

# More on Geometric Morphisms between Realizability Toposes

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## Abstract

Geometric morphisms between realizability toposes are studied in terms of morphisms between partial combinatory algebras (pcas). The morphisms inducing geometric morphisms (the *computationally dense* ones) are seen to be the ones whose ‘lifts’ to a kind of completion have right adjoints. We characterize topos inclusions corresponding to a general form of relative computability. We characterize pcas whose realizability topos admits a geometric morphism to the effective topos.

**Keywords:** realizability toposes, partial combinatory algebras, geometric morphisms, local operators.

## Introduction

The study of geometric morphisms between realizability toposes was initiated by John Longley in his thesis [12]. Longley started an analysis of partial combinatory algebras (the structures underlying realizability toposes; see section 1.1) by defining a 2-categorical structure on them.

Longley’s “applicative morphisms” characterize regular functors between categories of assemblies that commute with the global sections functors to  $\text{Set}$ . Longley was thus able to identify a class of geometric morphisms with adjunctions between partial combinatory algebras. The geometric morphisms thus characterized satisfy two constraints:

- 1) They are *regular*, that is: their direct image functors preserve regular epimorphisms.
- 2) They restrict to geometric morphisms between categories of assemblies.

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Restriction 1 was removed by Pieter Hofstra and the second author in [4], where a new class of applicative morphisms was defined, the *computationally dense* ones; these are exactly those applicative morphisms for which the induced regular functor on assemblies has a right adjoint (but the morphism itself need not have a right adjoint in the 2-category of partial combinatory algebras).

Restriction 2 was removed by Peter Johnstone in his recent paper [7], where he proved that *every* geometric morphism between realizability toposes satisfies this condition.

Moreover, Johnstone gave a much simpler formulation of the notion of computational density.

In the present paper we characterize the computationally dense applicative morphisms in yet another way: as those which, when “lifted” to the level of *order-pcas*, do have a right adjoint. We also have a criterion for when the geometric morphism induced by a computationally dense applicative morphism is an inclusion.

In a short section we collect some material on total combinatory algebras, and formulate a criterion for when a partial combinatory algebra is isomorphic to a total one.

We prove that every realizability topos which is a subtopos of Hyland’s *effective topos* is on a partial combinatory algebra of computations with an “oracle” for a partial function on the natural numbers. We employ a generalization of this “computations with an oracle for  $f$ ” construction to arbitrary partial combinatory algebras, described in [16] and denoted  $A[f]$ . Generalizing results by Hyland ([5]) and Phoa ([13]), we show that the inclusion of the realizability topos on  $A[f]$  into the one on  $A$  corresponds to the least local operator “forcing  $f$  to be realizable”.

The paper closes with some results about local operators in realizability toposes. We characterize the realizability toposes which admit a (necessarily essentially unique) geometric morphism to the effective topos, as those which have no De Morgan subtopos apart from  $\text{Set}$ .

In an effort to be self-contained, basic material is collected in section 1, which also establishes notation and terminology.

## 1 Background

### 1.1 Partial Combinatory Algebras

A *partial combinatory algebra* (or, as Johnstone calls them in [8, 7], *Schönfinkel algebra*) is a structure with a set  $A$  and a partial binary function on it, which we denote by  $a, b \mapsto ab$ . This map is called *application*; the idea is that every element of  $A$  encodes a partial function on  $A$ , and  $ab$  is the result of the function encoded by  $a$  applied to  $b$ .

The motivating example is the structure  $\mathcal{K}_1$  on the set of natural numbers, where  $ab$  is the outcome of the  $a$ -th Turing machine with input  $b$ .

Partial functions give rise to partial terms. In manipulating these we employ the following notational conventions:

- 1) The expression  $t \downarrow$  means that the term  $t$  is defined, or: denotes an element of  $A$ . We intend  $t \downarrow$  to also imply that  $s \downarrow$  for every subterm  $s$  of  $t$ .
- 2) We employ association to the left:  $abc$  means  $(ab)c$ . This economizes on brackets, but we shall be liberal with brackets wherever confusion is possible.
- 3) The expression  $s \preceq t$  means: whenever  $t$  denotes, so does  $s$ ; and in that case,  $s$  and  $t$  denote the same element of  $A$ . We write  $s \simeq t$  for the conjunction of  $s \preceq t$  and  $t \preceq s$ . The expression  $t = s$  means  $s \simeq t$  and  $t \downarrow$ .

