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Krivine's Classical Realizability from a Categorical Perspective

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In a sequence of papers (Krivine 2001; Krivine 2003; Krivine 2009) J.-L. Krivine has introduced his notion of *Classical Realizability* for classical second order logic and Zermelo-Fraenkel set theory. Moreover, in more recent work (Krivine 2008) he has considered forcing constructions on top of it with the ultimate aim of providing a realizability interpretation for the axiom of choice.
The aim of this paper is to show how Krivine's classical realizability can be understood as an instance of the categorical approach to realizability as started by Hyland in (Hyland 1982) and described in detail in (van Oosten 2008). Moreover, we will give an intuitive explanation of the iteration of realizability as can be found in (Krivine 2008).

1. Introduction

It is more or less straightforward how to interpret intuitionistic second order logic in a realizability model based on closed λ-terms. This was studied in detail by J.-L. Krivine and M. Parigot in the late 1980s, see (Krivine & Parigot 1990; Krivine 1990a). Around 1990 due to the seminal paper (Griffin 1990) it got clear to many researchers how to give a proof term assignment for classical logic via a λ-calculus with control operators which serve as realizers for classical principles like *reductio ad absurdum* or *Peirce's law*, see e.g. (Streicher & Reus 1998). Krivine was one of the first taking up Griffin's suggestion in his work on so-called "storage operators" (Krivine 1990b). Then beginning with his address (Krivine 2001) to the Logic Colloquium 2000 in Paris in a sequence of papers (Krivine 2003; Krivine 2009) Krivine developed his theory of *Classical Realizability* for extensions of classical second order logic and Zermelo-Fraenkel set theory. In more recent yet unpublished work (Krivine 2008) Krivine has embarked on the long term project of providing a realizability interpretation for full ZFC, i.e. Zermelo-Fraenkel set theory with the full Axiom of Choice. This is to be achieved by considering forcing interpretations inside classical realizability models. In (Krivine 2008) he has shown how to contract this two step model construction into one step.

Reading through Krivine's papers introducing classical realizability one gets the impression that his account is highly original (which it definitely is!) and does not see at all how it may fit into the structural (i.e. semantic) approach to realizability as initiated by M. Hyland in (Hyland 1982) and fully described in (van Oosten 2008). In particular, it is most puzzling that Krivine considers his classical realizability as "generalised forcing" because toposes of the form $\mathsf{Sh}(B)$ for a complete boolean algebra B are cocomplete whereas cocomplete realizability toposes are necessarily equivalent to **Set**. In order to clear up this confusion we introduce a notion of "abstract Krivine structure" (aks) and show how to construct a classical realizability model for each such aks. Moreover, we characterise those aks's which correspond to forcing over meet semilattices with a distinguished set of truth values. Next we show how any aks $\mathbb{A}$ gives rise to an *order combinatory algebra* (oca) with a filter of distinguished truth values which induces a tripos (see (van Oosten 2008) for explanation of these notions) which also gives rise to a model of ZF.

A pleasant aspect of triposes is that they give rise to a conceptually clear account of iteration of model constructions which is also explained in (van Oosten 2008). This framework we use for explaining the iterated model construction of (Krivine 2008).

2. A Recap of Krivine's Classical Realizability

In classical realizability as described e.g. in (Krivine 2009) one considers as realizers closed terms in an extension of the untyped λ-calculus. For realizing classical logic one needs at least the control operator cc.

Possibly open terms of this kind are given by the following grammar

$$t ::= x \mid \lambda x.t \mid ts \mid \mathsf{cc}\,t \mid \mathsf{k}_\pi$$

where π ranges over list or stacks (fr. *pile*) of terms and k is a constant turning stacks π into terms k_π. We write Λ for the set of closed terms and Π for the set of stacks of closed terms. A *process* is a pair $t \star \pi$ of a term and a stack. We write $\Lambda \star \Pi$ for the set of processes. On processes the relation $\succ$ of *head reduction* is defined inductively as follows

(pop)	$\lambda x.t * s.\pi$	$\succ$	$t[s/x] * \pi$
(push)	$ts * \pi$	$\succ$	$t * s.\pi$
(store)	$\mathsf{cc}\,t * \pi$	$\succ$	$t * \mathsf{k}_\pi.\pi$
(restore)	$\mathsf{k}_\pi * t.\pi'$	$\succ$	$t * \pi$

We write $\succeq$ for the reflexive transitive closure of $\succ$. The first two clauses allow one to compute weak head normal forms of λ-terms and constitute the core of Krivine's abstract machine (see (Streicher & Reus 1998) for background information). The remaining rules tell how to evaluate calls of the control operator cc and terms of the form k_π. Obviously cc is the control operator "call with current continuation" since in order to evaluate $\mathsf{cc}\,t$ one applies t to k_π (the current continuation) keeping the continuation π. When applying k_π to an argument t in context π' one evaluates t w.r.t. the restored context π throwing away the current context π'. Motivation and more explanation can be found in

(Streicher & Reus 1998) which, however, is based on the alternative control operator $\mathcal{C}$ whose meaning is given by the rule

$$\mathcal{C}\, t \succ tt \star \mathsf{k}_\pi$$

where t is applied to the "current continuation" k_π now in the empty context instead of the current context π.

All this is not a purely formal game since the above language can be interpreted in the recursively defined domain

$$D \cong \Sigma^{\mathsf{List}(D)} \cong \prod_{n\in\omega} \Sigma^{D^n}$$

where Σ is the two point lattice $\bot \sqsubset \top$. One can show that $D \cong \Sigma \times D^D$, i.e. D^D is a retract of D. In analogy[†] with (Streicher & Reus 1998) the semantic clauses are

$$\begin{aligned}
&[\![\lambda x.t]\!]\rho\langle\rangle = \top \\
&[\![\lambda x.t]\!]\rho\langle d, k\rangle = [\![t]\!]\rho[d/x]k \\
&[\![ts]\!]\rho k = [\![t]\!]\rho\langle[\![s]\!]\rho, k\rangle \\
&[\![\mathsf{cc}\, t]\!]\rho k = [\![t]\!]\rho\langle\mathsf{ret}(k), k\rangle \\
&[\![\mathsf{k}_\pi]\!]\rho = \mathsf{ret}([\![\pi]\!]\rho)
\end{aligned}$$

where

$$\begin{aligned}
&\mathsf{ret}(k)\langle\rangle = \top \\
&\mathsf{ret}(k)\langle d, k'\rangle = d(k)
\end{aligned}$$

and

$$\begin{aligned}
&[\![\langle\rangle]\!]\rho = \langle\rangle \\
&[\![t.\pi]\!]\rho = \langle[\![t]\!]\rho, [\![\pi]\!]\rho\rangle.
\end{aligned}$$

It is tempting to define a relation $\perp\!\!\!\perp \subseteq D \times \mathsf{List}(D)$ as

$$d \perp\!\!\!\perp k \qquad \text{iff} \qquad d(k) = \top$$

which can be lifted to syntax by putting $t \perp\!\!\!\perp \pi$ iff $[\![t]\!] \perp\!\!\!\perp [\![\pi]\!]$. Thus $\perp\!\!\!\perp$ is a set of processes which is *saturated* in the sense that

$$p \succ q \in \perp\!\!\!\perp \quad \text{implies} \qquad p \in \perp\!\!\!\perp$$

i.e. is closed under head expansion.[‡]

Saturated sets of processes are an essential ingredient for defining the classical realizability interpretation for second order logic as in (Krivine 2009). For a saturated set $\perp\!\!\!\perp$ and subsets X and Y of Π and Λ, respectively, we define

$$X^\perp = \{t \in \Lambda \mid \forall \pi \in X.\ t \perp\!\!\!\perp \pi\} \qquad Y^\perp = \{\pi \in \Pi \mid \forall t \in Y.\ t \perp\!\!\!\perp \pi\}$$

[†] In (Streicher & Reus 1998) there was employed the recursively defined domain $D \cong \Sigma^{D^\omega}$ which is isomorphic to Σ_∞ and thus validates $D \cong D^D$.

[‡] Actually the relation $\perp\!\!\!\perp$ under consideration is also closed under head reduction and even semantic equality.

and call a set S (of terms or stacks) *biorthogonally closed* iff $S^{\perp\perp} = S$. We write $\mathcal{P}_{\perp\!\!\perp}(\Lambda)$ and $\mathcal{P}_{\perp\!\!\perp}(\Pi)$ for the collections of biorthogonally closed sets of terms and stacks, respectively. In realizability models induced by $\perp\!\!\!\perp$ propositions A will be interpreted as $|A| \in \mathcal{P}_{\perp\!\!\perp}(\Lambda)$. However, it turns out as convenient to represent $|A|$ via a set $||A||$ of stacks with $|A| = ||A||^{\perp}$ which, however, will in general be different from $|A|^{\perp}$.

