Preface

One good example is worth a host of generalities (Hyland)

"It is too early for a book on realizability", Martin Hyland has said on various occasions. And certainly, the ultimate book about this topic will have to wait until the world of realizability is better understood.

In the meantime, however, one can try to do something about the publicity situation of the field, which leaves something to be desired. Whenever a starting Ph.D. student wishes to work in realizability, he or she has to fumble his/her way forward amidst a mass of scattered papers, unpublished notes, Ph.D. theses which are not all electronically available. It seemed to me that one coherent presentation might help.

The main purpose of this book is to introduce you to the effective topos $\mathcal{E}ff$ and related toposes. The effective topos is a strange thing, and understanding of its logic is rare, even among topos theorists; and although by now there is quite a collection of books on topos theory, none of them treats $\mathcal{E}ff$ in anything like the detail it deserves. True, I am aware that Peter Johnstone is writing the third volume of his monumental *Sketches of an Elephant*, in which there will be a part on this topos; but I am quite sure that the overlap between his and my treatment will be limited.

Instead of aiming for a structural approach, which would first establish as many abstract properties of the object one is studying and then try to view as much as possible as consequences of these, I decided to work *by example* (see the motto of this introduction). I am, as Alex Simpson once remarked, a "details person" and I believe very much in concrete understanding taking place in the mind by working through lots of examples; the concepts, hopefully, come afterwards. Of course, it would be silly to refrain from developing some theory first if this really makes things simpler, and I start by going over the theory of partial combinatory algebras, and then the theory of triposes. When studying the effective topos, it is by interpreting various theories in it, that I hope you get some sort of picture.

Quite a bit of work on realizability takes place entirely in the category of *assemblies*, the separated objects for the double negation local operator; and assemblies are a very nice place to be. But the effective topos is much more, and I have deliberately focused on the *higher order* features of $\mathcal{E}ff$ too, by going over interpretations of higher order arithmetic, set theory and synthetic domain theory, in an attempt to stay clear of Scott's lamented "first order disease" (preface to [18]).

It is my hope that at least the first three chapters of the book are written in a sufficiently leasurely text book style for a graduate student with some requisite preliminary knowledge (which I will detail below) to read it. In the fourth and last chapter, which is a bit of 'capita selecta', the style becomes more succinct and the aim is rather to summarize results and give a guide to the literature.

Preliminaries. The book is aimed at advanced undergraduate students or beginning Ph.D. students, who have at least some knowledge of the following topics:

1. Logic: a course presenting the notions of *language*, *theory*, *structures* and Gödel's Completeness Theorem. At least some acquaintance with Peano Arithmetic. However, it would be *very useful* if you have studied Intuitionistic logic too.

2. Category Theory: the first five chapters of MacLane's *Categories for the Working Mathematician*, as well as acquaintance with the notion of a topos, and the general idea of interpreting a theory in a category. As I said, there is by now quite a range of books on topos theory but if you really want to *learn* the subject, in my opinion the best book is still the old [80]. Yes, it is tough going.

3. Recursion Theory: the basics of computable functions, Kleene's T-predicate and normal form, the recursion theorem and the notion of Turing reducibility.

Style. There is only one aspect in which I have deliberately deviated from standard scientific usage: and that is, that being not a group of authors or of royal descent, I see no point in writing 'we' when I mean 'I' (which doesn't mean I don't use 'we', like in: '. . . now we shall see. . . '), and therefore also none in writing 'the reader' when I mean 'you'. But, I have not been very consistent in this.

Terminology and Notation. I have tried to be as conservative and uncontroversial as possible, but as this has become a somewhat contentious issue in Category Theory, I should maybe clarify my position. Over the last 20 years, a few attempts have been made to improve categorical terminology, with varying degrees of success. Category theory has a lot of redundant terminology (just think of the number of ways you can say that a category 'has finite limits'); despite this, words are often used with multiple meanings; and then, sometimes terminology is not very well chosen.

The most radical attempt to rename everything, was [48]. Unless you have read this book from cover to cover, it is impossible to find anything, because the index will be meaningless to you. In this particular case my advice would be: just *read* it from cover to cover (it is a wonderful book), but forget about the new terminology since it didn't catch on.

The choice of good terminology, it is too often forgotten, requires a bit of literary talent. Such talent is not displayed by calling a terminal object a 'terminator'. Good examples of imaginative terminology are Paul Taylor's 'prone' and 'supine' in the context of fibrations.

In general my position is that there is only one thing worse than bad terminology, and that is *continually changing* terminology. Therefore, even if I agree with some criticisms made, I stick in most cases to the old names. I do avoid 'left exact' because one can just say 'has finite limits'. Using the word 'cartesian' for this (as in [83]) creates confusion and is ineffective if one still has to write on p. 161 (l. -10): 'cartesian (i.e. preserves finite limits)'. For 'exact category' I see no alternative. If this is horrible, is 'effective regular' an improvement?