With these conventions, we define:

**Definition 1.1** A set  $A$  with a partial binary map on it is a *partial combinatory algebra* (pca) if there exist elements  $k$  and  $s$  in  $A$  which satisfy, for all  $a, b, c \in A$ :

- i)  $kab = a$
- ii)  $sab \downarrow$
- iii)  $sabc \preceq ac(bc)$

This definition is mildly nonstandard, since most sources require  $\simeq$  instead of  $\preceq$  in clause iii). However, in our paper [3] we show that in fact, every pca in our sense is isomorphic to a pca in the stronger sense (where the isomorphism is in the sense of applicative morphisms, see section 1.3), so the two definitions are essentially the same.

It is a consequence of definition 1.1 that for any term  $t$  which contains variables  $x_1, \dots, x_{n+1}$ , there is a term  $\langle x_1 \cdots x_{n+1} \rangle t$  without any variables, which has the following property: for all  $a_1, \dots, a_{n+1} \in A$  we have

$$\begin{aligned} & (\langle x_1 \cdots x_{n+1} \rangle t) a_1 \cdots a_n \downarrow \\ & (\langle x_1 \cdots x_{n+1} \rangle t) a_1 \cdots a_{n+1} \preceq t(a_1, \dots, a_{n+1}) \end{aligned}$$

Every pca  $A$  has *pairing* and *unpairing* combinators: there are elements  $\pi, \pi_0, \pi_1$  of  $A$  satisfying  $\pi_0(\pi ab) = a$  and  $\pi_1(\pi ab) = b$ .

Moreover, every pca has *Booleans*  $\mathsf{T}$  and  $\mathsf{F}$  and a *definition by cases* operator: an element  $u$  satisfying  $u\mathsf{T}ab = a$  and  $u\mathsf{F}ab = b$ ; such an element  $u$  is seen as operating on three arguments  $v, a, b$ , which operation is often denoted by

$$\text{if } v \text{ then } a \text{ else } b$$

In this paper we assume that  $\mathsf{T} = k$  and  $\mathsf{F} = k(\text{skk})$ , so  $\mathsf{T}ab = a$  and  $\mathsf{F}ab = b$ .

Finally, we mention that every pca  $A$  comes equipped with a copy  $\{\bar{n} \mid n \in \mathbb{N}\}$  of the natural numbers: the *Curry numerals*. For every  $n$ -ary partial computable function  $F$ , there is an element  $a_F \in A$ , such that for all  $n$ -tuples of natural numbers  $k_1, \dots, k_n$  in the domain of  $F$ ,  $a_F \bar{k}_1 \cdots \bar{k}_n = \overline{F(k_1, \dots, k_n)}$ . For more background on pcas we refer to [17], chapter 1.

## 1.2 Assemblies and Realizability Toposes

Every pca determines a category of *assemblies* on  $A$ , denoted  $\text{Ass}(A)$ . An object of  $\text{Ass}(A)$  is a pair  $(X, E)$  where  $X$  is a set and  $E$  associates to each element  $x$  of  $X$  a nonempty subset  $E(x)$  of  $A$ . A morphism  $(X, E) \rightarrow (Y, F)$  between assemblies on  $A$  is a function  $f : X \rightarrow Y$  of sets, for which there is an element  $a \in A$  which *tracks*  $f$ , which means that for every  $x \in X$  and every  $b \in E(x)$ ,  $ab \downarrow$  and  $ab \in F(f(x))$ .

The category  $\text{Ass}(A)$  has finite limits and colimits, is locally cartesian closed (hence regular), has a natural numbers object and a strong-subobject classifier (which is called a *weak subobject classifier* in [6]); hence it is a quasitopos.

There is an adjunction  $\text{Set} \begin{matrix} \xleftarrow{\Gamma} \\ \xrightarrow{\nabla} \end{matrix} \text{Ass}(A)$ ,  $\Gamma \dashv \nabla$ : here  $\Gamma$  is the global sections functor (or the forgetful functor  $(X, E) \mapsto X$ ) and  $\nabla$  sends a set  $X$  to the assembly  $(X, E)$  where  $E(x) = A$  for every  $x \in X$ .

The category  $\text{Ass}(A)$  is, except in the trivial case  $A = 1$ , not *exact*. Its exact completion as a regular category (sometimes denoted  $\text{Ass}(A)_{\text{ex/reg}}$ ) is a topos, the *realizability topos on  $A$* , which we denote by  $\text{RT}(A)$  with only one exception: the topos  $\text{RT}(\mathcal{K}_1)$  is called the *effective topos* and denoted  $\mathcal{E}ff$ . The effective topos was discovered by Martin Hyland around 1979 and described in the landmark paper [5]. The notation  $\mathcal{E}ff$  serves both to underline the special place of the effective topos among realizability toposes (as we shall see in this paper) and the special place of  $\mathcal{K}_1$  among pcas, and to acknowledge the seminal character of Hyland's work.

