PROOF THEORY
TUTORIAL SESSION 2

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Axioms

$$\frac{}{p \rightarrow p}\ \textsc{axiom}$$

Structural Rules

$$\frac{\Gamma, A, B, \Pi \rightarrow \Delta}{\Gamma, B, A, \Pi \rightarrow \Delta}\ \textsc{exchange}\mathrm{L} \qquad \frac{\Gamma \rightarrow \Delta, A, B, \Lambda}{\Gamma \rightarrow \Delta, B, A, \Lambda}\ \textsc{exchange}\mathrm{R}$$

$$\frac{A, A, \Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta}\ \textsc{contraction}\mathrm{L} \qquad \frac{\Gamma \rightarrow \Delta, A, A}{\Gamma \rightarrow \Delta, A}\ \textsc{contraction}\mathrm{R}$$

$$\frac{\Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta}\ \textsc{weakening}\mathrm{L} \qquad \frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A}\ \textsc{weakening}\mathrm{R}$$

Propositional Rules

$$\frac{\Gamma \rightarrow \Delta, A \qquad B, \Gamma \rightarrow \Delta}{A \supset B, \Gamma \rightarrow \Delta}\ {\supset}L \qquad \frac{A, \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \supset B}\ {\supset}R$$

$$\frac{A, B, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta}\ {\wedge}L \qquad \frac{\Gamma \rightarrow \Delta, A \qquad \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \wedge B}\ {\wedge}R$$

$$\frac{A, \Gamma \rightarrow \Delta \qquad B, \Gamma \rightarrow \Delta}{A \vee B, \Gamma \rightarrow \Delta}\ {\vee}L \qquad \frac{\Gamma \rightarrow \Delta, A, B}{\Gamma \rightarrow \Delta, A \vee B}\ {\vee}L$$

$$\frac{\Gamma \rightarrow \Delta, A}{\neg A, \Gamma \rightarrow \Delta}\ {\neg}L \qquad \frac{A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg A}\ {\neg}R$$

Cut Rule

$$\frac{\Gamma \rightarrow \Delta, A \qquad A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}\ \textsc{cut}$$

Figure 1: Sequent Calculus for Propositional Classical Logic

EXERCISE I — Sequent Calculus

In the lecture we discussed Gentzen's sequent calculus as depicted in Figure 1.

EXERCISE I.A — Elementary Formulae

Prove the following statements in this system.

(i) $\phi \supset \psi \supset (\phi \wedge \psi)$;

(ii) $(\chi \supset \phi) \supset (\chi \supset \psi) \supset (\chi \supset \phi \wedge \psi)$;

(iii) $(\phi \supset \alpha) \supset (\psi \supset \beta) \supset (\phi \wedge \psi) \supset (\alpha \wedge \beta)$;

(iv) $(\phi \supset \chi) \supset (\psi \supset \chi) \supset (\phi \vee \psi) \supset \chi$;

(v) $(\phi \supset \alpha) \supset (\psi \supset \beta) \supset (\phi \vee \psi) \supset (\alpha \vee \beta)$;

(vi) $(\phi \supset \psi \supset \chi) \supset (\phi \wedge \psi) \supset \chi$;

(vii) $\chi \vee (\phi \wedge \psi) \supset (\chi \wedge \phi) \vee (\chi \wedge \psi)$;

(viii) $\phi \supset \neg\neg\phi$;

(ix) $\neg(\phi \wedge \psi) \supset \neg\phi \vee \neg\psi$;

(x) $\neg(\phi \vee \psi) \supset \neg\phi \wedge \neg\psi$;

(xi) $\neg\neg\phi \supset \phi$;

(xii) $(\neg\psi \supset \neg\phi) \supset \phi \supset \psi$;

(xiii) $\phi \vee \neg\phi$;

(xiv) $\phi \wedge \neg\phi \supset \psi$.

A proof of (xiii) can be found in Gentzen (1935, p. 13).

EXERCISE I.B — Cut Elimintation

In the proof of cut elimination we treated the case of $\neg$ and $\wedge$, but neglected $\vee$ and $\supset$. Also, we only treated the easier left-hand side for conjunction. Prove all cases.

Example 1 (The Left-Hand Case of $\supset$). Suppose that $A \supset B$ is contained in Γ, and that the amount of connectives in the sequent $\Gamma \to \Delta$ is m. Let Γ' be Γ without occurrences of this formula. Realize that from the sequent $A \supset B, \Gamma' \to \Delta$ one can derive $\Gamma \to \Delta$ using only weak inferences. Moreover, from the sequents $\Gamma' \to \Delta, A$ and $B, \Gamma' \to \Delta$ one can prove $A \supset B, \Gamma' \to \Delta$ using the right-hand implication rule. Hence we know, by the inversion theorem and the assumption that $\Gamma \to \Delta$ is provable, that both these sequents are provable as well. It is clear that the amount of connectives in these sequents is at most $m - 1$.

By induction we now know of proofs L and R of $\Gamma' \to \Delta, A$ and $B, \Gamma' \to \Delta$ respectively, both without any use of CUT and containing fewer than 2^{m-1} strong inferences. Now consider the following derivation.

$$\cfrac{\cfrac{\cfrac{\mathsf{L}}{\Gamma' \to \Delta, A} \qquad \cfrac{\mathsf{R}}{B, \Gamma' \to \Delta}}{A \supset B, \Gamma' \to \Delta}\supset L}{\Gamma \to \Delta} \tag{1}$$

It is clear that (1) is a proof. Moreover, as L and R contain at most $2^{m-1} - 1$ strong inferences, we know that this entire proof contains at most $2 \cdot (2^{m-1} - 1) + 1 = 2^m - 1$ strong inferences.

EXERCISE I.C — Sharing

Consider the rule $\wedge$R and see that the *context* Γ is 'duplicated' to both premisses. The rules $\wedge$R, $\vee$L and $\supset$ L all have this same structure. Troelstra and Schwichtenberg (1996) say that these types of rules are *context-sharing*. A *context-indendent* (or *non-sharing*) version of the former rule would be the following.

$$\frac{\Gamma \rightarrow \Delta, A \qquad \Pi \rightarrow \Lambda, B}{\Gamma, \Pi \rightarrow \Delta, \Lambda, A \wedge B} \wedge \text{Rp} \tag{2}$$

Construct context-independent variants of the other context-dependent rules in the system. Also prove that the system obtained by replacing the context-dependent rules with their context-independent counterparts is equivalent in power to the system of Figure 1. A way to do this is by showing how the new and old rules can be derived from one another. By example, the 'old' right-conjunction rule is proven using the new conjunction rule of (2) as follows.

$$\cfrac{\cfrac{\cfrac{\Gamma \rightarrow \Delta, A \qquad \Gamma \rightarrow \Delta, B}{\Gamma, \Gamma \rightarrow \Delta, \Delta, A \wedge B} \wedge \text{Rp}}{\Gamma \rightarrow \Delta, \Delta, A \wedge B} \text{CONTRACTIONL}}{\Gamma \rightarrow \Delta, A \wedge B} \text{CONTRACTIONR}$$

More details are given by Troelstra and Schwichtenberg (1996, p. 55).

References

Gentzen, Gerhard (1935). "Untersuchungen über das logische Schließen." In: *Mathematische Zeitschrift* 39 (1). 10.1007/BF01201353, pages 176–210. ISSN: 0025-5874. URL: http://dx.doi.org/10.1007/BF01201353.

Troelstra, Anne Sjerp and Helmut Schwichtenberg (1996). *Basic Proof Theory*. Volume 43. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press. ISBN: 0-521-77911-1.