Solutions HW 1 **Graphs and Algorithms**

Date: 7/2/2012 Due: 2/3/2012

Problem 1. Graph Coloring:

1. (4 pts) Show that any graph with maximum degree d can be colored with $d+1$ colors.

   Solution: We do induction on the number of nodes. If $n \leq d+1$, this is trivial. Suppose the result holds for all graphs with $\leq n$ vertices. Then given a graph G on $n+1$ vertices and maximum degree d, remove some vertex v to obtain G'. G' has n vertices, and maximum degree at most d, and thus has a $d+1$ coloring by our hypothesis. Now simply assign v some color that is not used by its neighbors (such a color exists as $\deg(v) \leq d$).

2. (1 pt) Can you also color the graph with $d+1$ colors, if the average degree is at most d?

   Solution: No. Consider K_{d+2} plus many isolated vertices.

3. (2 pt) Suppose we wish to color the edges of a graph such that any two edges sharing a vertex are colored differently (this is called edge coloring). For a graph with maximum degree d, show that part 1 implies an edge coloring using $2d+1$ colors.

   Solution: Note that each edge e "conflicts" with at most $2(d-1)$ other edges (those that share a vertex with e). So by the previous part, we can color using $2(d-1)+1 = 2d-1 \leq 2d+1$ colors.

4. (4 pts) Give an efficient algorithm to determine whether a graph can be colored using two colors. Your algorithm should also find the 2-coloring (if the graph is 2-colorable).

   Solution: Recall that a graph is bipartite if and only if it has no odd cycles. Assume that the graph is connected, otherwise we consider each component separately. Pick a vertex v and do a breadth first search. Note that there are no edges between vertices at the same depth (otherwise this would give an odd cycle). So we can color the vertices in the odd layers as 1, and the even layers as 2.

5. (8 pts) It turns out that the problem of determining whether a graph is 3-colorable or not is NP-complete. Given a 3-colorable graph G, design a polynomial time algorithm to color G using $O(\sqrt{n})$ colors.

   Solution: We repeat the following until there are no vertices of degree more than $\sqrt{n}$ left. Pick a vertex v with $\deg(v) > \sqrt{n}$. Since the graph is 3-colorable, its neighborhood $N(v)$ must be 2-colorable (as v must be assigned a different color from every vertex in $N(v)$). So, we color v with a brand new color and $N(v)$ with 2 other new colors. Remove v and its neighborhood from G.

   Now at every iteration of this step we use 3 new colors, and at least $\sqrt{n}$ vertices. So, we cannot use more than $3\sqrt{n}$ colors in total. Now we are left with a graph with maximum degree $\leq \sqrt{n}$, and we can color it using another $\sqrt{n}+1$ brand new colors.

6. (1pt) Show that any k-colorable graph has an independent set of size at least n/k.

   Solution: In any valid coloring, vertices of the same color form an independent set. As there are k colors, some color is used for n/k vertices or more.

7. (5 pts) Suppose we are given an algorithm which, given any 3-colorable graph G, finds an independent set of G of size (say) $n/1000$. Show that this algorithm can be used to color G using $O(\log n)$ colors.

   Solution: We repeatedly find an independent set using this algorithm, give it a new color and remove these vertices (this can be done repeatedly since the resulting graph remains 3-colorable). The size of the graph shrinks by a factor of $1 - 1/1000$ at each step, and hence after i steps the number of vertices left is at most $n(1-1/1000)^i$. So the procedure terminates in $O(\log_{1000/999} n) = O(\log n)$ steps.

Problem 2. Graph Planarity:

1. (5 pts) Show that any planar graph with $n \geq 3$ vertices can have at most $3n - 6$ edges.

   Solution: Each face has at least 3 edges and each edge is shared by at most two faces, so $2m \geq 3f$. By Euler's formula, we also have that $f = m - n + 2$ and hence $m - n + 2 \geq 2m/3$, which gives the result.

2. (1 pt) Using the above, show that a planar graph is always 6-colorable.

   Solution: We use induction on n. As there are $3n - 6$ edges, the average degree is less than 6 and hence there is a vertex of degree 5 or less. Remove it, color the smaller graph, and put this vertex back and give it a color that is not used by its neighbors.

3. (2 pts) Show that a bipartite planar graph can have at most $2n - 4$ edges.

   Solution: Since there are no odd cycles, each face has 4 or more edges. So, in the previous argument we have $2m \geq 4f$ and $f = m - n + 2$. These together give the result.

4. (7 pts) A subset of vertices S is called a balanced separator if removing these vertices decomposes the graph into connected components of size at most $2n/3$. Show that this can be used to find a maximum size independent set in a planar graph in time $2^{O(\sqrt{n})}$, using divide and conquer strategy.

