Special points in Shimura varieties, an introduction.

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Introduction

In the Workshop the central theme is a conjecture, see (4.2). On this conjecture work has been done from various directions: mixed characteristic, G-functions, equidistribution, Galois-orbits. Special cases have been proved (André, Moonen, Edixhoven, Clozel, Ullmo, Zhang, Yafaev, and several others); the general case seems to be open.

The aim of the Workshop is twofold:

- to bring together specialists working in this area; we hope that interactions between these participants will create new insight; and
- in the first two days 15-16/XII an overview of this field and some applications will be given (as an "instructional conference"); we hope that non-specialists will get access in this way to the various methods involved, and we hope that this paves the way for interactions between specialists working in quite different areas.

In these informal notes I sketch a particular case of the conjecture: *the case of special points,* CM-*points, on the moduli space of polarized abelian varieties.*

Notions discussed in this talk will be in algebraic geometry over $\mathbb{C}$. Most of the talks in the Workshop will treat cases over fields of characteristic zero (usually a number field, or $\mathbb{C}$).

In these note you find:
a description of the moduli space $\mathcal{A}_g = \mathcal{A}_{g,1} \otimes \mathbb{C}$ of principally polarized abelian varieties over $\mathbb{C}$,
the definition of a special subvariety $S \subset \mathcal{A}_g$,
a formulation of the André-Oort conjecture, an analogy MM $\leftrightarrow$ AO, and
a discussion of a series of examples.
Some notions not defined in these notes and in my talk will find their natural place in the talks by Ben Moonen on Shimura varieties. I hope that understanding the setting in the particular case $\mathcal{A}_g$ will contribute in appreciating the general setting of Shimura varieties. The more general notions: Shimura variety, Mumford-Tate group, Hecke correspondence, and special subvarieties in Shimura varieties will be discussed later in the talks by Ben Moonen.

1 Moduli of abelian varieties

We write k for an *algebraically closed field.*

(1.1) As you know, "elliptic curves over k are classified by their j-invariant". One can make geometry out of this by constructing a scheme $\mathcal{A}_1 \to \mathrm{Spec}(\mathbb{Z})$, which has certain universal properties, which in particular imply that there is a map

$$j(k) : \{\text{ell.curves over } \ k\} \xrightarrow{\sim} \mathcal{A}_1(k).$$

In fact $\mathcal{A}_1 \cong \mathbb{A}^1_{\mathbb{Z}}$, and $\mathcal{A}_1(k) = k$.

Moreover, over $\mathbb{C}$ this can be understood via complex theory. Let $\mathfrak{h}_1$ denote the upper half plane $\mathfrak{h}_1 = \{z \in \mathbb{C} \mid \Im(z) > 0\}$. The group $\Gamma = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ acts on $\mathfrak{h}_1$ (in a well-known way), and this action is properly discontinuous; we have

$$\mathcal{A}_1(\mathbb{C}) \quad \cong \quad \Gamma\backslash\mathfrak{h}_1,$$

where $j(E) \in \mathbb{C} = \mathcal{A}_1(\mathbb{C})$ corresponds with

$$j(E) \quad \leftrightarrow \quad (z \bmod \Gamma) \qquad \text{iff } E(\mathbb{C}) \cong \mathbb{C}/(\mathbb{Z}{\cdot}1 + \mathbb{Z}{\cdot}z).$$

All this can be generalized to abelian varieties of higher dimension. However for $g \geq 2$ not every complex torus of that dimension is algebraizable, and the action of $\mathrm{SL}_{2g}(\mathbb{Z})$ is not properly discontinuous on the set of all possible periods. This is repaired by the notion of a polarization.

(1.2) Remark. For an open subscheme $S \subset \mathbb{A}^1$, a family of elliptic curves $\mathcal{E} \to S$ is called a *tautological family* if for every $s \in S(k) \subset k$ we have $j(\mathcal{E}_s) = s$.
Exercise. *Over a dense-open subset $S \subset \mathbb{A}^1$ which contains 0 or 1728 in $\mathbb{A}^1(K)$ for some field K there is no tautological family possible.*
Exercise. *Over $S := \mathbb{A}^1_{\mathbb{C}} - \{0, 1728\}$ there are exactly 4 tautological families up to isomorphism* (hence, such a family is not "universal").
For higher g analogous phenomena occur. All these difficulties disappear if one considers level structures.

(1.3) A divisor D on an abelian variety A gives a map $A(K) \to \mathrm{Pic}(A)$ by $\varphi_D(x) = [D_x - D] \in \mathrm{Pic}(A)$; here $x \in A(K)$, and D_x is the translate of D by x, and $[\ ]$ means the linear equivalence class. By the theory of abelian varieties and their duals we obtain a homomorphism $\varphi_D : A \to A^t$. If D is effective, the morphism φ_D is an isogeny iff D is ample. We say that $\lambda : A \to A^t$ is a *polarization* if over some field extension it can be given by an ample divisor. A polarization $\lambda : A \to A^t$ is called a *principal polarization* if it is moreover an isomorphism.

An important construction, classical in some cases, but in full generality due to Mumford proves the existence of a scheme $\mathcal{A}_{g,d} \to \mathrm{Spec}(\mathbb{Z})$ which "classifies" abelian varieties with a polarization of degree d^2 (here we omit a lot of important theory, and we ignore difficulties/subtleties about "universal families" the difference between coarse and fine moduli spaces, and so on ...). As a result we obtain:

$$j(k) : \{(A, \lambda) \ \text{ over } \ k \mid \dim(A) = g, \deg(\lambda) = d^2\}/\cong_k \quad \xrightarrow{\sim} \quad \mathcal{A}_{g,d}(k).$$

Let us denote $\mathcal{A}_{g,1} \otimes \mathbb{C}$ from now on by $\mathcal{A}_g$, the moduli space of principally polarized abelian varieties of dimension g over $\mathbb{C}$. It is a normal variety of dimension $g(g+1)/2$ over $\mathbb{C}$.

