Algorithms and Complexity (AC)

Marie Schmidt

(Based on slides by Gerhard Woeginger and Jesper Nederlof)

Landelijk Netwerk Mathematische Besliskunde

LNMB, Sep–Nov 2019

(Preliminary) program

9 Sep	: Introduction, basic concepts, time complexity and computational models, P versus NP
16 Sep	: reductions, NP-hardness and NP-completeness
23 Sep	: Pseudopolynomial time, strong/weak NP-hardness, co-NP
30 Sep	: Exercise set 1
30 Sep	: Approximation algorithms
7 Oct	: More on approximation algorithms
14 Oct	: Exercise set 2
14 Oct	: Exact algorithms for NP-hard problems
21 Oct	: More exact algorithms for NP-hard problems
28 Oct	: Exercise set 3
28 Oct	: Treewidth
4 Nov	: Randomized algorithms
11 Nov	: Exercise set 4
11 Nov	: **No lecture!!**

Website: `http://www.win.tue.nl/~jnederlo/LNMB/`

First 5 lectures: Marie Schmidt (`schmidt2@rsm.nl`), last 4 lectures: Jesper Nederlof (`j.nederlof@tue.nl`)

Program for the first three weeks

- Basic definitions: decision problems, graphs
- computational models and (worst-case) time complexity
- P versus NP
- Reductions
- NP-hardness
- A catalogue of NP-hard problems
- pseudo-polynomial time
- strong NP-hardness & weak NP-hardness
- co-NP, co-NP versus NP

And maybe more. . . ?

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Well-defined procedure that transforms an input into an output.

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Output: A permutation $(a'_1, a'_2, \ldots, a'_n)$ of the input sequence such that $a'_i \leq a'_2 \leq \ldots \leq a'_n$

Set $A := (a_1)$
for $i = 2, \ldots, n$ **do**
  update A by inserting a_i at the 'correct' position in sorted sequence A
end for
return A

Algorithm

Well-defined procedure that transforms an input into an output.

Example: Insertion Sort - for a machine

Input: A sequence of n numbers $(a_1, a_2, \ldots, a_n)$
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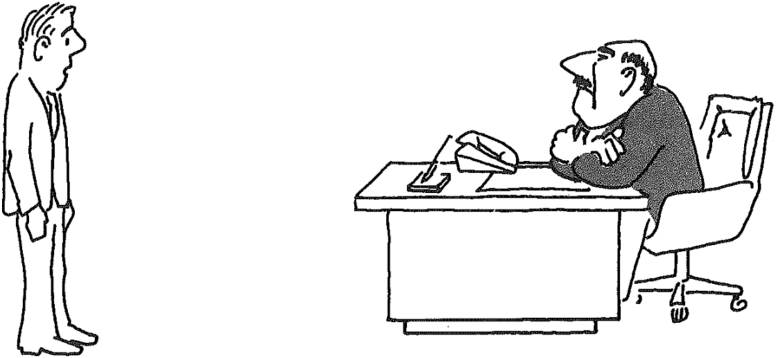
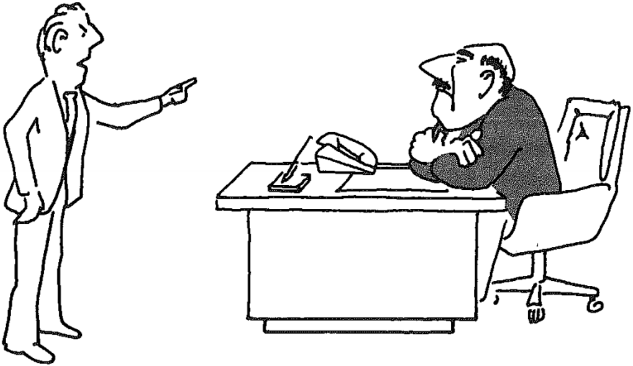
Set $A := (a_1)$
for $i = 2, \ldots, n$ **do**
  $key := A[j]$
  $i := j - 1$
  while $i > 1$ and $A[i] > key$ **do**
    $A[i+1] := A[i]$
    $i := i - 1$
  end while
  $A[i+1] := key$
end for
return A

When we analyze an algorithm, we are interested in:

- running time of the algorithm
- space (memory) needed by the algorithm (probably not treated in this course)
- for optimization problems: quality of the output
    - exact algorithm
    - approximation algorithm
    - heuristic algorithm



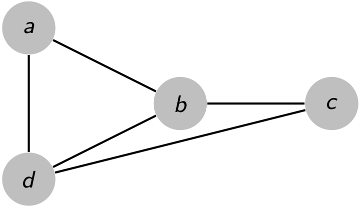
“I can’t find an efficient algorithm, I guess I’m just too dumb.”



“I can’t find an efficient algorithm, because no such algorithm is possible!”

Basic concepts: Graphs

Graph: pair (V, E) where V is set of *vertices* and E is a set of pairs of vertices called *edges*



Matching: set of non-adjacent edges (no two edges share a vertex), *perfect* if $|V|/2$ edges

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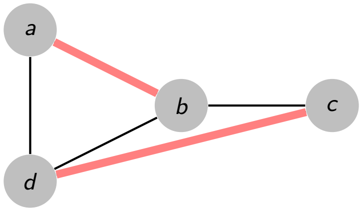

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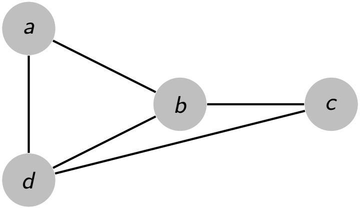

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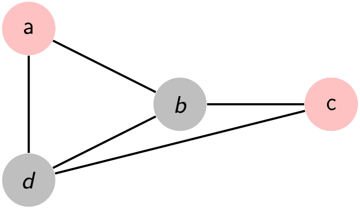

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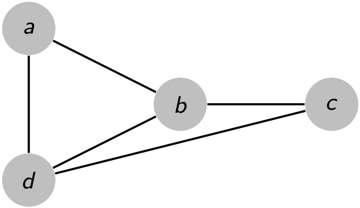

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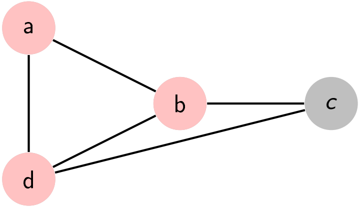

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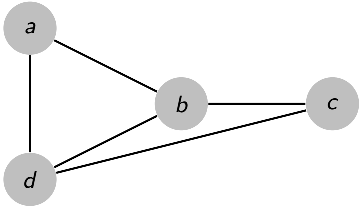

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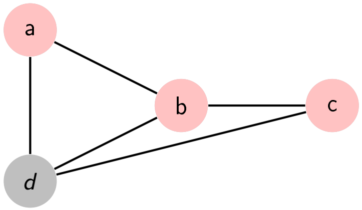

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Much more terminology: cycles, Hamiltonian cycles, trees, forests,...

