

Bonus Test
Proof Theory, 9-3-2011

Exercise 1 Determine which of the following propositional formulas are tautologies:

- a. $((p \wedge q) \vee (p \wedge \neg q)) \supset p$
- b. $(p \supset (r \vee s)) \supset ((p \supset r) \wedge (p \supset s))$
- c. $((p \supset q) \supset p) \supset p$
- d. $q \supset (\neg q \supset p)$
- e. $(\neg(r \supset q) \vee (r \wedge q)) \supset r$

Exercise 2 Give cut-free proofs of the following sequents (here, P and Q are predicate symbols):

- a. $\forall x(P(x) \supset Q(a)) \rightarrow (\forall x\neg P(x)), Q(a)$
- b. $\exists xP(x), Q(a) \rightarrow \exists x(P(x) \wedge Q(a))$

Solutions:

Exercise 1: yes, no, yes, yes, yes.

Exercise 2:

2a.

$$\begin{array}{c}
 \text{Weakening Right} \frac{Pb \rightarrow Pb}{Pb \rightarrow Pb, Qa} \quad \frac{Qa \rightarrow Qa}{Qa \rightarrow Qa} \text{Weakening Right} \\
 \neg \text{ Right} \frac{Pb \rightarrow Pb, Qa}{\rightarrow Pb, Qa, \neg Pb} \quad \frac{Qa \rightarrow Qa}{Qa \rightarrow Qa, \neg Pb} \text{Exchange Right} \\
 \text{Exchanges Right} \frac{\rightarrow Pb, Qa, \neg Pb}{\rightarrow \neg Pb, Qa, Pb} \quad \frac{Qa \rightarrow Qa, \neg Pb}{Qa \rightarrow \neg Pb, Qa} \supset \text{Left} \\
 \frac{\rightarrow \neg Pb, Qa, Pb}{Pb \supset Qa \rightarrow \neg Pb, Qa} \forall \text{Left} \\
 \frac{Pb \supset Qa \rightarrow \neg Pb, Qa}{\forall x(Px \supset Qa) \rightarrow \neg Pb, Qa} \forall \text{Right} \\
 \frac{\forall x(Px \supset Qa) \rightarrow \neg Pb, Qa}{\forall x(Px \supset Qa) \rightarrow \forall x\neg Px, Qa}
 \end{array}$$

2b.

$$\begin{array}{c}
 \text{Weakening Left} \frac{Pb \rightarrow Pb}{Qa, Pb \rightarrow Pb} \quad \frac{Qa \rightarrow Qa}{Qa \rightarrow Qa} \text{Weakening Left} \\
 \text{Exchange Left} \frac{Qa, Pb \rightarrow Pb}{Pb, Qa \rightarrow Pb} \quad \frac{Qa \rightarrow Qa}{Pb, Qa \rightarrow Qa} \wedge \text{Right} \\
 \frac{Pb, Qa \rightarrow Pb}{Pb, Qa \rightarrow Pb \wedge Qa} \exists \text{Right} \\
 \frac{Pb, Qa \rightarrow \exists x(Px \wedge Qa)}{\exists xPx, Qa \rightarrow \exists x(Px \wedge Qa)} \exists \text{Left}
 \end{array}$$