The complex space $\mathcal{A}_g(\mathbb{C})$, or the algebraic variety $\mathcal{A}_g$ associated with this is *a special case of the notion of a Shimura variety.* We will not define this notion. In the talks by Ben Moonen you will have the definition of:

Shimura variety

(1.4) We explain what the set $\mathcal{A}_g(\mathbb{C})$ is, and in which way this is an analytic space. An abelian variety A over $\mathbb{C}$ gives a complex torus $A(\mathbb{C}) = \mathbb{C}^g/\Lambda$. We can choose coordinates in $\mathbb{C}^g$ and $\tau \in (\mathbb{C}^g)^g$ such that

$$A(\mathbb{C}) = \mathbb{C}^g/\Lambda_\tau = T_\tau, \quad \Lambda_\tau = \mathbb{Z}{\cdot}e_1 \oplus \cdots \oplus \mathbb{Z}{\cdot}e_g \oplus \mathbb{Z}{\cdot}\tau_1 \oplus \cdots \oplus \mathbb{Z}{\cdot}\tau_g.$$

(1.5) Remark/Exercise. Let $g \geq 2$. Let $P \subset (\mathbb{C}^g)^g$ be the set of $\tau \in \mathbb{C}^g)^g$ such that $\{e_1, \cdots, e_g, \tau_1, \cdots, \tau_g\}$ is linearly independent over $\mathbb{R}$. Introduce an equivalence relation on P by

$$\tau \sim \sigma \quad \Leftrightarrow \quad T_\tau \cong T_\sigma.$$

The quotient space $P/\sim$ with the quotient topology is *not a Hausdorff space* (i.e. there is no good moduli space for Complex tori). See [4], Cor. 7.3.3.

(1.6) Define:

$$\mathfrak{h}_g := \{\tau \in (\mathbb{C}^g)^g = \mathrm{Mat}(g \times g, \mathbb{C}) \mid \tau = {}^t\tau, \ \ \Im(\tau) > 0\};$$

this is called the "Siegel upper half space"; if (A, λ) is a principally polarized abelian variety then there exist $\tau \in \mathbb{C}^g$, and isomorphism $A(\mathbb{C}) \cong T_\tau$ such that:

the matrix τ is symmetric, i.e. $\tau = {}^t\tau$;
the matrix $\Im(\tau)$ is positive definite.

Conversely, if $\tau \in \mathfrak{h}_g$, the complex torus T_τ is algebraizable, and the standard hermitian form gives a principal polarization, see [8], page 214. Moreover, $T_\tau \cong T_\sigma$ iff there exists

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z}) : \quad \sigma = (A\tau + B)(C\tau + D)^{-1}.$$

One shows that the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ operating in this way on $\mathfrak{h}_g$ is properly discontinuous, and that there is an isomorphism of complex spaces:

$$\mathcal{A}_g(\mathbb{C}) \quad \cong \quad \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathfrak{h}_g$$

where $[(A, \lambda)] = x \in \mathcal{A}_g(\mathbb{C})$ corresponds with

$$x \quad \leftrightarrow \quad (\tau \bmod \mathrm{Sp}_{2g}(\mathbb{Z})) \quad \text{iff} \quad A(\mathbb{C}) = \mathbb{C}^g/\Lambda_\tau.$$

(1.7) In the talks by Ben Moonen you will see a notation introduced $\mathbb{H}_g$; then, one can give an isomorphism $\mathfrak{h}_g \cong \mathbb{H}_g$, and we see how the notions above coincide in this way with later descriptions.

2 Complex multiplication

(2.1) Note that for an abelian variety A over a field K the endomorphism ring $\mathrm{End}(A)$ is torsion-free of finite rank as $\mathbb{Z}$-module. We write $\mathrm{End}^0(A) = \mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q}$, the endomorphism algebra of A. If $K \subset L$ is an extension of fields, $\mathrm{End}(A\otimes_K L)/\mathrm{End}(A)$ is torsion-free. For A over a field K there exists a finite extension $K \subset L$ such that for every $L \subset L'$ we have $\mathrm{End}(A\otimes_K L) = \mathrm{End}(A\otimes_K L')$.

Sometimes $\mathrm{End}(A\otimes_K L)$ is denoted by $\mathrm{End}_L(A)$, but certainly I will not use that notation.

If A is an abelian variety of dimension g over a field K, $\mathrm{rk}_{\mathbb{Z}}(\mathrm{End}(A)) \leq (2g)^2$. If the characteristic of K equals zero we have $\mathrm{rk}_{\mathbb{Z}}(\mathrm{End}(A)) \leq 2g^2$.

The endomorphism algebra $\mathrm{End}^0(A) = \mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q}$ is semi-simple and $\mathrm{End}(A)$ is an order in $\mathrm{End}^0(A)$.

(2.2) Albert algebras. Let (A,λ) be a polarized abelian variety over a field K. The endomorphism algebra $E := \mathrm{End}^0(A)$, is of finite $\mathbb{Q}$-dimension. The polarization equips the endomorphism algebra E with an anti-involution, $^* : E \to E$, called the Rosati involution: for $a \in E$ we have $a^* \in E$ uniquely given by $\lambda{\cdot}a^* = a^t{\cdot}\lambda$. It has the properties:

$a^{**} = a$ and $(a{\cdot}b)^* = b^*{\cdot}a^*$, an *anti-involution*, and

$a \mapsto \mathrm{Tr}(a{\cdot}a^*)$ is a *positive definite quadratic form* on $E\otimes_{\mathbb{Q}}\mathbb{R}$.

Such an algebra, of finite dimension over $\mathbb{Q}$, with anti-involution satisfying these properties is called an Albert algebra. These were classified by A. Albert. For references and for this classification, see [13], [8], [16].

