PROOF THEORY
TUTORIAL SESSION 3

Jeroen Goudsmit

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In the third lecture we discussed first order logic, as covered by Buss (1998) in Section 2.1 and 2.3. Below a brief recapitulation and a couple of exercises. Some of the formulations below are adapted from van Oosten and Moerdijk (2009).[1]

EXERCISE 1 — First Order Logic

We are concerned with expressions in a first order language. Such a language contains a set of *function symbols* of a given arity, and a set of *relation symbols* of a given arity. One can think of a nullary function symbol (a function symbol with arity zero) as a *constant symbol*.

Definition 1 (Language). A *language* L is a pair $\langle \mathrm{fun}(L), \mathrm{rel}(L) \rangle$, where $\mathrm{fun}(L)$ is the set of function symbols and $\mathrm{rel}(L)$ is the set of relation symbols of L. To each function symbol f and relation symbol R there is a natural number called its *arity*, written respectively as arity f and arity R.

Definition 2 (Terms and Formulae). Let L be a language and let **V** be a countably infinite set of variables. The set of *terms* over L, denoted by $\mathcal{T}(L)$, is defined inductively as follows.[2]

$\mathcal{T}(L) \ni f(t_1, \ldots, t_n)$	if f is a function symbol with arity n and $t_1, \ldots, t_n$ are terms
$\mathcal{T}(L) \ni x$	if x is a variable

The set of *formulae*, denoted by $\mathcal{L}(L)$, is defined inductively as below.

$\mathcal{L}(L) \ni t = s$	if t and s are terms
$\mathcal{L}(L) \ni R(t_1, \ldots, t_n)$	if f is a relation symbol with arity n and $t_1, \ldots, t_n$ are terms
$\mathcal{L}(L) \ni \phi \mathsf{C} \psi$	if ϕ and ψ are formulae and $\mathsf{C} = \wedge, \vee, \supset$
$\mathcal{L}(L) \ni \neg\phi$	if ϕ is a formula
$\mathcal{L}(L) \ni \mathsf{Q}\, \phi(x)$	if ϕ is a formula, x is a variable and $\mathsf{Q} = \forall x, \exists x$

Intuitively, placing a quantifier $\forall x$ in front of a formula ϕ "binds" the variable x in the thusly created expression. All variables which occur in ϕ that are not bound in such a way are called *free*, and we can define this formally below.

[1] In particular, some of the exercises below are taken directly from this text.

[2] Do note that this definition includes the equality relation within the definition of formulae. This is not the case in Buss (1998), but one can simply include the relation symbol $=$ and add the proper axioms to each theory. See van Oosten and Moerdijk (2009) for a hint as to what these axioms ought to be.

Definition 3 (Free Variables). Given a formula ϕ over the language L, we define the set of *free variables* as below, proceeding by induction on the structure of both terms and formulae.

$$
\begin{aligned}
\mathrm{FV}(x) &:= x \\
\mathrm{FV}(f(t_1,\dots,t_n)) &:= \textstyle\bigcup_{i=1}^{n} \mathrm{FV}(t_n) \\
\mathrm{FV}(t = s) &:= \mathrm{FV}(t) \cup \mathrm{FV}(s) \\
\mathrm{FV}(R(t_1,\dots,t_n)) &:= \textstyle\bigcup_{i=1}^{n} \mathrm{FV}(t_n) \\
\mathrm{FV}(\phi \mathsf{C} \psi) &:= \mathrm{FV}(\phi) \cup \mathrm{FV}(\psi) \quad \text{where } \mathsf{C} = \wedge, \vee, \supset \\
\mathrm{FV}(\neg\phi) &:= \mathrm{FV}(\phi) \\
\mathrm{FV}(\mathsf{Q}\,\phi) &:= \mathrm{FV}(\phi) - \{x\} \qquad \text{where } \mathsf{Q} = \forall x, \exists x
\end{aligned}
$$

A variable x which occurs in ϕ but is not free is said to be a *bound variable.* If ϕ contains no free variables, we say that ϕ is a *sentence.*

Buss (1998, Section 2.3.1) explains that one can define terms in a different manner, encoding the fact whether a variable is bound within the very syntax.

