Basics, Matchings, Hall's theorem

1. If maximum degree of G is d, then $\alpha(G) \geq n/(d+1)$ and $\chi(G) \leq d+1$.
2. Show by induction that a tree has exactly $n-1$ edges.
3. Show that removing an edge from a tree, breaks the graph into exactly two components.
4. Let C be any cycle in a graph, and suppose all the edge weights are distinct. Show that the heaviest edge on this cycle C cannot lie in any minimum spanning tree.

   [Hint: Suppose it does. Can you contradict optimality by constructing a strictly better spanning tree. The previous exercise might help.]
5. Show that G is bipartite if and only if it has no odd cycles. Equivalently, a graph is 2-colorable iff it has no odd cycles. Is a tree bipartite?
6. Give an efficient algorithm to detect bipartiteness.
7. Vertex Cover: Given a graph $G = (V, E)$, a subset of vertices S is a vertex cover if for each edge $e = (u, v) \in E$, at least one of the edge points of e lies in S. Show the following:
   (a) A set S is a vertex cover if and only if $V \setminus S$ is an independent set.
   (b) Show that size of maximum independent set is equal to n minus the size of the minimum vertex cover.
   (c) Show that the minimum vertex cover can be no smaller than size of the maximum matching.
   (d) Given a maximum matching M, if we pick both endpoints of edges in M, show that this form a valid vertex cover.
   (e) Use the above observations to obtain a 2-approximation algorithm for finding minimum vertex cover in a graph.
8. Given two vertices s and t, consider the problem of finding the shortest path from s to t. Convince yourself why there should be a polynomial time algorithm for this problem.

   [Hint: Consider an infection spreading along the edges at rate 1 starting from s. How soon will it reach t? Can you discretize this process?]
9. **Piercing countries:** There are two maps on opposite sides on a paper, with n countries each and all the n countries having equal area (assume that the entire paper is the map). Show that one can place n pins so that each country (on both sides) is pierced exactly once.
10. **Teachers to Courses:** There are several courses that need to be assigned teachers. Course i, must be assigned t_i teachers. For each teacher j, there is some subset of courses S_j that she is able to teach. Each teacher can be assigned to teach at most one course. Give a polynomial time algorithm to find a valid assignment of teachers to courses (or detect if this is infeasible)?

    [Hint: Make t_i copies of course i]

11. **Scheduling jobs on machines:** We are given n jobs and m machines and job j has size p_j. Find a way to assign the jobs to machines so as to minimize the total completion time of all the jobs. Here the completion of a job is defined as the time at which it completes in the schedule (i.e. if job j is assigned to machine i, its completion time is its own size p_j plus the size of the jobs that appear before it on machine i).

    [Hint: If j is k-th last job on machine i, how much does it contribute to the objective for machine i. Make an appropriate graph and find a min-cost matching of all jobs.]

12. Does the above solution work if for each job j there is some subset of machines S_j, such that j can only be scheduled on some machine in S_j.

13. **Chinese postman Problem:** Given a graph with costs/distances on the edges, we will design an algorithm for finding a minimum cost tour that visits each edge.

    Hint: a) Show this problem is same as adding edges to make graph eulerian.
    b) Show that additional edges form paths connecting odd degree vertices (plus possibly cycles).
    c) Define a complete auxiliary graph H with vertices as odd degree vertices in the original graph G, with edge cost (u, v) equal to the shortest path distance between u and v in original graph.
    d) Show that perfect matching here gives a solution and vice versa. (So we are using min-cost non-bipartite matching)