For a saturated set $\perp\!\!\!\perp$ of processes second order logic over a (typically countable) set M of individuals is interpreted as follows: n-ary predicate variables range over functions $M^n \to \mathcal{P}(\Pi)$ and formulas A are interpreted as $||A||_\rho \subseteq \Pi$ according to the clauses

$$||X(t_1,\dots,t_n)||_\rho = \rho(X)([\![t_1]\!]_\rho,\dots,[\![t_1]\!]_\rho)$$

$$||A {\to} B||_\rho = |A|_\rho.||B||_\rho$$

$$||\forall x A(x)|| = \bigcup_{a\in M} ||A||_{\rho[a/x]}$$

$$||\forall X A[X]||_\rho = \bigcup_{R\in\mathcal{P}(\Pi)^{M^n}} ||A||_{\rho[R/X]}$$

where ρ is a valuation sending individual variables to elements of M and n-ary predicate variables to elements of $\mathcal{P}(\Pi)^{M^n}$. The set of *realizers for* A is defined as $|A|_\rho = ||A||_\rho^{\perp}$. If A is closed we write simply $|A|$ and $||A||$ instead of $|A|_\rho$ and $||A||_\rho$, respectively, since the interpretation of A does not depend on ρ.

Notice that we have

$$|\forall x A| = \bigcap_{a\in M} \big|A[a/x]\big|$$

$$|\forall X A| = \bigcap_{R\in\mathcal{P}(\Pi)^{M^n}} \big|A[R/X]\big|$$

since we have

$$\left(\bigcup_{i\in I} X_i\right)^{\perp} = \bigcup_{i\in I} X_i^{\perp}$$

for arbitrary families $X : I \to \mathcal{P}(\Pi)$.

In general $|A{\to}B|$ is a **proper** subset of

$$|A|{\to}|B| = \{t{\in}\Lambda \mid \forall s{\in}|A|\ ts \in |B|\}$$

since in general

$$ts * \pi \in \perp\!\!\!\perp \quad \not\Rightarrow \quad t * s.\pi \in \perp\!\!\!\perp$$

but one easily checks that for every $t \in |A|{\to}|B|$ its η-expansion $\lambda x.tx \in |A{\to}B|$. But, of course, we have $|A{\to}B| = |A|{\to}|B|$ whenever $\perp\!\!\!\perp$ is also *closed under head reduction*, i.e. $\perp\!\!\!\perp \ni p \succ q$ implies $q \in \perp\!\!\!\perp$.

In (Krivine 1990a) one finds a proof term assigment for intuitionistic second order logic which for sake of convenience we recall in in Figure 1 where $A(F(\vec{x}))$ stands for the formula obtained from $A(X)$ by replacing every subformula of the form $X(\vec{t})$ by $F(\vec{t})$.

As proved in e.g. (Krivine 2009) the following soundness result holds: if one can derive $x_1{:}A_1,\dots,x_k{:}A_k \vdash u : B$ and $v_i \in |A_i|$ for $i = 1,\dots,k$ then $u[\vec{v}/\vec{x}] \in |B|$, i.e. proof terms are realizers. But, of course, there may be realizers which do not come from proofs in

Typing Rules for Intuitionistic Second Order Logic

$$\frac{}{\Gamma, x{:}A, \Delta \vdash x : A}$$

$$\frac{\Gamma, x{:}A \vdash u : B}{\Gamma \vdash \lambda x.u : A \to B} \qquad \frac{\Gamma \vdash u : A \to B \quad \Gamma \vdash v : A}{\Gamma \vdash uv : B}$$

$$\frac{\Gamma \vdash u : A(x)}{\Gamma \vdash u : \forall x A(x)}\ (x \text{ not free in } \Gamma) \qquad \frac{\Gamma \vdash u : \forall x A(x)}{\Gamma \vdash u : A(t)}$$

$$\frac{\Gamma \vdash u : A(X)}{\Gamma \vdash u : \forall X A(X)}\ (X \text{ not free in } \Gamma) \qquad \frac{\Gamma \vdash u : \forall X A(X)}{\Gamma \vdash u : A(F(\vec{x}))}$$

Fig. 1. Typing Rules for Second Order Intuitionistic Logic

intuitionistic second order logic. For example $\lambda x.\mathsf{cc}\, x$ realizes Peirce's law $((A \to B) \to A) \to A$ which can be seen as follows. Suppose $t \in |(A \to B) \to A|$ and $\pi \in ||A||$. We have to show that $\lambda x.\mathsf{cc}\, x \star t.\pi \in \perp\!\!\!\perp$. Since $\lambda x.\mathsf{cc}\, x \star t.\pi \succ \mathsf{cc}\, t \star \pi \succ t \star \mathsf{k}_\pi.\pi$ it suffices to show that $\mathsf{k}_\pi \in |A \to B|$. Suppose $s \in |A|$ and $\pi' \in ||B||$ then $\mathsf{k}_\pi \star s.\pi' \succ s \star \pi \in \perp\!\!\!\perp$ and thus $\mathsf{k}_\pi \star s.\pi' \in \perp\!\!\!\perp$. Thus, in particular, the term $\lambda x.\mathsf{cc}\, x$ realizes $(\neg A \to A) \to A$ where $\neg A \equiv A \to \bot$ with $\bot \equiv \forall X\, X$. Accordingly, the term $\lambda f.(\lambda x.\mathsf{cc}\, x)(\lambda y.fy)$ realizes $\neg\neg A \to A$ since $\lambda y.fy$ realizes $\neg A \to A$ whenever f realizes $\neg A \to \bot$. Thus untyped λ-calculus extended by cc allows one to represent proofs of classical second order logic as terms.

However, for realizing other non-logical axioms one may have to consider further extensions. For example in (Krivine 2003) in order to realize the axiom of *countable choice* Krivine has added a constant χ^* together with the reduction rule

$$\chi^* \star t.\pi \ \succ \ t \star n_t.\pi$$

where n_t is the Church numeral representation of a Gödel number for t.[§] This is an instance of Krivine's general point of view that new programming concepts should be motivated by their need to realize important non-logical axioms. In (Krivine 2008) for example (one cell) memory has been motivated by the need to realize Cohen forcing.

3. Abstract Krivine Structures

We have seen at the end of the previous section that one cannot work with one single language. For this reason it is necessary to axiomatize the kind of structure needed for performing Krivine's classical realizability interpretation. Such structures have been

[§] One might consider the assignment $t \mapsto n_t$ as a kind of quote construct as one finds in LISP. Thus χ^* may be understood as the program $\lambda x.\, x(\mathsf{quote}(x))$.

axiomatized in (Krivine 2008) including a form of λ-abstraction which is technically a bit cumbersome. Instead we here introduce a combinatory version which we call *abstract Krivine structure* (aks) and which is inspired by the notion of partial combinatory algebra (pca) on which ordinary realizability is based on as explained in detail in (van Oosten 2008).

Definition 3.1. (Abstract Krivine Structure)
An **abstract Krivine structure** (aks) is given by

— a set Λ of "terms" together with a binary application operation (written as juxtaposition) and distinguished elements $\mathsf{K}, \mathsf{S}, \mathsf{cc} \in \Lambda$
— a set Π of "stacks" together with a push operation (push) from $\Lambda \times \Pi$ to Π (written $t.\pi$) and a unary operation $\mathsf{k} : \Pi \to \Lambda$ (written as k_π)
— a saturated subset $\perp\!\!\!\perp$ of $\Lambda \times \Pi$

where *saturated* means that $\perp\!\!\!\perp$ satisfies the closure conditions

(S1)	$ts \star \pi \in \perp\!\!\!\perp$	whenever	$t \star s.\pi \in \perp\!\!\!\perp$
(S2)	$\mathsf{K} \star t.s.\pi$ in $\perp\!\!\!\perp$	whenever	$t \star \pi$ in $\perp\!\!\!\perp$
(S3)	$\mathsf{S} \star t.s.u.\pi \in \perp\!\!\!\perp$	whenever	$tu(su) \star \pi \in \perp\!\!\!\perp$
(S4)	$\mathsf{cc} \star t.\pi \in \perp\!\!\!\perp$	whenever	$t \star \mathsf{k}_\pi.\pi \in \perp\!\!\!\perp$
(S5)	$\mathsf{k}_\pi \star t.\pi' \in \perp\!\!\!\perp$	whenever	$t \star \pi \in \perp\!\!\!\perp$.

A strong abstract Krivine structure (saks) is an aks where (S1) can be strengthened to

(SS1) $\quad ts \star \pi \in \perp\!\!\!\perp \quad$ iff $\quad t \star s.\pi \in \perp\!\!\!\perp$.

