Exam Proof Theory

June 15, 2011, 13.00–16.00

SOLUTIONS *All exercises were worth 16 points; the provisional grade was computed by the formula* $g = \frac{s+4}{10}$*, where* s *is the total number of points; then bonus points were added to obtain the final grade*

Problem 1:
Give a *complete* cut-free proof (that is: give every inference step) of the sequent

$$\neg\forall x \neg R(x) \ \to\ \exists x R(x)$$

where R is a unary predicate symbol.

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{R(a) \to R(a)}{R(a) \to \exists x R(x)}\ \exists\text{ right}}{\to \exists x A(x), \neg R(a)}\ \neg\text{ right}}{\to \exists x R(x), \forall x \neg R(x)}\ \forall\text{ right}}{\neg\forall x \neg R(x) \to \exists x R(x)}\ \neg\text{ left}$$

The top formula is an axiom because $R(a)$ is an atomic formula. If you started with an application of $\neg$ right, obtaining $\to R(a), \neg R(a)$, you'd need to insert Exchange right in order to be able to continue; failure to do so would cost 2 points. Students who mistakenly applied $\forall$ right first (not allowed, since there is still a free variable floating around), or otherwise produced a wrong proof, could get at most 8 points.

Problem 2:
Let ϕ be the sentence

$$\forall y(\exists v R(v) \supset \exists w(S(y,w) \vee \forall x T(x,w)))$$

where R, S, T are predicate symbols.

a) Give a prenex normal form for ϕ

b) Give a Skolemization of ϕ

c) Give a Herbrandization of ϕ

Solution: for instance

a) (5 points) $\forall y \forall v \exists w \forall x (R(v) \supset (S(y,w) \vee T(x,w)))$

b) (5 points) $\forall y \forall v \forall x (R(v) \supset (S(y, f(y,v)) \vee T(x, f(y,v))))$

c) (6 points) $\exists w (R(c) \supset (S(d,w) \vee T(g(w),w)))$

Problem 3:
Give a definition (by induction on ϕ) of the notions of a *positive subformula* and a *negative subformula* of a formula ϕ, such that the following statement is true:

> Whenever a sequent $A_1, \dots, A_s \to B_1, \dots, B_t$ appears in a cut-free proof with end-sequent $\Gamma \to \Delta$, then every A_i occurs as a positive subformula of some formula in Γ or as a negative subformula of a formula in Δ, and the same (with roles reversed) for the B_j.

[Hint: there is some subtlety required with the clauses for the quantifiers]

Solution: As formulated, this exercise also had a trivial solution: *every* subformula of ϕ is both positive and negative. The one student noting this was also smart enough to come up with the intended solution which is: define, by induction on ϕ, what the positive and negative subformulas of ϕ are:

> If ϕ is atomic, ϕ is a positive subformula of ϕ and ϕ does not have negative subformulas;
>
> if ϕ is $\psi \vee \chi$ or $\psi \wedge \chi$ then the positive subformulas of ϕ are ϕ itself or the positive subformulas of either ψ or χ; the negative subformulas of ϕ are the negative subformulas of either ψ or χ;
>
> if ϕ is $\psi \supset \chi$ then the positive subformulas of ϕ are ϕ itself, the positive subformulas of χ and the negative subformulas of ψ; the negative subformulas of ϕ are the negative subformulas of χ and the positive subformulas of ψ;
>
> if ϕ is $\neg\psi$ then the positive subformulas of ϕ are ϕ itself and the negative subformulas of ψ; the negative subformulas of ϕ are the positive subformulas of ψ

if ϕ is $\exists x\psi(x)$ or $\forall x\psi(x)$ then the positive subformulas of ϕ are ϕ itself and the positive subformulas of $\psi(t)$ for some term t; the negative subformulas of ϕ are the negative subformulas of $\psi(t)$ for some term t.

What was the 'subtlety'? This exercise was modelled after an exercise in Girard's book (p. 115). Girard's definition in the case of quantifiers is: if ϕ is $\forall x\psi(x)$ then a positive subformula of ϕ is ϕ itself or a positive subformula of $\psi(a)$ where a is a free variable; a negative subformula of ϕ is a negative subformula of $\psi(t)$ for some term t (suggesting that the definition for $\exists x\psi(x)$ is dual, with roles reversed). However, this is *wrong* as the following proof shows:

$$\dfrac{\dfrac{R(t) \to R(t)}{R(t) \to \exists x R(x)}}{\forall y R(y) \to \exists x R(x)}$$

where t is a term which is not a variable. Then the LHS occurrence of $R(t)$ in the axiom is neither a negative subformula of $\exists x R(x)$ nor (in Girard's definition) a positive subformula of $\forall y R(y)$.

