k < \exp(x) - \sum_{0\leqslant i\leqslant k-1} \frac{x^i}{i!} < \frac{3}{k!}x^k \quad , \tag{3}$$

for every $x \in (0, 0.49)$. In particular,

$$\left|\exp(x) - \sum_{0\leqslant i\leqslant k-1} \frac{x^i}{i!}\right| < \frac{3}{k!}x^k \quad \text{for every } x \in (0, 0.49) \quad . \tag{4}$$

As for (I), we require k to be large enough to have $\frac{3}{k!}x^k < 10^{-11}$ for every $x \in (0.48, 0.49) \subseteq (0, 0.49)$, i.e., we require k to satisfy $\frac{3}{k!}0.49^k < 10^{-11}$. The smallest such k is $k = 12$. Since

[1] If we would develop exp around a rational number x_0 inside the interval we are interested in, we'd need fewer than eleven terms to achieve the desired accuracy (w.r.t. arithmetic with arbitrary elements of $\mathbb{R}$). But we would then stray from our path to a set of 'certificates' for the p_i-inequalities consisting of rational computations only: Taylor's theorem would require us to know $\exp(x_0)$ in order to compute the coefficients of the approximating polynomial. Since $\exp(x_0)$ is irrational for every rational x_0 (e.g., [4]), another approximation would be necessary, resulting in additional complexity outweighing the gain in simplicity due to a lower-degree polynomial. Same for developing around an irrational number of the form $\log(x_0)$ with rational x_0 inside the respective intervals (which would keep the constant term rational yet necessitate approximations for what value to substitute into the variable). So developing around 0 seems the only sensible choice for our purposes of deriving rational certificates. The price of the ease of evaluating the constant term $\exp(0)$ is a higher number of terms in order to 'bend' the Taylor polynomial to within the required accuracy at points far from 0.

$\frac{3}{12!}0.49^{12} < 0.11998784433 \cdot 10^{-11}$ and $0.39995948109 \cdot 10^{-12} < \frac{1}{12!}0.49^{12}$, (3) implies (I).(2), and hence (I).(1).

As for (II), for every $x \in (-0.05, 0)$, there exists $\xi_x \in (-0.05, 0)$ such that $\exp(x) = \sum_{0 \leqslant i \leqslant k-1} \frac{x^i}{i!} + \frac{\exp(\xi_x)}{k!}x^k$. Since $\frac{1}{2} < \exp(-0.05) < \exp(\xi_x) < \exp(0) = 1$, we know that for every even k, and any $x \in (-0.05, 0)$ we have $x^k > 0$ and

$$\frac{1}{2k!}x^k < \exp(x) - \sum_{0 \leqslant i \leqslant k-1} \frac{x^i}{i!} < \frac{1}{k!}x^k \quad , \tag{5}$$

while for every odd k and any $x \in (-0.05, 0)$ we have $x^k < 0$ and

$$\frac{1}{k!}x^k < \exp(x) - \sum_{0 \leqslant i \leqslant k-1} \frac{x^i}{i!} < \frac{1}{2k!}x^k \quad . \tag{6}$$

In particular we now know that for every k (of whatever parity) and any $x \in (-0.05, 0)$,

$$\Big|\exp(x) - \sum_{0 \leqslant i \leqslant k-1} \frac{x^i}{i!}\Big| < \frac{1}{k!}|x|^k \quad . \tag{7}$$

We require k to be large enough to have $\frac{1}{k!}|x|^k < 10^{-10}$ for every $x \in (-0.05, 0)$, i.e., we require k to satisfy $\frac{1}{k!}0.05^k < 10^{-10}$. The smallest such k is $k = 6$. Since $k = 6$ is even, (5) together with $1.0850694444 \cdot 10^{-11} < \frac{1}{2}\frac{1}{6!}0.05^6$ and $\frac{1}{6!}0.05^6 < 2.1701388889 \cdot 10^{-11}$ imply (II).(2), and hence (II).(1). In particular we know that $\sum_{0 \leqslant i \leqslant 5} \frac{x^i}{i!}$ underestimates $\exp(x)$ for every $x \in (-0.05, 0)$. ■

Lemma 7 (verified bounds for t_0) *There exists exactly one real number $t_0 \in (0, 1)$ with $Y(t_0) = 1$, and it satisfies*

$$0.6263716633 - 10^{-10} < t_0 < 0.6263716633 + 10^{-10} \quad . \tag{8}$$

Proof: Since all factors in denominators within $Y(t)$ are non-zero for $t \in (0, 1)$, the function $t \mapsto Y(t)$ is continuous as a composition of continuous functions. By Lemma 4, it is moreover strictly monotone increasing in $(0, 1)$. Therefore the claim follows (existence from continuity, uniqueness from monotonicity) via the Intermediate Value Theorem if we can show that

(1) $Y(0.6263716633 - 10^{-10}) < 1 \quad ,$

(2) $Y(0.6263716633 + 10^{-10}) > 1 \quad .$

A finite certificate for (1) is given by the calculation

$$Y(0.6263716633 - 10^{-10}) = -1 + h_1(0.6263716633 - 10^{-10}) \cdot \exp(h_2(0.6263716633 - 10^{-10}))$$

(by the upper bounds in (1) and (3) in Lemma 5, and since exp is monotone increasing)
$$< -1 + 2.0941746326 \cdot \exp(-0.0460123253)$$

(by the upper bound in (1))
$$< -1 + 2.0941746326 \cdot \left(2.1701388889 \cdot 10^{-11} + \sum_{0 \leqslant i \leqslant 5} \frac{(-0.0460123253)^i}{i!}\right)$$
$$= 0.99999999995544408263310738324518270587020818583224485385349606 8 + \frac{1}{3} \cdot 10^{-62} < 1 \quad , \tag{9}$$

while a finite certificate for (2) is given by the calculation

$$
\begin{aligned}
& Y(0.6263716633 + 10^{-10}) = -1 + h_1(0.6263716633 + 10^{-10}) \cdot \exp(h_2(0.6263716633 + 10^{-10})) \\
&\text{(by the lower bounds in (2) and (4) in Lemma 5, and since exp is monotone increasing)} \quad > -1 + 2.0941746334 \cdot \exp(-0.0460123254) \\
&\text{(by the lower bound in (2))} \quad > -1 + 2.0941746334 \cdot \\
&\quad \left(1.0850694444 \cdot 10^{-11} + \sum_{0 \leqslant i \leqslant 5} \frac{(-0.0460123254)^i}{i!} \right) \\
&\quad = 1.000000000095741729766895140580 0\backslash \\
&\quad 480697033915364640304336242832 > 1 \ ,
\end{aligned} \tag{10}
$$

where in each case \ denotes that a number contiguously continues in the next line. ■

The following defines the function t from [1], with explicit values for the 'suitable small neighborhood of 1' [1, p. 317, paragraph 2]:

Definition 8 *For every $y \in (0.9999999996, 1.00000000009)$ we define $t(y)$ to be the unique $t \in (0.6263716633 - 10^{-10}, 0.6263716633 + 10^{-10})$ with $Y(t) = y$.*

Let us note that $t_0 = t(1)$.