Problems

Problem instance:

- specification of problem data

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Example: Instance of decision version of clique

$V = \{a, b, c, d, e, f, g\}$; $k = 4$
$E = \{\{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}, \{b, d\}, \{b, e\}, \{c, e\}, \{d, e\},$
$\{d, f\}, \{e, f\}, \{e, g\}, \{f, g\}\}$;

Basic concepts: Input size and asymptotics

Problem size:
- length (number of symbols) of reasonable encoding of instance (often denoted as n)

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Example for encodings

- Graph: adjacency list; adjacency matrix
- Set: list of elements; bit vector
- Number: decimal; binary; hex; unary

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big-Oh notation

$f(n)$ is $O(g(n))$ denotes
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big-Omega, big-Theta

$f(n)$ is $\Omega(g(n))$ denotes that $\exists n_0, C$ such that $\forall n > n_0$ $f(n) \geq C \cdot g(n)$.
$f(n)$ is $\Theta(g(n))$ denotes that $f(n)$ is $O(g(n))$ and $\Omega(g(n))$.

Different types of algorithmic problems:

- Optimization problems (min/max)
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Instance: a graph $G = (V, E)$; a bound k
Question: does G contain a clique of size (at least) k?

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Instance: a graph $G = (V, E)$; a bound k
Question: does G contain a clique of size (at least) k?

Example (neither optimization nor decision problem) SORTING

Instance: A sequence of n numbers $(a_1, a_2, \ldots, a_n)$
Task: A permutation $(a'_1, a'_2, \ldots, a'_n)$ of the input sequence such that $a'_i \leq a'_2 \leq \ldots \leq a'_n$

Observation

Every discrete optimization problem can be rewritten into
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use bisection search on the interval of objective values

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Let G be a graph on n vertices.

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Let G be a graph on n vertices.

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Let G be a graph on n vertices.

Does G contain a clique of size at least $n/2$? – YES
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Does G contain a clique of size at least $13n/16$? – YES

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Does G contain a clique of size at least $13n/16$? – YES
Etc.

Search takes logarithmic number of steps –> fast and simple

Time complexity of an algorithm

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$\rightarrow$ depends on computational model

Our choice: Random-access-machine (RAM) model

executes operations one after another (no concurrent operations)

Elementary steps $\hat{=}$ assumption: can be executed in constant time

- arithmetic: add, subtract, multiply, divide, remainder, floor, ceiling
- data movement: load, store, copy
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Why do we use RAM:

- similar to how a computer works & approximates running time of computer well
- easier to analyze than many alternatives

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BUT: there are alternatives, e.g.,

- alternative computational models
- time complexity in *output length*
- average case time complexity

What is the worst-case time complexity of InsertionSort?

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Both for encoding length, and for time complexity, we make use of big-Oh notation.

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Note: Determining / proving the worst-case time complexity of an algorithm can be difficult!

Turing machines

- Alternative mathematical models of computation
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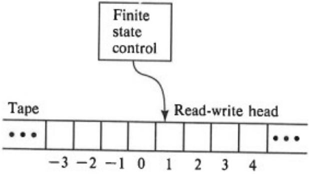
Not this.

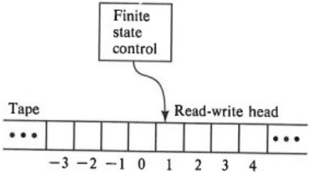


But this!

Deterministic one-tape Turing machine (DTM)

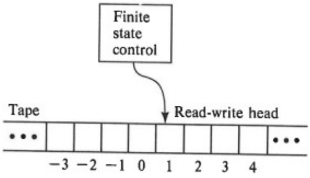
A DTM consists of

1. a *finite state control*
2. a *read-write head*
3. a *tape*: two-way infinite sequence of tape squares



A program for a DTM specifies:

1. a finite set Γ of tape symbols, including a subset $\Sigma \subset \Gamma$ of *input symbols* and a distinguished *blank symbol* $b \in \Gamma \setminus \Sigma$
2. a finite set Q of *states*, inclusing a distinguished *start state* q_0 and two distinguished *halt states* q_Y and q_N
3. a *transition function* $\delta : (Q \setminus \{q_Y, q_N\}) \times \Gamma \to Q \times \Gamma \times \{-1, 1\}$



Operation of a DTM program

Input: finite string $x \in \Sigma$

Initialize: write string in tape squares 1 to $|x|$, one symbol per square (all other tape squares are blank), state $q = q_0$, read-write head scans tape square 1

while $q \notin \{q_Y, q_N\}$ **do**

look up $(q', s'\Delta) := \delta(q, s)$ for current state q and read-write head pointing at square with symbol s

erase s

write s' in its place

move one square to the left if $\Delta = -1$, one square to the right if $\Delta = 1$

set $q := q'$

end while

if $q = q_Y$ **then**

return YES

else

return NO

end if

Each iteration of the while-loop counts as a step

A program for a DTM machine

$\Gamma = \{0, 1, b\}$, $\Sigma = \{0, 1\}$, $Q = \{q + 0, q_1, q_2, q_Y, q_N\}$

q	0	1	b
q_0	$(q_0, 0, +1)$	$(q_0, 1, +1)$	$(q_1, b, -1)$
q_1	$(q_2, b, -1)$	$(q_3, b, -1)$	$(q_N, b, -1)$
q_2	$(q_Y, b, -1)$	$(q_N, b, -1)$	$(q_N, b, -1)$
q_3	$(q_N, b, -1)$	$(q_N, b, -1)$	$(q_N, b, -1)$

$\delta(q, s)$

- Let's try this out!

