Solutions to exercises of lecture 5

Exercise 5.1. Recall that in the Traveling Salesman problem, we are given a graph $G = (V, E)$ with an integer weight w_e and the question is to find a Hamiltonian cycle $C \subseteq E$ minimizing $\sum_{e \in C} w_e$. Can you solve it in $O^*(n!)$ time?

Solution: Every Hamiltonian cycle is described by a permutation $v_1, \dots, v_n$ of the vertices so we can simply iterate over all permutations $v_1, \dots, v_n$ of V and see which one minimizes $w_{(v_n,v_1)} + \sum_{i=1}^{n-1} w_{(v_i,v_{i+1})}$

Exercise 5.2. In the k-coloring problem we are given a graph G and integer k and need to determine whether G has a k-coloring. Do you expect this problem parameterized by k to be FPT?

Solution: Not assuming $P \neq NP$: an $O(f(k)n^c)$ time algorithm for this, for some constant c, would imply an $O(n^c)$ time algorithm for 3-coloring.

Exercise 5.3. Find an algorithm detecting cliques of size at least k in $O(n^k k^2)$ time, why is this running time not sufficient to prove the problem to be FPT?

Solution: We cannot write $O(n^k)$ as $f(k)\texttt{poly}(n)$, since the latter implies a fixed exponent of n while in the first the exponent depends on k.

Exercise 5.4. Show that if G has a FVS of size at most k, it has a $k+2$-coloring. Can you give an example of a graph with a FVS of size at most k but no $k+1$ coloring?

Solution: use colors $1, \dots, k$ to color all vertices in the FVS with a distinct color, use $k+1, k+2$ for a two-coloring of the forest (which is easily seen to exist by fixing one color and propagating). A complete graph on $k+2$ vertices would be such an example.

Exercise 5.5. Give an $O^*(2^{n/2})$ time, $O^*(2^{n/4})$ space algorithm for Subset Sum using the 4SUM algorithm.

Solution: Assume n is a multiple of 4, construct an integer a_i for every $W \subseteq \{1, \dots, n/4\}$, b_i for every $X \subseteq \{n/4+1, \dots, n/2\}$,$c_i$ for every $Y \subseteq \{n/2+1, \dots, 3n/4\}$,$d_i$ for every $Z \subseteq \{3n/4+1, \dots, n\}$, set the target of the 4SUM problem to be t. This 4SUM instance has a solution if and only if the subset sum instance has one since every subset $S \subseteq \{1, \dots, n\}$ can be written as $W \cup X \cup Y \cup Z$.

Exercise 5.6. Can you solve 4-coloring in $O^*(2^n)$ time? What about 3-coloring in $O^*((2-\epsilon)^n)$ time, for some $\epsilon > 0$ (Hint: use that $\binom{n}{k} \leq 2^{0.92n}$ for $k \leq n/3$)?

Solution: For 4-coloring, we may iterate over all vertex sets $X \subseteq V$ that could have the first two colors. Given such X we just need to see whether both $G[X]$ and $G[V \setminus X]$ are 2-colorable.

For the second question, note that one color class must be of size at most $n/3$ so in Algorithm `3colv2` we may iterate over all sets of size at most $n/3$ instead.

Exercise 5.7. Solve Vertex Cover in $O^*(1.4656^k)$ time.

Solution: Adjust `vc2` as follows: if there exists no vertex of degree at least 3, we have a set of cycles, paths and isolated vertices and an optimal solution is computed in polynomial time by a simple greedy argument. Otherwise, branch as in Line 4 of `vc2`. If $T(k)$ denotes the number of leaves in the branching tree we see that $T(0) = 1$ and for $k > 0$

$$T(k) \leq \max_{d \geq 3} T(k-1) + T(k-d).$$

We see that $T(k)$ is bounded by 1.4656^k since $1.4656^{-1} + 1.4656^{-3} \leq 1$

Exercise 5.8. Recall the definition of NP. Why can any problem instance $x \in \{0,1\}^n$ of a language in NP be solved in $2^{\mathtt{poly}(|x|)}$ time?

Solution: NP: there exists a polynomial time verifier V, (e.g., an algorithm that runs in time polynomial in x with the following property: there exists a *certificate* c such that $V(x,c)$ returns true if and only if $x \in L$). Since V runs in time polynomial in the input, $|c|$ needs to be polynomial in $|x|$, so given an instance x, we can iterate over all $2^{|c|} = 2^{\mathtt{poly}(|x|)}$ possible c and see whether $V(x,c)$ gives true somewhere.

Exercise 5.9. An algorithm running in time $n^{\lg(n)^c}$ for some constant c is called *quasi-polynomial*. Recently, in a big breakthrough[1] László Babai showed that the 'Graph Isomorphism problem' can be solved in quasi-polynomial time. Graph Isomorphism is not known to be NP-complete. Can you explain why a quasi-polynomial time algorithm for an NP-complete problem would be a *huge* result (Hint: recall the definition of NP-completeness)?