Remark 9 (correctness of Definition 8) *Definition 8 does indeed define a function*

$$t\colon (0.9999999996, 1.00000000009) \to (0.6263716633 - 10^{-10}, 0.6263716633 + 10^{-10}) \quad . \tag{11}$$

Proof: Uniqueness of the $t(y)$ from Definition 8 follows from Lemma 4, while for existence we have to show that the argument in the proof of Lemma 7 can be carried out with any $y \in (0.9999999996, 1.00000000009)$ replacing the 1 in the conditions (1) and (2) of Lemma 7. This follows from (9) and (10): since
$0.99999999995544408263310738324518270587020818583224485385349606 8 + \frac{1}{3} \cdot 10^{-62}$
< 0.9999999996 and
$1.000000000095741729766895140580048069703391536464030433624283 2 > 1.00000000009$,
each of these calculations can be used as is for proving the existence of any $t(y)$ with $y \in (0.9999999996, 1.00000000009)$. ■

Definition 10 (R; cf. [1, (2.6)]) *With t as in Definition 8, we define the function*

$$
\begin{aligned}
R\colon (0.9999999996, 1.00000000009) &\longrightarrow \mathbb{R} \\
y &\longmapsto R(y) := \frac{(3 \cdot t(y) + 1)\,(-t(y) + 1)^3}{16 \cdot t(y)^3} \quad .
\end{aligned} \tag{12}
$$

Lemma 11 *With $\xi(t) := \frac{(3 \cdot t+1)(-t+1)^3}{16 \cdot t^3}$,*

(1) $0.03819109771 < \xi(0.6263716633 - 10^{-10}) < 0.03819109772 \quad ,$

(2) $0.03819109762 < \xi(0.6263716633 + 10^{-10}) < 0.03819109763 \quad .$

Proof: Finite statements about integers. Same comments as in the proof of Lemma 5 apply. ■

Lemma 12 (some pointwise bounds for $B_0(t)$) *With B_0 as in Definition 1.(1),*

(1) $0.00073969957 < B_0(0.6263716633 - 10^{-10}) < 0.00073969958 \quad ,$

(2) $0.00073969956 < B_0(0.6263716633 + 10^{-10}) < 0.00073969957$.

Proof: Finite statements about integers. The same comments as in the proof of Lemma 5 apply. ■

Lemma 13 (uniform bounds for $B_0(t)$) *With B_0 as in Definition 1.(1),*

$$0.00073969896 < B_0(t) < 0.00073970019 \tag{13}$$

for every $t \in I := (0.6263716633 - 10^{-10}, 0.6263716633 + 10^{-10})$.

Proof: If we had a proof that B_0 is monotone decreasing in I, then (13) would follow from the slightly stronger pointwise bounds in Lemma 12—but the (known) continuity of B_0 alone is of course not enough to use Lemma 12. Unfortunately, a complete proof of this monotonicity seems to require at least as much work as the proof of (13) that follows.

The plan of the proof is the following: for each of the seven summands in B_0 we will derive both upper and lower bounds which uniformly hold in I. In the end, we add these bounds to derive the bounds in (13).

In the following paragraph, we prove the uniform bounds

$$0.22495616614 < \tfrac{(3t-1)^2(t+1)^6\log(t+1)}{512t^6} < 0.22495616711 \quad \text{for every } t \in I \quad . \tag{14}$$

Since $3 \cdot t > 1$ for every $t \in I$, the function $t \mapsto (3t-1)^2$ is evidently monotone increasing in I. So are the two functions $t \mapsto (t+1)^6$ and $t \mapsto \log(t+1)$. Therefore, $t \mapsto (3t-1)^2(t+1)^6\log(t+1)$ is monotone increasing in I as a product of three such functions. Hence, for every $t \in I$,

$$(3t-1)^2(t+1)^6\log(t+1) < (3t-1)^2(t+1)^6\log(t+1)\Big|_{t=0.6263716633+10^{-10}} < 6.95601448698 \tag{15}$$

and

$$(3t-1)^2(t+1)^6\log(t+1) > (3t-1)^2(t+1)^6\log(t+1)\Big|_{t=0.6263716633-10^{-10}} > 6.95601447059 \quad . \tag{16}$$

The function $t \mapsto 512t^6$ is evidently monotone increasing in I. Hence, for every $t \in I$,

$$512t^6 > 512t^6\Big|_{t=0.6263716633-10^{-10}} > 30.92164387643 \tag{17}$$

and

$$512t^6 < 512t^6\Big|_{t=0.6263716633+10^{-10}} < 30.92164393568 \quad . \tag{18}$$

Since (16) and (18) hold in all of I, it follows that, for every $t \in I$,

$$\tfrac{(3t-1)^2(t+1)^6\log(t+1)}{512t^6} > \tfrac{6.95601447059}{30.92164393568} > 0.22495616614 \quad , \tag{19}$$

proving the lower bound in (14).

Since (15) and (17) hold in all of I, it follows that, for every $t \in I$,

$$\tfrac{(3t-1)^2(t+1)^6\log(t+1)}{512t^6} < \tfrac{6.95601448698}{30.92164387643} < 0.22495616711 \quad , \tag{20}$$

proving the upper bound in (14).

In the following paragraph, we prove the uniform bounds

$$-0.28456395530 < \tfrac{(3t^4-16t^3+6t^2-1)\log(3t+1)}{32t^3} < -0.28456395528 \quad \text{for every } t \in I \quad . \tag{21}$$

Since $2+\sqrt{3} > 1$, $2-\sqrt{3} < 0.5$ and $\frac{\mathrm{d}}{\mathrm{d}t}(12t^3-48t^2+12t) = 36t^2-96t+12 = 12t(t-(2+\sqrt{3}))(t-(2-\sqrt{3}))$, it is evident that $\frac{\mathrm{d}}{\mathrm{d}t}(12t^3-48t^2+12t) < 0$ for every $t \in I$, i.e., $t \mapsto 3t^4-16t^3+6t^2-1$ is strictly monotone decreasing in I, so

$$\begin{aligned} 3t^4-16t^3+6t^2-1 &> 3t^4-16t^3+6t^2-1\Big|_{t=0.6263716633+10^{-10}} \\ &= -2.1161809442159711262496568523624448554192 \quad \text{for every } t \in I \quad , \end{aligned} \tag{22}$$

and

$$\begin{aligned} 3t^4-16t^3+6t^2-1 &< 3t^4-16t^3+6t^2-1\Big|_{t=0.6263716633-10^{-10}} \\ &= -2.1161809425425888723475949656101944348672 \quad \text{for every } t \in I \quad . \end{aligned} \tag{23}$$

The function $t \mapsto \log(3t+1)$ is evidently strictly monotone increasing in I, hence

$$\log(3t+1) > \log(3t+1)\Big|_{t=0.6263716633-10^{-10}} > 1.05748295164 \quad \text{for every } t \in I \quad , \tag{24}$$

and

$$\log(3t+1) < \log(3t+1)\Big|_{t=0.6263716633+10^{-10}} < 1.05748295186 \quad \text{for every } t \in I \quad . \tag{25}$$

The function $t \mapsto 32t^3$ is evidently monotone increasing in I. Hence, for every $t \in I$,

$$32t^3 > 32t^3\Big|_{t=0.6263716633-10^{-10}} = 7.864050340179393384432870014976 \quad , \tag{26}$$

and

$$32t^3 < 32t^3\Big|_{t=0.6263716633+10^{-10}} = 7.864050347712349427668874499328 \quad . \tag{27}$$