In order to give meaning to a formula we develop a system of semantics. In this classical case the type of meaning we wish to assign to a sentence is that of truth; a sentence is either true of false. To be able to interpret the terms, we need a structure which reflects the syntactic structure of the terms of the language.

Definition 4 (Structure). Let L be a language. A *structure* $\mathcal{M}$ over L is a triple $\langle M, F, R\rangle$ where

(i) M is a non-empty set;

(ii) F is a family of functions to M indexed by $\mathrm{fun}(L)$ such that $F_f : M^{\mathrm{arity}\, f} \to M$ for each function symbol f;

(iii) R is a family of relations indexed by $\mathrm{rel}(L)$ such that $R_P \subseteq M^{\mathrm{arity}\, P}$ for each relation symbol P.

We write $f^{\mathcal{M}}$ for F_f and $P^{\mathcal{M}}$ for R_P for function symbols f and relation symbols P. We often will write $\mathcal{M}$ to mean the underlying set M.

In order to interpret terms, one needs to assign a value to the variables occurring within this term. So to interpret a given term t in the structure $\mathcal{M}$ it suffices to have a partial map from the set of variables to $\mathcal{M}$, defined on the variables occurring in t. We call such a map an *object assignment* of L on $\mathcal{M}$. Given an object assignment $\sigma : \mathbf{V} \to \mathcal{M}$, a variable x and $m \in \mathcal{M}$ we define $\sigma(m/x)$ as the map which sends x to m and which behaves exactly as σ on all other variables.

Definition 5 (Interpretation). Let L be a language, let $\mathcal{M}$ be a structure over L and let σ be an object assignment of L on $\mathcal{M}$. We define the interpretation of a term t of L in M under σ, written as $t^{\mathcal{M}}[\sigma]$, inductively as below.

$$
\begin{aligned}
x^{\mathcal{M}}[\sigma] &:= \sigma(x) \\
f(t_1,\dots,t_n)^{\mathcal{M}}[\sigma] &:= f^{\mathcal{M}}\big({t_1}^{\mathcal{M}}[\sigma],\dots,{t_n}^{\mathcal{M}}[\sigma]\big)
\end{aligned}
$$

Equipped with the above interpretation of terms we can inductively define the interpretation of a formula ϕ of L in $\mathcal{M}$ under the object assignment σ, denoted $\mathcal{M} \models \phi[\sigma]$.

$$
\begin{aligned}
\mathcal{M} \models t = s[\sigma] &\text{ if and only if } t^{\mathcal{M}}[\sigma] = s^{\mathcal{M}}[\sigma] \\
\mathcal{M} \models P(t_1,\dots,t_n)[\sigma] &\text{ if and only if } \big({t_1}^{\mathcal{M}}[\sigma],\dots,{t_n}^{\mathcal{M}}[\sigma]\big) \in P^{\mathcal{M}} \\
\mathcal{M} \models \phi \wedge \psi[\sigma] &\text{ if and only if } \mathcal{M} \models \phi[\sigma] \text{ and } \mathcal{M} \models \phi[\sigma] \\
\mathcal{M} \models \phi \vee \psi[\sigma] &\text{ if and only if } \mathcal{M} \models \phi[\sigma] \text{ or } \mathcal{M} \models \phi[\sigma] \\
\mathcal{M} \models \phi \supset \psi[\sigma] &\text{ if and only if } \mathcal{M} \models \phi[\sigma] \text{ implies } \mathcal{M} \models \phi[\sigma] \\
\mathcal{M} \models \neg\phi[\sigma] &\text{ if and only if } \mathcal{M} \models \phi[\sigma] \text{ is not true} \\
\mathcal{M} \models \forall x\, \phi[\sigma] &\text{ if and only if } \mathcal{M} \models \phi[\sigma(m/x)] \text{ holds for all } m \in \mathcal{M} \\
\mathcal{M} \models \exists x\, \phi[\sigma] &\text{ if and only if } \mathcal{M} \models \phi[\sigma(m/x)] \text{ holds for some } m \in \mathcal{M}
\end{aligned}
$$