(2.3) Simple abelian varieties. An abelian variety B over a field K is said to be *simple*, or, if confusion could occur, K-simple, if there is no abelian subvariety $C \subset B$ such that $0 \neq C \neq B$.

If B is a simple abelian variety over a field K, its endomorphism algebra $\mathrm{End}^0(A)$ is a division algebra; its center either is a totally real field, or it is a CM-field.

For an abelian variety A over a field K there exist simple abelian varieties $B_1, \cdots, B_t$ and an isogeny $A \sim B_1 \times \cdots \times B_t$ (Poincaré-Weil complete reducibility).

(2.4) Definition.

- *A simple abelian variety B over a field of dimension g is said to admit* sufficiently many complex multiplications *if* $\mathrm{End}^0(B)$ *contains a subfield of degree* $2g$ *over* $\mathbb{Q}$.
- [We write smCM as abbreviation for this.] [Other terminology: CM-abelian variety.]
- *An abelian variety A over a field is said to admit* sufficiently many complex multiplications *if there is an isogeny $A \sim \sum B_j$, where every B_j is simple and admits* smCM.
- [We will say "A admits smCM over K" if confusion is possible.]
- *Let (A,λ) be a polarized abelian variety. Suppose that A admits* smCM. *We say that the moduli point $[(A,\lambda)] = x \in \mathcal{A}_g$* is a CM-point; *instead of this terminology we will, equivalently, say that $x \in \mathcal{A}_g$* is a special point:

$$\boxed{\text{CM-point in } \mathcal{A}_g \quad = \quad \text{special point in } \mathcal{A}_g.}$$

(2.5) Remarks. 1) If B is simple over a field of characteristic zero, which admits smCM then $E = \mathrm{End}^0(B)$ is a field, $[E : \mathbb{Q}] = 2g$, which is a CM-field (i.e. a quadratic totally complex extension of a totally real subfield).
2) An abelian variety which admits smCM will also be called a CM-abelian variety. An abelian variety in characteristic zero is called of CM-type if it admits smCM, and if moreover the action of $\mathrm{End}^0(A)$ on the tangent space of A is specified.
3) It might happen that A admits smCM over K, and that $A \otimes_K L$ admits smCM over L, such that $\mathrm{End}(A) \neq \mathrm{End}(A \otimes_K L)$. However if the characteristic of K is zero, and A admits smCM over K, and $A \otimes_K L$ is simple for every $K \subset L$, then $\mathrm{End}(A) = \mathrm{End}(A \otimes_K L')$ for every $K \subset L'$.
4) Sometimes we see the phrase: "A is defined over K and is CM", by which the author means that A is defined over K, and that $A \otimes_K L$ admits smCM for some extension $K \subset L$; be careful. For example you will see the phrase: "E is an elliptic curve over $\mathbb{Q}$ with CM"; however no elliptic curve defined over $\mathbb{Q}$ admits smCM over $\mathbb{Q}$.
5) For an elliptic curve E over a field K the property $\mathbb{Z} \neq \mathrm{End}(E)$ is equivalent with "E admits smCM". However for abelian varieties of dimension at least two the analogous statement is not true. Sometimes we find "A as CM" by which the author means $\mathbb{Z} \neq \mathrm{End}(A)$. I think this is misleading; note that "A as CM" in this sense does not imply that A admits smCM.
6) If A is an abelian variety over a finite field, then it admits smCM, as Tate showed. If A is an abelian variety over a field of characteristic zero which admits smCM, then it can be defined over a number field, a finite extension of $\mathbb{Q}$. If A is an abelian variety which admits smCM, then it is isogenous with an abelian variety defined over a finite extension of the prime field, as Grothendieck proved.
7) It might happen that A is defined over K, that A is K-simple and that $A \otimes_K L$ is not L-simple for some extension.

In the talks by Ben Moonen you will have the definition of:

Mumford-Tate group

This notion and the property below will get their natural place once we use Hodge theory.

(2.6) Property. *Let A be an abelian variety over $\mathbb{C}$ of dimension g; the following properties are equivalent:*

(1) *A admits* smCM;
(2) *there is a commutative semi-simple subalgebra $E \subset \mathrm{End}^0(A)$ with $\dim_{\mathbb{Q}} E = 2g$;*
(3) *the Mumford-Tate group of A is a torus.*

(2.7) Remark. A moduli point $x \in \mathcal{A}_g$ belonging to a CM-abelian variety is called a CM-point. For an arbitrary Shimura variety there is no notion of "smCM", but there is a notion of a Mumford-Tate group. Therefore, in order to be able to generalize the notion of CM-point to arbitrary Shimura varieties we now speak of "special points" instead.

(2.8) A remark on terminology. Whenever working over a field, a *variety* V defined over K will be an algebraic set (affine, quasi-projective, projective, abstract), which is geometrically *irreducible* and *reduced.* However, the notion of "Shimura variety" usually does not follow this convention.

3 Special points, special subvarieties

(3.1) Let H be an algebraic group defined over a field contained in $\mathbb{R}$. We consider the real Lie group $H(\mathbb{R})$, and we write $H(\mathbb{R})^+$ for the connected component (in the "real topology") containing the identity point.

Definition. A *special subvariety*, or a special subset, S of $\mathcal{A}_g = \mathcal{A}_{g,1} \otimes \mathbb{C}$ is:

- a (Zariski-) closed (algebraic) subvariety $S \subset \mathcal{A}_g$, such that
- there exist an algebraic subgroup $H \subset \mathrm{Sp}(2g)$ defined over $\mathbb{Q}$, and
- a CM-point $x = (\tau \bmod \mathrm{Sp}_{2g}(\mathbb{Z})) \in A(\mathbb{C}) = \mathbb{C}^g/\Lambda_\tau$ with
- $$\Pi(H(\mathbb{R})^+ \cdot \tau) = S \subset \mathcal{A}_g(\mathbb{C});$$
  here $\Pi : \mathfrak{h}_g \to \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathfrak{h}_g = \mathcal{A}_g(\mathbb{C})$ is the natural projection.