It follows that, for every $t \in I$,

$$\begin{aligned} &\frac{(3t^4-16t^3+6t^2-1)\log(3t+1)}{32t^3} \\ \text{(by (22))} \quad &> \frac{(-2.1161809442159711262496568523624448554192)\cdot\log(3t+1)}{32t^3} \\ \text{(by (25); we recall that multiplying with a negative number flips an inequality)} \quad &> \frac{(-2.1161809442159711262496568523624448554192)\cdot 1.05748295186}{32t^3} \\ \text{(by (27))} \quad &> \frac{(-2.1161809442159711262496568523624448554192)\cdot 1.05748295186}{7.864050347712349427668874499328} \\ &> -0.28456395530 \quad , \end{aligned} \tag{28}$$

proving the lower bound in (21), and also that, for every $t \in I$,

$$\begin{aligned} &\frac{(3t^4-16t^3+6t^2-1)\log(3t+1)}{32t^3} \\ \text{(by (23))} \quad &< \frac{(-2.1161809425425888723475949656101944348672)\cdot\log(3t+1)}{32t^3} \\ \text{(by (24); we recall that multiplying with a negative number flips an inequality)} \quad &< \frac{(-2.1161809425425888723475949656101944348672)\cdot 1.05748295164}{32t^3} \\ \text{(by (26))} \quad &< \frac{(-2.1161809425425888723475949656101944348672)\cdot 1.05748295164}{7.864050340179393384432870014976} \\ &< -0.28456395528 \quad , \end{aligned} \tag{29}$$

which proves the upper bound in (21).

In the following paragraph, we prove the uniform bounds

$$0.00029614190 < \tfrac{(3t+1)^2(-t+1)^6\log(2t+1)}{1024t^6} < 0.00029614191 \quad \text{for every } t \in I \quad . \tag{30}$$

While it is evident that $t \mapsto (3t+1)^2$ is strictly monotone increasing, and $t \mapsto (-t+1)^6$ strictly monotone decreasing in I, it is not evident whether the product $t \mapsto (3t+1)^2(-t+1)^6$ decreases or increases in I. To decide this, we note that $\frac{\mathrm{d}}{\mathrm{d}t}(3t+1)^2(-t+1)^6 = 24(-1+t)^5 t(1+3t)$, and from this factorization it *is* evident that $\frac{\mathrm{d}}{\mathrm{d}t}(3t+1)^2(-t+1)^6 < 0$ for every $t \in I$, hence that $t \mapsto (3t+1)^2(-t+1)^6$ is indeed strictly monotone decreasing in I. Therefore,

$$\begin{aligned}(3t+1)^2(-t+1)^6 &< (3t+1)^2(-t+1)^6\Big|_{t=0.6263716633-10^{-10}} \\ &< 0.02255053559 \quad \text{for every } t \in I \quad ,\end{aligned} \tag{31}$$

and

$$\begin{aligned}(3t+1)^2(-t+1)^6 &> (3t+1)^2(-t+1)^6\Big|_{t=0.6263716633+10^{-10}} \\ &> 0.02255053553 \quad \text{for every } t \in I \quad .\end{aligned} \tag{32}$$

Moreover, since function $t \mapsto \log(2t+1)$ evidently is strictly monotone increasing in I, we know that

$$\begin{aligned}\log(2t+1) &> \log(2t+1)\Big|_{t=0.6263716633-10^{-10}} \\ &> 0.81214872970 \quad \text{for every } t \in I \quad ,\end{aligned} \tag{33}$$

and

$$\begin{aligned}\log(2t+1) &< \log(2t+1)\Big|_{t=0.6263716633+10^{-10}} \\ &< 0.81214872989 \quad \text{for every } t \in I \quad .\end{aligned} \tag{34}$$

Furthermore, since the function $t \mapsto 1024t^6$ evidently is strictly monotone increasing in I, we know that

$$\begin{aligned}1024t^6 &> 1024t^6\Big|_{t=0.6263716633-10^{-10}} \\ &> 61.84328775287 \quad \text{for every } t \in I \quad ,\end{aligned} \tag{35}$$

and

$$\begin{aligned}1024t^6 &< 1024t^6\Big|_{t=0.6263716633+10^{-10}} \\ &< 61.84328787136 \quad \text{for every } t \in I \quad .\end{aligned} \tag{36}$$

It follows that, for every $t \in I$,

$$\begin{aligned} & \tfrac{(3t+1)^2(-t+1)^6\log(2t+1)}{1024t^6} \\ \text{(by (32))} \quad & > \tfrac{0.02255053553\cdot\log(2t+1)}{1024t^6} \\ \text{(by (33))} \quad & > \tfrac{0.02255053553\cdot 0.81214872970}{1024t^6} \\ \text{(by (36))} \quad & > \tfrac{0.02255053553\cdot 0.81214872970}{61.84328787136} \\ & > 0.00029614190 \quad ,\end{aligned} \tag{37}$$

proving the lower bound in (30), and also that, for every $t \in I$,

$$\begin{aligned} & \tfrac{(3t+1)^2(-t+1)^6 \log(2t+1)}{1024t^6} \\ \text{(by (31))} \quad & < \tfrac{0.02255053559 \cdot \log(2t+1)}{1024t^6} \\ \text{(by (34))} \quad & < \tfrac{0.02255053559 \cdot 0.81214872989}{1024t^6} \\ \text{(by (35))} \quad & < \tfrac{0.02255053559 \cdot 0.81214872989}{61.84328775287} \\ & < 0.00029614191 \quad , \end{aligned} \tag{38}$$

which proves the upper bound in (30).

Since $t \mapsto \frac{1}{4}\log(t+3)$ is evidently strictly monotone increasing in I, we know that, for every $t \in I$,

$$\begin{aligned} \tfrac{1}{4}\log(t+3) &> \tfrac{1}{4}\log(t+3)\Big|_{t=0.6263716633-10^{-10}} \\ &> 0.32205815164 \quad \text{for every } t \in I \quad , \end{aligned} \tag{39}$$

and

$$\begin{aligned} \tfrac{1}{4}\log(t+3) &< \tfrac{1}{4}\log(t+3)\Big|_{t=0.6263716633+10^{-10}} \\ &< 0.32205815165 \quad \text{for every } t \in I \quad . \end{aligned} \tag{40}$$

Since $t \mapsto \frac{1}{2}\log(t)$ is evidently strictly monotone increasing in I, we know that, for every $t \in I$,

$$\begin{aligned} \tfrac{1}{2}\log(t) &> \tfrac{1}{2}\log(t)\Big|_{t=0.6263716633-10^{-10}} \\ &> -0.23390568644 \quad \text{for every } t \in I \quad , \end{aligned} \tag{41}$$

and

$$\begin{aligned} \tfrac{1}{2}\log(t) &< \tfrac{1}{2}\log(t)\Big|_{t=0.6263716633+10^{-10}} \\ &< -0.23390568627 \quad \text{for every } t \in I \quad . \end{aligned} \tag{42}$$

As to the summand $\frac{3}{8}\log(16)$ in B_0, there are the bounds

$$1.03972077083 < \tfrac{3}{8}\log(16) < 1.03972077084 \quad . \tag{43}$$

In the following paragraph, we prove the uniform bounds

$$0.02472734758 < \tfrac{(217t^6+920t^5+972t^4+1436t^3+205t^2-172t+6)(-t+1)^2}{2048t^4(3t+1)(t+3)} < 0.02472734762 \quad \text{for every } t \in I \quad . \tag{44}$$

