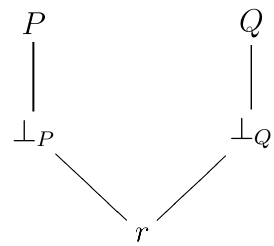
Exercises

Proof Theory, 06-04-2011

Exercise Let P and Q be partial orders with least elements $\bot_P$ and $\bot_Q$ respectively. We define the partial order $P \oplus Q$ as the disjoint union of P and Q with a new least element r added:



Suppose $\{D(p) \,|\, p \in P\}$, $\{D(q) \,|\, q \in Q\}$, $\{f_{pp'} \,|\, p \leq p' \text{ in } P\}$, $\{f_{qq'} \,|\, q \leq q' \text{ in } Q\}$, $\{[g]_p \,|\, g \in \mathcal{L}, p \in P\}$, etcetera, make P and Q into Kripke structures for a language $\mathcal{L}$.

Define: $D(r) = D(\bot_P) \times D(\bot_Q)$, $f_{r\bot_P}$ and $f_{r\bot_Q}$ are the first and second projections. For an n-ary function symbol g let

$$[g]_r((\xi_1, \eta_1), \ldots, (\xi_n, \eta_n)) = ([g]_{\bot_P}(\xi_1, \ldots, \xi_n), [g]_{\bot_Q}(\eta_1, \ldots, \eta_n))$$

and for an n-ary relation symbol R define: $((\xi_1, \eta_1), \ldots, (\xi_n, \eta_n)) \in [R]_r$ if and only if both $(\xi_1, \ldots, \xi_n) \in [R]_{\bot_P}$ and $(\eta_1, \ldots, \eta_n) \in [R]_{\bot_Q}$.

a. Show that this definition makes $P \oplus Q$ into a Kripke structure for $\mathcal{L}$.
b. Show that for every atomic formula $\phi(a_1, \ldots, a_n)$ and n-tuple $((\xi_1, \eta_1), \ldots, (\xi_n, \eta_n))$ of elements of $D(r)$, we have: $r \Vdash \phi[((\xi_1, \eta_1), \ldots, (\xi_n, \eta_n))]$ if and only if both $\bot_P \Vdash \phi[(\xi_1, \ldots, \xi_n)]$ and $\bot_Q \Vdash \phi[(\eta_1, \ldots, \eta_n)]$.
c. Show that the equivalence of part b extends to all Harrop formulas.
d. Use c to give a Kripke-model theoretic proof of the statement that if the sequent $\Gamma \to A \vee B$ is LJ-provable, where Γ consists of Harrop formulas, then at least one of the sequents $\Gamma \to A$, $\Gamma \to B$ is LJ-provable.
[Hint: use the completeness theorem for Kripke models]