Recall that a combinatory algebra is a set A with a binary application operation (denoted by juxtaposition) and distinguished elements k and s of A satisfying the equations $\mathsf{k}xy = x$ and $\mathsf{s}xyz = xz(yz)$. Notice that an aks is not equationally defined but instead the axioms (S1-5) state that $\perp\!\!\!\perp$ is "closed under head expansion". In other words the notion of an abstract Krivine structure is free from an equality given in advance. However, we could define a notion of observational equivalence $t \sim s$ as

$$\forall \pi \in \Pi.\ t \star \pi \in \perp\!\!\!\perp \Leftrightarrow s \star \pi \in \perp\!\!\!\perp$$

Later in subsection 5.1 we will show how any aks can be organized into a so-called *order combinatory algebra* (oca).

Next we will show how any aks gives rise to a model of classical second order logic in a way analogous to section 2. Again a proposition A will be interpreted as $||A|| \subseteq \Pi$ and its set of realizers is defined as

$$|A| = ||A||^{\perp} = \{t \in \Lambda \mid \forall \pi \in ||A||.\ \ t \star \pi \in \perp\!\!\!\perp\}$$

The interpretation is given by the following clauses

$$||R(\vec{t})|| = R\big([\![\vec{t}]\!]\big)$$

$$||A{\to}B|| = |A|.||B|| = \big\{t.\pi \mid t \in |A|, \pi \in ||B||\big\}$$

$$||\forall x A(x)|| = \bigcup_{a\in M} ||A(a)||$$

$$||\forall X A(X)|| = \bigcup_{R\in\mathcal{P}(\Pi)^{M^n}} ||A(R)||$$

where M is the underlying set of the model and formulas are closed but may contain (constants for) elements of M or $\mathcal{P}(\Pi)^{M^n}$. One could define propositions more restrictively as

$$\mathcal{P}_{\perp\!\!\!\perp}(\Pi) = \{X \in \mathcal{P}(\Pi) \mid X = X^{\perp\perp}\}$$

and this would not change the meaning of $|A|$ for closed formulas though it would change the meaning of $||A||$. But like in section 2 it turns out as convenient to postpone the biorthogonal closure.

Notice that $\mathcal{P}_{\perp\!\!\!\perp}(\Pi)$ is in 1-1-correspondence with

$$\mathcal{P}_{\perp\!\!\!\perp}(\Lambda) = \{X \in \mathcal{P}(\Lambda) \mid X = X^{\perp\perp}\}$$

via $(-)^{\perp}$. Then in case the aks under consideration is strong we have

$$|R(\vec{t})| = R(\llbracket\vec{t}\rrbracket)$$

$$|A{\to}B| = |A|{\to}|B| = \{t \in L \mid \forall s \in |A|.\ ts \in |B|\}$$

$$|\forall x A(x)| = \bigcap_{a\in M} |A(a)|$$

$$|\forall X A(X)| = \bigcap_{R\in\mathcal{P}_{\perp\!\!\!\perp}(\Lambda)^{M^n}} |A(R)|$$

allowing one to redefine the realizability interpretation according to a more traditional pattern.

Again if the aks under consideration is not strong then in general we only have

$$|A{\to}B| \subseteq |A|{\to}|B| = \{t \in \Lambda \mid \forall s \in |A|.\ ts \in |B|\}$$

but elements of $|A|{\to}|B|$ can be uniformly transformed into elements of $|A{\to}B|$ via the combinator $\mathsf{E} = \mathsf{S}(\mathsf{KI})$ with $\mathsf{I} = \mathsf{SKK}$.

Lemma 3.2. If $t \in |A|{\to}|B|$ then $\mathsf{E}t \in |A{\to}B|$.

Proof. One easily checks that

$$\mathsf{I} \star t.\pi \in \perp\!\!\!\perp \;\Leftarrow\; t \star \pi \in \perp\!\!\!\perp$$

and, therefore, we have

$$\mathsf{E}t \star s.\pi \in \perp\!\!\!\perp \;\Leftarrow\; \mathsf{KI}s(ts) \star \pi \in \perp\!\!\!\perp \;\Leftarrow\; \mathsf{I} \star ts.\pi \in \perp\!\!\!\perp \;\Leftarrow\; ts \star \pi \in \perp\!\!\!\perp$$

Then for $s \in |A|$, $\pi \in ||B||$ we have $\mathsf{E}t \star s.\pi \in \perp\!\!\!\perp$ because $ts * \pi \in \perp\!\!\!\perp$ since $t \in |A|{\to}|B|$. Thus $\mathsf{E}t \in |A{\to}B|$ as desired. $\square$

In a sense $\mathsf{E}t$ is a combinator version of the η-expansion $\lambda x.tx$, i.e. E corresponds to the λ-term $\lambda y.\lambda x.yx$.

4. Cohen Forcing as an Instance of Abstract Krivine Structures

Already in (Krivine 2001) Krivine emphasizes that he considers classical realizability as a generalization of Cohen's forcing. We will make this precise by showing that Cohen forcing is the commutative case of classical realizability. Notice that in case of realizability induced by a partial combinatory algebra $\mathbb{A}$ this does not make sense since if in $\mathbb{A}$ application is commutative and associative we have $x = kxy = kyx = y$ and thus $\mathbb{A}$ is trivial.

Usually a notion of forcing is given by a conditional meet-semilattice, i.e. a poset with a greatest element 1 such that the infimum xy of x and y exists provided they have a lower bound. For our purposes we consider the superficially more general situation of a meet-semilattice $\mathbb{P}$ together with a down closed subset $\mathcal{D}$. Such a situation induces an aks as follows.

Lemma 4.1. Let $\mathbb{P}$ be a meet-semilattice and $\mathcal{D}$ a down closed subset of $\mathbb{P}$. This induces a saks where $\Lambda = \Pi = \mathbb{P}$, application and the push operation are given by the meet operation of $\mathbb{P}$, the constants are interpreted as 1 and $\perp\!\!\!\perp = \{(p, q) \in \mathbb{P}^2 \mid pq \in \mathcal{D}\}$.

Now for such an aks the set $\mathcal{P}_{\perp\!\!\!\perp}(\Pi)$ of propositions coincides with the set of all subsets of $\mathbb{P}$ of the form $X^\perp = \{p \in \mathbb{P} \mid \forall q \in X.\, pq \in \mathcal{D}\}$ for some $X \subseteq \mathbb{P}$. Notice that sets of the form $X^\perp$ are always down closed and contain $\mathcal{D}$ as a subset. In case $X \subseteq \mathbb{P}$ is already down closed $X^\perp$ can be computed in a way familiar from Cohen forcing.

Lemma 4.2. If $X \subseteq \mathbb{P}$ is down closed then $X^\perp = \{p \in \mathbb{P} \mid \forall q \leq p\ (q \in X \Rightarrow q \in \mathcal{D})\}$.

Proof. Suppose $p \in X^\perp$ and $q \in X$ with $q \leq p$. Then $q = qp \in \mathcal{D}$. For the reverse direction suppose $p \in \mathbb{P}$ with $\forall q \leq p\ (q \in X \Rightarrow q \in \mathcal{D})$. Then for $q \in X$ we have $pq \in X$ since X is down closed and so $pq \in \mathcal{D}$ by assumption on p. $\square$

Moreover, it is an easy exercise to show that

Lemma 4.3. For down closed $X, Y \subseteq \mathbb{P}$ we have

$$X \to Y = \{p \in \mathbb{P} \mid \forall q \in X.\, p \wedge q \in Y\} = \{p \in \mathbb{P} \mid \forall q \leq p\ (q \in X \Rightarrow q \in Y)\}$$

and thus $Z \subseteq X \to Y$ iff $Z \cap X \subseteq Y$.

Using Lemma 4.2 one easily sees that for downclosed $X \subseteq \mathbb{P}$ we have $X = X^{\perp\perp}$ iff $\mathcal{D} \subseteq X$ and $p \in X \setminus \mathcal{D}$ whenever for all $q \leq p$ with $q \notin \mathcal{D}$ there exists $r \leq q$ with $r \in X \setminus \mathcal{D}$. Thus $\mathcal{P}_{\perp\!\!\!\perp}(\Pi)$ is via $(-) \setminus \mathcal{D}$ in 1-1-correspondence with those subsets A of the poset $\mathbb{P}_\uparrow = \mathbb{P} \setminus \mathcal{D}$ that are *regular* in the sense that $p \in A$ whenever $\forall q {\leq} p\ \exists r {\leq} q\ r \in A$. Lemma 4.2 and 4.3 tell us that under this correspondence negation and implication are as in Cohen forcing (or Kripke models).

Now we can characterise those aks's which arise from Cohen forcing.