We have $\frac{\mathrm{d}}{\mathrm{d}t}\,(217t^6+920t^5+972t^4+1436t^3+205t^2-172t+6)(-t+1)^2 = 2(t-1)(868t^6+2569t^5+616t^4+1646t^3-1744t^2-463t+92)$, and since $2(t-1) < 0$ for every $t \in I$, to prove that $t \mapsto (217t^6+920t^5+972t^4+1436t^3+205t^2-172t+6)(-t+1)^2$ is strictly monotone increasing in I it suffices to show that $868t^6+2569t^5+616t^4+1646t^3-1744t^2-463t+92 < 0$ for every $t \in I$. This is equivalent to

$$868t^6+2569t^5+616t^4+1646t^3+92 \quad < \quad 1744t^2+463t \quad \text{for every } t \in I \quad . \tag{45}$$

Since both $t \mapsto 868t^6+2569t^5+616t^4+1646t^3+92$ and $t \mapsto 1744t^2+463t$, are strictly monotone increasing in I, we have, for every $t \in I$,

$$
\begin{aligned}
868t^6 + 2569t^5 + 616t^4 + 1646t^3 + 92 &< 868t^6 + 2569t^5 + 616t^4 + 1646t^3 + 92 \Big|_{t=0.6263716633+10^{-10}} \\
&< 891.450148292474 \\
&< 974.2535871037253045145 6 \\
&= 1744t^2 + 463t \Big|_{t=0.6263716633-10^{-10}} \quad < \quad 1744t^2 + 463t \quad ,
\end{aligned} \tag{46}
$$

proving (45). Since we now know that $t \mapsto (217t^6+920t^5+972t^4+1436t^3+205t^2-172t+6)(-t+1)^2$ is strictly monotone increasing in I, it follows that, for every $t \in I$,

$$
\begin{aligned}
&(217t^6 + 920t^5 + 972t^4 + 1436t^3 + 205t^2 - 172t + 6)(-t + 1)^2 \\
>&(217t^6 + 920t^5 + 972t^4 + 1436t^3 + 205t^2 - 172t + 6)(-t + 1)^2 \Big|_{t=0.6263716633-10^{-10}} \\
>&81.3892822256
\end{aligned} \tag{47}
$$

and

$$
\begin{aligned}
&(217t^6 + 920t^5 + 972t^4 + 1436t^3 + 205t^2 - 172t + 6)(-t + 1)^2 \\
<&(217t^6 + 920t^5 + 972t^4 + 1436t^3 + 205t^2 - 172t + 6)(-t + 1)^2 \Big|_{t=0.6263716633+10^{-10}} \\
<&81.3892822381 \quad .
\end{aligned} \tag{48}
$$

Since $t \mapsto 2048t^4(3t + 1)(t + 3)$ is evidently strictly monotone increasing in I, it follows that, for every $t \in I$,

$$
2048t^4(3t + 1)(t + 3) > 2048t^4(3t + 1)(t + 3)\Big|_{t=0.6263716633-10^{-10}} > 3291.4683555 \tag{49}
$$

and

$$
2048t^4(3t + 1)(t + 3) < 2048t^4(3t + 1)(t + 3)\Big|_{t=0.6263716633+10^{-10}} < 3291.4683606 \quad . \tag{50}
$$

It follows that, for every $t \in I$,

$$
\begin{aligned}
& \tfrac{(217t^6+920t^5+972t^4+1436t^3+205t^2-172t+6)(-t+1)^2}{2048t^4(3t+1)(t+3)} \\
\text{(by (47))} \quad & > \tfrac{81.3892822256}{2048t^4(3t+1)(t+3)} \\
\text{(by (50))} \quad & > \tfrac{81.3892822256}{3291.4683606} > 0.02472734758 \quad ,
\end{aligned} \tag{51}
$$

proving the lower bound in (44), and

$$
\begin{aligned}
& \tfrac{(217t^6+920t^5+972t^4+1436t^3+205t^2-172t+6)(-t+1)^2}{2048t^4(3t+1)(t+3)} \\
\text{(by (48))} \quad & < \tfrac{81.3892822381}{2048t^4(3t+1)(t+3)} \\
\text{(by (49))} \quad & < \tfrac{81.3892822381}{3291.4683555} < 0.02472734762 \quad ,
\end{aligned} \tag{52}
$$

proving the upper bound in (44).

We now add our uniform bounds for the summands in B_0 to prove the uniform bounds in (13). In doing so, we have to pay attention which summand appears with a minus-sign in the definition of B_0.

From the lower bound in (14), the upper bounds in (21) and (30), the lower bound in (39), and the upper bounds in (41), (43) and (44), it follows that, for every $t \in I$,

$$\begin{aligned} B_0(t) &> 0.22495616614 - (-0.28456395528) - 0.00029614191 \\ &\quad + 0.32205815164 - (-0.23390568627) - (1.03972077084) - (0.02472734762) \\ &\quad = 0.00073969896 \quad , \end{aligned} \tag{53}$$

proving the lower bound in (13).

From the upper bound in (14), the lower bounds in (21) and (30), the upper bound in (39) and the lower bounds in (41), (43) and (44) it follows that, for every $t \in I$,

$$\begin{aligned} B_0(t) &< 0.22495616711 - (-0.28456395530) - 0.00029614190 \\ &\quad + 0.32205815165 - (-0.23390568644) - (1.03972077083) - (0.02472734758) \\ &\quad = 0.00073970019 \quad , \end{aligned} \tag{54}$$

proving the upper bound in (13). This completes the proof of Lemma 13. ■

Lemma 14 (bounds for $B_0(t_0)$) *With B_0 as in Definition 1.*(1),

$$0.00073969896 < B_0(t_0) < 0.00073970019 \tag{55}$$

Proof: In view of Lemma 7, the bounds in (55) follow from the uniform bounds in Lemma 13. ■

Lemma 15 (pointwise bounds for $B_2(t_0)$) *With B_2 as in Definition 1.*(2),

(1) $-0.0014914312 - 10^{-10} < B_2(0.6263716633 - 10^{-10}) < -0.0014914312 + 10^{-10}$,

(2) $-0.0014914312 - 10^{-10} < B_2(0.6263716633 + 10^{-10}) < -0.0014914312 + 10^{-10}$.

Proof: Left to the reader. The same comments as in the proof of Lemma 5 apply. ■

Lemma 16 (uniform bounds for $B_2(t)$) *With B_2 as in Definition 1.*(2),

$$-0.001491431277 < B_2(t) < -0.001491431155 \quad . \tag{56}$$

for every $t \in I := (0.6263716633 - 10^{-10}, 0.6263716633 + 10^{-10})$.

Proof: The plan of the proof is the same as for Lemma 13: for each of the four summands in B_2, derive both upper and lower bounds which uniformly hold in I. In the end, we add these bounds to derive the bounds in (56).