Acknowledgements. Since this book contains a large part of my work done in the last two decades, I should thank the people who were influential to me during that period.

I feel privileged in having had *Anne Troelstra* as thesis advisor. His work [161] has been immensely useful to me for a long time. This is such a neat arrangement of systems and interpretations (with a lot of original work in between), that in my view he is the Linnaeus of realizability. He might be pleased with this title, having had a passion for botany all his life.

I have learned very much from discussions with great people in the subject; the greatest of all is *Martin Hyland*. But I also mention *Pino Rosolini* and *Dana Scott*, whom I visited and who in different periods of my development, opened my eyes to new visions. I have moreover been extremely fortunate to be working in the group of *Ieke Moerdijk*, who always has ideas and questions which stimulate research.

Since part of what I report in this book has been joint work with others, I thank my coauthors *Lars Birkedal*, *Martin Hofmann*, *Pieter Hofstra*, *Claire Kouwenhoven*, *Alex Simpson* and *Thomas Streicher*. Especially with the last two I have had long discussions at times.

Albert Visser and *Dick de Jongh* have generated questions which find their way into this book. It is clear from the text to what extent I am indebted to *John Longley*, whose work I admire. *Bas Terwijn* helped me with the proof of 3.2.32.

Also inspiring has been reading work of, and discussing with, *Andrej Bauer*, *Benno van den Berg*, *Peter Lietz* and *Matías Menni*.

Introduction

In 1945, Stephen Cole Kleene published a paper ([88]) in which he showed that the partial recursive functions (the theory of which he had been developing himself during the 1930-ies) could be used to give an interpretation of the logic of Brouwer's Intuitionism. By means of this interpretation any classical mathematician, whatever his philosophical views, could study intuitionistic logic.

Every partial recursive function can be assigned a *Gödel number* or *index*, in such a way that, if we denote by φ_e the partial recursive function with index e, and write $\varphi_e(x)\downarrow$ for 'x is in the domain of φ_e', we have the following properties:

> The partial function sending the pair (e, f) to $\varphi_e(f)$ is partial recursive as function of e, f;
>
> there are primitive recursive functions S^m_n with the property that for every e and every $m + n$-tuple $x_1, \ldots, x_m, y_1, \ldots, y_n$, we have
>
> $$\varphi_{S^m_n(e,x_1,\ldots,x_m)}(y_1, \ldots, y_n) \simeq \varphi_e(x_1, \ldots, x_n, y_1, \ldots, y_m)$$
>
> where $\simeq$ means: either side is defined iff the other is; and if defined, they are equal.

There is, moreover, a primitive recursive bijection $\langle\cdot,\cdot\rangle : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$. We say that the number $\langle a, b\rangle$ *codes the pair* (a, b). From the code of a pair one can, primitive-recursively, recover both components of the pair.

Kleene's interpretation was given in the form of a relation between numbers and sentences of arithmetic, which he called 'realizes': a number could realize (that is: witness, carry information to the truth of) a statement. The relation is defined by recursion on the logical structure of the sentence. Two significant clauses (for the full definition, see section 3.1) read:

> n realizes $(\phi \vee \psi)$ if and only if n codes a pair (a, b) and either $a = 0$ and b realizes ϕ, or $a \neq 0$ and b realizes ψ;
>
> n realizes $(\phi \to \psi)$ if and only if for every m such that m realizes ϕ, $\varphi_n(m)\!\downarrow$ and $\varphi_n(m)$ realizes ψ.

Kleene proved that the axioms and rules of first-order intuitionistic arithmetic are sound for this interpretation. That is: if ϕ is a consequence of these axioms and rules, then there is a number n which realizes ϕ.

The converse does *not* hold: there are sentences which have a realizer, yet are unprovable in intuitionistic arithmetic. This makes Kleene's interpretation an interesting model.

In subsequent years, many variations of this idea were developed, and interpretations given. At the same time (the 1950-ies and 1960-ies) a school of 'recursive constructive mathematics' was working; these people investigated which theorems of mathematics would remain valid if 'everything is recursive'. For example, a group would be a subset G of $\mathbb{N}$ together with partial recursive functions $(\cdot)^{-1} : G \to G$ and $\cdot : G \times G \to G$ satisfying the group axioms, and a group homomorphism $G \to H$ would have to be a partial recursive function. Which theorems of algebra would remain?

Or, more sophisticatedly, a group would be a set G together with, for every $x \in G$, a set of numbers $E(x)$ thought of as carrying 'recursive information' about x; a function from such a construct (G, E) to another (H, E') would be a function $f : G \to H$ for which there exists a partial recursive function ϕ such that for every $x \in G$ and every $n \in E(x)$, $\phi(n)\!\downarrow$ and $\phi(n) \in E'(f(x))$. Such objects (G, E) are now called *assemblies*, f is a morphism of assemblies and ϕ *tracks* f.