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- Let's try this out!
- What does this program do?

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- Let's try this out!
- What does this program do?
- How many steps do we need?

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$\delta(q, s)$

- Let's try this out!
- What does this program do?
- How many steps do we need?
- How many steps would we need at most?
- How much space do we need (at most)?

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Church-Turing thesis

Anything that can be calculated by an *effective method* can be computed by a deterministic Turing machine.

Non-deterministic Turing machine

Non-deterministic Turing machine (NDTM)

1. guessing module: write-only head
2. checking module: deterministic Turing machine

A program for a DTM specifies:

exactly the same as a DTM program:

1. finite set of tape symbols Γ of tape symbols, including blank symbol
2. finite set Q of *states*, i
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Terminology & definitions

Accepting computation: all computations that terminate in accepting state (q_Y).
Non-accepting computations: all computations that terminate in non-accepting-state (q_N) or do not terminate at all.

NDTM program M **accepts** x if *there is* an accepting computation for x on M.

The **time complexity** of an NDTM program for a string x is defined as the *minimum* running time over all accepting computations of x by M.
The worst-case time-complexity of an NDTM program is the maximum time complexity over all strings x of a certain length n that are accepted by n.

Non-deterministic algorithm

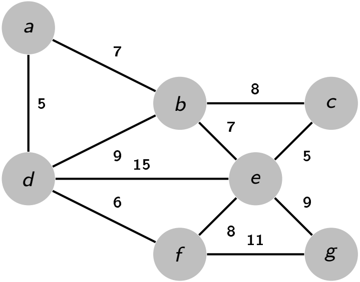
non-deterministic algorithm $\hat{=}$ program for a non-deterministic Turing machine

1. Oracle/guessing stage
2. Checking stage

time complexity of a non-deterministic algorithm
$\hat{=}$ time complexity of the corresponding program

Travelling Salesman Problem (TSP) - Decision version

Instance: cities $1, \ldots, n$; distances $d(i,j)$; a bound B
Question: does there exist a roundtrip of length at most B?



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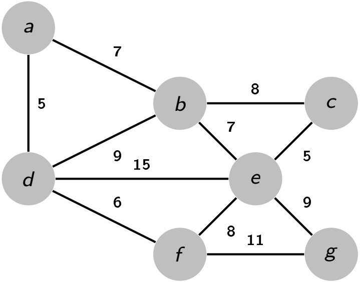

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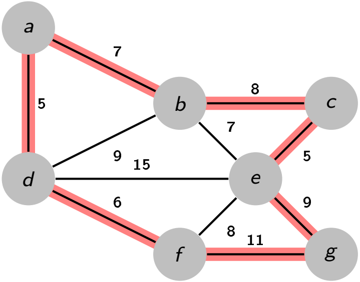

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Warnings:

1. The Church-Turing thesis relates to *deterministic* Turing machines.
2. A non-deterministic Turing machine is a *theoretical* construct, not an actual machine!

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Example of a minimization problem

- Given (adjacency list of) $G = (V, E)$ and $w_e \in \mathbb{R}$ for every $e \in E$,
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spanning: all vertices are incident to an edge

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Computes a Minimum Spanning Tree T using a *greedy* approach:

- Consider edges in ascending order of cost
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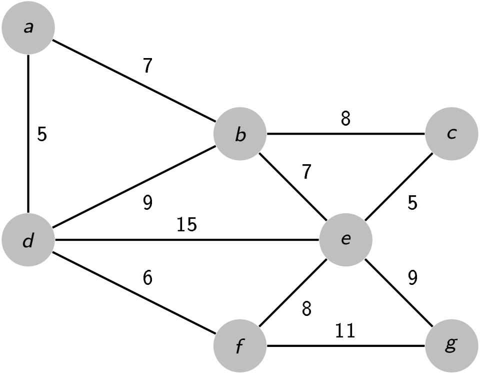

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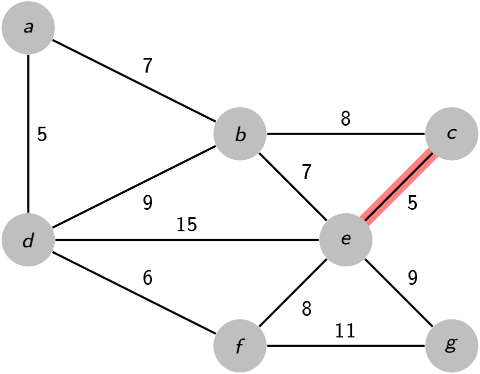

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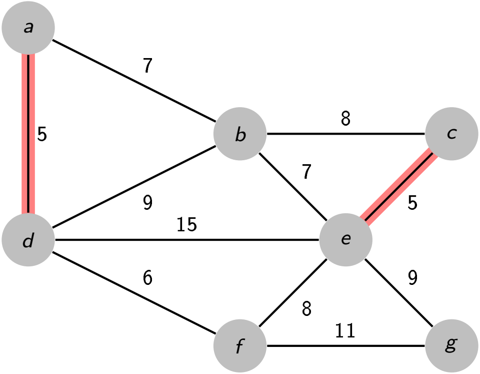

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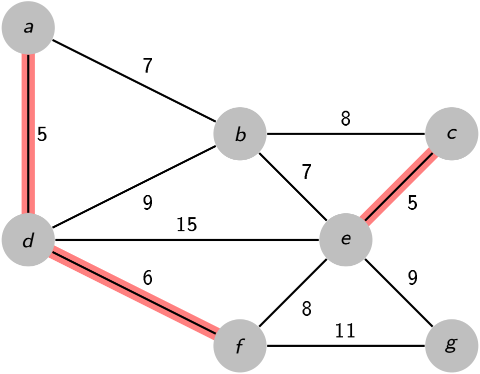

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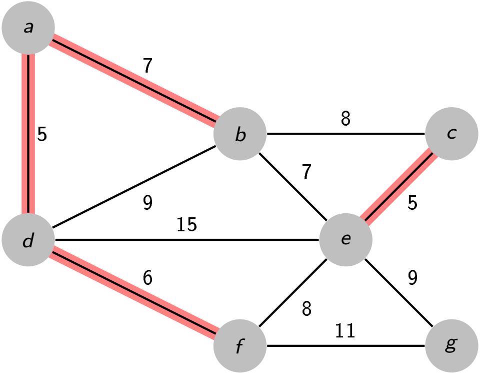