In the following paragraph, we prove the uniform bounds

$$0.01786492701 < \tfrac{(-t+1)^3(3t-1)(3t+1)(t+1)^3\log(t+1)}{256t^6} < 0.01786492706 \quad \text{for every } t \in I \quad . \tag{57}$$

Since $t \mapsto -1+3t^2$ is strictly monotone increasing in I, it follows that $-1+3t^2 > -1+3\cdot(0.6263716633-10^{-10})^2 = 0.17702438137980270272 > 0$ for every $t \in I$, and now it is evident from $\frac{\mathrm{d}}{\mathrm{d}t}\,(-t+1)^3(3t-1)(3t+1)(t+1)^3 = -24(-1+t)^2t(1+t)^2(-1+3t^2)$ that $\frac{\mathrm{d}}{\mathrm{d}t}\,(-t+1)^3(3t-1)(3t+1)(t+1)^3 < 0$ for every $t \in I$, i.e., that $t \mapsto (-t+1)^3(3t-1)(3t+1)(t+1)^3$ is strictly monotone decreasing in I, so

$$\begin{aligned} (-t+1)^3(3t-1)(3t+1)(t+1)^3 &> (-t+1)^3(3t-1)(3t+1)(t+1)^3\Big|_{t=0.6263716633+10^{-10}} \\ &> 0.56791522564 \quad \text{for every } t \in I \quad , \end{aligned} \tag{58}$$

and

$$(-t+1)^3(3t-1)(3t+1)(t+1)^3 < (-t+1)^3(3t-1)(3t+1)(t+1)^3\Big|_{t=0.6263716633-10^{-10}}$$
$$< 0.56791522584 \quad \text{for every } t \in I \quad . \tag{59}$$

Since $t \mapsto \log(t+1)$ is strictly monotone increasing in I, it follows that

$$\log(t+1) > \log(t+1)\Big|_{t=0.6263716633-10^{-10}}$$
$$> 0.48635156016 \quad \text{for every } t \in I \quad , \tag{60}$$

and

$$\log(t+1) < \log(t+1)\Big|_{t=0.6263716633+10^{-10}}$$
$$< 0.48635156029 \quad \text{for every } t \in I \quad . \tag{61}$$

Since $t \mapsto 256t^6$ is strictly monotone increasing in I, it follows that

$$256t^6 > 256t^6\Big|_{t=0.6263716633-10^{-10}} > 15.46082193821 \quad \text{for every } t \in I \quad , \tag{62}$$

and

$$256t^6 < 256t^6\Big|_{t=0.6263716633+10^{-10}} < 15.46082196784 \quad \text{for every } t \in I \quad . \tag{63}$$

It follows that, for every $t \in I$,

$$\begin{aligned}
& \tfrac{(-t+1)^3(3t-1)(3t+1)(t+1)^3\log(t+1)}{256t^6} \\
\text{(by (58))} \quad & > \tfrac{0.56791522564\cdot\log(t+1)}{256t^6} \\
\text{(by (60))} \quad & > \tfrac{0.56791522564\cdot 0.48635156016}{256t^6} \\
\text{(by (63))} \quad & > \tfrac{0.56791522564\cdot 0.48635156016}{15.46082196784} \\
& > 0.01786492701 \quad ,
\end{aligned} \tag{64}$$

proving the lower bound in (57), and also that, for every $t \in I$,

$$\begin{aligned}
& \tfrac{(-t+1)^3(3t-1)(3t+1)(t+1)^3\log(t+1)}{256t^6} \\
\text{(by (59))} \quad & < \tfrac{0.56791522584\cdot\log(t+1)}{256t^6} \\
\text{(by (61))} \quad & < \tfrac{0.56791522584\cdot 0.48635156029}{256t^6} \\
\text{(by (62))} \quad & < \tfrac{0.56791522584\cdot 0.48635156029}{15.46082193821} \\
& < 0.01786492706 \quad ,
\end{aligned} \tag{65}$$

proving the upper bound in (57).

In the following paragraph, we prove the uniform bounds

$$0.02019321732 < \tfrac{(-t+1)^3(3t+1)\log(3t+1)}{32t^3} < 0.02019321738 \quad \text{for every } t \in I \quad . \tag{66}$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}(-t+1)^3(3t+1) = -12(-1+t)^2 t < 0$ for every $t \in I$, we know that $t \mapsto (-t+1)^3(3t+1)$ is strictly monotone decreasing in I, hence

$$(-t+1)^3(3t+1) > (-t+1)^3(3t+1)\Big|_{t=0.6263716633+10^{-10}}$$
$$> 0.15016835728 \quad \text{for every } t \in I \quad , \tag{67}$$

and

$$(-t+1)^3(3t+1) < (-t+1)^3(3t+1)\Big|_{t=0.6263716633-10^{-10}}$$
$$< 0.15016835750 \quad \text{for every } t \in I \quad . \tag{68}$$

Since $t \mapsto \log(3t+1)$ is strictly monotone increasing, it follows that for every $t \in I$,

$$\log(3t+1) > \log(3t+1)\Big|_{t=0.6263716633-10^{-10}} > 1.05748295164 \quad \text{for every } t \in I \quad , \tag{69}$$

and

$$\log(3t+1) < \log(3t+1)\Big|_{t=0.6263716633+10^{-10}} < 1.05748295186 \quad \text{for every } t \in I \quad . \tag{70}$$

Since $t \mapsto 32t^3$ is strictly monotone increasing in I, it follows that

$$32t^3 > 32t^3\Big|_{t=0.6263716632} > 7.86405034017 \tag{71}$$

and

$$32t^3 < 32t^3\Big|_{t=0.6263716634} < 7.86405034771 \quad . \tag{72}$$

It follows that, for every $t \in I$,

$$\begin{aligned}
& \frac{(-t+1)^3(3t+1)\log(3t+1)}{32t^3} \\
\text{(by (67))} \quad & > \frac{0.15016835728 \cdot \log(3t+1)}{32t^3} \\
\text{(by (69))} \quad & > \frac{0.15016835728 \cdot 1.05748295164}{32t^3} \\
\text{(by (72))} \quad & > \frac{0.15016835728 \cdot 1.05748295164}{7.86405034771} \\
& > 0.02019321732 \quad ,
\end{aligned} \tag{73}$$

proving the lower bound in (66), and also that, for every $t \in I$,

$$\begin{aligned}
& \frac{(-t+1)^3(3t+1)\log(3t+1)}{32t^3} \\
\text{(by (68))} \quad & < \frac{0.15016835750 \cdot \log(3t+1)}{32t^3} \\
\text{(by (70))} \quad & < \frac{0.15016835750 \cdot 1.05748295186}{32t^3} \\
\text{(by (71))} \quad & < \frac{0.15016835750 \cdot 1.05748295186}{7.86405034017} \\
& < 0.02019321738 \quad ,
\end{aligned} \tag{74}$$

proving the upper bound in (66).