If 'everything is recursive' then certainly all functions from natural numbers to natural numbers are; what about functions on functions from $\mathbb{N}$ to $\mathbb{N}$, etcetera? Well, every function $\mathbb{N} \to \mathbb{N}$ has an index e, so a function from functions $\mathbb{N} \to \mathbb{N}$ to $\mathbb{N}$ is a partial recursive function acting on indices. But it has to be *extensional*: if e and e' are indices of the same function (and every function has infinitely many indices), they should be sent to the same number. This leads to the construction of the type structure HEO of hereditary effective operations, which I define now.

We have *types*: a type o for natural numbers; if σ and τ are types then we have a type $(\sigma \to \tau)$ of functions from things of type σ to things of type τ. For every type σ we define a set HEO_σ of natural numbers and an equivalence relation $\equiv_\sigma$ on this set, as follows:

$$\begin{array}{rcl} \mathrm{HEO}_o & = & \mathbb{N} \\ n =_o m & \text{iff} & n = m \\ \mathrm{HEO}_{\sigma\to\tau} & = & \{e \,|\, \forall f \in \mathrm{HEO}_\sigma(\varphi_e(f)\!\downarrow \wedge\, \varphi_e(f) \in \mathrm{HEO}_\tau) \wedge \\ & & \forall f f' \in \mathrm{HEO}_\sigma(f \equiv_\sigma f' \Rightarrow \varphi_e(f) \equiv_\tau \varphi_e(f'))\} \\ e \equiv_{\sigma\to\tau} e' & \text{iff} & \forall f \in \mathrm{HEO}_\sigma(\varphi_e(f) \equiv_\tau \varphi_{e'}(f)) \end{array}$$

Then HEO gives some interpretation of the hierarchy of higher-type functions from a recursive point of view.

Now the objects $(\mathrm{HEO}_\sigma, \equiv_\sigma)$ can also be seen as assemblies, in the following way. Given two assemblies (X, E) and (Y, E') we can form an assembly of functions $(X, E) \Rightarrow (Y, E')$: this is an assembly (V, E'') where V is the set of all functions $f : X \to Y$ which are tracked by a partial recursive function, and $E''(f)$ is the set of indices of partial recursive functions which track f. Let us now form a type structure of assemblies: we take for o the assembly $A_o = (\mathbb{N}, n \mapsto \{n\})$ and, supposing that for types σ and τ we have defined assemblies A_σ and A_τ, then at type $\sigma \to \tau$ we take the assembly $A_{\sigma\to\tau} = A_\sigma \Rightarrow A_\tau$. So for each σ we have an assembly $A_\sigma = (X_\sigma, E_\sigma)$. With this notation we can verify that for every σ, there is a bijection between X_σ and the set of equivalence classes of HEO_σ under $\equiv_\sigma$ such that for every $x \in X_\sigma$, the set $E(x)$ is equal to the $\equiv_\sigma$-equivalence class to which x corresponds.

Inspired by this phenomenon, Martin Hyland constructed a topos in which HEO is really the structure of *all* higher type functionals over the natural numbers.

This topos is the *effective topos*.

At the same time, his student Andy Pitts worked out a general theory on how to obtain such toposes: *tripos theory*. It turned out that various modifications of the realizability interpretation that had been studied in the past, also gave rise to toposes, and these came often out of standard topos-theoretic constructions applied to the efective topos.

The enormous advantage of the topos-theoretic approach is, that there is a uniform notion of truth for any higher-order language, and mathematical constructions like function space, power set and so on are already given to you. You don't have to wonder about what real-valued functions are, and in the effective topos *all* functions from the reals to the reals are continuous.

The natural numbers are not the only set which can act as a set of 'realizers'. The properties of indices of partial recursive functions, given at the beginning of this introduction, mean that $\mathbb{N}$ is an example of a so-called 'partial combinatory algebra'. This notion, first formally defined by Feferman but going back to the work of Schönfinkel, embodies exactly what one needs in order to construct a 'realizability topos'.

In this book, I start with a chapter on partial combinatory algebras. Then, tripos theory is developed. In the third, and most voluminous chapter, I present the effective topos in detail. This topos harbors interpretations of a number of theories, and in some cases I have found it necessary to also introduce you briefly into the particular theory at hand. Finally, chapter 4 gives a number of variations, with emphasis on how these can be topos-theoretically constructed.

If you are only interested in the effective topos, you can skip chapter 1, but you'll need at least some parts of chapter 2 (which give the tripos-to-topos construction and the internal logic of the constructed topos out of that of the tripos).

Contents