Theorem 4.4. A aks arises up to isomorphism from a down closed subset of a meet-semilattice iff it is strong and satisfies the following requirements

(1) $\mathsf{k} : \Pi \to \Lambda$ is a bijection

(2) the application operation endows Λ with the structure of a commutative idempotent monoid
(3) application coincides with the push operation (when identifying Λ and Π via k).

Proof. It is clear that all these conditions are necessary. Suppose we are given a saks satisfying the conditions above. By condition (2) the set Λ is a meet-semilattice $\mathbb{P}$ under application. For $\mathcal{D}$ we take the subset $\{t \in \Lambda \mid (t, 1) \in \perp\!\!\!\perp\}$ of $\mathbb{P} = \Lambda$. Notice that $\mathcal{D}$ is down closed due to condition (3). Since the aks is strong by assumption we have

$$ts \in \mathcal{D} \quad \text{iff} \quad (ts, 1) \in \perp\!\!\!\perp \quad \text{iff} \quad (t, s1) \in \perp\!\!\!\perp \quad \text{iff} \quad (t, s) \in \perp\!\!\!\perp$$

which finishes the argument. □

This explains in which sense Krivine considers forcing as commutative realizability.

5. Classical Realizability Tripos and Topos

The aim of this section is to show that with any aks one may associate a tripos, the so-called **Krivine tripos**, giving rise to a model of higher order classical logic extending the model of second order classical logic of Section 3.

5.1. *Abstract Krivine Structures as Order Combinatory Algebras*

Hofstra and van Oosten's notion of order partial combinatory algebra (opca) (Hofstra & van Oosten 2003) generalizes both pca's and complete Heyting algebras (cHa's) as explained in (van Oosten 2008). For our purposes we just need the following non-partial version which also covers the case of complete Heyting algebras.

Definition 5.1. (Order Combinatory Algebra)
A **order combinatory algebra** (oca) is a triple $(\mathbb{A}, \leq, \bullet)$ where $\leq$ is a partial order on $\mathbb{A}$ and $\bullet$ is a binary monotone operation on $\mathbb{A}$ such that there exist $k, s \in \mathbb{A}$ with

$$k \bullet a \bullet b \leq a \qquad s \bullet a \bullet b \bullet c \leq a \bullet c \bullet (b \bullet c)$$

for all $a, b, c \in \mathbb{A}$.

With every aks we may associate the oca whose underlying set is $\mathcal{P}_{\perp\!\!\!\perp}(\Pi)$, where $a \leq b$ iff $a \supseteq b$ and application is defined as

$$a \bullet b = \{\pi \in \Pi \mid \forall t \in |a|, s \in |b|.\ t * s.\pi \in \perp\!\!\!\perp\}^{\perp\perp}$$

where $|a| = a^{\perp}$ and similarly for b. Obviously $a \leq b$ iff $|a| \subseteq |b|$. Notice that in case the aks under consideration is strong we have

$$|a \bullet b| = \{ts \mid t \in |a|, s \in |b|\}^{\perp\perp}$$

which explains how we have arrived at the definition of $\bullet$. For showing that $\big(\mathcal{P}_{\perp\!\!\!\perp}(\Pi), \leq, \bullet\big)$ is actually an oca we have to identify appropriate $k, s \in \mathcal{P}_{\perp\!\!\!\perp}(\Lambda)$ satisfying the conditions

(1) $k \bullet x \bullet y \leq x$ and
(2) $s \bullet x \bullet y \bullet z \leq x \bullet z \bullet (y \bullet z)$

for all $x, y, z \in \mathcal{P}_{\perp\!\!\!\perp}(\Pi)$. The most immediate choice for k and s is $\{\mathsf{K}\}^\perp$ and $\{\mathsf{S}\}^\perp$, respectively, because then $|k| = \{\mathsf{K}\}^{\perp\perp}$ and $|s| = \{\mathsf{S}\}^{\perp\perp}$.

One can show by brute force that these choices of k and s validate the conditions (1) and (2). But instead we here give a more elegant argument suggested to us by Benno van den Berg. First we define

$$x \to y = \{t.\pi \mid t \in |x|, \pi \in y\}^{\perp\perp}$$

for $x, y \in \mathcal{P}_{\perp\!\!\!\perp}(\Pi)$ and observe that

Lemma 5.2. From $x \leq y \to z$ it follows that $x \bullet y \leq z$.

Proof. Suppose $x \leq y \to z$. Then we have $\forall t \in |x|.\forall s \in |y|.\forall \pi \in z.\ t \star s.\pi \in \perp\!\!\!\perp$ from which it follows that $z \subseteq x \bullet y$. Thus $x \bullet y \leq z$ as desired. □

Moreover, we have

Lemma 5.3. If $t \in |x|$ and $s \in |y|$ then $ts \in |x \bullet y|$.

Proof. Suppose $t \in |x|$ and $s \in |y|$. Let $\pi \in x \bullet y$. Then $t \star s.\pi \in \perp\!\!\!\perp$ and thus $ts * \pi \in \perp\!\!\!\perp$ by property (S1) of $\perp\!\!\!\perp$. □

For later use we observe that the inverse of the implication of Lemma 5.2 holds in the following restricted sense.

Lemma 5.4. If $x \bullet y \leq z$ then for all $t \in |x|$ we have $\mathsf{E}t \in |y \to z|$.

Proof. Suppose $x \bullet y \leq z$, i.e. $\forall u \in |x \bullet y|.\forall \pi \in z.\, ts \star \pi \in \perp\!\!\!\perp$. Thus, by Lemma 5.3 we have $\forall t \in |x|.\forall s \in |y|.\forall \pi \in z.\, ts \star \pi \in \perp\!\!\!\perp$. Since $ts \star \pi \in \perp\!\!\!\perp$ implies $\mathsf{E}t \star s.\pi \in \perp\!\!\!\perp$ it follows that $\forall t \in |x|.\forall s \in |y|.\forall \pi \in z.\, \mathsf{E}t \star s.\pi \in \perp\!\!\!\perp$. Thus $\forall t \in |x|.\, \mathsf{E}t \in |y \to z|$ as desired. □

Now we are ready to show that (1) and (2) hold for $k = \{\mathsf{K}\}^\perp$ and $s = \{\mathsf{S}\}^\perp$.

ad (1) : For showing that $k \bullet x \bullet y \leq x$ it suffices by Lemma 5.2 (applied twice) to show that $k \leq x \to y \to x$. But, obviously, we have $\mathsf{K} \in |x \to y \to x|$ and thus $k = \{\mathsf{K}\}^{\perp\perp} \subseteq |x \to y \to x|$.

ad (2) : For showing that $s \bullet x \bullet y \bullet z \leq x \bullet z \bullet (y \bullet z)$ it suffices by (multiple application of) Lemma 5.2 to show that $s \leq x \to y \to z \to (x \bullet z \bullet (y \bullet z))$. Thus it suffices to show that $\mathsf{S} \in |x \to y \to z \to (x \bullet z \bullet (y \bullet z))|$. For this purpose suppose $t \in |x|$, $s \in |y|$, $u \in |z|$ and $\pi \in x \bullet z \bullet (y \bullet z)$. Applying Lemma 5.3 iteratively we get $tu(su) \in |x \bullet z \bullet (y \bullet z)|$ and thus $tu(su) \star \pi \in \perp\!\!\!\perp$. By property (S3) of $\perp\!\!\!\perp$ it follows that $\mathsf{S} * t.s.u.\pi \in \perp\!\!\!\perp$ as desired.

In order to define the Krivine tripos induced by an aks we have to identify an appropriate filter in the induced oca in the sense of the following

Definition 5.5. (Filter on an Order Combinatory Algebra)
A **filter** on an oca $(\mathbb{A}, \leq, \bullet)$ is a subset Φ of $\mathbb{A}$ closed under $\bullet$ and containing (some choice of) k and s (for $\mathbb{A}$).

We have the following examples of filters

(1) In case of a saks induced by a downclosed set $\mathcal{D}$ in a meet-semilattice $\mathbb{P}$ a natural choice of a filter is $\{\mathbb{P}\}$.
(2) For an arbitrary aks $\Phi = \{a \in \mathcal{P}_{\perp\!\!\!\perp}(\Pi) \mid |a| \neq \emptyset\}$ is a filter on the induced oca $\mathcal{P}_{\perp\!\!\!\perp}(\Pi)$.

We now collect a few facts about filtered oca's $(\mathbb{A}, \Phi)$ from (van Oosten 2008; Hofstra 2006) which will be needed subsequently for verifying the construction of the Krivine tripos in subsection 5.2. For sake of convenience we often write xy instead of $x \bullet y$ for $x, y \in \mathbb{A}$. A *polynomial* over $\mathbb{A}$ is a term built from elements of $\mathbb{A}$ and a (countable) set of variables via the application operation $\bullet$.