In the following paragraph, we prove the uniform bounds

$$0.00059228380 < \frac{(3t+1)^2(-t+1)^6\log(2t+1)}{512t^6} < 0.00059228381 \quad \text{for every } t \in I \quad . \tag{75}$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}\,(3t+1)^2(-t+1)^6 = 24(t-1)^5t(3t+1) < 0$ for every $t \in I$, we know that $t \mapsto (3t+1)^2(-t+1)^6$ is strictly monotone decreasing in I, hence

$$(3t+1)^2(-t+1)^6 > (3t+1)^2(-t+1)^6\Big|_{t=0.6263716633+10^{-10}} > 0.02255053553 \quad \text{for every } t \in I \quad , \tag{76}$$

and

$$(3t+1)^2(-t+1)^6 < (3t+1)^2(-t+1)^6\Big|_{t=0.6263716633-10^{-10}} < 0.02255053560 \quad \text{for every } t \in I \quad . \tag{77}$$

Since $t \mapsto \log(2t+1)$ is strictly monotone increasing in I, it follows that

$$\log(2t+1) > \log(2t+1)\Big|_{t=0.6263716633-10^{-10}} > 0.81214872970 \quad \text{for every } t \in I \tag{78}$$

and

$$\log(2t+1) < \log(2t+1)\Big|_{t=0.6263716633+10^{-10}} < 0.81214872989 \quad \text{for every } t \in I \quad . \tag{79}$$

Since $t \mapsto 512t^6$ is strictly monotone increasing in I, it follows that

$$512t^6 > 512t^6\Big|_{t=0.6263716633-10^{-10}} > 30.92164387643 \quad \text{for every } t \in I \tag{80}$$

and

$$512t^6 < 512t^6\Big|_{t=0.6263716633+10^{-10}} < 30.92164393568 \quad \text{for every } t \in I \quad . \tag{81}$$

It follows that, for every $t \in I$,

$$\begin{aligned} & \tfrac{(3t+1)^2(-t+1)^6\log(2t+1)}{512t^6} \\ \text{(by (76))} \quad & > \tfrac{0.02255053553\cdot\log(2t+1)}{512t^6} \\ \text{(by (78))} \quad & > \tfrac{0.02255053553\cdot 0.81214872970}{512t^6} \\ \text{(by (81))} \quad & > \tfrac{0.02255053553\cdot 0.81214872970}{30.92164393568} \\ & > 0.00059228380 \quad , \end{aligned} \tag{82}$$

proving the lower bound in (75), and, for every $t \in I$,

$$\begin{aligned} & \tfrac{(3t+1)^2(-t+1)^6\log(2t+1)}{512t^6} \\ \text{(by (77))} \quad & < \tfrac{0.02255053560\cdot\log(2t+1)}{512t^6} \\ \text{(by (79))} \quad & < \tfrac{0.02255053560\cdot 0.81214872989}{512t^6} \\ \text{(by (80))} \quad & < \tfrac{0.02255053560\cdot 0.81214872989}{30.92164387643} \\ & < 0.00059228381 \quad , \end{aligned} \tag{83}$$

proving the upper bound in (75).

In the following paragraph, we prove the uniform bounds

$$0.000244575293 < \tfrac{(t-1)^4(185t^4+698t^3-217t^2-160t+6)}{1024t^4(3t+1)(t+3)} < 0.000244575295 \quad \text{for every } t \in I \quad . \tag{84}$$

For every $t \in I$, evidently $(t-1)^3 < 0$. Moreover, since both $t \mapsto 740t^4 + 2073t^3 + 92$ and $t \mapsto 1698t^2 + 183t$ are strictly monotone increasing in I, we have, for every $t \in I$,

$$\begin{aligned} 740t^4 + 2073t^3 + 92 & < 740t^4 + 2073t^3 + 92\Big|_{t=0.6263716633+10^{-10}} \\ & = 715.3525597141428299499534408273089356632640 \\ & < 780.82181422656832973952 \\ & = 1698t^2 + 183t\Big|_{t=0.6263716633-10^{-10}} \\ & < 1698t^2 + 183t \quad , \end{aligned} \tag{85}$$

i.e., $(740t^4 + 2073t^3 - 1698t^2 - 183t + 92) < 0$ for every $t \in I$. Taken together, it follows that $\frac{\mathrm{d}}{\mathrm{d}t}(t-1)^4(185t^4 + 698t^3 - 217t^2 - 160t + 6) = 2(t-1)^3(740t^4 + 2073t^3 - 1698t^2 - 183t + 92) > 0$ for every $t \in I$, hence $t \mapsto (t-1)^4(185t^4 + 698t^3 - 217t^2 - 160t + 6)$ is strictly monotone increasing in I, so

$$\begin{aligned} &(t-1)^4(185t^4 + 698t^3 - 217t^2 - 160t + 6) \\ >&(t-1)^4(185t^4 + 698t^3 - 217t^2 - 160t + 6)\Big|_{t=0.6263716633-10^{-10}} \\ >&0.40250592053 \quad \text{for every } t \in I \quad , \end{aligned} \tag{86}$$

and

$$\begin{aligned} &(t-1)^4(185t^4 + 698t^3 - 217t^2 - 160t + 6) \\ <&(t-1)^4(185t^4 + 698t^3 - 217t^2 - 160t + 6)\Big|_{t=0.6263716633+10^{-10}} \\ <&0.40250592191 \quad \text{for every } t \in I \quad . \end{aligned} \tag{87}$$

Since $t \mapsto 1024t^4(3t+1)(t+3)$ is strictly monotone increasing, we furthermore know

$$\begin{aligned} &1024t^4(3t+1)(t+3) \\ >&1024t^4(3t+1)(t+3)\Big|_{t=0.6263716633-10^{-10}} \\ >&1645.7341777 \quad \text{for every } t \in I \quad , \end{aligned} \tag{88}$$

and

$$\begin{aligned} &1024t^4(3t+1)(t+3) \\ <&1024t^4(3t+1)(t+3)\Big|_{t=0.6263716633+10^{-10}} \\ <&1645.7341803 \quad \text{for every } t \in I \quad . \end{aligned} \tag{89}$$

It follows that, for every $t \in I$,

$$\begin{aligned} & \tfrac{(t-1)^4(185t^4+698t^3-217t^2-160t+6)}{1024t^4(3t+1)(t+3)} \\ \text{(by (86))} \qquad & > \tfrac{0.40250592053}{1024t^4(3t+1)(t+3)} \\ \text{(by (89))} \qquad & > \tfrac{0.40250592053}{1645.7341803} \\ & > 0.000244575293 \quad , \end{aligned} \tag{90}$$

proving the lower bound in (84), and, for every $t \in I$,

$$\begin{aligned} & \tfrac{(t-1)^4(185t^4+698t^3-217t^2-160t+6)}{1024t^4(3t+1)(t+3)} \\ \text{(by (87))} \qquad & < \tfrac{0.40250592191}{1024t^4(3t+1)(t+3)} \\ \text{(by (88))} \qquad & < \tfrac{0.40250592191}{1645.7341777} \\ & < 0.000244575295 \quad , \end{aligned} \tag{91}$$

proving the upper bound in (84).

From the lower bound in (57), the upper bound in (66), and the lower bounds in (75) and (84), it follows that, for every $t \in I$,

$$\begin{aligned} B_2(t) &> 0.01786492701 - 0.02019321738 + 0.00059228380 + 0.000244575293 \\ &= -0.001491431277 \quad , \end{aligned} \tag{92}$$

proving the lower bound in (56).

