PROOF THEORY
TUTORIAL SESSION 1

Jeroen Goudsmit

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EXERCISE I — Frege Proofs

Axioms

K : $p \supset (q \supset p)$	**S*** : $(p \supset q) \supset (p \supset q \supset r) \supset (p \supset r)$
Il : $p \supset (p \vee q)$	**Ir** : $(q \supset (p \vee q))$
Pl : $(p \wedge q) \supset p$	**Pr** : $(p \wedge q) \supset q$
I : $(p \supset r) \supset (q \supset r) \supset (p \vee q \supset r)$	**P** : $p \supset q \supset p \wedge q$
Ni : $(p \supset q) \supset (p \supset \neg q) \supset \neg p$	**Ne** : $\neg\neg p \supset p$

Inference Rules

$$\frac{p \qquad p \supset q}{q}\ \textit{modus ponens}$$

Figure 1: Frege Proof System for Propositional Classical Logic

In the lecture we discussed the proof system depicted in Figure 1. For convenience, all axioms are given a label. These labels correspond to the names from combinatorial logic. This exercise is about constructing proofs in this system.

EXERCISE I.A — Implicational Fragment

Prove the following, within the Frege proof system of Figure 1. Do *not* use the Deduction Theorem, but construct the proofs by hand.

(i) $p \supset p$, call this **Id**

(ii) $q \supset p \supset p$, call this **K***

(iii) $q \supset (p \supset q \supset r) \supset p \supset r$

(iv) $p \supset (p \supset q) \supset q$

EXERCISE 1.B — Some Laws

The Deduction Theorem is a useful tool in proving the existence of proofs in the Frege proof system. Recall that it is formulated as below.

Theorem 1 (Deduction Theorem). *Suppose there is a proof of q under the assumptions Γ, p. Now there also exists a proof of $p \supset q$ under the assumptions Γ.*

Proof. The proof proceeds by induction on the structure of the assumed proof. In the base case we have a formula ϕ, which is either in Γ or it is p. If it is in Γ, the proof "*modus ponens* on $\phi \supset p \supset \phi$ (an instance of **K**) and ϕ" satisfies our demands. Otherwise, the proof **Id** of Exercise 1.1.(i) does.

For the induction step, we have a proof of "*modus ponens* of $\phi \supset \psi$ on ϕ". By induction we know of proofs of $p \supset \phi$ and $p \supset \phi \supset \psi$. The desired proof is obtained by two applications of *modus ponens* and the previously constructed proof on $(p \supset \phi) \supset (p \supset \phi \supset \psi) \supset p \supset \psi$, which is an instance of $\mathbf{S}^*$. □

Lemma 1. *There is a proof* **C** *of $(p \supset q) \supset (q \supset r) \supset p \supset r$ in the Frege system of Figure 1.*

Proof. Below we construct a proof of $p \supset r$, under the assumptions $p \supset q$ and $q \supset$.

1.	$p \supset q$	by assumption
2.	$(q \supset r) \supset p \supset q \supset r$	**K**
3.	$q \supset r$	by assumption
4.	$p \supset (q \supset r)$	*modus ponens* on 2 and 3
5.	$(p \supset q) \supset (p \supset q \supset r) \supset p \supset r$	$\mathbf{S}^*$
6.	$(p \supset q \supset r) \supset p \supset r$	*modus ponens* on 5 and 1
7.	$p \supset r$	*modus ponens* on 6 and 4

Now using the Deduction Theorem, we know of a proof $(q \supset r) \supset p \supset r$ under the assumption of $p \supset q$. Another application of the deduction theorem yields the desired. □

Prove the following laws, using the Deduction Theorem.

(i) $(r \supset a) \supset (r \supset b) \supset r \supset (a \wedge b)$

(ii) $(p \supset a) \supset (q \supset b) \supset (p \wedge q) \supset (a \wedge b)$

(iii) $(p \wedge q) \vee r \supset (p \vee r) \wedge (q \vee r)$

(iv) $(p \supset q \supset r) \supset (p \wedge q) \supset r$

(v) $((p \wedge q) \supset r) \supset p \supset q \supset r$

(vi) $(q \supset (p \wedge \neg p)) \supset \neg q$

(vii) $\neg q \supset q \supset (p \wedge \neg p)$

(viii) $p \supset \neg\neg p$

The laws (i) and (ii) describe how to form implications into a conjunction. Can you think of laws about implications starting at a disjunction?