Hand-in exercises for the course Foundations of Mathematics

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1 Exercises

Exercise 1 [To be handed in November 21, 2023]

a) (4 pts) Show that $\mathbb{N}$ can be written as a disjoint union $\bigcup_{i\in\mathbb{N}} C_i$ with all C_i infinite (here, "disjoint union" means that $C_i \cap C_j = \emptyset$ whenever $i \neq j$).

b) (6 pts) We write $\Delta(X, Y)$ (the "symmetric difference") for the set

$$(X - Y) \cup (Y - X).$$

Show that for every sequence $(A_n \mid n \in \mathbb{N})$ of subsets of $\mathbb{N}$, there is a subset B of $\mathbb{N}$ such that for all n, $\Delta(B, A_n)$ is infinite.

Exercise 2 [To be handed in November 28,2023] By $\mathcal{P}(\mathbb{R})$ we denote, as usual, the power set of $\mathbb{R}$. Let P be the poset of all pairs (A, g) such that $A \subseteq \mathbb{R}$ and $g : A \to \mathcal{P}(\mathbb{R})$ is a function and the following conditions hold:

i) If $a_1, a_2 \in A$ and $a_1 \neq a_2$ then $g(a_1) \cap g(a_2) = \emptyset$.

ii) $0 \in A$ and $0 \in g(0)$.

iii) If $a_1, a_2 \in A$, $r_1 \in g(a_1)$ and $r_2 \in g(a_2)$, then $a_1 + a_2 \in A$ and $r_1 + r_2 \in g(a_1 + a_2)$.

The set P is preordered by: $(A, g) \leq (B, h)$ if $A \subseteq B$ and for all $a \in A$, $g(a) \subseteq h(a)$.

a) (4 pts) Show that the poset P satisfies the condition of Zorn's Lemma, that is: every chain in P has an upper bound. Conclude that P has a maximal element.

b) (4 pts) Let (A, g) be a maximal element of P. Prove that the set A is closed under addition and that, if $r \in A$, also $-r \in A$.

c) (2 pts) Let (A, g) be a maximal element of P. Show that $\bigcup_{a\in A} g(a) = \mathbb{R}$ and there is a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$, $f(x+y) = f(x)+f(y)$ and for all $x \in \mathbb{R}$, $x \in g(f(x))$.

Exercise 3 [To be handed in December 5, 2023] This is Exercise 45 from the book:

Let L be a set. Write $\mathcal{P}^*(L)$ for the set of nonempty subsets of L. Suppose that $h : \mathcal{P}^*(L) \to L$ is a function such that the following two conditions are satisfied:

i) For each nonempty family $\{A_i \mid i \in I\}$ of elements of $\mathcal{P}^*(L)$, we have

$$h(\bigcup_{i \in I} A_i) \;=\; h(\{h(A_i) \mid i \in I\})$$

ii) For each $A \in \mathcal{P}^*(L)$, $h(A) \in A$.

Show that there is a unique relation $\leq$ on L , which well-orders L, and is such that for each nonempty subset A of L the element $h(A)$ is the least element of A.

Exercise 4 [To be handed in December 12, 2023] Two L-structures M and N are said to be *elementarily equivalent* (notation: $M \equiv N$) if they satisfy the same L-sentences: for any L-sentence ϕ we have $M \models \phi$ if and only if $N \models \phi$.

Let $L = \{<\}$ be the language of (strict) posets. Let P be a poset such that for every $n \in \mathbb{N}$ there is an ascending sequence $p_1 < p_2 < \cdots < p_n$ of length n. Show that there is a poset Q with the properties:

i) $P \equiv Q$.

ii) In Q there is an *infinite* ascending sequence $q_1 < q_2 < \cdots$.

Exercise 5 [To be handed in January 16, 2024] Demonstrate by constructing proof trees:

i) (5 pts) $(\phi \to \exists x \psi) \vdash \exists x(\phi \to \psi)$ (here it is assumed that the variable x does not occur in ϕ.

ii) (5 pts) $(\phi \to \psi) \vee (\psi \to \phi)$.