From the upper bound in (57), the lower bound in (66), and the upper bounds in (75) and (84), it follows that, for every $t \in I$,

$$\begin{aligned} B_2(t) &< 0.01786492706 - 0.02019321732 + 0.00059228381 + 0.000244575295 \\ &= -0.001491431155 \quad , \end{aligned} \tag{93}$$

proving the upper bound in (56). This completes the proof of Lemma 16. ■

Lemma 17 (bounds for $B_2(t_0)$) *With B_2 as in Definition 1.*(1)*,*

$$-0.001491431277 < B_2(t_0) < -0.001491431155 \tag{94}$$

Proof: In view of Lemma 7, the bounds in (94) follow from the uniform bounds in Lemma 16. ■

Lemma 18

$$0.0381910976 = 0.0381910977 - 10^{-10} < R(1) < 0.0381910976 + 10^{-10} < 0.0381910977 \quad . \tag{95}$$

Proof: By Definition 10, we know that with t_0 as in Lemma 8 we have $R(1) = \frac{(3\cdot t_0+1)(-t_0+1)^3}{16\cdot t_0^3}$. It is routine to check that the function $t \mapsto \xi(t) := \frac{(3\cdot t+1)(-t+1)^3}{16\cdot t^3}$ is strictly monotone decreasing for $t \in (0,1)$, hence $R(1) = \xi(t_0)$ together with the bounds on t_0 from (8) in Lemma 7 implies

$$\xi(0.6263716633 - 10^{-10}) < R(1) < \xi(0.6263716633 + 10^{-10}) \quad , \tag{96}$$

so in (95) the lower bound follows from the lower bound in Lemma 11.(1), while the upper bound follows from the upper bound in Lemma 11.(2). ■

Lemma 19 (exact formula for ν in terms of t_0) *With $\rho = \gamma^{-1}$ as in [1, p. 310], C the exponential generating function of connected labelled planar graphs, and with B_0 and B_2 as in Definition 1, and with R as in [1, (2.6)] and B_0 and B_2 as in Definition 1,*

$$\nu := C(\rho) = R(1) + B_0(t_0) + B_2(t_0) \quad . \tag{97}$$

Proof: See [1, p. 321, (4.7)], together with the equation immediately above that. ■

Lemma 20 (verified bounds for ν) *The real number ν defined in [1] satisfies*

$$0.037439365283 < \nu < 0.037439366735 \quad . \tag{98}$$

Proof: The lower bound follows from

$$\begin{aligned} \nu &\overset{(97)}{=} R(1) + B_0(t_0) + B_2(t_0) \\ \text{(by Lemmas 18, 14 and 17)} \quad &> 0.0381910976 + 0.00073969896 + (-0.001491431277) \\ &= 0.037439365283 \end{aligned} \tag{99}$$

and the upper bound from

$$\begin{aligned} \nu &\overset{(97)}{=} R(1) + B_0(t_0) + B_2(t_0) \\ \text{(by Lemmas 18, 14 and 17)} \quad &< 0.0381910977 + 0.00073970019 + (-0.001491431155) \\ &= 0.037439366735 \quad . \end{aligned} \tag{100}$$

■

Definition 21 ($A(t)$, $\rho(t)$) *With*

$$A(t) := \tfrac{(3t-1)(t+1)^3\log(t+1)}{16t^3} + \tfrac{(3t+1)(-t+1)^3\log(2t+1)}{32t^3} + \tfrac{(-t+1)(185t^4+698t^3-217t^2-160t+6)}{64t(3t+1)^2(t+3)} \qquad (101)$$

and

$$r(t) := \tfrac{1}{16}(3t+1)^{\frac{1}{2}}(t^{-1}-1)^3\exp(A(t)) \qquad (102)$$

we define

$$\rho := r(t_0) \qquad (103)$$

Proof: See [1, p. 310]. ■

Lemma 22 (uniform bounds for $A(t)$) *With A as in Definition 21,*

$$0.48968967248 < A(t) < 0.48968967363 \qquad (104)$$

for every $t \in I := (0.6263716633 - 10^{-10}, 0.6263716633 + 10^{-10})$.

Proof: The structure of the proof is analogous to the proofs of Lemmas 13 and 16.

In the following paragraph, we prove the uniform bounds

$$0.46777725975 < \tfrac{(3t-1)(t+1)^3\log(t+1)}{16t^3} < 0.46777726082 \quad \text{for every } t \in I \quad . \qquad (105)$$

Because of $\frac{\mathrm{d}}{\mathrm{d}t}\,(3t-1)(t+1)^3 = 12t(t+1)^2 > 0$ for every $t \in I$, we know that $t \mapsto (3t-1)(t+1)^3$ is strictly monotone increasing in I and hence

$$(3t-1)(t+1)^3 > (3t-1)(t+1)^3\Big|_{t=0.6263716632} > 3.78185681259 \quad \text{for every } t \in I \quad , \qquad (106)$$

and

$$(3t-1)(t+1)^3 < (3t-1)(t+1)^3\Big|_{t=0.6263716634} < 3.78185681657 \quad \text{for every } t \in I \quad . \qquad (107)$$

Since $t \mapsto 16t^3$ is strictly monotone increasing,

$$16t^3 > 16t^3\Big|_{t=0.6263716632} > 3.93202517008 \quad \text{for every } t \in I \quad , \qquad (108)$$

and

$$16t^3 < 16t^3\Big|_{t=0.6263716634} < 3.93202517386 \quad \text{for every } t \in I \quad . \qquad (109)$$

From (106), (60) and (109) follows $((3t-1)(t+1)^3 \cdot \log(t+1))/(16t^3) > 3.78185681259 \cdot 0.48635156016 \,/\, 3.93202517386 > 0.46777725975$ for every $t \in I$, proving the lower bound in (105). From (107), (61) and (108) follows $((3t-1)(t+1)^3 \cdot \log(t+1))/(16t^3) < 3.78185681657 \cdot 0.48635156029 \,/\, 3.93202517008 < 0.46777726082$ for every $t \in I$, proving the upper bound in (105).

In the following paragraph, we prove the uniform bounds

$$0.01550842571 < \tfrac{(3t+1)(-t+1)^3\log(2t+1)}{32t^3} < 0.01550842575 \quad \text{for every } t \in I \quad . \qquad (110)$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}\,(3t+1)(-t+1)^3 = -12t(t-1)^2 < 0$ for every $t \in I$, we know that $t \mapsto (3t+1)(-t+1)^3$ is strictly monotone decreasing in I, so

$$(3t+1)(-t+1)^3 > (3t+1)(-t+1)^3\Big|_{t=0.6263716634} > 0.15016835728 \quad \text{for every } t \in I \quad , \qquad (111)$$

and

$$(3t+1)(-t+1)^3 < (3t+1)(-t+1)^3\Big|_{t=0.6263716632} < 0.15016835750 \quad \text{for every } t \in I \quad . \tag{112}$$

From (111), (33) and (72) it follows that $((3t+1)(-t+1)^3 \cdot \log(2t+1))/(32t^3) > 0.15016835728 \cdot 0.81214872970 \,/\, 7.86405034771 > 0.01550842571$, proving the lower bound in (110). From (112), (34) and (71) it follows that $((3t+1)(-t+1)^3 \cdot \log(2t+1))/(32t^3) < 0.15016835750 \cdot 0.81214872989 \,/\, 7.86405034017 < 0.01550842575$.

