**Bonus Test**
Proof Theory, 6-4-2011
with solutions

**Exercise 1** LJ denotes the sequent calculus for intuitionistic logic.

Prove: if the sequent

$$\rightarrow \forall x \exists y A(x, y)$$

is provable in LJ, then there is a term $t(x)$ such that the sequent $\rightarrow \forall x A(x, t(x))$ is provable in LJ.

**Solution**: if the sequent in question has a proof, it has a cut-free proof by the cut-elimination theorem for LJ. If $P$ is a cut-free proof of $\rightarrow \forall x \exists y A(x, y)$ then the last step in that proof must have been either a $\forall$ right, or a Weakening right. In the second case, we have a proof of the empty sequent $\rightarrow$, and we can get the conclusion (for *any* term in the language) by Weakening right; in the first case, we have a proof of $\rightarrow \exists y A(x, y)$ and hence a term $t(x)$ such that $\rightarrow A(x, t(x))$ is provable; an application of $\forall$ right then gives the result,

**Exercise 2** We are given the propositional Kripke structure with underlying partial order $\{0 < 1\}$, and the following truth assignment:

$$\sigma_p = \{1\},\ \sigma_q = \sigma_r = \emptyset$$

Determine for which of the following formulas $\phi$ we have $0 \Vdash \phi[\sigma]$ (no proofs are required):
a. $\neg\neg p \supset p$
b. $((p \supset q) \supset p) \supset q$
c. $((p \wedge q) \supset r) \supset (p \supset (q \supset r))$
d. $((p \supset q) \supset r) \supset (p \supset (q \supset r))$

**Solution**:
a. No. $0 \Vdash \neg\neg p$ but $0 \nVdash p$.
b. No. Note that $0 \Vdash \neg q$; since $0 \Vdash \neg\neg p$, $0 \Vdash \neg(p \supset q)$, so $0 \Vdash (p \supset q) \supset p$. However, $0 \nVdash q$.
c. Yes. This formula is provable in LJ.
d. Yes. Since $0 \Vdash \neg q$, $0 \Vdash q \supset r$ hence $0 \Vdash p \supset (q \supset r)$ and $0 \Vdash ((p \supset q) \supset r) \supset (p \supset (q \supset r))$.