If $\mathbb{A}$ is an oca then for every polynomial $t[\vec{x}, x]$ there exists a polynomial $\lambda^* x.t$ whose free variables are included in the list $\vec{x}$ such that

$$(\lambda^* x.t)a \leq t[\vec{x}, a]$$

for all $a \in \mathbb{A}$. Moreover, if all constants of t are in Φ then $\lambda^* x.t \in \Phi$ provided $\vec{x} \in \Phi$. For example $k' = \lambda^* x.\lambda^* y.y \in \Phi$.

Using these facts we can define in every oca $\mathbb{A}$ pairing and projection operations

$$p = \lambda^* x.\lambda^* y.\lambda^* z.zxy \quad p_1 = \lambda^* z.zk \quad p_2 = \lambda^* z.zk'$$

validating the laws

$$p_1(pxy) \leq x \qquad p_2(pxy) \leq y$$

It is clear from their construction that $p, p_1, p_2 \in \Phi$.

5.2. *The Krivine Tripos*

Given an oca $\mathbb{A} = (\mathbb{A}, \leq, \bullet)$ and a filter Φ on $\mathbb{A}$ one may associate with it a **Set**-indexed preorder $[-, \mathbb{A}]_\Phi$ as follows

— $[I, \mathbb{A}]_\Phi = \mathbb{A}^I$ is the set of all functions from set I to $\mathbb{A}$
— endowed with the *entailment* relation

$$\phi \vdash_I \psi \quad \text{iff} \quad \exists a \in \Phi \forall i \in I.\ a \bullet \phi_i \leq \psi_i$$

— for $u : J \to I$ the *reindexing map* $[u, \mathbb{A}]_\Phi = u^* : \mathbb{A}^I \to \mathbb{A}^J$ sends ϕ to $u^*\phi = (\phi_{u(j)})_{j \in J}$.

It is easy to see that $\vdash_I$ actually defines a preorder on $\mathbb{A}^I$. Let $e = \lambda^* x.x \in \Phi$. Then for all $\varphi \in \mathbb{A}^I$ we have $\forall i \in I.\ e \bullet \varphi_i \leq \varphi_i$ and thus $\varphi \vdash_I \varphi$. Suppose $\varphi \vdash_I \psi$ and $\psi \vdash_I \theta$. Then there exists $a, b \in \Phi$ such that $a \bullet \varphi_i \leq \psi_i$ and $b \bullet \psi_i \leq \theta_i$ for all $i \in I$. Then for $c = \lambda^* x.b \bullet (a \bullet x) \in \Phi$ we have

$$c \bullet \varphi_i \leq b \bullet (a \bullet \varphi_i) \leq b \bullet \psi_i \leq \theta_i$$

for all $i \in I$. Thus $\varphi \vdash_I \theta$.

Suppose $u : J \to I$ is a map in **Set** and $\varphi \vdash_I \psi$. Then there exists $a \in \Phi$ with $\forall i \in I.\ a \bullet \phi_i \leq \psi_i$. Thus, *a fortiori* we have $\forall j \in J.\ a \bullet \phi_{u(j)} \leq \psi_{u(j)}$, i.e. $u^*\varphi \vdash_J u^*\psi$. Thus u^* preserves entailment.

The following lemma provides a characterisation of $\vdash_I$ which will turn out as useful when proving that $[-, \mathbb{A}]_\Phi$ is a tripos.

Lemma 5.6. For all sets I we have

$$\varphi \vdash_I \psi \quad \text{iff} \quad \exists t \in \Lambda.\forall i \in I.\, t \in |\varphi_i|{\to}|\psi_i| \quad \text{iff} \quad \exists t \in \Lambda.\forall i \in I.\, t \in |\varphi_i \to \psi_i|$$

for all $\varphi, \psi \in [I, \mathbb{A}]_\Phi$.

Proof. Suppose $\varphi \vdash_I \psi$. Then there exists $a \in \Phi$ such that $\forall i \in I.\ a \bullet \phi_i \leq \psi_i$. By Lemma 5.3 for all $i \in I$, $t \in |a|$ and $s \in |\varphi_i|$ we have $ts \in |a \bullet \phi_i| \subseteq |\psi_i|$. Let $t \in |a|$. Then for all $i \in I$ we have $t \in |\varphi_i| \to |\psi_i|$.

Suppose for some $t \in \Lambda$ we have $t \in |\varphi_i| \to |\psi_i|$ for all $i \in I$. Then by Lemma 5.4 we have $\mathsf{E}u \in |\varphi_i \to \psi_i|$ for all $i \in I$.

Suppose there exists a $t \in \Lambda$ such that $\forall i \in I.\ t \in |\varphi_i \to \psi_i|$. Then we have

$$\forall i \in I.\ \{t\}^{\perp\perp} \subseteq |\varphi_i \to \psi_i|$$

and thus for $a = \{t\}^\perp$ we have

$$\forall i \in I.\forall u \in |a|.\forall v \in |\varphi_i|.\forall \pi \in \psi_i.\ u \star v.\pi \in \perp\!\!\!\perp$$

from which it follows that $\forall i \in I.\ a \bullet \varphi_i \leq \psi_i$ and thus $\varphi \vdash_I \psi$ since $a = \{t\}^\perp \in \Phi$. □

The following lemma will be useful in the proof of Theorem 5.10.

Lemma 5.7. Let I be set and $\varphi, \psi, \theta \in [I, \mathbb{A}]_\Phi$. We write $\varphi{\to}\psi$ for the family $(\varphi_i{\to}\psi_i)_{i \in I}$. Then $\theta \vdash_I \varphi{\to}\psi$ iff there exists an $a \in \Phi$ such that $\forall i \in I.\, a \bullet \theta_i \bullet \varphi_i \leq \psi_i$.

Proof. Suppose $\theta \vdash_I \varphi \to \psi$. Then there is an $a \in \Phi$ such that $\forall i \in I.\, a \bullet \theta_i \leq \varphi_i{\to}\psi_i$. By Lemma 5.2 it follows that $\forall i \in I.\, a \bullet \theta_i \bullet \varphi_i \leq \psi_i$.

For the reverse direction suppose $a \in \Phi$ with $\forall i \in I.\, a{\bullet}\theta_i{\bullet}\varphi_i \leq \psi_i$. Then by Lemma 5.3 we have

$$\forall i \in I.\forall t \in |a \bullet \theta_i|.\forall s \in |\varphi_i|.\, ts \in |a \bullet \theta_i \bullet \varphi_i| \subseteq |\psi_i|$$

and thus

$$\forall i \in I.\forall t \in |a \bullet \theta_i|.\, \mathsf{E}t \in |\varphi_i{\to}\psi_i|$$

by Lemma 5.4. Thus by Lemma 5.6 there is a $b \in \Phi$ with

$$\forall i \in I.\, b \bullet (a \bullet \theta_i) \leq \varphi_i{\to}\psi_i$$

Let $c \in \Phi$ with $c \bullet x \leq b \bullet (a \bullet x)$ for all $x \in \mathbb{A}$. Thus for all $i \in I$ we have

$$c \bullet \theta_i \leq b \bullet (a \bullet \theta_i) \leq \varphi_i{\to}\psi_i$$

i.e. $\theta \vdash_I \varphi{\to}\psi$ as desired. □

Furthermore for every set I we will need an "equality predicate" $\mathrm{eq}_I : I \times I \to \mathbb{A}$ on I defined as

$$\mathrm{eq}_I(i,j) = \begin{cases} \{\mathsf{I}\}^\perp & \text{if } i = j \\ \Pi & \text{otherwise} \end{cases}$$

which has the following remarkable properties.

Lemma 5.8. For every $i \in I$ we have $\mathsf{I} \in |\mathrm{eq}_I(i,i)|$ and $eq_I(i,i) \bullet a \leq a$ for all $a \in \mathbb{A}$. If $i, j \in I$ with $i \neq j$ then $\mathrm{eq}_I(i,j) \bullet a \leq b$ for all $a, b \in \mathbb{A}$.

Proof. Obviously, we have $|\mathrm{eq}_I(i,i)| = \{\mathsf{I}\}^{\perp\perp}$. Thus $\mathsf{I} \in |\mathrm{eq}_I(i,i)|$ since $\mathsf{I} \in \{\mathsf{I}\} \subseteq \{\mathsf{I}\}^{\perp\perp}$. Let $a \in \mathbb{A}$. By Lemma 5.2 for showing $\mathrm{eq}_I(i,i) \bullet a \leq a$ it suffices to show that $\mathrm{eq}_I(i,i) \leq a{\to}a$ which holds since $\mathsf{I} \in |a{\to}a|$ and thus $\{\mathsf{I}\}^{\perp\perp} \subseteq |a{\to}a|$.