Exercise 6 [To be handed in January 23, 2024] This is Exercise 129 from the book:
Let L be a language, T an L-theory and L' an extension of the language L.

a) (4 pts) Show that the poset of all L'-theories which are conservative extensions of T, ordered by inclusion, satisfies the hypothesis of Zorn's Lemma.

b) (6 pts) By Zorn's Lemma, there is a maximal L'-theory U which is conservative over T. Show that for every L'-sentence $\psi \notin U$, there are an L-sentence ϕ and an L'-sentence $\gamma \in U$ such that $\gamma \wedge \psi \models \phi$ and $T \not\models \phi$.

2 Solutions

Exercise 1 a): according to the proof of Proposition 1.1.4ii) there is a bijective function $\psi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Let $\pi_0 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the first projection. Define C_i to be $\{n \in \mathbb{N} \,|\, \pi_0(\psi(n)) = i\}$. Clearly, C_i is infinite, $C_i \cap C_j = \emptyset$ for $i \neq j$ and $\bigcup_{i\in\mathbb{N}} C_i = \mathbb{N}$.
b): Given the sequence $(A_n \,|\, n \in \mathbb{N})$, let χ_n be the characteristic function of A_n. We use the sequence $(C_n \,|\, n \in \mathbb{N})$ from part a). We define the set B by its characteristic function χ:

$$\chi(k) \;=\; 1 - \chi_n(k)$$

where n is the unique number such that $k \in C_n$. We see that $C_n \subseteq \Delta(B, A_n)$ (note that $x \in \Delta(B, A_n)$ if and only if exactly one of $\{\chi(x), \chi_n(x)\}$ equals 1), so this set is infinite for all n.

Exercise 2 a): First we observe that the pair (A, g) with $A = \{0\}$ and $g(0) = \{0\}$ is an element of P, which is therefore nonempty; so the empty chain has an upper bound. Now suppose $C = (A_i, g_i)_{i\in I}$ is a nonempty chain in P. We define:

$$A - \bigcup_{i\in I} A_i \quad g(a) = \bigcup_{a\in A_i} g_i(a) \quad \text{for } a \in A$$

We check that (A, g) satisfies conditions i)–iii), for then it will be an upper bound for C, as is immediate.

i) Suppose $a_1, a_2 \in A$, $a_1 \neq a_2$. Then

$$\begin{array}{rcl} g(a_1) \cap g(a_2) & = & (\bigcup_{a_1\in A_i} g_i(a_1)) \cap (\bigcup_{a_2\in A_j} g_j(a_2)) \\ & = & \bigcup_{a_1\in A_i, a_2\in A_j} g_i(a_1) \cap g_j(a_2) \end{array}$$

Since C is a chain, we may suppose $(A_i, g_i) \leq (A_j, g_j)$. Then $g_i(a_1) \cap g_j(a_2) \subseteq g_j(a_1) \cap g_j(a_2) = \emptyset$. So condition i) holds.

ii) this condition holds by nonemptiness of C.

iii) If $a_1, a_2 \in A$, $r_1 \in g(a_1), r_2 \in g(a_2)$ then again by the chain property of C, we have $a_1, a_2 \in A_i$, $r_1 \in g_i(a_1)$, $r_2 \in g_i(a_2)$ for some $i \in I$, so $r_1 + r_2 \in g_i(a_1 + a_2) \subseteq g(a_1 + a_2)$ and on the way we have checked that $a_1 + a_2 \in A$. We conclude that $(A, g) \in P$.

We conclude by Zorn's Lemma that P has a maximal element.

b): Suppose $a_1, a_2 \in A$ but $a_1 + a_2 \notin A$. Then by condition iii) for A, $g(a_1)$ and $g(a_2)$ cannot both be nonempty. This means that if we define (A', g') as follows:

$$A' = A \cup \{a_1 + a_2\} \quad g'(a) = \begin{cases} g(a) & \text{if } a \in A \\ \emptyset & \text{if } a = a_1 + a_2 \end{cases}$$

then conditions i)–iii) are satisfied, so (A', g') is an element of P and a proper extension of (A, g), which violates the maximailty of the latter. We conclude that $a_1 + a_2 \in A$.

