PROOF THEORY

TUTORIAL SESSION 1 — SOME SOLUTIONS

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EXERCISE 1 — Frege Proofs

EXERCISE 1.A — Implicational Fragment

(i) $p \supset p$, call this **Id**

(ii) $q \supset p \supset p$, call this $\mathbf{K}^*$

(iii) $q \supset (p \supset q \supset r) \supset p \supset r$

(iv) $p \supset (p \supset q) \supset q$

Proof of (i):

1.	$(p \supset p \supset p) \supset (p \supset (p \supset p) \supset p) \supset (p \supset p)$	$\mathbf{S}^*$
2.	$p \supset p \supset p$	**K**
3.	$(p \supset (p \supset p) \supset p) \supset (p \supset p)$	*modus ponens* on 1 and 2
4.	$p \supset (p \supset p) \supset p$	**K**
5.	$p \supset p$	*modus ponens* on 3 and 4

Proof of (ii)

1.	$(p \supset p) \supset q \supset p \supset p$	**K**
2.	$p \supset p$	**Id**
3.	$q \supset p \supset p$	*modus ponens* on 1 and 2

Proof of (iii)

1.	$(q \supset p \supset q) \supset (q \supset (p \supset q) \supset (p \supset q \supset r) \supset p \supset r) \supset q \supset (p \supset q \supset r) \supset p \supset r$	$\mathbf{S}^*$
2.	$q \supset p \supset q$	**K**
3.	$(q \supset (p \supset q) \supset (p \supset q \supset r) \supset p \supset r) \supset q \supset (p \supset q \supset r) \supset p \supset r$	*modus ponens* on 1 and 2
4.	$((p \supset q) \supset (p \supset q \supset q) \supset p \supset r) \supset q \supset (p \supset q) \supset (p \supset q \supset q) \supset p \supset r$	**K**
5.	$(p \supset q) \supset (p \supset q \supset q) \supset p \supset r$	$\mathbf{S}^*$
6.	$q \supset (p \supset q) \supset (p \supset q \supset q) \supset p \supset r$	*modus ponens* on 4 and 5
7.	$q \supset (p \supset q \supset r) \supset p \supset r$	*modus ponens* on 3 and 6

Proof of (iv)

1.	$(p \supset (p \supset q) \supset p \supset q) \supset (p \supset ((p \supset q) \supset p \supset q) \supset (p \supset q) \supset q) \supset p \supset (p \supset q) \supset q$	$\mathbf{S}^*$
2.	$p \supset (p \supset q) \supset p \supset q$	$\mathbf{K}^*$
3.	$(p \supset ((p \supset q) \supset p \supset q) \supset (p \supset q) \supset q) \supset p \supset (p \supset q) \supset q$	*modus ponens* on 1 and 2
4.	$p \supset ((p \supset q) \supset p \supset q) \supset (p \supset q) \supset q$	proof (iii)
5.	$p \supset (p \supset q) \supset q$	*modus ponens* on 4 and 5

EXERCISE I.B — Some Laws

(i) $(r \supset a) \supset (r \supset b) \supset r \supset (a \wedge b)$

(ii) $(p \supset a) \supset (q \supset b) \supset (p \wedge q) \supset (a \wedge b)$

(iii) $(p \wedge q) \vee r \supset (p \vee r) \wedge (q \vee r)$;

Proof of (iii):

1 .	$p \wedge q \supset p$	**Pl**
2 .	$p \supset p \vee r$	**Il**
3 .	$(p \wedge q \supset p) \supset (p \supset p \vee r) \supset p \wedge q \supset p \vee r$	**C**
4 .	$(p \supset p \vee r) \supset p \wedge q \supset p \vee r$	*modus ponens* on 3 and 1
5 .	$p \wedge q \supset p \vee r$	*modus ponens* on 4 and 2
6 .	$p \wedge q \supset q$	**Pr**
7 .	$q \supset q \vee r$	**Il**
8 .	$(p \wedge q \supset q) \supset (q \supset q \vee r) \supset p \wedge q \supset q \vee r$	**C**
9 .	$(q \supset q \vee r) \supset p \wedge q \supset q \vee r$	*modus ponens* on 8 and 6
10.	$p \wedge q \supset q \vee r$	*modus ponens* on 9 and 7
11.	$(p \supset p \vee r) \supset (q \supset q \vee r) \supset (p \wedge q) \supset (p \vee r) \wedge (q \vee r)$	proof (ii)
12.	$(q \supset q \vee r) \supset (p \wedge q) \supset (p \vee r) \wedge (q \vee r)$	*modus ponens* on 11 and 6
13.	$(p \wedge q) \supset (p \vee r) \wedge (q \vee r)$	*modus ponens* on 12 and 10
14.	$(r \supset p \vee r)$	**Ir**
15.	$(r \supset q \vee r)$	**Ir**
16.	$(r \supset p \vee r) \supset (r \supset q \vee r) \supset r \supset (p \vee r) \wedge (q \vee r)$	proof (i)
17.	$(r \supset q \vee r) \supset r \supset (p \vee r) \wedge (q \vee r)$	*modus ponens* on 16 and 14
18.	$r \supset (p \vee r) \wedge (q \vee r)$	*modus ponens* on 17 and 15
19.	$((p \wedge q) \supset (p \vee r) \wedge (q \vee r)) \supset (r \supset (p \vee r) \wedge (q \vee r)) \supset (p \wedge q) \vee r \supset (p \vee r) \wedge (q \vee r)$	**I**
20.	$(r \supset (p \vee r) \wedge (q \vee r)) \supset (p \wedge q) \vee r \supset (p \vee r) \wedge (q \vee r)$	*modus ponens* on 19 and 13
21.	$(p \wedge q) \vee r \supset (p \vee r) \wedge (q \vee r)$	*modus ponens* on 20 and 18