Suppose $i,j \in I$ with $i \neq j$ and $a,b \in \mathbb{A}$. Then $\mathrm{eq}_I(i,j) = \Pi \supseteq a{\to}b$ from which it follows that $\mathrm{eq}_I(i,j) \leq a{\to}b$ and thus $\mathrm{eq}_I(i,j) \bullet a \leq b$ by Lemma 5.2. □

Lemma 5.9. Let $a \in \mathbb{A}$ and $t \in \big|\{\mathsf{I}\}^{\perp}{\to}a\big|$. Then

i) $\mathsf{SI}t \in \big|\{\mathsf{I}\}^{\perp}{\to}a\big|$ and
ii) $\mathsf{SI}t \in |\Pi{\to}b|$ for all $b \in \mathbb{A}$.

Proof. Suppose $t \in \big|\{\mathsf{I}\}^{\perp}{\to}a\big|$.

For showing i) suppose $s \in \{\mathsf{I}\}^{\perp\perp}$ and $\pi \in a$. We have to show that $\mathsf{SI}t \star s.\pi \in \perp\!\!\!\perp$. Since $t \star s.\pi \in \perp\!\!\!\perp$ we have $ts \star \pi \in \perp\!\!\!\perp$. Thus $\mathsf{I} \star ts.\pi \in \perp\!\!\!\perp$ and accordingly $ts.\pi \in \{\mathsf{I}\}^{\perp}$. Thus $s \star ts.\pi \in \perp\!\!\!\perp$ and therefore also $\mathsf{I}s(ts) \star \pi \in \perp\!\!\!\perp$. Thus, by property (S3) of $\perp\!\!\!\perp$ we have $\mathsf{S} \star \mathsf{I}.t.s.\pi \in \perp\!\!\!\perp$ and therefore also $\mathsf{SI}t \star s.\pi$ as desired.

For showing ii) suppose $s \in \Pi^{\perp}$ and $\pi \in b$. We have to show that $\mathsf{SI}t \star s.\pi \in \perp\!\!\!\perp$. Since $s \in \Pi^{\perp}$ we have $s \star ts.\pi \in \perp\!\!\!\perp$. Thus also $\mathsf{I}s(ts) \star \pi \in \perp\!\!\!\perp$. By property (S3) of $\perp\!\!\!\perp$ we have also $\mathsf{S} \star \mathsf{I}.t.s \in \perp\!\!\!\perp$ and thus also $\mathsf{SI}t \star s.\pi \in \perp\!\!\!\perp$ as desired. □

Now we are ready to prove the main result of this section.

Theorem 5.10. In case $\mathbb{A} = \mathcal{P}_{\perp\!\!\!\perp}(\Pi)$ arises from an aks as given by $\perp\!\!\!\perp \subseteq \Lambda \times \Pi$ and $\Phi = \{a \in \mathcal{P}_{\perp\!\!\!\perp}(\Lambda) \mid |a| \neq \emptyset\}$ the indexed preorder $[-,\mathbb{A}]_\Phi$ is a **tripos**, i.e.

— all $[I,\mathbb{A}]_\Phi$ are pre-Heyting-algebras whose structure is preserved by reindexing
— for every $u : J \to I$ in **Set** the reindexing map u^* has a left adjoint $\exists_u$ and a right adjoint $\forall_u$ satisfying the (Beck-)Chevalley condition
— there is a *generic predicate* $\mathsf{T} \in [\Sigma,\mathbb{A}]_\Phi$ such that all other predicates can be obatined from T by appropriate reindexing.

Proof. Recall that we often denote application in the oca $\mathcal{P}_{\perp\!\!\!\perp}(\Pi)$ by juxtaposition.

We first show that $[I,\mathbb{A}]_\Phi$ has finite infima. Let $\top = \{\pi \in \Pi \mid \forall t \in \Lambda.\ t \star \pi \in \perp\!\!\!\perp\}$ which obviously is an element in $\mathcal{P}_{\perp\!\!\!\perp}(\Pi)$ and satisfies $a \leq \top$ for all $a \in \mathbb{A}$. Let $\top_I$ be the constant family in $[I,\mathbb{A}]_\Phi$ with value $\top$. If $\varphi \in [I,\mathbb{A}]_\Phi$ then for all $i \in I$ we have $(\lambda^*x.x)\varphi_i \leq |\top|$. Since $\lambda^*x.x \in \Phi$ we have $\varphi \vdash_I \top_I$. Thus $\top_I$ is a greatest element in $[I,\mathbb{A}]_\Phi$. For showing that $[I,\mathbb{A}]_\Phi$ has binary infima suppose $\varphi,\psi \in \mathbb{A}^I$. Let $\varphi \wedge \psi \in \mathbb{A}^I$ with $(\varphi\wedge\psi)_i = p\varphi_i\psi_i$ for all $i \in I$. Since $\forall i \in I.\, p_1(p\varphi_i\psi_i) \leq \varphi_i$ we have $\varphi \wedge \psi \vdash_I \varphi$ and since $\forall i \in I.\, p_2(p\varphi_i\psi_i) \leq \psi_i$ we have $\varphi\wedge\psi \vdash_I \psi$. Suppose $\theta \vdash_I \varphi,\psi$. Then there exist $a,b \in \Phi$ such that for all $i \in I$ we have $a\theta_i \leq \varphi_i$ and $b\theta_i \leq \psi_i$. For $c = \lambda^*x.p(ax)(bx) \in \Phi$ we have for all $i \in I$ that

$$c\theta_i \leq p(a\theta_i)(b\theta_i) \leq p\varphi_i\psi_i = (\varphi\wedge\psi)_i$$

and thus $\theta \vdash_I \varphi\wedge\psi$ as desired. Obviously, every reindexing u^* preserves $\top$ and $\wedge$.

Next we show that all $[I,\mathbb{A}]_\Phi$ have implication. Suppose $\varphi,\psi \in [I,\mathbb{A}]_\Phi$. We define $\varphi{\to}\psi$ as $(\varphi{\to}\psi)_i = \varphi_i{\to}\psi_i$ for $i \in I$. Suppose $\theta \vdash_I \varphi{\to}\psi$. Then there exists $a \in \Phi$ with $a\theta_i \leq \varphi_i{\to}\psi_i$ for all $i \in I$. Then by Lemma 5.2 we have $a\theta_i\varphi_i \leq \psi_i$ for all $i \in I$. Thus we have

$$a(p_1(p\theta_i\varphi_i))(p_2(p\theta_i\varphi_i)) \leq a\theta_i\varphi_i \leq \psi_i$$

for all $i \in I$. Thus for $f = \lambda^*x.a(p_1x)(p_2x) \in \Phi$ we have

$$f(\theta_i \wedge \varphi_i) \leq \psi_i$$

for all $i \in I$, i.e. $\theta \wedge \varphi \vdash_I \psi$. For the reverse direction suppose $\theta \wedge \varphi \vdash_I \psi$. Then there is an $a \in \Phi$ with $a(p\theta_i\varphi_i) \leq \psi_i$ for all $i \in I$. Then for $f = \lambda^*x.\lambda^*y.a(pxy) \in \Phi$ we have

$$f\theta_i\varphi_i \leq \psi_i$$

for all $i \in I$. Thus, by Lemma 5.7 it follows that $\theta \vdash_I \varphi{\to}\psi$. Thus we have shown that $\varphi{\to}\psi$ is actually the exponential in $[I, \mathbb{A}]_\Phi$. It follows from $\varphi{\to}\psi \vdash_I \varphi{\to}\psi$ that $(\varphi{\to}\psi) \wedge \varphi \vdash_I \psi$. Since for $u : J \to I$ we have $u^*(\varphi{\to}\psi) = u^*\varphi{\to}u^*\psi$ and u^* preserves $\wedge$ it follows that $(u^*\varphi{\to}u^*\psi) \wedge u^*\varphi = u^*((\varphi{\to}\psi) \wedge \varphi) \leq u^*\psi$. Thus reindexing preserves implication.