Remark / warning. In general for a subgroup $H \subset \mathrm{Sp}(2g)$ and a CM-point $x \in A(\mathbb{C})$ the set $\Pi(H(\mathbb{R})^+ \cdot \tau)$ is not Zariski closed. Here is an easy example; consider the subgroup $H \subset \mathrm{SL}_2$ consisting of matrices $\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}$. Clearly this is an algebraic subgroup defined over $\mathbb{Q}$. However, for any $x \in A(\mathbb{C})$ the set $\Pi(H(\mathbb{R})^+ \cdot \tau)$ is not Zariski closed.

Remark We did not give the usual definition of a special subvariety. However, once we have the theory of Shimura varieties at our disposal, using [9], Th. II.3.1, or [10], 4.3, we see that the definition above coincides with the usual definition of a special subvariety applied to $\mathcal{A}_g$.

(3.2) Any zero-dimensional special subset of $\mathcal{A}_g$ consists of a CM-point, and conversely for any CM-point x, the set $S = \{x\}$ is a special subset.

(3.3) PEL-type. Let (A, λ) be a principally polarized abelian variety, and let $R \subset \mathrm{End}(A)$ be a subalgebra such that for any $f \in R$ we have $f^* \in R$. Consider all $[(B, \mu)] = y \in \mathcal{A}$ such that there exists an embedding $R \hookrightarrow \mathrm{End}(B)$ where the image is invariant under the Rosati involution on $\mathrm{End}(B)$ given by μ; see 7.3 for a more precise explanation. An irreducible component of the set of such point is a special subset; it is called a PEL-*type Shimura subvariety* (P = polarization, E = endomorphisms, L = linear). We will see examples of such special subsets.

For $g < 4$ any special subset is not a PEL-type Shimura subvariety. However, as Mumford showed, for $g = 4$ there do exist special subsets not given in this way. Note that "in general" a special subset is not a PEL-type Shimura subvariety.

A description of Shimura subvarieties of PEL-type is a matter of (very clever) linear algebra: describe which lattices allow a form which give a polarization, and which allow a faithful action by R in the desired way. This is completely carried out in [21].

In the talks by Ben Moonen you will have the definition of:

Shimura subvariety, Hecke correspondences

Using that definition, and the notion of a Hecke correspondence one can define the notion of a special subvariety of a Shimura variety. That notion generalizes the notion of special subset of $\mathcal{A}_g$ given above to the case of general Shimura varieties.

(3.4) Terminology / Equivalent:

- A subvariety of Hodge type (terminology used in $\mathcal{A}_g$).
- A special subvariety.
- An irreducible component of a translate by a Hecke correspondence of a sub-Shimura variety.
- In some cases people use the term "modular subvariety".

4 The André-Oort conjecture

(4.1) Property. *Let V be a Shimura variety, and let $S \subset V$ be a special subvariety. The set of special points in S is dense in S.*
By "dense" we mean Zariski dense in S, but the set of special points is also dense in $S(\mathbb{C})$ in the classical topology.

One can wonder whether the "converse" of this is true:

(4.2) Conjecture (Yves André and Frans Oort). *Let V be a Shimura variety; let $\Lambda \subset V(\mathbb{C})$ be a set of special points. Then* (?) *the Zariski closure of Λ inside V is a finite union of special subvarieties.* Or: *If a subvariety $S \subset V$ contains a Zariski-dense set of special points, then* (?) *S is a special subvariety.*

This was given in [1], pp. 214-216, where *curves* in $\mathcal{A}_g$ having a dense subset of CM-points were expected to be modular curves. Independently I started thinking of these things around 1990 in connection with the Coleman conjecture, and I formulated (between 1990 and 1993) a conjecture on arbitrary subvarieties of $\mathcal{A}_g$ see [17], 2. Curiously, Yves André and I independently were motivated by the Manin-Mumford conjecture, but for quite different reasons. Also see [9], IV.1.2 and [12], 6.4.

5 The Manin-Mumford conjecture, an analogy

Torsion points are dense in a complex torus, are dense in an abelian variety. In a translate over a torsion point of an abelian subvariety the torsion points are dense. The Manin-Mumford conjecture is the converse of this. The original form:

(5.1) The Manin-Mumford conjecture, now a theorem, proved by M. Raynaud. *Let T be a complex torus, i.e. $T = \mathbb{C}^g/\Lambda$, where $\mathbb{Z}^{2g} \cong \Lambda \subset \mathbb{C}^g$ is a lattice. Let T_{tors} be the set of torsion points* points of finite order, i.e. T_{tors} is the union of all images $\frac{1}{n}\Lambda \to \mathbb{C}^g/\Lambda$, union over all $n \in \mathbb{Z}_{>0}$, i.e. $T_{\mathrm{tors}} = \mathbb{Q}{\cdot}\Lambda/\Lambda \subset \mathbb{C}^g/\Lambda$. *Let $S \twoheadrightarrow S' \subset T$ be the image of a non-constant morphism of a Riemann surface into T. Suppose that $S' \cap T_{\mathrm{tors}}$ is* non finite. *Then S' is a*

Riemann surface, its genus is one, it is the translate by a torsion point t of a one-dimensional subtorus $T' \subset T$:

$$S' \quad = \quad t + T'.$$

This conjecture has been proved by Raynaud in 1983, see [19].

