Exercise 1 **Graphs and Algorithms**

Basics, Matchings, Hall's theorem

1. **Teachers to Courses:** There are several courses that need to be assigned teachers. Course i, must be assigned t_i teachers. For each teacher j, there is some subset of courses S_j that she is able to teach. Each teacher can be assigned to teach at most one course. Give a polynomial time algorithm to find a valid assignment of teachers to courses (or detect if this is infeasible)?

   [Hint: Make t_i copies of course i]

2. **Scheduling jobs on machines:** We are given n jobs and m machines and job j has size p_j. Find a way to assign the jobs to machines so as to minimize the total completion time of all the jobs. Here the completion of a job is defined as the time at which it completes in the schedule (i.e. if job j is assigned to machine i, its completion time is its own size p_j plus the size of the jobs that appear before it on machine i).

   [Hint: If j is k-th last job on machine i, how much does it contribute to the objective for machine i. Make an appropriate graph and find a min-cost matching of all jobs.]

3. Does the above solution work if for each job j there is some subset of machines S_j, such that j can only be scheduled on some machine in S_j.

4. **Chinese postman Problem:** Given a graph with costs/distances on the edges, we will design an algorithm for finding a minimum cost tour that visits each edge.

   Hint: a) Show this problem is same as adding edges to make graph eulerian.
   b) Show that additional edges form paths connecting odd degree vertices (plus possibly cycles).
   c) Define a complete auxiliary graph H with vertices as odd degree vertices in the original graph G, with edge cost (u, v) equal to the shortest path distance between u and v in original graph.
   d) Show that perfect matching here gives a solution and vice versa. (So we are using min-cost non-bipartite matching)

5. Show that the following is always a valid co-NP witness for the shortest $s - t$ path problem. For each vertex v, set $y_v = d(s, v)$ where $d(s, v)$ is the shortest distance from s to v. That is, is the shortest path is indeed bigger than some target D, then this gives a valid certificate that any polynomial time verifier can check and convince itself.

   [Hint: Show that $|y_u - y_v|$ is at most the length of edge (u, v), and recall the telescoping argument for why this was sufficient.]

6. Prove Hall's theorem using max-flow min-cut theorem.

7. **Directed Chinese Postman:** Given a directed graph, a postman tour is one that traverses each (directed) edge at least once. Show using circulations how to find a minimum cost tour.

   [Hint: Need to determine which edges to repeat to make the graph Eulerian. Also show that a directed graph is Eulerian iff for each vertex v, its indegree is equal to its outdegree.]

8. **Teachers to Courses (2)**: There are n lecturers and m courses. For each lecturer i there is a subset S_i of courses that she can teach. For each course j, a minimum of c_j (di fferent) lecturers must be assigned. For each lecturer i, we have a bound a_i on the maximum number

of courses that she can be assigned to. Design an algorithm using flow to find an assignment of lectures to courses (if it exists) that satisfies all the requirements.

[Hint: What can go wrong if you try to use matching as in the problem 1 above]