Next we show that $[-, \mathbb{A}]_\Phi$ has universal quantification. For $\alpha : J \to I$ and $\varphi \in [J, \mathbb{A}]_\Phi$ we define $\forall_u(\varphi)$ in $[I, \mathbb{A}]_\Phi$ as

$$\forall_\alpha(\varphi)_i = \left(\bigcup_{j \in J} \mathrm{eq}_I(\alpha(j), i){\to}\varphi_j\right)^{\perp\perp}$$

for all $i \in I$. Notice that

$$\left|\forall_\alpha(\varphi)_i\right| = \left(\bigcup_{j \in J} \mathrm{eq}_I(\alpha(j), i){\to}\varphi_j\right)^{\perp} = \bigcap_{j \in J} |\mathrm{eq}_I(\alpha(j), i){\to}\varphi_j|$$

Suppose $\psi \in [I, \mathbb{A}]_\Phi$. We have to show that

$$\alpha^*\psi \vdash_J \varphi \qquad \text{iff} \qquad \psi \vdash_I \forall_\alpha(\varphi)$$

Suppose $\psi \vdash_I \forall_\alpha(\varphi)$. Then there is a $c \in \Phi$ with $c\psi_i \leq \mathrm{eq}_I(\alpha(j), i){\to}\varphi_j$ for all $i \in I$ and $j \in J$. Thus, in particular, we have $c\psi_{\alpha(j)} \leq \mathrm{eq}_I(\alpha(j), \alpha(j)){\to}\varphi_j$ for all $j \in J$. By Lemma 5.6 it follows that $\psi_{\alpha(j)} \vdash_{j \in J} \mathrm{eq}_I(\alpha(j), \alpha(j)){\to}\varphi_j$ and thus $\mathrm{eq}_I(\alpha(j), \alpha(j)) \vdash_{j \in J} \psi_{\alpha(j)}{\to}\varphi_j$ by that part of propositional logic we have already established for $[I, \mathbb{A}]_\Phi$. Thus, by Lemma 5.6 there is a $t \in \Lambda$ such that

$$\forall j \in J. \forall s \in |\mathrm{eq}_I(\alpha(j), \alpha(j))|.\, ts \in |\psi_{\alpha(j)}{\to}\varphi_j|$$

from which it follows that

$$\forall j \in J.\, t\mathsf{I} \in |\psi_{\alpha(j)}{\to}\varphi_j|$$

since by Lemma 5.8 we have $\mathsf{I} \in |\mathrm{eq}_I(\alpha(j), \alpha(j))|$ for all $j \in J$. Thus we have

$$\forall j \in J. \forall s \in |\psi_{\alpha(j)}|.\, t\mathsf{I}s \in |\psi_{\alpha(j)}|{\to}|\varphi_j|$$

from which it follows by Lemma 5.6 that $\alpha^*\psi \vdash_J \varphi$ as desired.

Suppose $\alpha^*\psi \vdash_J \varphi$. Then there exists an $a \in \Phi$ such that $\forall j \in J.\, a\psi_{\alpha(j)} \leq \varphi_j$. Then $b = \lambda^*x.\lambda^*y.y(ax) \in \Phi$. Suppose $i \in I$ and $j \in J$. If $\alpha(j) = i$ then by Lemma 5.8

$$b\psi_i\mathrm{eq}_I(\alpha(j), i) \leq \mathrm{eq}_I(\alpha(j), i)(a\psi_i) \leq a\psi_i \leq a\psi_i = a\psi_{\alpha(j)} \leq \varphi_j$$

and otherwise we have

$$b\psi_i \mathrm{eq}_I(\alpha(j), i) \leq \mathrm{eq}_I(\alpha(j), i)(a\psi_i) \leq \varphi_j$$

again by Lemma 5.8. Thus we have shown that

$$\forall i \in I, j \in J.\, b\psi_i \mathrm{eq}_I(\alpha(j), i) \leq \varphi_i$$

from which it follows by Lemma 5.7 that there is a $c \in \Phi$ with

$$\forall i \in I, j \in J.\, c\psi_i \leq \mathrm{eq}_I(\alpha(j), i) {\to} \varphi_j$$

Thus we have

$$\forall i \in I.\ c\psi_i \leq \bigcap_{j \in J} \mathrm{eq}_I(\alpha(j), i) {\to} \varphi_j = \forall_\alpha(\varphi)_i$$

i.e. $\psi \vdash_I \forall_\alpha(\varphi)$ as desired.

For showing that $\forall$ satisfies the (Beck-)Chevalley condition suppose

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] & J \arrow[d, "\alpha"] \\ K \arrow[r, "\beta"'] & I \end{tikzcd}$$

is a pullback in **Set** and $\varphi \in [J, \mathbb{A}]_\Phi$. We have to show that $\beta^* \forall_\alpha \varphi \cong \forall_p q^* \varphi$. Notice that $\beta^* \forall_\alpha \varphi \vdash_K \forall_p q^* \varphi$ holds anyway by abstract nonsense. Thus it suffices to show that $\forall_p q^* \varphi \vdash_K \beta^* \forall_\alpha \varphi$. For this by Lemma 5.6 it suffices to show that for every $k \in K$ the term SI sends elements of $(\forall_p q^* \varphi)_k|$ to elements of $|(\beta^* \forall_\alpha \varphi)_k|$. Suppose $k \in K$. We have

$$|(\forall_p q^* \varphi)_k| = \bigcap_{z \in P} \left|\mathrm{eq}_K(p(z), k) {\to} \varphi_{q(z)}\right|$$

and

$$|(\beta^* \forall_\alpha \varphi)_k| = \bigcap_{j \in J} |\mathrm{eq}_I(\alpha(j), \beta(k)) {\to} \varphi_j|$$

Suppose $t \in \bigcap_{z \in P} \left|(\mathrm{eq}_K(p(z), k) {\to} \varphi_{q(z)}\right|$ and $j \in J$. Suppose $\alpha(j) = \beta(k)$. Then there is a $z \in P$ with $p(z) = k$ and $q(z) = j$. By assumption on t we have $t \in \left|(\mathrm{eq}_K(p(z), k) {\to} \varphi_{q(z)}\right|$ and thus $t \in |\mathrm{eq}_I(\alpha(j), \beta(k)) {\to} \varphi_j|$ since $\mathrm{eq}_K(p(z), k) = \mathrm{eq}_I(\alpha(j), \beta(k))$. Thus by Lemma 5.9 i) we have $\mathsf{SI}t \in |\mathrm{eq}_I(\alpha(j), \beta(k)) {\to} \varphi_j|$ since $\mathrm{eq}_I(\alpha(j), \beta(k)) = \{\mathsf{I}\}^\perp$. Otherwise if $\alpha(j) \neq \beta(k)$ then $\mathsf{SI}t \in |\mathrm{eq}_I(\alpha(j), \beta(k)) {\to} \varphi_j|$ by Lemma 5.9 ii) since $\mathrm{eq}_I(\alpha(j), \beta(k)) = \Pi$. Thus, in any case $\mathsf{SI}t \in |\mathrm{eq}_I(\alpha(j), \beta(k)) {\to} \varphi_j|$.

Next we show that there exists a generic predicate T. Let $\Sigma = \mathbb{A}$ and $\mathsf{T} = \mathrm{id}_\mathbb{A} \in [\mathbb{A}, \mathbb{A}]_\Phi$. Then for $\varphi \in [I, \mathbb{A}]_\Phi$ we have $\varphi = \varphi^* \mathsf{T}$ as desired.

It is well known that the remaining logical structure can be obtained from the already established one by second order encoding à la Russell-Prawitz. $\square$

For every tripos the equality predicate on I is given by $\exists_{\delta_I}(\top_I)$ where $\delta_I = \langle \mathrm{id}_I, \mathrm{id}_I \rangle$ is the diagonal on I and $\exists_{\delta_I} \dashv \delta_I^*$. We observe that this notion of equality on I coincides with the one given by eq_I.

Lemma 5.11. For every set I and $\rho \in [I\times I, \mathbb{A}]_\Phi$ we have

$$\mathrm{eq}_I \vdash_{I\times I} \rho \quad \text{iff} \quad \top_I \vdash_I \delta_I^* \rho$$

and thus $\exists_{\delta_I}(\top_I) \cong \mathrm{eq}_I$.

Proof. Suppose $\mathrm{eq}_I \vdash_{I\times I} \delta_I^*\rho$. Then by Lemma 5.6 there is a term t with $\forall i,j \in I.\forall s \in |\mathrm{eq}_I(i,j)|.\, ts \in \rho(i,j)$. Then for all $i \in I$ the term $\mathsf{K}(t\mathsf{I})$ sends elements of $|\top|$ to elements of $|\rho(i,i)|$. Thus $\top_I \vdash_I \delta_I^*\rho$ by Lemma 5.6.

For the reverse direcetion suppose $\top_I \vdash_I \delta_I^*\rho$. Then there exists $a \in \mathbb{A}$ such that $a\top \leq \rho(i,i)$ for all $i \in I$. Thus by Lemma 5.8 we have $\mathrm{eq}_I(i,j)a \leq \rho(i,j)$ for all $i,j \in I$. Let $b \in \mathbb{A}$ with $bxy \leq yx$. Then we have $ba\mathrm{eq}_I(i,j) \leq eq_I(i,j)a \leq \rho(i,j)$ for all $i,j \in I$. Thus by Lemma 5.6 we have $\mathrm{eq}_I \vdash_{I\times I} \rho$ as desired. □

As described in (van Oosten 2008) the tripos $[-,\mathbb{A}]_\Phi$ induces a topos $\mathbf{Set}\big[[-,\mathbb{A}]_\Phi\big]$ which we may call the **classical realizability topos** induced by the abstract Krivine structure under consideration or simply **Krivine topos**.

