Tight bounds for counting colorings parameterised by cutwidth via matrix ranks

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(slides by Carla Groenland and Isja Mannens)
Utrecht University

Joint work with Isja Mannens, Jesper Nederlof and Krisztina Szilágyi
STACS 2022

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- Consider a divide and conquer scheme for comp. problem PROB
- Suppose it divides global solutions in families A, B of partial solutions
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- For $\oplus p$-PROB, this is the rank of $rk_p(M)$ over $\mathbb{Z}_p$.

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- Rows and columns are indexed by forests on vertex set $[k]$. Matrix indicates whether the (multi-set) union forms a tree.

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  [CLN'18] $rk(M^k_{match}) \geq 4^k \rightarrow$ tight lower bound for $\#\textsc{Ham Cyc}/pw$.

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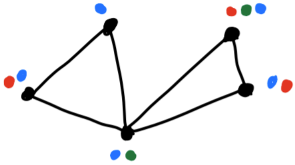
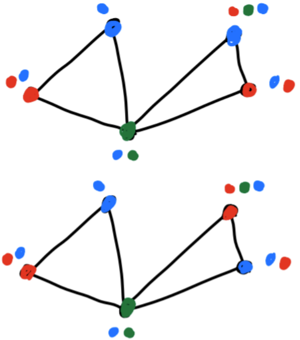
q-list coloring

Given: Graph $G = (V, E)$ and for each $v \in V$, a list $L(v) \subseteq \{1, \dots, q\}$.

Want: $c(v) \in L(v)$ for all $v \in V$ such that $c(u) \neq c(v)$ for $uv \in E$.





#q-LIST COL MOD p

Given. G graph, lists $L(v) \subseteq \{1, \dots, q\}$ for all $v \in V(G)$, $k \in \mathbb{F}_p$.

Output. What is the number of list colorings of G modulo p?

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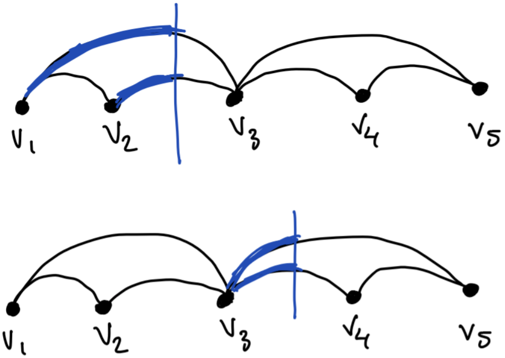
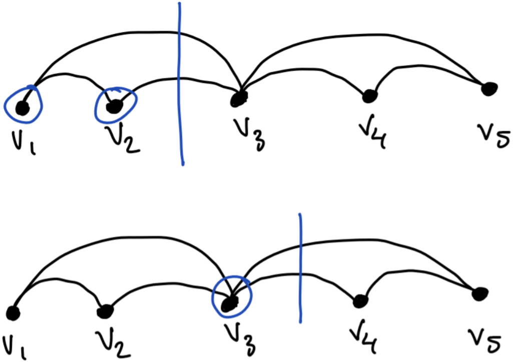
Cutwidth



$\max_\sigma \min_i$ # edges crossing ith cut

Pathwidth



$-1 + \max_\sigma \min_i$ # left endpoints of edges crossing ith cut

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Hypothesis used for Lower bound

Strong Exponential Time Hypothesis (SETH)

For all $\epsilon > 0$, there is some $k \geq 3$ such that k-SAT cannot be solved in time $O((2-\epsilon)^n)$.

q-coloring parameterised by width

q-COL: does a given graph admit a q-coloring?

Treewidth. Dynamic programming: $q^{\mathrm{tw}}\mathrm{poly}(n)$.

Also extends to $\#q$-LIST COL.

No $(q-\epsilon)^{\mathrm{tw}}\mathrm{poly}(n)$ algorithm under SETH.

(Lokshtanov, Marx and Saurabh, 2011)

Cutwidth. Randomised algorithm: $2^{\mathrm{ctw}}\mathrm{poly}(n)$.