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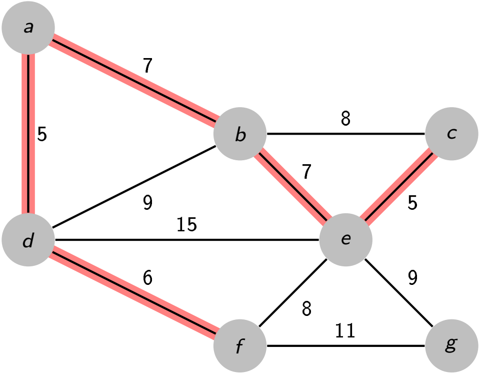

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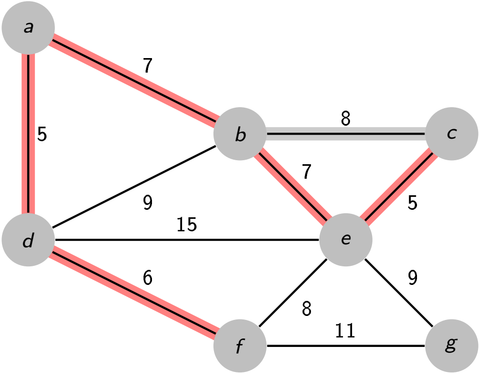

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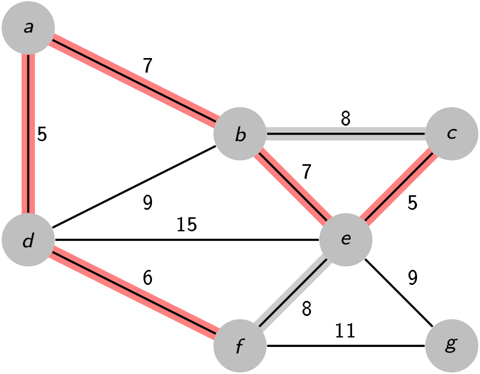

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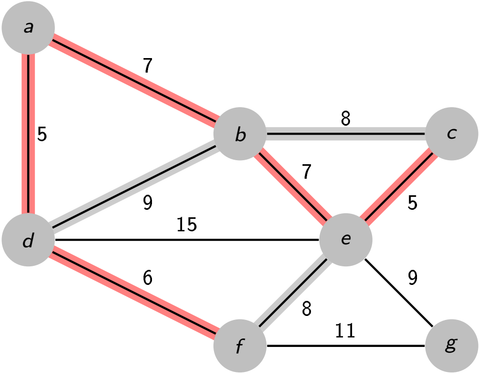

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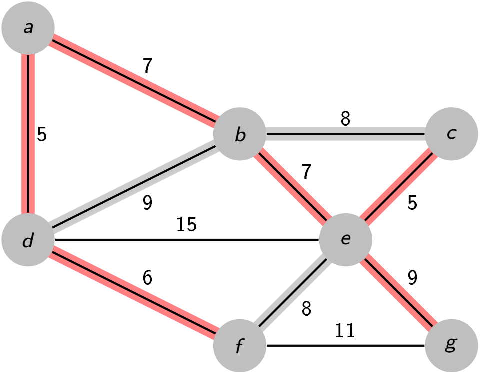

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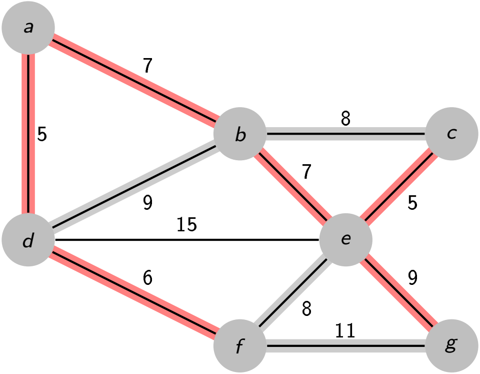

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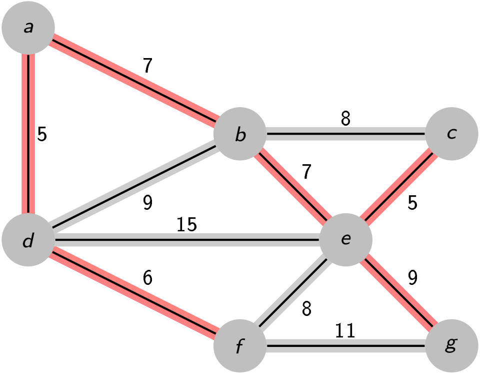

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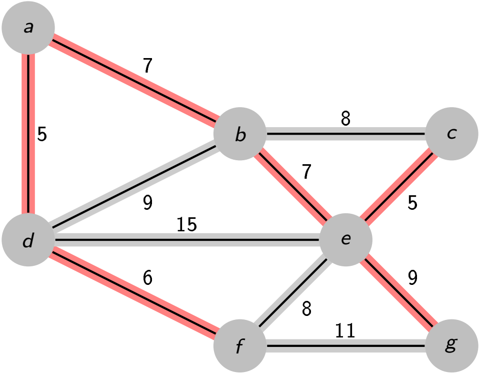

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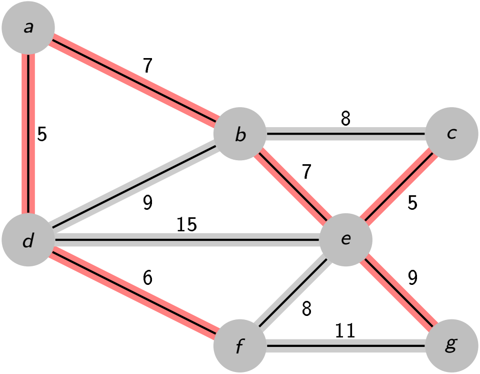

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Run-time $O(|E|^2)$ (if implemented naïvely); decision version in P

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A decision problem X lies in the complexity class P,

