Recognizing (p,q)-cluster graphs

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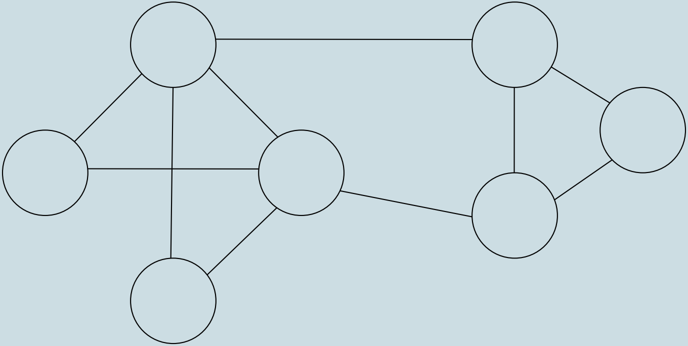
36th International Workshop on Graph Theoretic

Outline

Cluster Editing

- We are given a graph which should be a disjoint union of cliques, but it is not due to faulty data.



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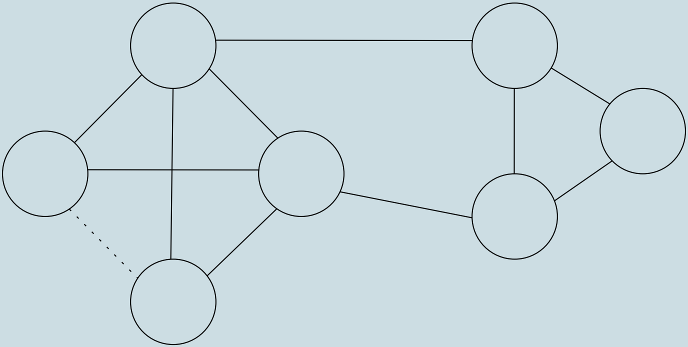

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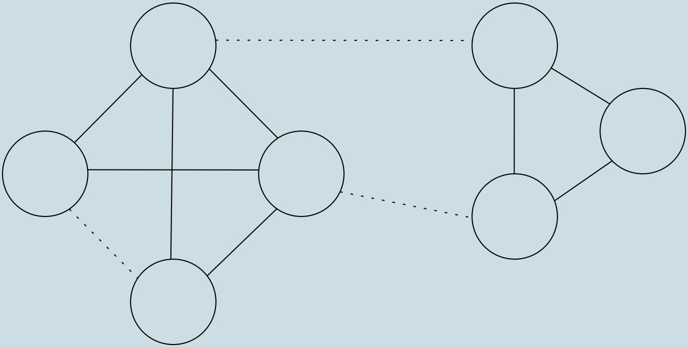

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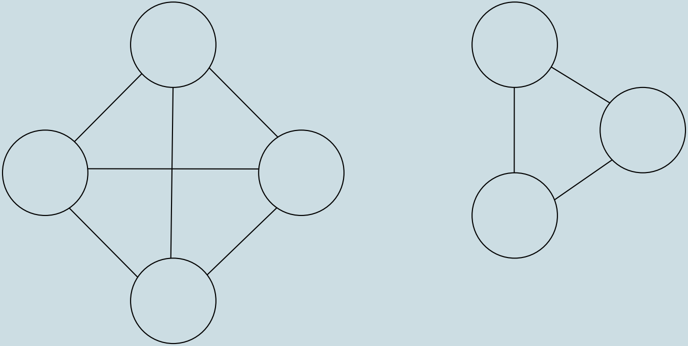

Cluster Editing

- The classical Cluster Editing problem is:
  - Given a graph G and integer k, is it possible to add and remove at most k edges to obtain a disjoint union of cliques?
- This problem, and many variants, are well studied and known to be NP-complete and FPT parameterized by k.

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  - Probably lots of editing has to be done → FPT algorithms are slow.
  - There might be classes with very few friends → Unbalanced.
- Today: Minimize the maximum of the number of friendships broken and the number of non-friends over all classes.

(p,q)-cluster graphs

Definition ((p,q)-clustering)

Given a graph $G = (V, E)$, integers p, q

Asked a partition $C_1, \ldots, C_l$ of V s.t. for every C_i:

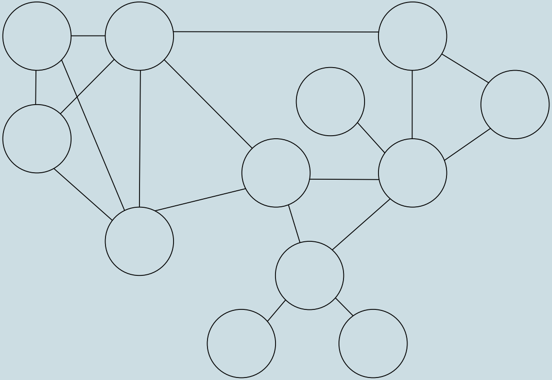
- the number of edges with exactly one endpoint in C_i is at most p, and
- the number of non-adjacent pairs $u, v \in C_i$ (with $u \neq v$) is at most q.