   Solution: Consider the set S, and consider all possible subsets of S that could form an independent set. There are at most $2^{|S|} = 2^{c\sqrt{n}}$ of them. For each such valid choice $X \subset S$, recursively find the maximum independent set in the components $C_1, \ldots, C_k$ that result upon removing S (subject to the constraint that X is already picked). That is, when considering the independent sets in C_i, discard all the neighbors of X in C_i.

   To analyze the running time, we can actually afford ourselves a lot of slack. Let $T(n)$ denote the running time for a graph on n vertices. Even if we assume that we get n components, each of which has size $2n/3$. The running time can be bounded as

$$T(n) \leq 2^{c\sqrt{n}} \cdot n \cdot T(2n/3) + n^{O(1)} \cdot 2^{c\sqrt{n}}$$

where the latter $n^{O(1)} \cdot 2^{c\sqrt{n}}$ term is the time for finding a separator plus the overhead incurred (to remove neighbors of X in C_i's) while considering each choice of X.

As $2^{c\sqrt{n}} \geq n^{O(1)}$ for all large enough n, we can write this as $T(n) \leq 2^{2c\sqrt{n}}T(2n/3)$ and hence $T(n) \leq 2^{2c\sqrt{n}+\sqrt{2n/3}+\sqrt{4n/9}\ldots} = 2^{O(\sqrt{n})}$.

Problem 3. Girth versus Degree. The girth of a graph is defined the length of the smallest cycle.

1. (4 pts) Show that any graph with minimum degree d, where $d > 2$, has girth at most $1 + 2\log_{d-1} n$ (i.e. has some cycle of length $\leq 1 + 2\log_{d-1} n$).

   Solution: Let $\ell = \log_{d-1} n$. For the sake of contradiction, suppose the minimum length cycle C has length at least $\geq 2\ell + 2$. Let v be any vertex on C. Consider all the nodes at distance ℓ or less from v. They must form a tree, otherwise we have a cycle of length $\leq 2\ell + 1$, contradicting our assumption. But since the minimum degree is d, the number of leaves in this tree is at least $d(d-1)^{\ell-1} > d - 1^{\ell} = n$ which is not possible.

2. (4 pts) Show that a graph with average degree d has girth at most $1 + 2\log_{\lfloor d/4 \rfloor} n$.

   Solution: Repeatedly remove the vertices of degree $d/4$ or less. Note that removing a vertex cannot decrease the girth as we do not add any new cycles. This removes at most $nd/4$ edges. The graph had $nd/2$ edges initially as the average degree was d. So in the resulting graph there are still $nd/4$ edges left (so it is non-empty) and the minimum degree is at least $d/4$. We now use the first part.

3. (1 pt) Show that a graph with girth $\log n$ or more can have at most $O(n)$ edges.

   Solution: By the previous part, the girth $g \leq 1 + 2\log_{d/4} n$ and hence $(d/4)^{(g-1)/2} \leq n$ which implies that $d \leq 4n^{2/(g-1)}$. Thus if $g = \log n$, then the average degree $d = O(1)$.

4. (1pt) Given an example of a graph that has girth 4, and $\Omega(n^2)$ edges.

   Solution: Complete bipartite Graph.

Problem 4. Kneser Graphs: Given integers p and k, consider the graph on $n = \binom{p}{k}$ vertices, where each vertex corresponds to a k-element subset of $\{1, \ldots, p\}$. There is an edge (v, w) between two vertices v and w if and only if the subsets corresponding to v and w are disjoint.

1. (2 pt) Show that the maximum clique in this graph has size at most $\lfloor p/k \rfloor$.

   Solution: In a clique all the corresponding sets of the vertices must be pairwise disjoint. Since each set has size k, there cannot be more than p/k such sets.

2. (2 pt) Can you find a coloring using p colors.

   Solution: For a set S, color it by the smallest element contained in S. This is a valid coloring because if there is an edge between two sets, they are disjoint, and hence the smallest element in them is different.

3. (6 pts) Can you improve this to obtain a coloring using $p - 2k + 2$ colors.

   Solution: Consider the sets that lie completely in $\{1, \ldots, 2k - 1\}$. They must form an independent set since no two such sets can be disjoint. So, we use one color to color all such sets. For the remaining sets (these contain at least one element $\geq 2k$), give each set S the color of the smallest element $\geq 2k$ contained in it. Note that this gives a valid coloring, and that the number of colors used is $1 + (p - 2k + 1) = p - 2k + 2$.