In the conjecture one can as well consider an algebraic curve C mapping to an abelian variety A over $\mathbb{C}$, with image C', requesting that $C'(\mathbb{C}) \cap A_{\mathrm{tors}}(\mathbb{C})$ is infinite and concluding that C' is nonsingular and of genus one. In this form the conjecture can be generalized:

(5.2) Theorem (Lang's conjecture). *Let $V \subset A$ be a subvariety of an abelian variety over an algebraically closed field k of characteristic zero. Suppose that $V(k) \cap A_{\mathrm{tors}}(k)$ is Zariski-dense in V. Then V is the translate of an abelian subvariety $B \subset A$ over a torsion point.*
This has been proved by Raynaud, see [20]. For references, and for a fourth proof of this conjecture, see [18].

Note that there are several different versions of Lang's conjecture.

(5.3) An analogy. The moduli space $\mathcal{A}_g$ "locally feels like a group". Moreover this can be made precise in several cases. For example the moduli space $\mathcal{A}_g \to \mathrm{Spec}(\mathbb{Z})$ completed at an ordinary point in characteristic p, by the theory of Serre-Tate parameters, is formal torus.

Under such an analogy one can give the CM-points an interpretation as torsion points in that group structure. In fact, the torsion points in the Serre-Tate parameters exactly are the CM-points.

Therefore, a variety containing a dense set of CM-points "locally feels like a subgroup" or "feels like a homogeneous space under a group"; hence we see the analogy between a special subvariety and a translate over a torsion point of an abelian subvariety:

Shimura variety	abelian variety
special point	torsion point
special subvariety	abelian subvariety translated over a torsion point
the conjecture by André	MM: the Manin-Mumford conjecture
the AO conjecture, see (4.2)	Lang's generalization of MM

The fact that the Manin-Mumford conjecture is true, as well as certain generalizations, plus this analogy MM $\leftrightarrow$ AO, is encouraging for expecting AO to be true. However, up to now, we have not been able to "transplant" directly the methods which prove MM to the situation of AO; the analogy mentioned does not hint at a mathematical implication.

(5.4) The Manin-Mumford conjecture, the André-Oort conjecture, and also the "Hecke orbit conjecture" (not discussed here) all fall into the following pattern:

suppose given a variety V (geometric properties), and
a subset $\Gamma \subset V$ (arithmetic properties);
what can be said about (the geometry of) the closure of Γ *in* V*?*

In MM, here Γ is a set of torsion points: it is a finite union of translates of abelian subvarieties;
in AO, here Γ is a set of special points: it should be a finite union of special subvarieties;
in HO, here Γ is the Hecke orbit of a point: it is the locus defined by a Newton polygon.

6 The Coleman conjecture

(6.1) In [5], Conjecture 6, we find:
Conjecture. Define

$$\Delta_g := \{C \mid \text{genus}(C) = g, \quad \text{Jac}(C) \quad \text{admits smCM}\}/\cong_{\mathbb{C}};$$

here C stand for a complete, irreducible algebraic curve over $\mathbb{C}$. Then:

$$g \geq 4 \quad \stackrel{?}{\Longrightarrow} \quad \#(\Delta_g) < \infty.$$

(6.2) Explanation. Consider $\mathcal{T}_g^0 \subset \mathcal{A}_g = \mathcal{A}_{g,1} \otimes \mathbb{C}$, the open Torelli locus (the set of Jacobians of algebraic curves). Write $\mathcal{T}_g$ for the Zariski closure of $\mathcal{T}_g^0$. For $g \leq 3$ we have $\mathcal{T}_g = \mathcal{A}_g$; for $0 < g \leq 3$ there are infinitely many Jacobians which admit smCM. For $g \geq 4$ we have $\mathcal{T}_g \subsetneqq \mathcal{A}_g$, and one could wonder whether this proper locally closed subset $\mathcal{T}_g^0$ intersects the (dense) set of CM-points in only finitely many ways.

(6.3) Status. As we will see: Δ_4, Δ_6 and Δ_7 are infinite. Whether this conjecture by Coleman is true for $g = 5$ and / or for $g \geq 8$ seems to be unknown.

One can try to approach the Coleman conjecture by "splitting it up into two parts": the AO-conjecture, and a question about existence of special subvarieties contained in the Torelli locus.

(6.4) *Suppose* (4.2) *holds for $\mathcal{A}_g$; then the following are equivalent:*

(1) *There exists no special subvariety $S \subset \mathcal{A}_g$, of positive dimension, such that $S \subset \mathcal{T}_g$ and $S \cap \mathcal{T}_g^0 \neq \emptyset$.*

(2) *For this value of g the conjecture of Coleman is true: $\#(\Delta_g) < \infty$.*

Indeed, assume (1) is not true for a specific value of g. As special points are dense in a special subset, and $\mathcal{T}_g^0$ is dense-open in $\mathcal{T}_g$, it follows that in this case we have $\#(\Delta_g) = \infty$.

The Zariski closure of Δ_g by (4.2) is a finite union of special subvarieties. If $\#(\Delta_g) = \infty$, at least one of these components is positive dimensional, and we have contradicted (1) for this g.

(6.5) On might expect that for $g \gg 0$ statement (1) is correct; assuming the AO-conjecture then the Coleman conjecture would follow for that value of g. However we do not see much evidence for (1) to be true; it seems a difficult question.

7 An example

In this section we describe some examples of subsets of some $\mathcal{A}_g$ defined by families of curves which admit some complex multiplication (but not all fibers admit smCM).

(7.1) For $N \in \mathbb{Z}_{>1}$ and for $\lambda \in \mathbb{C} - \{0,1\} = \mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}$ we define $C_{N,\lambda}$ for the complete non-singular curve defined as the normalization of the completion of the affine curve defined by:

$$Y^N = X(X-1)(X-\lambda).$$

These curves were studied by Shimura, in [7] and in [3].