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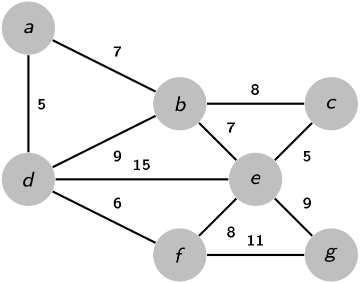

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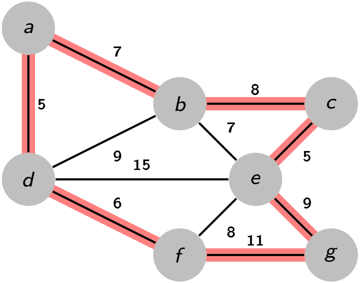

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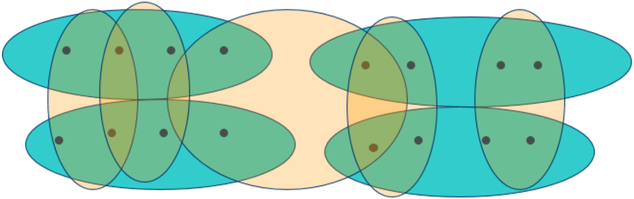

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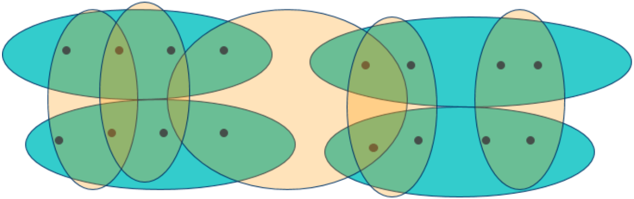

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P versus NP

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  P stands for Polynomial Time
- NP = huge class of problems that fulfill some soft condition
  NP contains lots of interesting and important decision problems
  NP stands for Non-deterministic Polynomial Time

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- NP = huge class of problems that fulfill some soft condition
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Big open question

P=NP ????

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Big open question

P=NP ????

Answer YES:

- would trigger a revolution in computing
- if a short solution exists, it can be found quickly

Answer NO:

- that's what most people expect
- even very short solutions may be very hard to find

To prove that a problem

- can be solved in $O(n \log n)$, $O(n^3)$, etc
- is in P
- is in NP

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is straightforward (although not always easy):

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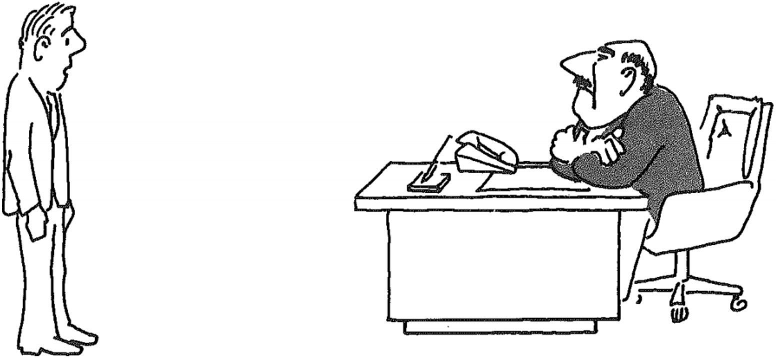
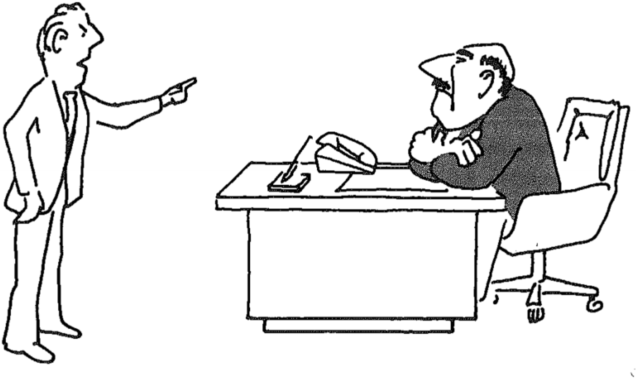
How do we prove that a problem can**not** be solved in a certain time?



“I can’t find an efficient algorithm, I guess I’m just too dumb.”



"I can't find an efficient algorithm, because no such algorithm is possible!"

Proving lower bounds (short note)

Example: Find maximum element from unsorted list

Input: A list of numbers $m_1, m_2, \ldots m_n$, a number M.
Question: Is there an element $\geq M$ in the list.

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- requires time $\Omega(n)$
- thus: $\Theta(n)$

Note: Most problems need time $\Omega(n)$ to be solved.
Can you think of one that does not?

Sorting

Input: A sequence of n numbers $(a_1, a_2, \ldots, a_n)$
Task: Create a permutation $(a'_1, a'_2, \ldots, a'_n)$ of the input sequence such that $a'_i \leq a'_2 \leq \%ldots \leq a'_n$

- Insertion sort needs $O(n^2)$ in the worst case.

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There are $n!$ different permutations of n numbers.

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$$\log_2(n!) = \log_2(n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1) = \log_2(n) + \log_2(n-1) + \ldots + \log(2) + \log(1)$$

$$= \sum_{i=1}^{n} \log_2(i) = \sum_{i=1}^{\frac{n}{2}-1} \log_2(i) \sum_{i=\frac{n}{2}}^{n} \log_2(i) \geq 0 + \sum_{i=\frac{n}{2}}^{n} \log_2(\frac{n}{2}) = \frac{n}{2} \log_2(\frac{n}{2}) = \Omega(n \log(n))$$

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For a more extensive proof, see here
`https://www.cs.cmu.edu/~avrim/451f11/lectures/lect0913.pdf`

3-Satisfiability (3-SAT)

Instance:
a set of logical variables $X := \{x_1, \ldots, x_n\}$ and a set of clauses C of three literals over X

Question: does there exist a truth assignment for X that simulsatisfies all clauses in C?

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From stackexchange
(https://cstheory.stackexchange.com/questions/1060/best-upper-bounds-on-sat?rq=1 and
https://cstheory.stackexchange.com/questions/93/what-are-the-best-current-lower-bounds-on-3sat)
(retrieved 13.9.19)

- Best found non-randomized algorithm (for 3-SAT) seems to be 1.32793^n
- Best found randomized algorithm similar ($O(1.321^n)$?)
- No one so far has been able to prove $\Omega(n^2)$

Lower bounds on problem complexity tend to be rare / weak / difficult to prove.

