Solutions to exercises of lecture 12

Excercise 12.1. Why does the linearly independent path problem contain the colorful path problem as a special case? How does the approach outlined in Section **??** count the number of k-paths modulo 2 in this case?

Solution: Given an instance (G, c) of the colorful path problem, we can create an equivalent instance (G, c') of the linearly independent path problem by letting $c'(v) \in \mathbb{Z}_2^k$ being the unit vector with a 1 on the $c(v)$'th coordinate and 0's otherwise. The vectors associated with the vertices visited are linearly independent if and only if all colors occur at most once. In the approach in Section 12.3, $ORTH(y)$ now is the set of vertices with a vector orthogonal to y which means vertices v such that the $c(v)$'th coordinate of y equals 0. So if y is interpreted as a set in the natural way Y, $ORTH(y)$ is the set of all vertices v such that $c(v) \notin Y$, so actually we are using the inclusion exclusion formula as in Exercise 11.6 (accept that modulo 2 there $(-1)^x = 1$ so we omit it).

Excercise 12.2. Modify/use Algorithm `linindkpaths`$'$ to solve k-path in undirected graphs.

Solution: Simply replace every undirected edge with two directed edges in both sides. Note that the algorithm does not immediately work in the undirected setting (by modifying the definition and algorithm for $w^W_{ORTH(y)}(k)$) since in the undirected setting a path gives rise to two walks (one being the reverse of the other) so the sum will always be even.

Excercise 12.3. An alternative way of using the isolation lemma would have been to assign random weights to every vertex rather than edge. Why this is not a good idea?

Solution: If for example $k = n$, all k-paths will visit the same set of vertices so will not be able to isolate one path. We cannot apply to isolation lemma because a path cannot be identified given only its set of visited vertices.

Excercise 12.4. A triangle in an undirected graph $G = (V, E)$ is a triple of vertices (u, v, w) with a $(u, v), (v, w), (u, w) \in E$. A k-triangle-packing is a collection of triangles $T_1, \dots, T_k$ that are mutually disjoint. Given an algorithm that given G and integer k determines whether G has a triangle packing in $O^*(8^k)$ time with constant one-sided error probability.

Solution: We follow the vector coding approach. Assign to every vertex $v \in V$ a vector $c(v) \in \mathbb{Z}_2^{3k}$, and call a triangle-packing $(T_1, \dots, T_k)$ linearly independent if the $3k$ vectors assigned to the vertices in these triangles are. By Section 12.2 it is sufficient to decide whether a linearly independent triangle-packing exists in $O^*(2^{3k})$ time.

Let $\mathcal{T} \subseteq 2^E$ be all edge-sets of linearly independent triangle-packings. Apply the isolation lemma with $U = E$ and $\mathcal{F} = \mathcal{T}$ we see that if we assign a random weight $\omega : E \to \{1, \dots, 2|E|\}$ to every edge uniformly and independently at random, then with probability at least $1/2$, ω isolates $\mathcal{F}$. For convenience, we'll use $\omega(T) = \omega(u,v) + \omega(v,w) + \omega(u,w)$ to denote the weight of a triangle $\{u, v, w\}$.

Now we focus on counting the number of $P \in \mathcal{T}$ with $\omega(P) = W$ for fixed W modulo 2.

$$S_A^W = \left| \left\{ \{T_1, \dots, T_k\} : T_i \text{ is a triangle in } G[A] \ \wedge \sum_{i=1}^{k} \omega(T_i) = W \right\} \right|,$$

then by the same proof as in Section 12.3 we have that

$$|\{P \in \mathcal{T} : \omega(P) = W\}| \equiv \sum_{y \in \mathbb{Z}_2^{3k}} S^W_{ORTH(y)}.$$

Thus now it suffices to compute S_A^W in polynomial time for any given $A \subseteq V$ and $W \leq 2|E|$. Instead of computing S_A^W we'll first compute

$$Q_A^W(l) = \left| \left\{ (T_1, \dots, T_l) : T_i \text{ is a triangle in } G[A] \ \wedge \sum_{i=1}^{l} \omega(T_i) = W \right\} \right|,$$

but note the only difference is that the order of the T_i's matter here so we can simply compute S_A^W from $Q_A^W(k)$ with $S_A^W = Q_A^W(k)/k!$. To compute Q_A^W note that $Q_A^W(0) = 1$ if $W = 0$ and $Q_A^W(0) = 0$ otherwise and that for $l > 0$ we have

$$Q_A^W(l) = \sum_{T \text{ triangle of } G[A]} Q_A^{W - \omega(T)}(l-1),$$

so we can compute $Q_A^W(l)$ indeed in polynomial time.

Excercise 12.5. Give a polynomial time algorithm that computes the parity of the number of perfect matchings of a bipartite (multi-)graph. Assume that computing the determinant of a matrix can be done in polynomial time.

Solution: Number the vertices on both sides with integer $1, \dots, n$ (if the sizes are different, there is no perfect matching). For $1 \leq i, j \leq n$ define A_{ij} to be one if there is an edge from i to j. Then

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{sgn(\sigma)} \prod_{i=1}^{n} a_{ij} \equiv_2 \sum_{\sigma \in S_n} (-1)^{sgn(\sigma)} \prod_{i=1}^{n} a_{ij} = |\{\sigma : a_{i,\sigma(i)} = 1 \text{ for all } i\}|,$$

equals the number of perfect matchings. Note that we can simply reduce the multi-graph case to the graph case since we may remove parallel edges as long as we maintain the parity of the number of edges between every pair: this does not influence the parity of the number of perfect matchings.

Excercise 12.6. Let $G = (V, E)$ be a bipartite graph with parts V_1, V_2. Show that $|\{X \subseteq V_1 : N(X) = V_2\}| \equiv_2 |\{X \subseteq V : X \text{ independent set}\}|$. Hint: Group the independent sets on $X \cap V_1$, how many sets $Y \subseteq V_2$ such that $X \cup Y$ is an independent set are there?

Solution:

$$\begin{aligned}
|\{X \subseteq V : X \text{ independent set}\}| &= \sum_{X \subseteq V_1} |\{Y \subseteq V_2 : X \cup Y \text{ independent set}\}| \\
&= \sum_{X \subseteq V_1} 2^{|V_2 \setminus N(X)|} \\
&\equiv_2 \sum_{\substack{X \subseteq V_1 \\ |V_2 \setminus N(X)| = 0}} 1 \\
&= |\{X \subseteq V_1 : N(X) = V_2\}|.
\end{aligned}$$