Exercises

Proof Theory, 23-3-2011

Exercise 1 Show that one can identify the Skolem function for $\exists x A$ with the Herbrand function for $\forall x \neg A$, and the Skolem function for $\exists x \neg A$ with the Herbrand function for $\forall x A$ (compare the axioms).

Show also that, under this identification, we have:

$$A^H \;\leftrightarrow\; \neg(\neg A)^S$$

Exercise 2 Explain why the following proof illustrates the need for the negations convention on p. 51 of the notes (2nd paragraph of 2.5.3):

$$
\begin{array}{cl}
Pa \to Pa \qquad Pb \to Pb & \\
\hline
Pa, Pb \to Pa \wedge Pb & \wedge \text{ right} \\
\hline\hline
\forall x Px, \forall x Px \to Pa \wedge Pb & \forall \text{ left twice, Exch left} \\
\hline
\forall x Px \to Pa \wedge Pb & \text{Contr left} \\
\hline\hline
\forall x Px \to \exists y \exists v (Py \wedge Pv) & \exists \text{ right, twice} \\
\hline
\to \forall x Px \supset \exists y \exists v (Py \wedge Pv) & \supset \text{ right}
\end{array}
$$

Exercise 3 Prove the following form of Herbrand's theorem, given in Girard's book:

Let A be a prenex formula; e.g., $A = \exists x \forall y \exists z \forall t R(x, y, z, t)$ with R quantifier-free. Let f, g be two new function symbols, f unary, g binary.

Then A is provable in LK if and only if there exist terms $U_1, \ldots, U_n, W_1, \ldots, W_n$ (containing the function symbols $f.g$) such that the formula

$$R(U_1, f(U_1), W_1, g(U_1, W_1)) \vee \cdots \vee R(U_n, f(U_n), W_n, g(U_n, W_n))$$

is a propositional tautology (i.e. is provable using only the propositional inference rules)

Exercise 4 Consider the following proof of the 'paradox of the beer drinker' (there is a person such that, if he drinks beer then everybody drinks beer): $\exists y \forall x (\neg By \vee Bx)$:

$$
\begin{array}{c}
Ba \to Ba \\ \hline
\to Ba, \neg Ba \\ \hline
\to Ba, \neg Ba, Bb \\ \hline
\to Ba, \neg Ba \vee Bb \\ \hline
Ba, \forall x(\neg Ba \vee Bx) \\ \hline
\forall x(\neg Ba \vee Bx), Ba \\ \hline
\to \exists y \forall x(\neg By \vee Bx), Ba \\ \hline
\to \exists y \forall x(\neg By \vee Bx), Ba, \neg Bc \\ \hline\hline
\to \exists y \forall x(\neg By \vee Bx), \neg Bc \vee Ba \\ \hline
\to \exists y \forall x(\neg By \vee Bx), \forall x(\neg Bc \vee Bx) \\ \hline
\to \exists y \forall x(\neg By \vee Bx), \exists y \forall x(\neg By \vee Bx) \\ \hline
\to \exists y \forall x(\neg By \vee Bx)
\end{array}
$$

Apply the procedure of the general form of Herbrand's theorem to this proof. What is the strong $\vee$-expansion? What prenex normal form do you get?