We obtain a family $\mathcal{C}_N \to \mathbb{P}^1 - \{0,1,\infty\}$. We write

$$S_N \subset \mathcal{A}_g$$

for the Zariski closure of the image of the moduli map

$$\mathbb{P}^1 - \{0,1,\infty\} \to \mathcal{A}_g, \quad \lambda \mapsto \mathrm{Jac}(C_{N,\lambda})$$

given by this family (we will determine g in terms of N).

(7.2) We write $\lambda \in \mathbb{C} - \{0,1\}$.

(1) For $3 \nmid N$ we have $\mathrm{genus}(C_{N,\lambda}) = N - 1$.

(2) For $N = 3M$ we have $\mathrm{genus}(C_{N,\lambda}) = N - 2$.

(3) For $N = 2$ we have $C_2 = \mathcal{A}_1$.

(4) **Theorem** (De Jong and Noot). *For $N = 5$, respectively $N = 7$, the set $S_N \subset \mathcal{A}_{N-1}$ is a* special subvariety.

(5) **Corollary.** $\Delta_4 = \infty$ and $\Delta_6 = \infty$.

(6) **Proposition** (Belhaj Dahmane). *For $N \geq 8$ not divisible by 3, we have that $S_N \subset \mathcal{A}_{N-1}$ is not a special subvariety.* See [3], 2.8.3.

(7) **Conjecture.** *For $3 \nmid N$, and $N \geq 8$, we have $\#(S_N \cap \Delta_{N-1}) < \infty$.*

(8) **Proposition.** *For $N = 9$ the set $S_9 \subset \mathcal{A}_7$ is a* special subvariety.

(9) **Corollary.** $\Delta_7 = \infty$

(10) **Conjecture.** We expect that S_{3M} is not special for any $3M \geq 12$.

(11) **Conjecture.** We expect that $\#(S_N \cap \Delta_{N-2}) < \infty$ for any $N = 3M \geq 12$.

(7.3) Multiplication by a CM-field. We describe one special case of a PEL-type Shimura variety, according to [21]. Notation as e.g. used in [13] will be followed, also see [16], which however is slightly differs from the notation used in [21]. We fix (e.g. see [21], page 156 and page 175, notation adapted):

$g \in \mathbb{Z}_{>0}$; this will be the dimension of the abelian varieties in consideration;
a CM-field L; we write $[L : \mathbb{Q}] = e$;
we choose a numbering of the embeddings

$$\mathrm{Hom}(L, \mathbb{C}) \quad = \quad \{\chi_1, \cdots, \chi_e, \overline{\chi_1}, \cdots, \overline{\chi_e}\}, \quad \Phi = \{\chi_1, \cdots, \chi_e\}, \quad \overline{\Phi} = \{\overline{\chi_1}, \cdots, \overline{\chi_e}\},$$

we fix a representation $\iota : L \to \mathrm{End}(\mathbb{C}^g)$, and
we write r_j for the multiplicity of χ_j in ι,
respectively s_j for the multiplicity of $\overline{\chi_j}$ in ι.

After fixing g, L, ι we write $S^{(\iota)} \subset \mathcal{A}_g$ for the PEL-type Shimura variety classifying all triples (A, λ, i), where (A, λ) is a principally polarized abelian varieties of dimension g, and where $i : L \hookrightarrow \mathrm{End}(A)$ is λ-linear, i.e. the Rosati involution defined by λ is complex conjugation on $L \cong i(L)$, and i gives the representation ι on the tangent space of A. We suppose that $S^{(\iota)}$ is non-empty.

(7.4) Theorem, [21], page 175. *With notation fixed above,*

$$\dim(S^{(\iota)}) \quad = \quad \sum_{1 \le j \le e} r_j s_j.$$

(7.5) Exercise. Prove this formula, either using "linear algebra", or supposing that we are in mixed characteristic, that we have a prime of good, ordinary reduction for a point of $S^{(\iota)}$, and using Serre-Tate theory.

(7.6) Lemma. *We fix $N \in \mathbb{Z}_{\ge 4}$ with $3 \nmid N$. We fix $\lambda \in \mathbb{C} - \{0, 1\}$. We write $C = C_{N,\lambda}$. Then: genus$(C) = N - 1$; a basis for $\Gamma(C, \Omega^1)$ is:*

$$\{\frac{XdX}{Y^{N-1}}, \cdots, \frac{XdX}{Y^{\lceil \frac{2N+1}{3} \rceil}}, \frac{dX}{Y^{N-1}}, \cdots, \frac{dX}{Y^{\lceil \frac{N+1}{3} \rceil}}\}.$$

Here $\lceil c \rceil$ is the smallest integer $\ge c \in \mathbb{R}$.

Examples:

N=2, $\quad \{dX/Y\}$

N=5

$$\{\frac{XdX}{Y^4}, \frac{dX}{Y^4}, \frac{dX}{Y^3}, \frac{dX}{Y^2}\},$$

N=7

$$\{\frac{XdX}{Y^6}, \frac{XdX}{Y^5}, \frac{dX}{Y^6}, \frac{dX}{Y^5}, \frac{dX}{Y^4}, \frac{dX}{Y^3}\},$$

N=8

$$\{\frac{XdX}{Y^7}, \frac{XdX}{Y^6}, \frac{dX}{Y^7}, \cdots, \frac{dX}{Y^3}\},$$

N=11

$$\{\frac{XdX}{Y^{10}}, \frac{XdX}{Y^9}, \frac{XdX}{Y^8}, \frac{dX}{Y^{10}}, \cdots, \frac{dX}{Y^4}\}.$$

Proof. In the equation

$$Y^N = X(X-1)(X-\lambda) \quad \text{write} \quad X = \frac{1}{Z},\ ZY = \eta$$

and obtain

$$\eta^N = Z^{N-3}(1-Z)(1-\lambda Z).$$

As N and $N-3$ are coprime, normalization gives exactly one point $P \in C$ above $Q = (\eta = 0, z = 0)$; hence (we work in characteristic zero) by the Hurwitz-Zeuthen formula:

$$2g-2 \quad = \quad N\cdot(-2) + 4\cdot(N-1); \quad \text{this gives} \quad g = N-1.$$