$\rightarrow$ We look at a different approach.

"I can't find an efficient algorithm, but neither can all these famous people."

Reductions

Definition

For two decision problems X and Y, we say that X (polynomially) **reduces** to Y (and we write $X \leq_p Y$)
if there exists a polynomial time transformation f that translates instance of X into instances of Y with $I \in$YES(X) $\iff$ $f(I) \in$YES(Y).

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Often, we omit the word 'polynomially' and just say that X reduces to Y.

Intuition:

- X can be modelled as a special case of Y
- the 'computational hardness' of X is upper bounded by Y's
- If Y is easy, then also X is easy
- If X is difficult, then also Y is difficult

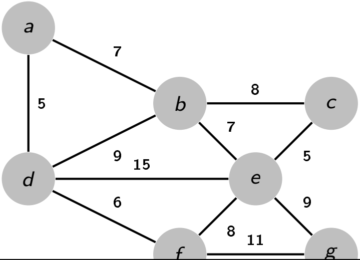
Hamiltonian cycle / TSP

Hamiltonian cycle (HC)

Instance: an undirected graph $G = (V, E)$
Question: does G contain a Hamiltonian cycle?
(a simple cycle that visits every vertex exactly once)

Travelling Salesman Problem (TSP)

Instance: cities $1, \ldots, n$; distances $d(i, j)$; a bound B
Question: does there exist a roundtrip of length at most B?



Theorem

HC $\leq_p$ TSP.

Proof: .

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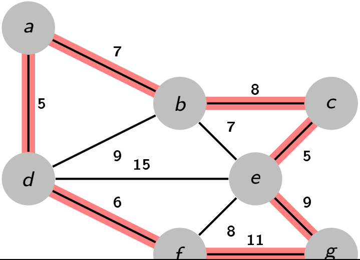

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Instance: a graph $G = (V, E)$; an integer k
Question: does G contain a clique of size (at least) k?

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Theorem

SAT $\leq_p$ CLIQUE.

Proof: Given a set of clauses $\{c_1, c_2, \ldots, c_m\}$, over $x_1, \ldots, x_n$ define instance instance of clique (our function f):

$$V = \{(l, i) | l \text{ is a literal in } c_i\}$$
$$E = \{\{(l, i), (l', i')\} | l \neq \neg l' \wedge i \neq i'\}$$
$$k = m$$

Lemma

Reducibility is a transitive relation:
$X \leq_p Y$ and $Y \leq_p Z$ implies $X \leq_p Z$

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Proof: by putting the two tranformations into series

NP-hardness

Definition

A decision problem X is *NP-hard*,
if **all** problems $Y \in NP$ can be reduced to it
(that is, if $Y \leq_p X$ holds for all $Y \in NP$)

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A decision problem X is *NP-complete*,
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Intuition:

- NP-complete problems are the hardest problems in NP
- Recall: NP is huge and contains tons of important problems
- Some people consider NP-complete problems to be intractable.

NP-hardness

Theorem

If one NP-complete problem X has a polynomial time algorithm then all NP-complete problems have polynomial time algorithms (and hence P=NP)

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NP-hardness

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Why? Can reduce to X and then solve produced instance of X.

"I can't find an efficient algorithm, but neither can all these famous people."

Cook-Levin theorem (1971)

SAT is NP-complete.

Cook-Levin theorem (1971)

SAT is NP-complete.

- Stephen Cook (born 1939):
  American-Canadian computer scientist and mathematician
- Leonid Levin (born 1948):
  Russian computer scientist, discovered the result somewhat earlier

Proof of Cook-Levin

Variable	Range	Intended meaning
$Q[i,k]$	$0 \leq i \leq p(n)$, $0 \leq k \leq \lvert Q \rvert$	at time i, M is in state k
$H[i,j]$	$0 \leq i \leq p(n)$, $-p(n) \leq j \leq p(n)+1$	at time i, the read-write head of M scans tape square j
$S[i,j,k]$	$0 \leq i \leq p(n)$, $-p(n) \leq j \leq p(n)+1$, $0 \leq k \leq \lvert \Gamma \rvert$	at time i, the entry on tape square j is s_k

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clause group	restriction imposed
G_1	at each time i, M is in exactly one state
G_2	at each time i, the read-write head is scanning exactly one tape square
G_3	at each time i, each tape square contains exactly one symbol from Γ
G_4	at time 0, the computation is in the initial configuration of its checking stage for input x
G_5	By time $p(n)$, M has entered state q_y and hence has accepted x
G_6	For each time i the configuration of M at time $i+1$ follows by a single application of the transition function δ from the configuration at time i

Proof of Cook-Levin

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Proof of Cook-Levin

G_1: at each time i, M is in exactly one state

$Q[i,0] \vee Q[i,1] \vee \ldots \vee Q[i,r]$ for all $0 \leq i \leq p(n)$
$\neg Q[i,j] \vee \neg Q[i,j']$ for all $0 \leq i \leq p(n), 0 \leq j \leq j' \leq r$

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Proof of Cook-Levin

G_2: at each time i, the read-write head is scanning exactly one tape square

$H[i,-p(n)] \vee H[i,-p(n)+1] \vee \ldots \vee H[i,p(n)+1]$ for all $0 \leq i \leq p(n)$
$\neg H[i,j] \vee \neg H[i,j']$ for all $0 \leq i \leq p(n), -p(n) \leq j \leq j' \leq$

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Proof of Cook-Levin

G_4: at time 0, the computation is in the initial configuration of its checking stage for input x

Variable	Range	Intended meaning
$Q[i,k]$	$0 \leq i \leq p(n)$, $0 \leq k \leq \|Q\|$	at time i, M is in state k
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Proof of Cook-Levin

G_4: at time 0, the computation is in the initial configuration of its checking stage for input x

$Q[0,0], H[0,1], S[0,0,0]$
$S[0,1,k_1], S[0,2,k_2], \ldots, S[0,n,k_n],$ with $x = (s_{k_1}, s_{k_2}, \ldots, s_{k_n})$
$S[0,n+1,0], S[0,n+2,0], \ldots, S[0,p(n)+1,0]$

