Bonus Test

Proof Theory, 18-5-2011

Exercise 1 Show that $I\Sigma_1$ proves that there are infinitely many prime numbers; i.e. the sentence

$$\forall x \exists y \, (y > x \wedge \mathrm{Prime}(y))$$

Solution: by Σ_1-induction one proves that for all x,

$$(1) \qquad \exists y \forall i \leq x (i > 0 \supset i|y)$$

and either x is prime or there is, by Δ_0-minimization, a least y such that $y > 1$ and $y|x$, which least y must be prime; hence

$$(2) \qquad \exists w (\mathrm{Prime}(w) \wedge w|x)$$

Now given x, take y as in (1); and then take w as in (2) for $y + 1$. Then w is a prime number $> x$.

Exercise 2 Let $G : \mathbb{N} \to \mathbb{N}$ be primitive recursive. Show that the function H, defined by

$$H(n) \;=\; \underbrace{G(\cdots(G}_{G(n)\text{ times}}(n)\cdots)$$

is primitive recursive too.

Solution: Let $K(n, y)$ be defined by

$$K(n, y) \;=\; \underbrace{G(\cdots(G}_{y\text{ times}}(n)\cdots)$$

Then $K(n, y)$ is primitive recursive, since $K(n, 0) = n$ and $K(n, y + 1) = G(K(n, y))$. Now $H(n) = K(n, G(n))$ so by composition, H is primitive recursive.