As described in (van Oosten 2008) for any tripos P over a topos $\mathcal{S}$ there is a 'constant objects' functor ∇_{P} from $\mathcal{S}$ to the topos $\mathcal{S}[\mathsf{P}]$ induced by P. This functor sends $I \in \mathcal{S}$ to the object $\big(I, \exists_{\delta_I}(\top_I)\big)$. In our particular case this gives an embedding ∇ of **Set** into the classical realizability topos sending a set I to (I, eq_I).

6. Forcing within Classical Realizability

Let P be a meet-semilattice. We write pq as shorthand for $p \wedge q$. Let C be an upward closed subset of P. With every $X \subseteq P$ one associates¶

$$|X| = \{p \in P \mid \forall q.\,(\mathsf{C}(pq) \to X(q))\}$$

Such subsets of P are called propositions. We say

$$p \text{ forces } X \qquad \text{iff} \qquad p \in |X|$$

and want

$$\begin{array}{lll} p \text{ forces } X \to Y & \text{iff} & \forall q.\,(|X|(q) \to |Y|(pq)) \\ p \text{ forces } \forall i \in I.X_i & \text{iff} & \forall i \in I.\; p \text{ forces } X_i \end{array}$$

to hold. Obviously, we have

$$\begin{array}{l} p \text{ forces } X \to Y \quad \text{iff} \\ \forall q.\,(|X|(q) \to |Y|(pq)) \;\; \text{iff} \\ \forall q.\,(|X|(q) \to \forall r.\,(\mathsf{C}(pqr) \to Y(r))) \quad \text{iff} \\ \forall q,r.\,(\mathsf{C}(pqr) \to |X|(q) \to Y(r)) \quad \text{iff} \\ p \in \big|\{qr \mid |X|(q) \to Y(r)\}\big| \end{array}$$

and

$$p \text{ forces } \forall i \in I.X_i \quad \text{iff} \quad p \in \big|\textstyle\bigcap_{i\in I} X_i\big|$$

¶ Traditionally, one would associate with X the set $X^\perp = \{p \in P \mid \forall q \in X.\, \neg\mathsf{C}(pq)\}$. But, classically, we have $|X| = (P \setminus X)^\perp$.

As in (Krivine 2008) we want to consider this construction inside a classical realizability topos. That this gives a topos again follows from Pitts' *iteration theorem* explained in (van Oosten 2008) and in (Hofstra 2008). It says that for any tripos P over a topos $\mathcal{S}$ and any tripos Q over $\mathcal{S}[\mathsf{P}]$ the resulting topos $\mathcal{S}[\mathsf{P}][\mathsf{Q}]$ is again induced by a tripos provided the functor $\nabla_\mathsf{Q} : \mathcal{S}[\mathsf{P}] \to \mathcal{S}[\mathsf{P}][\mathsf{Q}]$ preserves epis, namely by the tripos $(\nabla_\mathsf{Q}\nabla_\mathsf{P})^*\mathsf{Sub}_{\mathcal{S}[\mathsf{P}][\mathsf{Q}]}$. The requirement on ∇_Q is certainly satisfied in our case because Q is localic over $\mathcal{S}[\mathsf{P}]$. Alas, it is not obvious from general reasons that the tripos $(\nabla_\mathsf{Q}\nabla_\mathsf{P})^*\mathsf{Sub}_{\mathcal{S}[\mathsf{P}][\mathsf{Q}]}$ is induced by an appropriate aks. That this is the case nevertheless has been shown in (Krivine 2008). Our aim now is to reveal the intuition behind his construction.

Actually, in most cases P will not be a meet-semilattice inside a classical realizability topos **but** it will be so "from point of view" of $\mathsf{C} \subseteq P$. That means that as e.g. in (Krivine 2008) we are given an external[‖] set P, a distinguished element $1 \in P$, a binary operation on P (denoted by juxtaposition) and a predicate[††] $\mathsf{C} : P \to \mathcal{P}_{\perp\!\!\!\perp}(\Lambda)$ such that the following conditions hold in the classical realizability topos

$$\mathsf{C}(p(qr)) \leftrightarrow \mathsf{C}((pq)r)$$

$$\mathsf{C}(pq) \leftrightarrow \mathsf{C}(qp)$$

$$\mathsf{C}(p) \leftrightarrow \mathsf{C}(pp)$$

$$\mathsf{C}(1p) \leftrightarrow \mathsf{C}(p)$$

$$\big(\mathsf{C}(p) \leftrightarrow \mathsf{C}(q)\big) \to \big(\mathsf{C}(pr) \leftrightarrow \mathsf{C}(qr)\big)$$

together with

$$\mathsf{C}(pq) \to \mathsf{C}(p)$$

expressing that C is upward closed. On P we may define a congruence

$$p \simeq q \equiv \forall r. \, (C(rp) \leftrightarrow C(rq))$$

w.r.t. which P is a commutative idempotent monoid, i.e. a meet-semilattice, inside the classical realizability topos of which C is an upward closed subset whose complement contains at most one elemnt.

A term t realizes p forces $X \to Y$ iff

$$\forall q, r. \forall u {\in} \mathsf{C}(pqr). \forall s {\in} |X|(q). \forall \pi {\in} Y(r). \; t * u.s.\pi \in \perp\!\!\!\perp$$

Thus, one may want to define a notion of a pair (t, p) realizing $X \to Y$. For this purpose one has to find a new structure whose term part is $\Lambda \times P$. For this purpose Krivine has defined in (Krivine 2008) application and push as

$$(t,p)(s,q) = (ts, pq) \quad (t,p).(s,\pi) = (t * s, pq)$$

and a new relation $\perp\!\!\!\perp\!\!\!\perp \subseteq (\Lambda \times P) * (\Pi \times P)$ as

$$(t,p) * (\pi, q) \in \perp\!\!\!\perp\!\!\!\perp \;\text{ iff }\; \forall u \in \mathsf{C}(pq) \; t * \pi^u \in \perp\!\!\!\perp$$

‖ i.e. P is an object of **Set**

†† which induces a predicate $\mathsf{C}^\perp$ on P in the classical realizability topos

where π^u is obtained from π by inserting u at its bottom. Propositions w.r.t. this new aks are now subsets $\Pi \times P$ understood as functions from $P \to \mathcal{P}(\Pi)$. Now given such propositions X and Y we have

$$(t,p) \in |X \to Y| \quad \text{iff}$$

$$\forall (s,q) \in |X|.\forall (r,\pi) \in Y.\ (t,p) * (s,q).(\pi,r) \in \perp\!\!\!\perp \quad \text{iff}$$

$$\forall (s,q) \in |X|.\forall (r,\pi) \in Y.\forall u \in \mathsf{C}(pqr).\ t * s.\pi^u \in \perp\!\!\!\perp$$

in accordance with the above explication of t realizes p forces $X \to Y$. The only difference is that the realizer u of $\mathsf{C}(pqr)$ is now placed at a distinguished position, namely the bottom of the stack.

In order to jump back and forth between

$$t \text{ realizes } p \text{ forces } A \quad \text{and} \quad (t',p) \in |A|$$

in (Krivine 2008) there have been introduced "read" and "write" constructs in the original aks, namely commands χ and χ' whose operational semantics is given by

$$\begin{array}{llll} \text{(read)} & \chi * t.\pi^s \in \perp\!\!\!\perp & \text{whenever} & t * s.\pi \in \perp\!\!\!\perp \\ \text{(write)} & \chi' * t.s.\pi \in \perp\!\!\!\perp & \text{whenver} & t * \pi^s \in \perp\!\!\!\perp \end{array}$$

Using these one can transform t into t' and *vice versa*. Krivine concludes from this that for **realizing forcing one needs global memory**.

7. Conclusion

We have identified a notion of abstract Krivine structure as an axiomatic account of Krivine's classical realizability. Based on this we have shown in which precise sense Cohen forcing is the commutative case of classical realizability.

We have shown how Krivine's work on classical realizability can be seen as an instance of the categorical approach to realizability as initiated by Martin Hyland. This has been achieved by associating with every abstract Krivine structure an order pca which together with the filter of nonempty propositions induces a boolean tripos, i.e. a categorical model of higher order classical logic. This tripos gives rise to the ensuing classical realizability topos. This view allows us to get a more structural understanding of forcing within classical realizability using Pitts' Iteration Theorem.

We leave it as an open question whether techniques of Algebraic Set Theory (van den Berg & Moerdijk 2009) can be used for showing that every abstract Krivine structure gives rise to a model for ZF.

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