The second statement is proved by a similar trick: suppose $a_0 \in A$ but $-a_0 \notin A$. Let $A' = A \cup \{-a_0\}$, $g'(a) = g(a)$ for $a \in A$ and $g(-a_0) = \emptyset$. Again, conditions i)–iii) hold, so $(A', g') \in P$ and (A', g') extends (A, g); again violating maximality.

In fact, we could have shown directly that $A = \mathbb{R}$, from which the two statements to be proved, follow at once. Indeed, if $(\mathbb{R}, g')$ is such that $g'(a) = g(a)$ for $a \in A$, and $g'(a) = \emptyset$ otherwise, then it is easy to see that $(\mathbb{R}, g')$ is an element of P which extends (A, g). By maximality we must have equality, whence $A = \mathbb{R}$.

c) This part of the exercise turned out to be far more laborious than usual for a hand-in exercise; which is why we decided to award 5 points to parts a) and b) each, and give up to 2 bonus points for the students who had done (a substantial part of) c).

Assume (A, g) is a maximal element of P. Let us first see that if $Z = \bigcup_{a \in A} g(a)$ is equal to $\mathbb{R}$, then there is a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$, $f(x + y) = f(x) + f(y)$ and for all $x \in \mathbb{R}$, $x \in g(f(x))$. This is almost trivial: define, for $x \in Z$, $f(x)$ to be the unique a such that $x \in g(a)$ (unicity of a follows from condition i) of elements of P). Conditions ii) and iii) now imply the required properties of the function f. Note that we always have such a function $f : Z \to \mathbb{R}$ satisfying $f(0) = 0$ and $f(x+y) = f(x)+f(y)$, by the same definition.

Now for the proof that $Z = \mathbb{R}$. First we prove:

Claim 1 The set Z is closed under the function $x \mapsto -x$.

Proof: Assume $x_0 \in Z$ but $-x_0 \notin Z$. Let a_0 be such that $x_0 \in g(a_0)$. Define the function $g' : A \to \mathbb{R}$ by

$$g'(a) \;=\; \{x - x_0 \,|\, x \in g(a + a_0)\}$$

We check that (A, g') is an element of P which satisfies $(A, g) \leq (A, g')$. Since $-x_0 \in g'(-a_0)$ (because $0 \in g(0)$) we then have $(A, g) < (A, g')$, which violates the maximality of (A, g). We check conditions i)–iii) for (A, g'), as well as

iv): $g(a) \subseteq g'(a)$, for all $a \in A$.

i): Suppose $x - x_0 \in g'(a) \cap g'(a')$. Then $x \in g(a + a_0)$ and $x \in g(a' + a_0)$ so by i) for (A, g), $a = a'$. We see that (A, g') satisfies i).

ii): $g'(0) = \{x - x_0 \,|\, x \in g(a_0)\}$. Since $x_0 \in g(a_0)$ we have $0 = x_0 + (-x_0) \in g'(0)$, so this checks ii).

iii): Suppose $x - x_0 \in g'(a)$, $x' - x_0 \in g'(a')$. We need to see that $x + x' - 2x_0 \in g'(a + a')$. We have $x \in g(a + a_0), x' \in g(a' + a_0)$. By iii) for (A, g) this gives

$$x + x' \in g(a + a' + 2a_0)$$

Applying the definition of g' twice, we get: $x + x' - x_0 \in g'(a + a' + a_0)$, whence $x + x' - 2x_0 \in g'(a + a')$. Which is what we set out to show.

iv): If $x - x_0 \in g(a)$, then $x = (x - x_0) + x_0 \in g(a + a_0)$, so $x - x_0 \in g'(a)$. This shows $g(a) \subseteq g'(a)$.