We now say that ϕ is *valid* in $\mathcal{M}$, written as $\mathcal{M} \models \phi$, if $\mathcal{M} \models \phi[\sigma]$ is true for all σ. Finally, ϕ is said to be *true* when ϕ is valid in all structures. Given a set of sentences Γ, we say that $\mathcal{M}$ is a model for Γ whenever $\mathcal{M} \models \gamma$ for all $\gamma \in \Gamma$.

In Exercise 1.1, 1.3, 1.4 and 1.5 we make use of the binary relation symbol of equality. It is implicitly understood that all theories there include the proper axioms for equality.

EXERCISE I.A — Rationals

Consider the language L which has the binary function symbol $\cdot$, written for convenience in infix form. There are the obvious L-structures $\mathbb{Z}$ and $\mathbb{Q}$, where $\cdot$ is interpreted as usual multiplication. Give formulae $\phi_0(x)$ and $\phi_1(x)$ such that $S \models \phi_i(x)$ holds exactly if $x = i$ for $S = \mathbb{Z}, \mathbb{Q}$.[3] Find a sentence which is true in $\mathbb{Z}$ but not in $\mathbb{Q}$.

EXERCISE I.B — Contraction- and Cut Free

Can you find a contraction- and cut-free proof of the sequent $\exists x\, A(x) \to B$ from the sequent $A(a), A(b) \to B$? If not, why?

EXERCISE I.C — Partially Ordered Sets

Let L be the language of posets, which has the binary relation symbol $\leq$. Write down the axioms for reflexivity, transitivity and anti-symmetry in this language. Realize that $x < y$ holds precisely if $x \leq y$ and $x \neq y$. Consider the language L' which extends L by the symbol for strict inequality $<$. Define the proper axiom(s). Prove that this new theory does not entail density, i.e. show that the following formula is not provable.

$$\forall x\, \forall y\, \big(x \leq y \supset \exists z (x \leq z \wedge z \leq z)\big).$$

EXERCISE I.D — Groups

Let L be the language which has one nullary function symbol e and a binary function symbol $\cdot$. Each group can easily be made into an L-structure. Write down the axioms for a group in L, and call this theory Γ. Make sure that G is a group precisely if the corresponding L-structure is a model of Γ. (i) Define a theory such that only the infinite groups are models of this theory. (ii) Find a theory whose only models are abelian groups.

EXERCISE I.E — Commutative (Unitary) Rings

Let L be the language of rings, which has the nullary function symbols 0 and 1 and the binary function symbols $+$ and $\cdot$. Each ring can be made into a L structure. (i) Find a theory Γ such that the models of this theory are precisely the commutative unitary rings. (ii) Find a formula ϕ such that $R \models \Gamma, \phi$ precisely if R is a local ring. (iii) Extend the language of L to L' by adding an unary relation symbol I. As a consequence, I characterizes a subset of any L'-structure. Find the theory $\Gamma \subseteq \Delta$ needed to ensure that I characterizes an ideal of the Δ-models. (iv) Find a theory $\Gamma \subseteq \Pi$ such that Π-models make I into a prime ideal. (v) Find a theory $\Gamma \subseteq \Lambda$ such that Λ-models make I into a maximal ideal.

References

Buss, Samuel R. (1998). “Handbook of Proof Theory”. In: edited by Samuel R. Buss. Volume 137. Studies in Logic and the Foundations of Mathematics. Elsevier. Chapter Introduction to Proof Theory, pages 1–78.
Van Oosten, Jaap and Ieke Moerdijk (2009). *Sets, Models and Proofs*. URL: http://www.staff.science.uu.nl/~ooste110/syllabi/setsproofs09.pdf.

[3] By $\phi(x)$ we mean that the set of free variables of ϕ is contained in $\{x\}$.