A local parameter T at $P \in C$ satisfies $Z \sim T^N$, $\eta \sim T^{N-3}$, hence $Y \sim T^{-3}$; here we write $\sim$ for formal power series equality up to a unit, hence up to a non-zero multiplicative constant and up to higher order terms. As the function field K of C is separable and finite over $\mathbb{C}(X)$, a K-basis for differentials on C is given by dX. A differential form

$$\omega \quad = \quad \frac{X^\alpha dX}{Y^\beta}$$

is regular on $C - \{P\}$, i.e. on the affine curve we started with, if $\beta \le N-1$ and $\alpha \ge 0$. Note that

$$\omega \quad \sim \quad T^{3\beta}\cdot T^{-N\alpha}\cdot T^{-2N}\cdot T^{N-1}\cdot dT.$$

Hence

$$\omega \text{ is regular at } P \in C \quad \text{iff} \quad 3\beta \ge N\alpha + N + 1. \qquad \square$$

(7.7) Lemma. *We fix $N = 3M \in \mathbb{Z}_{\ge 3}$. We fix $\lambda \in \mathbb{C} - \{0,1\}$. We write $C = C_{N,\lambda}$. Then:* genus$(C) = N-2$; *a basis for $\Gamma(C, \Omega^1)$ is:*

$$\{\frac{XdX}{Y^{N-1}}, \cdots, \frac{XdX}{Y^{2M+1}}, \frac{dX}{Y^{N-1}}, \cdots, \frac{dX}{Y^{M+1}}\}.$$

Examples:

N=3 $\quad \{dX/Y^2\}$,

N=6 $\quad \{\frac{XdX}{Y^5}, \frac{dX}{Y^5}, \frac{d}{Y^4}, \frac{dX}{Y^3}\}$,

N=9 $\quad \{\frac{XdX}{Y^8}, \frac{XdX}{Y^7}, \frac{dX}{Y^8}, \cdots, \frac{dX}{Y^4}\}$,

N=12 $\quad \{\frac{XdX}{Y^{11}}, \frac{XdX}{Y^{10}}, \frac{XdX}{Y^9}, \frac{dX}{Y^{11}}, \cdots, \frac{dX}{Y^5}\}$.

Proof. We have morphisms $C_{N,\lambda} \to C_{3,\lambda} \to \mathbb{P}^1$. We show that the last one is unramified above $(z=0) = \infty \in \mathbb{P}^1$, giving $Q_1, Q_2, Q_3 \in C_{3,\lambda}$ above this point, and we show the first morphism is totally ramified above each of the $Q_j \in C_{3,\lambda}$.

In fact, in the equation $U^3 = X(X-1)(X-\lambda)$ substituting $X = 1/Z$ and $ZU = \xi$, we obtain $\xi^3 = (1-Z)(1-\lambda\cdot Z)$, and we derive the first claim. We conclude that genus$(C_{3,\lambda}) = 1$.

For $Y^N = X(X-1)(X-\lambda)$ we write $X = 1/Z$, and $ZY = \eta$, obtaining

$$\eta^N = Z^{N-3}(1-Z)(1-\lambda Z).$$

We write $N = 3M \geq 6$; note that $M = N/3$ and $M-1$ are coprime; a local parameter T at a point Q_j satisfies $Z \sim T^M$, and $\eta \sim T^{M-1}$. As Z is a local parameter in each of the Q_j this proves the claim that $C_{N,\lambda} \to C_{3,\lambda}$ totally ramifies at the points Q_j. The Zeuthen-Hurwitz theorem gives:

$$2g-2 \quad = \quad N\cdot(-2) + 3\cdot(N-1) + 3\cdot(M-1); \quad \text{hence} \quad g = N-2.$$

A differential form

$$\omega \quad = \quad \frac{X^\alpha dX}{Y^\beta}$$

is regular on $C - \{P\}$, i.e. on the affine curve we started with, if $\beta \leq N-1$ and $\alpha \geq 0$. Note that

$$\omega \quad \sim \quad T^\beta \cdot T^{-M\alpha} \cdot T^{-M-1} dT.$$

Hence

$$\omega \text{ is regular at } P \in C \quad \text{iff} \quad \beta \geq M\alpha + M + 1. \qquad \square$$

(7.8) Remark/Exercise. Consider the curve $C = C_{3,\zeta_6}$. Show that genus$(C) = 1$. Note that an affine piece of C can be given by the equation $Y^3 = X - 1$. Note that there are automorphisms of C: one given by

$$\alpha(Y) = \omega\cdot Y, \quad \alpha(X) = X, \quad \text{one given by } \beta(Y) = Y, \quad \beta(X) = \omega\cdot X;$$

here $\omega = \zeta_3$. Let E be the elliptic curve defined by C. Compute End(E).

(7.9) By (7.6) and (7.7) we have proved (1) and (2). We know that (3) is true: this is just the "Legendre family" for elliptic curves; for a proof (6) we refer [3], but for a motivation, see below; note that (6) + (4.2) would give (7); we expect (10) to be true, and then by (4.2) it would follow that (11) is true. We sketch a proof for the Theorem (4) of Johan de Jong and Rutger Noot, we then draw the conclusion (5); we give motivation for (6) and (10); we give a proof for (8) and (9).