This proves Claim 1. Now we turn to the full result:

Claim 2: $Z = \mathbb{R}$.
Proof: By maximality of (A, g) we may assume that $Z = \bigcup_{a \in A} g(a)$ is closed under addition and the function $x \mapsto -x$. Now suppose $x_0 \in \mathbb{R} - Z$. We consider two cases:
Case 1: For no natural number $k > 0$ we have $kx_0 \in Z$. We consider the set

$$Z' = \{x + kx_0 \mid x \in Z, k \in \mathbb{Z}\}$$

Note that every element of Z' can be *uniquely* expressed as $x + kx_0$, for if $x + kx_0 = x' + k'x_0$ then we would have $(k - k')x_0 \in Z$, which by our assumption can only happen if $k = k'$ and therefore $x = x'$. Note, that here we use the fact, previously proved, that Z is closed under the function $x \mapsto -x$.

Choose $\alpha \in \mathbb{R}$ arbitrary, let $A' = \{a + k\alpha \mid a \in A\}$ and define $g'(a + k\alpha) = \{x + kx_0 \mid x \in g(a)\}$. We see that (A', g') extends (A, g) in P, and arrive at the familiar contradiction.
Case 2: For some $m \in \mathbb{N}_{>0}$ we have $mx_0 \in Z$, let m be minimal with this property. Define

$$Z' = \{x + kx_0 \mid x \in Z, k \in \mathbb{N}, 0 < k < m\}$$

Again, every element of Z' can be uniquely written as $x + kx_0$ for $x \in Z$ and $0 < k < m$. Pick a_0 such that $mx_0 \in g(a_0)$. Let

$$A' = \{a + \frac{k}{m}a_0 \mid a \in A, 0 < k < m\}$$

Define $g' : A' \to \mathbb{R}$ by $g'(a + \frac{k}{m}a_0) = \{x + kx_0 \mid x \in g(a)\}$. Again, $(A, g) < (A', g')$.

Exercise 3. The relation $\leq$ is completely determined by the condition that $h(A)$ be the least element of A: for $x \leq y$ if and only if x is the least element of $\{x, y\}$, if and only if $x = h(\{x, y\})$. So, let us *define* $x \leq y$ by $x = h(\{x, y\})$, and show that $\leq$ is a well-order.

First we show that $\leq$ is a partial order:
Since $h(A) \in A$ always (condition ii)), we have $h(\{x, x\}) = h(\{x\}) = x$, so $x \leq x$ and $\leq$ is reflexive.
Suppose $x \leq y$ and $y \leq z$, so $h(\{x, y\}) = x$ and $h(\{y, z\}) = y$. Then

$$\begin{aligned} h(\{x, z\}) &= h(\{h(\{x, y\}), h(\{z\})\}) \\ &= h(\{h(\{x, y, z\})\}) \\ &= h(\{h(\{x\}), h(\{y, z\})\}) \\ &= h(\{x, y\}) \\ &= x \end{aligned}$$

(using condition i) twice) so $x \leq z$ and $\leq$ is transitive.
Finally, if $x \leq y$ and $y \leq x$ then $x = h(\{x, y\}) = h(\{y, x\}) = y$, so $\leq$ is antisymmetric. We conclude that $\leq$ is a partial order.

For the well-order property, we show that indeed, $h(A)$ is the least element of A, if $A \subseteq L$ is nonempty. For $a \in A$ we have

$$h(\{h(A), a\}) \;=\; h(\{h(A), h(\{a\})\}) \;=\; h(A \cup \{a\}) \;=\; h(A)$$

so $h(A) \leq a$ and $h(A)$ is the least element of A.

Exercise 4. The theory of *strict* posets, in the language $L_{spos} = \{<\}$, has the axioms:

$$\forall xyz(x < y \wedge y < z \to x < z)$$
$$\forall x \neg(x < x)$$

Call this theory T_{spos}.

Consider the language $L = L_{spos} \cup \{c_0, c_1, \ldots\}$, where the c_i are new constants. Let T be the L-theory which has the following axioms:

i) the axioms of T_{spos}

ii) the axioms $c_i < c_{i+1}$ for all $i \in \mathbb{N}$

iii) all L_{spos}-sentences which are true in P.