Variable	Range	Intended meaning
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Proof of Cook-Levin

G_5: by time $p(n)$, M has entered state q_y

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$Q[p(n), 1]$

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Proof of Cook-Levin

G_6: Changes according to transition function

Variable	Range	Intended meaning
$Q[i,k]$	$0 \leq i \leq p(n)$, $0 \leq k \leq \lvert Q \rvert$	at time i, M is in state k
$H[i,j]$	$0 \leq i \leq p(n)$, $-p(n) \leq j \leq p(n)+1$	at time i, the read-write head of M scans tape square j
$S[i,j,k]$	$0 \leq i \leq p(n)$, $-p(n) \leq j \leq p(n)+1$, $0 \leq k \leq \lvert \Gamma \rvert$	at time i, the entry on tape square j is s_k

Proof of Cook-Levin

G_6: Changes according to transition function

$$\neg H[i,j] \vee \neg Q[i,k] \vee \neg S[i,j,l] \vee H[i+1,j+\Delta]$$
$$\neg H[i,j] \vee \neg Q[i,k] \vee \neg S[i,j,l] \vee Q[i+1,k']$$
$$\neg H[i,j] \vee \neg Q[i,k] \vee \neg S[i,j,l] \vee S[i+1,j,l']$$

with for $q \in Q \setminus \{q_Y, q_N\}$: $\delta(q_k, s_l) = (q_{k'}, s_{l'}, \delta)$ and
for $q \in \{q_Y, q_N\}$: $\delta = 0$, $k' = k$, $l' = l$

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NP-hardness: 3-SAT

3-SAT

Instance: a set of logical variables $X := \{x_1, \ldots, x_n\}$ and a set of clauses C of three literals over X

Question: does there exist a truth assignment for X that simultaneously satisfies all clauses in C?

Theorem

3-SAT is NP-hard (and NP-complete).

NP-hardness: 3-SAT

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Question: does there exist a truth assignment for X that simultaneously satisfies all clauses in C?

Theorem

3-SAT is NP-hard (and NP-complete).

Proof: By reduction from SAT. Let $I = (X, C)$ an instance of SAT. We construct the following instance (X', C') of 3-SAT:

- $X_0 := X$
- For each clause c_j we construct a set of variables X_j and additional clauses C_j (with 3 literals each)
- $X' := \bigcup_{j=0}^{|C|} X_j$, $C' := \bigcup_{j=1}^{|C|} C_j$

NP-hardness: Integer programming

Integer programming (ILP)

Instance: an integer matrix A; an integer vector b
Question: does there exist an integer vector y with $Ay \leq b$?

Theorem

ILP is NP-hard (and NP-complete).

Proof:

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Proof: by reduction from SAT. Let (X, C) with $X = x_1, \ldots, x_n$ and $C = \{c_1, c_2, \ldots, c_m\}$ be an instance of SAT.

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Define A and b, use decision vars $y_j \in \{0, 1\}$ to indicate if $t(x_j) = true$.

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We define matrix A as

$$a_{ij} = \begin{cases} \\ \\ \end{cases}$$

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$$a_{ij} = \begin{cases} -1 & \text{if } x_j \text{ is in } c_i \\ & \\ & \end{cases}$$

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and $b_i = \#$negated literals in c_i -1.

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and $b_i = \#$negated literals in c_i -1.
encode $y_j \in \{0, 1\}$ as $0 \leq y_j \leq 1$.

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Define A and b, use decision vars $y_j \in \{0, 1\}$ to indicate if $t(x_j) = \mathit{true}$.
We define matrix A as

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and $b_i = \#$negated literals in c_i -1.
encode $y_j \in \{0, 1\}$ as $0 \leq y_j \leq 1$.
To show: There is a satisfying truth assignment for $(X, C) \Leftrightarrow$ there is a vector y fulfilling $Ay \leq b$

NP-hardness: Integer programming

Integer programming (ILP)

Instance: an integer matrix A; an integer vector b
Question: does there exist an integer vector y with $Ay \leq b$?

Theorem

SAT $\leq_p$ ILP, and therefore ILP is NP-hard (and NP-complete).

Consequence: Every problem in NP can be modelled as an ILP.

NP-hardness: Clique

Clique

Instance: a graph $G = (V, E)$; an integer k
Question: does G contain a clique of size (at least) k?

Theorem

CLIQUE is NP-hard (and NP-complete).

Proof: SAT is NP-hard and SAT $\leq_p$ CLIQUE.

NP-hardness: Independent set

Independent set (IS)

Instance: a graph $G = (V, E)$; an integer k
Question: does G contain an independent set of size (at least) k?
(a set of vertices that does not span any edge)

Theorem

IS is NP-hard (and NP-complete).

Proof:

NP-hardness: Independent set

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Proof: By reduction from CLIQUE:

NP-hardness: Independent set

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Theorem

IS is NP-hard (and NP-complete).

Proof: By reduction from CLIQUE:
Given an instance $(G = (V, E), k)$ of clique, construct the following instance of IS:
$V' := V$, $E' := \{\{i,j\} : i \neq j \in V, \{i,j\} \notin E\}$, $k' := k$.

NP-hardness: Independent set

Independent set (IS)

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Proof: By reduction from CLIQUE:
Given an instance $(G = (V, E), k)$ of clique, construct the following instance of IS:
$V' := V$, $E' := \{\{i,j\} : i \neq j \in V, \{i,j\} \notin E\}$, $k' := k$.
Show:
$X \subset V$ is a clique in $G \Leftrightarrow X$ is an independent set in G'

NP-hardness: Exact cover

Exact cover (Ex-Cov)

Instance: a ground set X; subsets $S_1, \ldots, S_m$ of X

Question: do there exist some subsets S_i that form a partition of X?

Theorem

(Ex-Cov) is NP-hard (and NP-complete).