Does not extend to $\#q$-LIST COL.

No $(2-\epsilon)^{\mathrm{ctw}}\mathrm{poly}(n)$ algorithm under SETH.

(Nederlof, Jansen, 2018)

Present paper builds on some insights from this paper

Our result

For n-vertex graphs of cutwidth ctw, there is an algorithm running in time

$$\begin{cases} q^{ctw}\mathrm{poly}(n) & \text{if } p \text{ does not divide } q-1, \\ (q-1)^{ctw}\mathrm{poly}(n) & \text{if } p \text{ divides } q-1. \end{cases}$$

Furthermore, under SETH there is no $(q-\epsilon)^{ctw}\mathrm{poly}(n)$ resp. $(q-1-\epsilon)^{ctw}\mathrm{poly}(n)$ algorithms in these cases.

Coloring compatibility matrix

$\mathrm{col}(X)$ = set of list q-colorings of $G[X]$.

Bipartite graph (X, Y, E), $x \in \mathrm{col}(X)$ and $y \in \mathrm{col}(Y)$.
Compatible: $x \sim y$ if $x(u) \neq y(v)$ for all $uv \in E$.

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$rk_p(M)$ depends on whether p divides $q - 1$.

Rank of coloring compatibility matrix

Bipartite graph (X, Y, E), for $x \in \mathrm{col}(X)$ and $y \in \mathrm{col}(Y)$ q-colorings

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$rk_p(M) \leq (q-1)^{|E|}$ if p divides $q-1$ and $\leq q^{|E|}$ otherwise.

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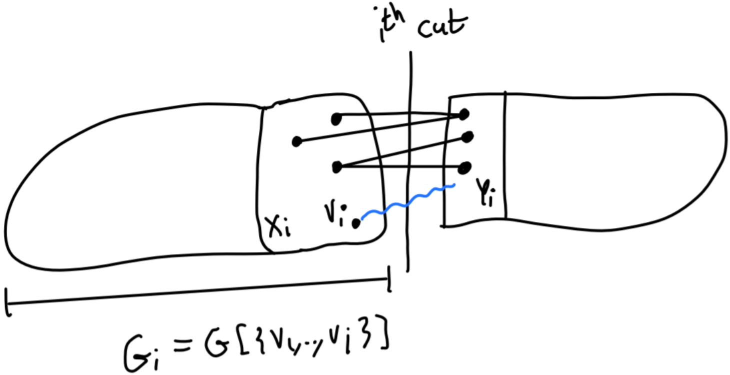
$$\begin{array}{c c} & \begin{array}{cccc} y(v)=1 & y(v)=2 & y(v)=3 & y(v)=4 \end{array} \\ \begin{array}{c} x(u)=1 \\ x(u)=2 \\ x(u)=3 \\ x(u)=4 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Cutwidth order $v_1, \ldots, v_n$.

$G_i = G[\{v_1, \ldots, v_i\}]$, the ith cut gives a bipartite graph (L_i, R_i).

$X_i = L_i \cup \{v_i\}$, $Y_i = N[X_i] = R_i$.



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For $x \in \mathrm{col}(X_i)$,

$$\begin{aligned}
T_i[x] &= \text{number of extensions of } x \text{ to } G_i \\
&= \text{number of } c \in \mathrm{col}(G_i - X_i) \text{ with } c \sim x \\
&= \sum_{c \in \mathrm{col}(G_i - X_i) : c \sim x} T_{i-1}[z_{|X_{i-1}}].
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Table size $|\mathrm{col}(X_i)| = q^{|X_i|} \leq q^{ctw+1}$.

Representative sets

Idea. *Compute smaller table 'representative' of bigger table.*

Coloring compatibility matrix M_i of bipartite graph on between X_i and Y_i.
For $y \in \mathrm{col}(Y_i)$,

$$\sum_{x \in \mathrm{col}(X_i)} T_i[x] M_i[x, y] = (T_i^t M_i)[y]$$

gives the number of $c \in \mathrm{col}(G_i)$ compatible with y.