Solution: NP-complete: L is NP-complete if for every other problem L' in NP there exists a polynomial time reduction from L' to L, e.g., an algorithm R such that for every input x, $x \in L'$ if and only if $R(x) \in L$. Since R is polynomial time, $|R(x)|$ is polynomial in $|x|$ thus if L is solved in time $|x|^{\lg(|x|)^c}$, then this gives an

$$|R(x)|^{\lg(|R(x)|)^c} = (|x|^{c'})^{\lg(|x|^{c'})^c} = |x|^{c'c' \lg(|x|)^c},$$

[1](see e.g., `http://www.quantamagazine.org/20151214-graph-isomorphism-algorithm/`)

time algorithm for problem L'. So this would be a huge result because it implies a quasi-polynomial time algorithm for any NP-complete problem.

Exercise 5.10. Show that Feedback Vertex Set is NP-hard. In particular, show that given an instance (G, k) of vertex cover, we can compute in polynomial time an equivalent instance (G', k) of feedback vertex set.

Solution: Given an instance $G = (V, E), k$ of vertex cover, add a vertex v_e for every edge (u, v) with neighbors $\{u, v\}$. There always exists an optimal FVS in which no added vertex is picked since v_e can be replaced with either u or w if $e = \{u, w\}$, and such a FVS is a FVS of the new graph if and only if it is a vertex cover of the old graph since for every edge $e = (u, w)$ it needs to hit the triangle u, w, v_e.

Alternatively, one could apply the degree 2 reduction rule to obtain a multigraph in which all edges occur twice.

Exercise 5.11. The n'th Fibonacci number f_n is defined as follows: $f_1 = 1$, $f_2 = 1$ and for $n > 2$, $f_n = f_{n-1} + f_{n-2}$. What is the running time of the following algorithm to compute f_n?

Algorithm `FIB1`(n)
Output: f_n
1: **if** $n = 1$ or $n = 2$ **then return** 1
2: **return** `FIB1`$(n-1)$+`FIB1`$(n-2)$.

Solution: The running time is at most $O(nf(n))$. To see this, note that the number of leaves is exactly $f(n)$. If you insist to be more precise to get rid of the n factor, note that the branching tree has no degree 2 vertices, and for any such tree the number of internal vertices is at most the number of leaves.

Exercise 5.12. In the Set Partition problem we are given $F_1, \ldots, F_m \subseteq U$ and need to find a subset of the sets that partition U. Can you do this in $O^*(2^{m/2})$ time?

Solution: Assume m is even by adding an empty set. Enumerate $L = \{\bigcup_{i \in X} F_i : X \subseteq \{1, \ldots, m/2\}\}$ and $R = \{\bigcup_{i \in X} F_i : X \subseteq \{m/2+1, \ldots, m\}\}$. For every $Y \in L$ check whether $U \setminus Y$ is in R, return yes if so and no otherwise.

Exercise 5.13. In this exercise we'll look at the d-Hitting Set problem: given sets $F_1, \ldots, F_m \subseteq U$ of size d each, where $|U| = n$, we need to find a subset $X \subseteq U$ with $|X| = k$ that 'hits' every set in the sense that $F_i \cap X \neq \emptyset$ for every i.

1. By which other name do you know 2-Hitting Set? Why is it equivalent?
2. Can you solve 3-Hitting Set in time $O^*(3^k)$?
3. Can you solve 3-Hitting Set in time $O^*(2.4656^k)$
   - Hint: Use iterative compression. Suppose you are also given a hitting set of size $k+1$, can you solve the problem in time $O^*(\sum_{i=1}^{k+1} \binom{k+1}{i} 1.4656^i)$. This equals $O^*(2.4656^k)$ by the binomial theorem.

Solution: Vertex Cover. The elements are the vertices, the sets the edges and there is a direct correspondence.

Pick a set of size at most 3 and branch on one of the three elements that needs to be included.

Suppose we have a 3-hitting set H of size $k+1$. Guess the subset $X \subseteq H$ that will be in the solution. Remove all sets that intersect with X and remove all elements from H. Since H was a hitting set, all sets are now of size at most 2. Pick elements in sets of size 1 so only sets of size 2 remain, and we have an instance of vertex cover. This instance of vertex cover can be solved in time $O^*(1.4656^{k-X})$ using the algorithm of Exercise 5.7. So indeed the running time of one compression step becomes $O^*(\sum_{i=1}^{k+1} \binom{k+1}{i} 1.4656^i)$.

Exercise 5.14. Give an algorithm that determines whether a given 3-CNF-Sat formula is satisfiable in time $O^*((2-\epsilon)^n)$, for some $\epsilon > 0$.

Solution: Use the following branching algorithm: pick a clause of maximum size and branch on all assignments of its variables satisfying it. For example, if the clause is $\neg v_2 \vee v_4 \vee \neg v_6$, we recurse on the CNF-formula obtained by setting v_2, v_4, v_6 to all 8 assignments except $1, 0, 1$. This results in 7 new recursive calls on formula's with at most $n-3$ variables, so we can use the following recurrence for the number of leaves of the branching tree $T(0) = 1$ and for $n > 0$

$$T(n) \leq \max\{7 \cdot T(n-3), 3 \cdot T(n-2), T(n-1)\}.$$

Setting $T(n) = 7^{n/3} < 1.913$ works since

$$\max\{7 \cdot 7^{\frac{n-3}{3}}, 3 \cdot 7^{\frac{n-2}{3}}, 7^{\frac{n-1}{3}}\} \leq 7^{n/3}.$$