Proof:

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Proof: by reduction from IS.
Let (G, k) with $G = (V, E)$ be an instance of IS.
Define an instance of (Ex-Cov) as follows: $X := E \cup \{1, \ldots, k\}$
and subsets
$S_{ih} := \{\{i,j\} : \{i,j\} \in E\} \cup \{h\}$ for $i \in V,\ h = 1, \ldots k$
$S_{\{i,j\}} := \{\{i,j\}\}$ for $\{i,j\} \in E$

NP-hardness: Exact cover

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$S_{\{i,j\}} := \{\{i,j\}\}$ for $\{i,j\} \in E$
Show:
If S is a solution to (Ex-Cov), $i : S_{ih} \in S$ is an independent set of size k.
If $X = \{x_1, x_2, \ldots, x_k\} \subset V$ is an independent set, $\bigcup_{j=1}^{k} S_{x_j j} \cup \{\{i,j\} : i,j \notin X\}$ is a solution to (Ex-Cov).

NP-hardness: Subset Sum

Subset Sum (SS)

Instance: positive integers $a_1, \ldots, a_n$; a bound b

Question: does there exist an index set $J \subseteq \{1, \ldots, n\}$ with $\sum_{j \in J} a_j = b$?

Theorem

SS is NP-hard (and NP-complete).

Proof:

NP-hardness: Subset Sum

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Theorem

SS is NP-hard (and NP-complete).

Proof: by reduction from Ex-Cov.

NP-hardness: Subset Sum

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Theorem

SS is NP-hard (and NP-complete).

Proof: by reduction from Ex-Cov.
Let $(X = \{x_1, \ldots, x_m\}, \{S_1, \ldots, S_n\})$ be an instance of Ex-Cov.
Define numbers a_j as $a_j := \sum_{i=1}^{m} c_{ij} \cdot d_i$ with $c_{ij} = 1$ if $x_i \in S_j$ and $d_i = (n+1)^{i-1}$.
Set $b := \sum_{i=1}^{m} (n+1)^{i-1}$.

NP-hardness: Subset Sum

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Set $b := \sum_{i=1}^{m} (n+1)^{i-1}$.

Show:
J is the index set of a solution to Ex-Cov $\Leftrightarrow$ J is the index set of a solution to SS.

NP-hardness: Subset Sum

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Question: does there exist an index set $J \subseteq \{1, \ldots, n\}$ with $\sum_{j \in J} a_j = b$?

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Set $b := \sum_{i=1}^{m} (n+1)^{i-1}$.

Show:
J is the index set of a solution to Ex-Cov $\Leftrightarrow$ J is the index set of a solution to SS.
Also: argue why this is a *polynomial-time* transformation.

NP-hardness: 2-Partition

2-PARTITION

Instance: positive integers $a_1, \dots, a_n$ with $\sum_{i=1}^{n} a_i = 2A$.

Question: does there exist an index set $I \subseteq \{1, \dots, n\}$ with $\sum_{i \in I} a_i = A$?

Theorem

2-PARTITION is NP-hard (and thus NP-complete).

Proof:

NP-hardness: 2-Partition

2-PARTITION

Instance: positive integers $a_1, \ldots, a_n$ with $\sum_{i=1}^{n} a_i = 2A$.

Question: does there exist an index set $I \subseteq \{1, \ldots, n\}$ with $\sum_{i \in I} a_i = A$?

Theorem

2-PARTITION is NP-hard (and thus NP-complete).

Proof: by reduction from SS.

Vertex cover (VC)

Instance: a graph $G = (V, E)$; an integer k
Question: does G contain a vertex cover of size (at most) k?
(a set of vertices that touches every edge)

Theorem

VC is NP-hard (and thus NP-complete).

Proof:

Vertex cover (VC)

Instance: a graph $G = (V, E)$; an integer k
Question: does G contain a vertex cover of size (at most) k?
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Theorem

VC is NP-hard (and thus NP-complete).

Proof: by reduction from IS.

NP-hardness: Hamiltonian cycle / TSP

Directed Hamiltonian cycle (dir-HC)

Instance: a directed graph (V, E)
Question: does this graph contain a directed Hamiltonian cycle?

Theorem

Dir-HC is NP-complete.

NP-hardness: Hamiltonian cycle / TSP

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Instance: a directed graph (V, E)
Question: does this graph contain a directed Hamiltonian cycle?

Theorem

Dir-HC is NP-complete.

Proof: Easy to see: in NP.
To show NP-hard: reduction from VC.
Given instance $G = (V, E), k$ of VC. Define $G' = (V', E')$:

NP-hardness: Hamiltonian cycle / TSP

Directed Hamiltonian cycle (dir-HC)

Instance: a directed graph (V, E)
Question: does this graph contain a directed Hamiltonian cycle?

Theorem

Dir-HC is NP-complete.

Proof: Easy to see: in NP.
To show NP-hard: reduction from VC.
Given instance $G = (V, E), k$ of VC. Define $G' = (V', E')$:

$$
\begin{aligned}
V' &= \{(i,j), \{i,j\}, (j,i) | \{i,j\} \in E\} \cup \{1, \dots, k\} \\
E' &= \{((i,j), \{i,j\}), (\{i,j\}, (i,j)), ((j,i), \{i,j\}), (\{i,j\}, (j,i)) | \{i,j\} \in E\} \\
&\quad \cup \{((i,j), q), (q, (i,j)), ((j,i), q), (q, (j,i)) | \{i,j\} \in E, q = 1, \dots, k\} \\
&\quad \cup \{((h,i), (i,j)) | \{h,i\} \in E, \{i,j\} \in E, h \neq j\} \\
&\quad \cup \{(i,j), (j,i) | 1 \leq i < j \leq k\}
\end{aligned}
$$

NP-hardness: Hamiltonian cycle / TSP

Theorem

HC is NP-hard (and thus NP-complete).

Proof: Reduction from dir-HC.

Theorem

TSP is NP-complete.

Proof: already seen: in NP and TSP$\leq_p$ HC.

Recommended reading

Garey and Johnson. 'Algorithms and Complexity'
Lenstra and Rinnooy Kan. Computational complexity of discrete optimization problems.
Annals of Discrete Mathematics 4 (pp 121-140), 1979.
Electronic copy available on website

Cormen, Leiserson, Rivest and Stein 'Introduction to Algorithms':

- Chapter 1-3 (basics)
- Chapter 23 (minimum spanning trees)
- Chapter 34 (P, NP, NP-completeness, Cook-Levin theorem, reductions)