Definition ((p,q)-cluster graphs)

If (G, p, q) is a YES-instance, then G is called a (p,q)-cluster graph.

(p,q)-cluster graphs

Is this graph a (2,3)-cluster graph (2 missing, 3 redundant)?



(p,q)-cluster graphs

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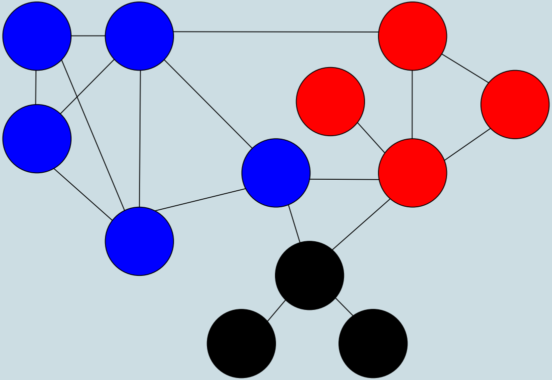

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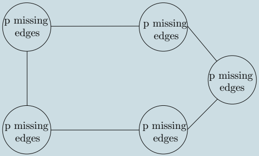
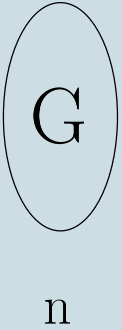


NP-completeness of (p,q)-clustering

Reduction from Clique: Let (G, k) be an instance of Clique. Create



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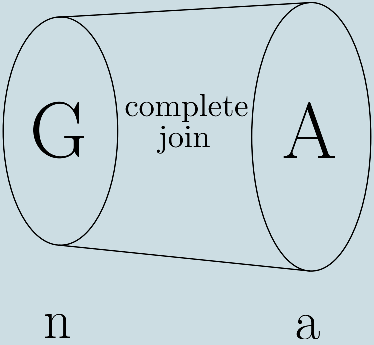

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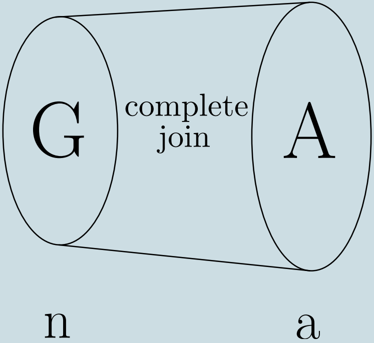

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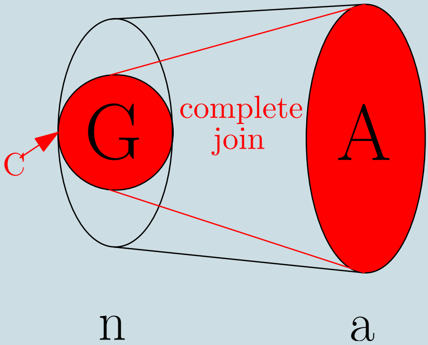

$a \geq (n-k)k+1 \qquad q = (n-k+1)a-1$

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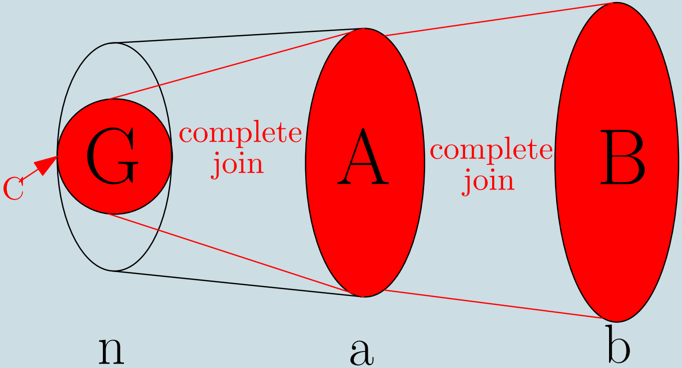

$a = (n-k)k+1$ $\quad q = (n-k+1)a-1$ $\quad b = q-a+2$ $\quad p = bk$

(0,q)-clustering

Can we partition the vertices of G into cliques such that each clique has at most q edges to other cliques?

- Solvable in polynomial time!

Definition

A high degree vertex is a vertex of degree at least $q+1$. A good clique is a clique having at most q edges to other cliques.

Lemma

G is a $(0,q)$-cluster graph iff all its high degree vertices are in good cliques.

Lemma

Every good clique containing a high degree vertex is the closed neighborhood of a vertex.

Proof.

- Consider a good clique C of size c with high degree vertex v

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- This doesn't happen since C is a good clique, so there is a vertex w without leaving edges.
- Then $N[w]=C$ since C is a clique.

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(0,q)-clustering

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These two facts together give us a polynomial time algorithm!

Further remarks

- We introduced (p,q)-clustering.
- NP-complete when p, q are part of the input.
- Special cases $(p, 0)$, $(p, 1)$, $(p, 2)$ and $(0, q)$ are polynomial time solvable.
- We also proved $(1, 1)$, $(1, 2)$ and $(1, 3)$ to be polynomial time solvable.
- Very recently, Lokshtanov and Marx found $f(p)n^c$ and $f(q)n^c$ algorithms.