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T_i' is M_i-**representative** for T_i if $T_i^t M_i \equiv_p (T_i')^t M_i$.

Reducing the table size

T_i' is M_i-representative for T_i if $T_i^t M_i \equiv_p (T_i')^t M_i$.

If we know $T_i'[x] = 0$, we do not need to compute it.

Linear algebra $\implies$ there exists such T_i' with $|\mathrm{supp}(T_i')| \leq \mathrm{rank}_p(M_i)$.

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	1	2	3
$1 : a$	0	1	1
$2 : b$	1	0	1
$3 : c$	1	1	0

	1	2	3
$1 : a + c$	0	1	1
$2 : b + c$	1	0	1
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- Can we efficiently compute T'_i?
- Can we exploit the zeros?

Maintain $R \subseteq X_i$ such that $T'_i[x] = 0$ if $x(v) = q$ for some $v \in R$.

T'_i *fully reduced* if $\{v \in X_i : \deg(v) = 1\} \subseteq R$.

Table size

Number of $x : X \to \{1, \ldots, q\}$ with $x(v) \neq q$ for all $v \in R$ is at most

$$(q-1)^{|R|} q^{|X|-|R|}.$$

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For each x we have a table entry $T_i[x]$.
Since $|X \setminus R| \leq \frac{1}{2}(ctw - |R|)$, this is at most $(q-1)^{ctw}$.

Upper bound sketch

Algorithm. Initialise for $i = 1$. For $i = 2, \dots, n$,

- Ensure T'_{i-1} fully reduced.
- Compute T'_i from T'_{i-1}.

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Lemma 1. If $v \in X_i$ degree 1, then can compute T' representative for T with set of reduced vertices $R \cup \{v\}$ in time $O((q-1)^{|R|} q^{|X_i|-|R|})$.

(Proof of Lemma 1 uses p divides $q-1$.)

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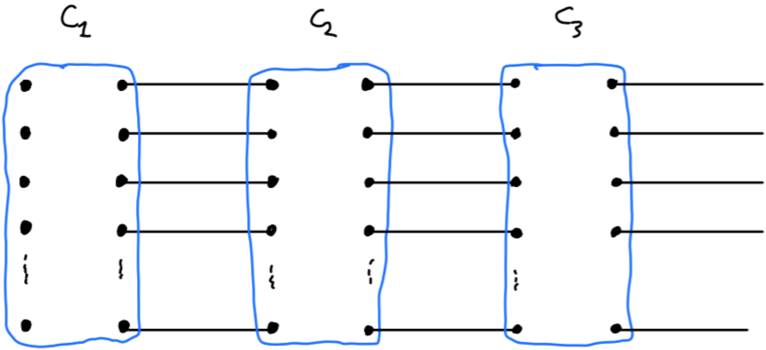
(Proof of Lemma 1 uses p divides $q-1$.)

Lemma 2. If T'_{i-1} representative for T_{i-1} and fully reduced, then can compute T'_i representative for T_i in time $O((q-1)^{ctw})$.

Lower bound

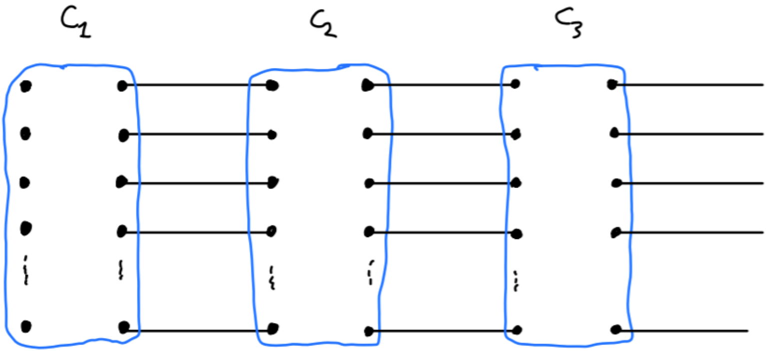
Under SETH, $\#\mathrm{CSP}(q, r)$ mod p cannot be solved in $(q-\epsilon)^n \mathrm{poly}(n, m)$ for some r [Lampis, '20 (and others?)]

n variables, m constraints
Constraints $\{1, \ldots, q\}^r \to \{0, 1\}$ depend on at most r variables.
Count number of satisfying assignments mod p.