I claim that T is consistent. For this, in view of the Compactness Theorem, we look at a finite subtheory of T. Such a theory is contained in the theory which has the axioms of i) and iii), and finitely many axioms of ii), say $\{c_i < c_{i+1} \,|\, 0 \leq i \leq n\}$ for some $n \in \mathbb{N}$. Call this theory T_n; it is a theory in the language $L_{spos} \cup \{c_i \,|\, 0 \leq i \leq n+1\}$.

Now we can make P into a model of T_n by picking an ascending sequence $p_0 < \cdots < p_{n+1}$ in P and defining $c_i^P = p_i$. So every theory T_n is consistent; by the Compactness Theorem we conclude that the theory T is consistent. Let Q be a model of T. Then Q is a poset by i), which has an infinite ascending sequence $c_0^Q < c_1^Q < \cdots$ by ii), and which satisfies the same L_{spos}-sentences as P, by iii).

Exercise 5
a)

$$\dfrac{{}^\dagger\neg\exists x(\phi\to\psi)^3 \qquad \dfrac{\dfrac{\dfrac{\dfrac{{}^\dagger\neg\exists x(\phi\to\psi)^3 \quad \dfrac{\dfrac{\dfrac{\dfrac{{}^\dagger\neg\phi^1 \quad {}^\dagger\phi^2}{\bot}\,\neg E}{\psi}\,\bot E}{\phi\to\psi}\,\to I,2}{\exists x(\phi\to\psi)}\,\exists I}{\bot}\,\neg E}{\phi}\,\bot E,1 \quad \phi\to\exists x\psi^5}{\exists x\psi}\,\to E \qquad \dfrac{\dfrac{{}^\dagger\psi(u)^4}{\phi\to\psi(u)}\,\to I}{\exists x(\phi\to\psi)}\,\exists I}{\exists x(\phi\to\psi)}\,\exists E,4}{\dfrac{\bot}{\exists x(\phi\to\psi)}\,\bot E,3}\,\neg E$$

b)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{{}^{\dagger}\phi^{1}}{\psi \to \phi}\,{\to} I}{(\phi \to \psi) \vee (\psi \to \phi)}\,{\vee} I \qquad {}^{\dagger}\neg((\phi \to \psi) \vee (\psi \to \phi))^{2}}{\bot}\,\neg E}{\neg \phi}\,\neg I, 1 \qquad {}^{\dagger}\phi^{3}}{\bot}\,\neg E}{\psi}\,\bot E}{\phi \to \psi}\,{\to} I, 3}{(\phi \to \psi) \vee (\psi \to \phi)}\,{\vee} I \qquad {}^{\dagger}\neg((\phi \to \psi) \vee (\psi \to \phi))^{2}}{\bot}\,\neg E}{(\phi \to \psi) \vee (\psi \to \phi)}\,\bot E, 2$$

Exercise 6

a) Call this poset P. Let $\mathcal{C} \subseteq P$ be a chain of L'-theories which are conservative extensions of T. We consider $\bigcup \mathcal{C}$. If for some L-sentence ϕ we have $\bigcup \mathcal{C} \vdash \phi$ then since proof trees are finite, there is a finite subset U of $\bigcup \mathcal{C}$ such that $U \vdash \phi$. By the chain property, there is $T'' \in \mathcal{C}$ such that $U \subseteq T''$. Since T'' is conservative over T, we see $T \vdash \phi$. We conclude that $\bigcup \mathcal{C}$ is conservative over T, hence an element of P and therefore an upper bound of $\mathcal{C}$ in P. The poset P satisfies the hypothesis of Zorn's Lemma and has therefore a maximal element.

b) Let U be a maximal element of the poset P of part a). If ψ is an L'-sentence outside U, then by maximality of U the L'-theory $U \cup \{\psi\}$ is no longer conservative over T: there is an L-sentence ϕ such that $U \cup \{\psi\} \vdash \phi$ and $T \nvdash \phi$. Again, there is a finite subset $U' \subseteq U$ such that $U' \cup \{\psi\} \vdash \phi$; if γ is the conjunction of all elements of U', then $\gamma \wedge \psi \vdash \phi$ and $T \nvdash \phi$.