

# Arithmetic is Categorical

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The core of the argument in the following proof is due to Charles McCarty ([3]); we present a reinterpretation and mild generalization of it.

Let  $\mathbf{I}\Sigma_1^i$  be the theory in the language  $\{0, S, +, \cdot, \leq\}$  axiomatized by the axioms of  $Q_{\leq}$  (see [1]) and induction for  $\Sigma_1$ -formulas; but based on intuitionistic logic.

**Theorem 0.1** *In the effective topos  $\mathcal{E}ff$  there exists (up to isomorphism) precisely one model of  $\mathbf{I}\Sigma_1^i$ , namely the standard model  $N$  (the canonical structure on the natural numbers object).*

**Proof.** We recall that  $\mathbf{I}\Sigma_1^i$  proves decidability of all  $\Delta_0$ -formulas. Hence every model of  $\mathbf{I}\Sigma_1^i$  must be a decidable object in  $\mathcal{E}ff$ , and therefore isomorphic to a modest set  $(X, E)$ .

Since such a model  $(X, E)$  has an element 0 and an injective endofunction  $S$ , there is an embedding from  $N$  into it: a function  $i : \mathbb{N} \rightarrow X$  such that for some total recursive function  $t$  we have  $t(n) \in E(n)$  for all  $n \in \mathbb{N}$ . The decidability of  $(X, E)$  means that there is a partial recursive function  $d$  which is defined on the set  $(\bigcup_{x \in X} E(x))^2$ , and satisfies:

$$d(k, l) = 0 \Leftrightarrow \text{there is } x \in X \text{ with } k, l \in E(x)$$

Now if  $x \in X$  is in the image of  $i$  then for each  $a \in E(x)$  there is a unique  $n \in \mathbb{N}$  such that  $d(a, t(n)) = 0$ ; and this  $n$  can be found recursively in  $a$ . We conclude:

*The map  $i$  embeds  $N$  as  $\neg\neg$ -closed subobject in  $(X, E)$*

Therefore, if the function  $i$  is surjective, it is an isomorphism.

For the sake of a contradiction, suppose  $i$  is not an isomorphism. Then there is an element  $c \in X$  which is not in the image of  $i$ , and by decidability of the linear order and the fact that  $i$  embeds  $N$  as downwards closed subset (which is all provable in  $\mathbf{I}\Sigma_1^i$ ) we have  $\mathcal{E}ff \models \forall n : N. i(n) < c$ .

Now, we can copy McCarty's argument. Since  $\mathbf{I}\Sigma_1^i$  proves the representability and totality of all primitive recursive functions, let  $\exists z T'(e, x, y, z)$  and  $\exists w U'(x, i, w)$  be  $\Sigma_1$ -formulas (so  $T'$  and  $U'$  are  $\Delta_0$ ) representing the Kleene  $T$ -predicate  $T(e, x, y)$  and result extracting function  $U(x) = i$ , respectively. Define the subobject  $A$  of  $(X, E)$  internally by

$$A = \{x < c \mid \forall y < c \neg \exists z < c \exists w < c (T'(x, x, y, z) \wedge U'(y, 1, w))\}$$

Then since  $A$  is given by a  $\Delta_0$ -formula,  $A$  is a decidable subobject of  $(X, E)$  and hence  $i^{-1}(A)$  is a decidable subobject of  $N$ ; which means that

$$R = \{n \in \mathbb{N} \mid \mathcal{E}ff \models n \in i^{-1}(A)\}$$

is a recursive subset of  $\mathbb{N}$ .

Moreover, for the following subsets of  $\mathbb{N}$ :

$$\begin{aligned} A_0 &= \{n \in \mathbb{N} \mid \varphi_n(n) = 0\} \\ A_1 &= \{n \in \mathbb{N} \mid \varphi_n(n) = 1\} \end{aligned}$$

we have  $A_0 \subset R$  and  $A_1 \cap R = \emptyset$ .

So,  $R$  is a recursive separation of the sets  $A_0$  and  $A_1$ , but it is well-known that this is impossible. ■

**Corollary 0.2** *Let IZF be intuitionistic set theory (as formulated in, e.g., [2]). Then IZF does not prove that there is a model of Peano Arithmetic. Moreover, IZF does not prove that there is a model of  $\text{IS}_1^i$  which is not a model of full Heyting Arithmetic.*

**Proof.** In [4], it is shown that the Friedman-McCarty realizability interpretation of IZF can be seen as an interpretation of IZF in  $\mathcal{E}ff$ . Any model as in the corollary would thus give rise to one such model in  $\mathcal{E}ff$ , which we have shown not to exist. ■

We conclude that whoever predicates his notion of truth on the effective topos, must accept the following nonstandard conclusions:

- a) Peano Arithmetic is ‘inconsistent’ (it has no models)
- b) Heyting Arithmetic is essentially finitely axiomatized (it is equivalent to  $\text{IS}_1^i$ ).

## References

- [1] Samuel R. Buss. First-order Proof Theory of Arithmetic. In S.R. Buss, editor, *Handbook of Proof Theory*. Elsevier, 1998.
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- [3] Charles McCarty. Variations on a Thesis: Intuitionism and Computability. *The Notre Dame Journal of Formal Logic*, 28(4):536–580, 1987.
- [4] J. van Oosten. *Realizability: an Introduction to its Categorical Side*, volume 152 of *Studies in Logic*. North-Holland, 2008.