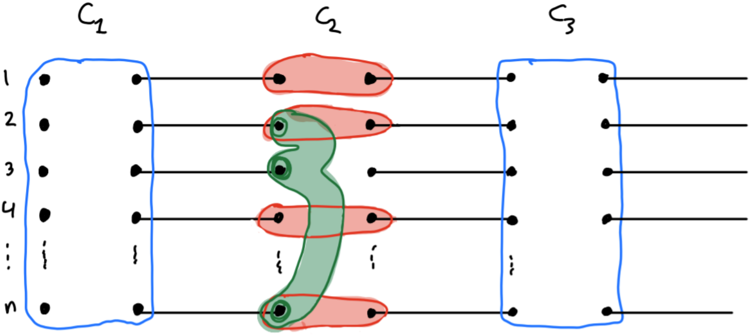
One column per constraint; one row per variable.

Cutwidth $= n + O(1)$, number of vertices $= \mathrm{poly}(n, m)$.

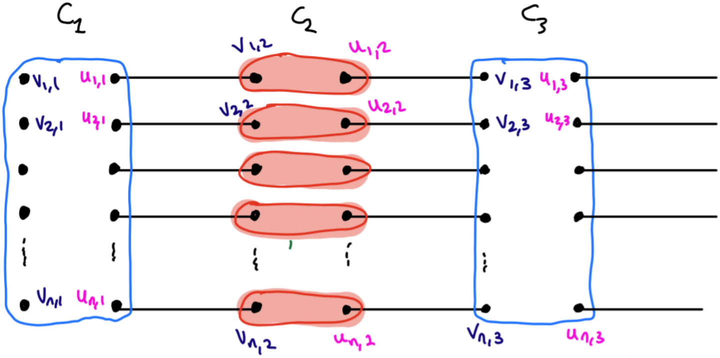


Want: number of q-colorings equals number of satisfying assignments.

"Identify a q-coloring with an assignment; check and copy."



Constraint 2 depends on variables 2, 3, n



Colour of $v_{i,2}$ = colour of $v_{i,3}$ for all i

Exploiting invertibility

Fix c_2, c_3 colorings of $v_{1,2}$ and $v_{1,3}$ respectively. The number of extensions of these colorings to the red graph equals

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$$\sum_{c_2'} f[c_2, c_2']M[c_2', c_3] = fM = M^{-1}M = I,$$

if we can set $f[c_2, c_2'] = M^{-1}[c_2', c_2]$.

Counting connected edge sets

Given $G = (V, E)$, how many $X \subseteq E(G)$ are there for which (V, X) is connected?

Tutte polynomial can link this to number of q-colorings.

$\implies$ under SETH there is no $(p - \epsilon)^{ctw}\text{poly}(n)$ algorithm.

'Correct' running times: $p^{ctw}\text{poly}(n)$, $p^{pw}\text{poly}(n)$, $p^{tw}\text{poly}(n)$.

Summary

Running time for counting modulo p the number of q-list colorings of n-vertex graphs of cutwidth ctw is

$$\begin{cases} q^{ctw}\text{poly}(n) & \text{if } p \text{ does not divide } q-1, \\ (q-1)^{ctw}\text{poly}(n) & \text{if } p \text{ divides } q-1. \end{cases}$$

The rank over $\mathbb{F}_p$ of the matrix $J_q - I_q$ (zeros on the diagonal, ones everywhere else) is

$$\begin{cases} q & \text{if } p \text{ does not divide } q-1, \\ q-1 & \text{if } p \text{ divides } q-1. \end{cases}$$