In the following paragraph, we prove the uniform bounds

$$0.00640398702 < \tfrac{(-t+1)(185t^4+698t^3-217t^2-160t+6)}{64t(3t+1)^2(t+3)} < 0.00640398706 \quad \text{for every } t \in I \quad . \tag{113}$$

Since both $t \mapsto 2745t^2$ and $t \mapsto 925t^4 + 2052t^3 + 114t + 166$ are strictly monotone increasing in I, we have $2745t^2 > 2745t^2\Big|_{t=0.6263716632} > 1076.97730896 > 884.07553334 > 925t^4 + 2052t^3 + 114t + 166\Big|_{t=0.6263716634} > 925t^4 + 2052t^3 + 114t + 166$ for every $t \in I$, hence $\frac{\mathrm{d}}{\mathrm{d}t}(-t+1)(185t^4 + 698t^3 - 217t^2 - 160t + 6) = -925t^4 - 2052t^3 + 2745t^2 - 114t - 166 > 0$ for every $t \in I$. Therefore, $t \mapsto (-t+1)(185t^4 + 698t^3 - 217t^2 - 160t + 6)$ is strictly monotone increasing in I, so

$$\begin{aligned} &(-t+1)(185t^4 + 698t^3 - 217t^2 - 160t + 6) \\ >&(-t+1)(185t^4 + 698t^3 - 217t^2 - 160t + 6)\Big|_{t=0.6263716632} \\ >&7.71707734263 \quad \text{for every } t \in I \quad , \end{aligned} \tag{114}$$

and

$$\begin{aligned} &(-t+1)(185t^4 + 698t^3 - 217t^2 - 160t + 6) \\ <&(-t+1)(185t^4 + 698t^3 - 217t^2 - 160t + 6)\Big|_{t=0.6263716634} \\ <&7.71707738122 \quad \text{for every } t \in I \quad . \end{aligned} \tag{115}$$

Since $t \mapsto 64t(3t+1)^2(t+3)$ is strictly monotone increasing in I, we have

$$64t(3t+1)^2(t+3) > 64t(3t+1)^2(t+3)\Big|_{t=0.6263716633-10^{-10}} > 1205.0426269 \quad \text{for every } t \in I \quad , \tag{116}$$

and

$$64t(3t+1)^2(t+3) < 64t(3t+1)^2(t+3)\Big|_{t=0.6263716633+10^{-10}} < 1205.0426279 \quad \text{for every } t \in I \quad . \tag{117}$$

From (114) and (117) it follows that $(-t+1)(185t^4+698t^3-217t^2-160t+6)/(64t(3t+1)^2(t+3)) > 7.71707734263 \,/\, 1205.0426279 > 0.00640398702$, proving the lower bound in (113). From (115) and (116) it follows that $(-t+1)(185t^4+698t^3-217t^2-160t+6)/(64t(3t+1)^2(t+3)) < 7.71707738122 \,/\, 1205.0426269 < 0.00640398706$, proving the upper bound in (113).

In view of Definition 21, the lower bounds in (105), (110) and (113) imply that for every $t \in I$,

$$A(t) > 0.46777725975 + 0.01550842571 + 0.00640398702 = 0.48968967248 \quad , \tag{118}$$

proving the lower bound in (104), while the upper bounds in (105), (110) and (113) imply that for every $t \in I$,

$$A(t) < 0.46777726082 + 0.01550842575 + 0.00640398706 = 0.48968967363 \quad , \tag{119}$$

proving the upper bound in (104). ■

Lemma 23 (bounds for $A(t_0)$) *With $A(t)$ as in Definition 21 and t_0 as in Definition 7,*

$$0.48968967248 < A(t_0) < 0.48968967363 \quad . \tag{120}$$

Proof: Immediate from Lemmas 7 and 22. ■

Lemma 24 (uniform bounds for $r(t)$) *With $r(t)$ as in Definition 21 and I as in Lemma 13,*

$$0.03672841251 < r(t) < 0.03672841266 \quad \textit{for every } t \in I \quad . \tag{121}$$

Proof: Since $t \mapsto \frac{1}{16}(3t+1)^{\frac{1}{2}}$ is evidently strictly monotone increasing in I,

$$\tfrac{1}{16}(3t+1)^{\frac{1}{2}} > \tfrac{1}{16}(3t+1)^{\frac{1}{2}}\Big|_{t=0.6263716633-10^{-10}} > 0.10604971913 \quad \text{for every } t \in I \tag{122}$$

and

$$\tfrac{1}{16}(3t+1)^{\frac{1}{2}} < \tfrac{1}{16}(3t+1)^{\frac{1}{2}}\Big|_{t=0.6263716633+10^{-10}} < 0.10604971915 \quad \text{for every } t \in I \quad . \tag{123}$$

Since $t \mapsto t^{-1} - 1$ is strictly monotone decreasing in I, so is $t \mapsto (t^{-1}-1)^3$, hence

$$(t^{-1}-1)^3 > (t^{-1}-1)^3\Big|_{t=0.6263716633+10^{-10}} > 0.21223798428 \quad \text{for every } t \in I \tag{124}$$

and

$$(t^{-1}-1)^3 < (t^{-1}-1)^3\Big|_{t=0.6263716633-10^{-10}} < 0.21223798483 \quad \text{for every } t \in I \quad . \tag{125}$$

Combining Lemma 23 with (I).(2) in Lemma 6, and since exp is strictly monotone increasing, it follows that, for every $t \in I$,

$$\begin{aligned} \exp(A(t)) &> \exp(0.48968967248) \\ &> 0.39995948109 \cdot 10^{-12} + \sum_{0 \leqslant i \leqslant 11} (0.48968967248)^i / i! \\ &> 1.63180974590 \quad , \end{aligned} \tag{126}$$

and, again for every $t \in I$,

$$\begin{aligned} \exp(A(t)) &< \exp(0.48968967363) \\ &< 0.11998784433 \cdot 10^{-11} + \sum_{0 \leqslant i \leqslant 11} (0.48968967363)^i / i! \\ &< 1.63180974778 \quad . \end{aligned} \tag{127}$$

It follows that, for every $t \in I$,

$$\begin{aligned} r(t) &= \tfrac{1}{16}(3t+1)^{\frac{1}{2}}(t^{-1}-1)^3 \exp(A(t)) \\ \text{(122)} \quad &> 0.10604971913 \cdot (t^{-1}-1)^3 \exp(A(t)) \\ \text{(124)} \quad &> 0.10604971913 \cdot 0.21223798428 \cdot \exp(A(t)) \\ \text{(126)} \quad &> 0.10604971913 \cdot 0.21223798428 \cdot 1.63180974590 \\ &> 0.03672841251 \quad , \end{aligned} \tag{128}$$

proving the lower bound in (121), and, for every $t \in I$,

$$
\begin{aligned}
r(t) &= \tfrac{1}{16}(3t+1)^{\frac{1}{2}}(t^{-1}-1)^3\exp(A(t)) \\
\text{(123)} \quad &< 0.10604971915 \cdot (t^{-1}-1)^3\exp(A(t)) \\
\text{(125)} \quad &< 0.10604971915 \cdot 0.21223798483 \cdot \exp(A(t)) \\
\text{(127)} \quad &< 0.10604971915 \cdot 0.21223798483 \cdot 1.63180974778 \\
&< 0.03672841266 \quad , \qquad (129)
\end{aligned}
$$

proving the upper bound in (121). ■

Proof of Lemma ??: Since $\rho = r(t_0)$ by Definition 21, it is immediate from Lemmas 7 and 24 that

$$0.03672841251 < \rho < 0.03672841266.$$

Recall that $G(\rho) = \exp(C(\rho)) = \exp(\nu)$. According to Lemma 20 we have $-0.037439366735 < -\nu < -0.037439365283$, so (II).(2) in Lemma 6 is applicable and implies, by strict monotonicity of exp,

$$\exp(-\nu) > 1.0850694444 \cdot 10^{-11} + \sum_{0 \leqslant i \leqslant 5} \frac{(-0.037439366735)^i}{i!} > 0.96325282112$$

and

$$\exp(-\nu) < 2.1701388889 \cdot 10^{-11} + \sum_{0 \leqslant i \leqslant 5} \frac{(-0.037439365283)^i}{i!} < 0.96325282254 \quad .$$

■

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