So, no subtlety, really. Some students managed to reproduce Girard's mistake; no points were deducted.

Some students tried to define the notion 'ψ is a positive/negative subformula of ϕ' by induction on ψ instead of ϕ. This is plainly wrong, since the sign of a subformula is not determined by its shape but by *its place within the ambient formula*.

Problem 4:
Let $\mathcal{L}$ be a language with just two binary predicate symbols R, S. Let ϕ be a negative $\mathcal{L}$-formula (that is: ϕ does not contain $\exists$ or $\vee$).

Prove: if the sequent $\to \phi$ has an LK-proof, then the sequent

$$\forall xy(\neg\neg R(x,y) \supset R(x,y)), \forall xy(\neg\neg S(x,y) \supset S(x,y)) \to \phi$$

has an LJ-proof (i.e., an intuitionistic proof).

Solution: Since $\to \phi$ has an LK-proof, its negative translation $\to (\phi)^-$ has an LJ-proof. And because ϕ does not contain $\exists$ or $\vee$, $(\phi)^-$ is just ϕ with a double negation $\neg\neg$ added before every atomic formula, that is: before every appearance of the relation symbols R and S.

Now the antecedent of the given sequent implies the equivalence of $R(s,t)$ and $\neg\neg R(s,t)$ (and same for S) for any terms s, t; so from $(\phi)^-$ and the antecedent of the sequent one can prove ϕ.

Problem 5:
Let $\mathcal{L}$ be a language with one constant c, one unary function symbol f and two unary predicate symbols R and S. Consider the $\mathcal{L}$-sentences:

$$\begin{array}{ll} \phi_1 & R(c) \\ \phi_2 & \forall x(R(x) \supset R(f(x))) \\ \phi_3 & \forall x \neg(R(x) \wedge S(x)) \\ \phi_4 & \neg\forall x \neg S(x) \end{array}$$

Prove that the sequent $\phi_1, \phi_2, \phi_3, \phi_4 \rightarrow \exists x S(x)$ has no LJ-proof.

Solution: This could be done in three ways. The first method was to observe that the sentences $\phi_1, \ldots, \phi_4$ are Harrop formulas. Hence, if there were an LJ-proof $\phi_1, \ldots, \phi_4 \rightarrow \exists x S(x)$ then there would be an LJ-proof of $\phi_1, \ldots, \phi_4 \rightarrow S(t)$ for some closed term t. However, the only closed terms are $c, f(c), f(f(c)), \ldots$ and clearly $\phi_1, \ldots, \phi_4$ imply $\neg S(t)$ for such t. So the only way the desired LJ-proof could exist is that $\phi_1, \ldots, \phi_4$ is inconsistent; but clearly, it has a model.

The second method was by constructing a Kripke-countermodel. The simplest such is a model on the poset $0 < 1$, with $X_0 = \{a\}$, $X_1 = \{a, b\}$; with $X_0 \rightarrow X_1$ the inclusion. Define $R_0 = R_1 = \{a\}$, $S_0 = \emptyset$, $S_1 = \{b\}$, let a be the interpretation of c, and interpret f by the function which has value a on every argument.

The third method was by arguing directly about a possible cut-free LJ-proof of $\phi_1, \ldots, \phi_4 \rightarrow \exists x S(x)$. This is possible because $S(x)$ is an atomic formula. However, there are many cases to consider and it is hard to make such a proof rigorous. I omit details.

Problem 6:
Recall that we showed in the course that there is a Δ_0-formula $\phi(x,y)$ of arithmetic for which the following holds:

$$\mathbb{N} \models \phi(n,m) \;\Leftrightarrow\; m = 2^n$$

for arbitrary $n, m \in \mathbb{N}$.

Which of the theories $\mathrm{I}\Delta_0$, $\mathrm{I}\Sigma_1$, $\mathrm{I}\Pi_1$ prove the sentence $\forall x \exists y \phi(x,y)$? Explain your answer.

Solution: $I\Delta_0$ does *not* prove the sentence; otherwise, by Parikh's theorem, it would prove $\forall x \exists y \leq t(x) \phi(x, y)$ for some term $t(x)$ of arithmetic. Then the function 2^x would be bounded by a polynomial; contradiction. This was worth 5 points.

$I\Sigma_1$ proves the sentence. In the lecture we proved that in $I\Sigma_1$ all primitive recursive functions are total, so we only have to see that 2^x is primitive recursive. But $2^0 = 1$ and $2^{x+1} = 2(2^x)$, so this is clear. This part gave 6 points.

$I\Pi_1$ also proves the sentence. This follows from the previous case and the observation, proved in the lecture, that $I\Pi_1 = I\Sigma_1$. This was worth 5 points.