Homework 3 **Graphs and Algorithms**

Treewidth, Probabilistic Arguments and Algebraic Methods

Date: 19/3/2016 Due: 01/4/2016

Note: for this homework you can earn a total of 70 pts, but 20 of these are bonus points i.e. 50 pts already corresponds to the mark 10 out of 10.

1. (25 pts) **Induced matchings:** An induced matching of a graph $G = (V, E)$ is a subset $M \subseteq V$ such that in $G[M]$ every vertex has degree 1.

   (a) (15pts) Find an algorithm that, given a graph G, a tree decomposition of width w of G and an integer l, determines in $O^*(2^{O(w)})$ time whether G has an induced matching M satisfying $|M| \geq l$. Hint: Follow the approach from Lecture 7, Section 5. Use dynamic programming; define for a bag i and partition of X_i into O, I_0, I_1 (referring to 'out', 'in with degree 0' and 'in with degree 1') a table entry $A[i, O, I_0, I_1]$ to be the maximum $|M|$ over all $M \subseteq V_i \setminus O$ such that every vertex in I_0 has no neighbor in M and every vertex in I_1 has exactly one neighbor in M.

   (b) (10pts) Use the approach from Exercise 7.11 to give an algorithm that takes as input a planar graph G and an integer l and determines whether G has an induced matching M satisfying $|M| \geq l$ in $O^*(2^{O(\sqrt{l})})$ time (if you failed the first part you may still use the algorithm asked as a subroutine).

2. (10 pts) **Tournaments with many Hamiltonian paths:** A Hamiltonian path in a directed graph is a path which visits each vertex exactly once. Show that there is a tournament on n vertices with at least $n!2^{1-n}$ Hamiltonian paths.

3. (20 pts) **General matchings via determinants.** Let G be a graph (not necessarily bipartite) on n vertices, where n is even. Consider the following $n \times n$ matrix A (with entries a_{ij}) defined using $O(n^2)$ variables x_{ij} as follows:

$$a_{ij} = \begin{cases} x_{ij} & (i,j) \in E(G) \text{ and } i > j \\ -x_{ji} & (i,j) \in E(G) \text{ and } i < j \\ 0 & (i,j) \notin E(G) \end{cases}$$

   Show that $det(A)$ is a non-zero polynomial if and only if G contains a perfect matching. [Hint: Each term corresponding to a permutation in the determinant can be viewed as a cover of vertices by oriented cycles. Furthermore, all the cycles must be of even length (whenever there is an odd cycle in some permutation, it cancels out with another permutation).]

4. (15 pts) **Disjoint pairs:** Give a randomized algorithm that takes $A_1, \ldots, A_m, B_1, \ldots, B_m \subseteq \{1, \ldots, n\}$ which are all of size k as input and determines whether there exist $1 \leq i, j \leq m$ such that $A_i \cap B_j = \emptyset$ in $4^k \cdot m \cdot poly(n)$ time. Your algorithm may have one-sided constant error probability: if no disjoint pairs exists it should always return no, and if a pairs exists it should return yes with at least constant probability. Hint: use the approach from Exercise 9.3, what can you say about (A, B) if they are separated by a set S?