(7.10) Proof of (4) and (5). Let us first treat the case (4, N=5). We have seen that $L = \mathbb{Q}(\zeta_5)$ act on the Jacobian of $C_{N,\lambda}$, and that it acts on the basis for the differential given above by multiplication by $\{\zeta, \zeta, \zeta^2, \zeta^3\}$; this computes

$$r_1 = 2, \quad r_2 = 1, \quad s_1 = 0, \quad s_2 = 1$$

for the action of $\mathbb{Q}(\zeta_5)$ on the tangent space of Jac$(C_{5,\lambda})$ for any $\lambda \in \mathbb{C} - \{0,1\}$. By the computation by Shimura, see (7.4), we know that for this action on the tangent space, the PEL-type Shimura variety $S^{(\iota)}$ has dimension

$$\dim(S^{(\iota)}) \quad = \quad \sum_{1\leq j\leq e} r_j s_j = 1.$$

Remark. Consider one point in S_5, the one given by $C_0 = C_{5,\zeta_6}$. Write $\zeta = \zeta_6$, and $\omega = \zeta_3$. Note that the transformation

$$x \mapsto (\omega - 1)x + 1 \quad \text{maps} \quad \{0, 1, \zeta, \infty\} \to \{1, \omega, \omega^2, \infty\}.$$

This shows that $C_0 = C_{5,\zeta_6}$ is the normalization of the curve given by $Y^2 = X^3 - 1$. This shows that

$$\mathbb{Q}(\zeta_{15}) \subset \mathrm{End}(C_0); \quad \text{hence } \mathrm{Jac}(C_0) \quad \text{admits smCM.}$$

We see that $S_5 \subset S^{(\iota)}$ is an irreducible component of $S^{(\iota)}$. Hence $S_5 \subset \mathcal{A}_4$ is a special subvariety; this proves (4, N=5). As the CM-points in $S^{(\iota)}$ are dense, the CM-points in S_5 are dense; this proves (5, N=5).

For the case (4, N=7) we see that

$$r_1 = 2, \quad r_2 = 2, \quad r_3 = 1, \quad s_1 = 0, \quad s_2 = 0, \quad s_3 = 1.$$

Again $\sum_{1 \leq j \leq e} r_j s_j = 1$. We conclude as before. □

(7.11) Remark. One can even show that all CM-fields appearing on S_5 give an infinite set of isomorphism classes; see [7], 2.7.

(7.12) (6) and (7). If $3 \nmid N$ and $N \geq 8$ the same computation gives $\sum_{1 \leq j \leq e} r_j s_j > 1$. This proves that $\dim(S_N) = 1 < \dim(S^{(\iota)})$ in this case. A closer analysis gives that indeed S_N is *not a special subvariety* is $3 \nmid N$ and $N \geq 8$, see [3], 2.8.3. If the AO-Conjecture would be known for this case, we could conclude (7). Is there a direct proof for this statement?

(7.13) For $N \in \mathbb{Z}_{>1}$ and for $d \mid N$ we have a morphism $C_{N,\lambda} \to C_{d,\lambda}$, given by $(x, y) \to (x, y^{N/d})$. This induces a homomorphism $J_{N,\lambda} \to J_{d,\lambda}$, where $J_{m,\lambda} = \mathrm{Jac}(C_{m,\lambda})$ for $m \in \mathbb{Z}_{>1}$. We write $J^{\mathrm{new}}_{N,\lambda}$ for the abelian variety which is the connected component of the intersection of the kernels of all homomorphisms $J_{N,\lambda} \to J_{d,\lambda}$ for all $1 < d \lneqq N$ and $d \mid N$. Note that for a prime number p we have $J^{\mathrm{new}}_{p,\lambda} = J_{p,\lambda}$. Note that:

$$J_{N,\lambda} \quad \sim \quad \prod_{1<d\leq N, \quad d|N} J^{\mathrm{new}}_{d,\lambda}.$$

Let us consider the case $N = 6$. We have $\mathrm{genus}(C_{6,\lambda}) = 4$ for every $\lambda \in \mathbb{C} - \{0, 1\}$. The vector space of regular differentials on every $C_{6,\lambda}$ is generated by $\{XdX/Y^5, dX/Y^5, dX/Y^4, dX/Y^3\}$. In the family $J_{6,\lambda}$ up to isogeny we have a family $J_{2,\lambda}$, corresponding with dX/Y^3, a "constant" family $J_{3,\lambda}$, corresponding with dX/Y^4, and a "constant" family $J^{\mathrm{new}}_{6,\lambda}$, corresponding with XdX/Y^5 and dX/Y^5 (this last abelian variety admits smCM, e.g. use [21], Prop. 14 on page 176). We see that $S_6 \subset \mathcal{A}_4$ contains infinitely many CM-points.

(7.14) (8) and 9. We see that $J_{9,\lambda} \sim J^{\mathrm{new}}_{9,\lambda} \times J_{3,\lambda}$. The differential dX/Y^6 comes from the isogeny factor $J_{3,\lambda}$. Using (7.7) we see that for the case (8, $N = 9$), the action of $\mathbb{Q}(\zeta_9)$ on $J^{\mathrm{new}}_{9,\lambda}$ is given by

$$r_1 = 2, \quad r_2 = 2, \quad r_3 = 1, \quad s_1 = 0, \quad s_2 = 0, \quad s_3 = 1.$$

Using the theorem (7.4) by Shimura we conclude (8), and (9) follows. □

8 A generalization ?

Yves André proposes a "generalized AO-Conjecture", which would contain MM and OA. See [2], pp. 8-9 (the end of Lecture III). Here is a special case of this expected generalization:

Theorem. *Let T be an irreducible curve, and let $f : \mathcal{E} \to T$ be a family of elliptic curves with non-constant moduli. Let $\sigma : T \to \mathcal{E}$ be a section which at the generic point is non-torsion. Then there are only finitely many $t \in T$ such that $\sigma(t) \in \mathcal{E}_t$ is torsion.* See [2] page 12: the end of Lecture IV.

It is not so obvious how to formulate this generalization over arbitrary Shimura varieties.

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