## 1.3 Morphisms of Pcas

In his thesis [12], John Longley laid the groundwork for the study of the dynamics of pcas, by defining a useful 2-category structure on the class of pcas.

**Definition 1.2** Let  $A$  and  $B$  be pcas. An *applicative morphism*  $A \rightarrow B$  is a total (or, as some people prefer, 'entire') relation from  $A$  to  $B$ , which we see as a map  $\gamma$  from  $A$  to the collection of nonempty subsets of  $B$ , which has a *realizer*, that is: an element  $r \in B$  satisfying the following condition: whenever  $a, a' \in A$  are such that  $aa' \downarrow$ , and  $b \in \gamma(a), b' \in \gamma(a')$ , then  $rbb' \downarrow$  and  $rbb' \in \gamma(aa')$ .

Given two applicative morphisms  $\gamma, \delta : A \rightarrow B$  we say  $\gamma \leq \delta$  if some element  $s$  of  $B$  satisfies: for every  $a \in A$  and  $b \in \gamma(a)$ ,  $sb \downarrow$  and  $sb \in \delta(a)$ .

Pcas, applicative morphisms and inequalities between them form a preordered-enriched category. Applicative morphisms have both good mathematical properties and a computational intuition: if a pca is thought of as a model of computation, then an applicative morphism is a simulation of one model into another.

A functor  $\text{Ass}(A) \xrightarrow{F} \text{Ass}(B)$  between categories of assemblies is called a  $\Gamma$ -*functor* if the diagram

$$\begin{array}{ccc} \text{Ass}(A) & \xrightarrow{\quad} & \text{Ass}(B) \\ & \searrow \Gamma & \swarrow \Gamma \\ & \text{Set} & \end{array}$$

commutes up to isomorphism.

Mathematically, applicative morphisms correspond to regular  $\Gamma$ -functors between categories of assemblies. To be precise, we have the following theorem:

**Theorem 1.3 (Longley)** *There is a biequivalence between the following two 2-categories:*

- 1) *the category of pcas, applicative morphisms and inequalities between them;*
- 2) *the category of categories of the form  $\text{Ass}(A)$ , regular  $\Gamma$ -functors and natural transformations.*

*One side of the biequivalence is given as follows: if  $\gamma : A \rightarrow B$  is an applicative morphism, the functor  $\gamma^* : \text{Ass}(A) \rightarrow \text{Ass}(B)$  which sends  $(X, E)$  to  $(X, \gamma \circ E)$  (composition of relations) is the corresponding regular  $\Gamma$ -functor.*

Since  $\text{RT}(A)$  is the ex/reg completion of  $\text{Ass}(A)$ , any functor of the form  $\gamma^*$  extends essentially uniquely to a regular functor  $\text{RT}(A) \rightarrow \text{RT}(B)$ , which we also denote by  $\gamma^*$ . So it makes sense to study geometric morphisms  $\text{RT}(A) \rightarrow \text{RT}(B)$  from the point of view of applicative morphisms  $A \rightarrow B$ : since the inverse image functor of any geometric morphism is regular, in order to study geometric morphisms  $\text{RT}(B) \rightarrow \text{RT}(A)$  one looks at those applicative morphisms  $\gamma : A \rightarrow B$  for which  $\gamma^*$  has a right adjoint.

The following definition is from [4]. Let us extend our notational conventions about application a bit: for  $a \in A, \alpha \subseteq A$  we write  $a\alpha \downarrow$  if  $ax \downarrow$  for every  $x \in \alpha$ , and in this case we write  $a\alpha$  for the set  $\{ax \mid x \in \alpha\}$ .

**Definition 1.4** An applicative morphism  $\gamma : A \rightarrow B$  is *computationally dense* if there is an element  $m \in B$  such that the following holds:

For every  $b \in B$  there is an  $a \in A$  such that for all  $a' \in A$ : if  $b\gamma(a') \downarrow$ , then  $aa' \downarrow$  and  $m\gamma(aa') \downarrow$  and  $m\gamma(aa') \subseteq b\gamma(a')$ .

**Theorem 1.5 ([4])** *An applicative morphism  $\gamma : A \rightarrow B$  induces a geometric morphism  $\text{RT}(B) \rightarrow \text{RT}(A)$  precisely when it is computationally dense.*

Obvious drawbacks of this theorem are the logical complexity of the definition of ‘computationally dense’ and the fact that, prima facie, the theorem only says something about geometric morphisms which are induced by a  $\Gamma$ -functor between categories of assemblies, in other words: geometric morphisms  $\text{RT}(B) \rightarrow \text{RT}(A)$  for which the inverse image functor maps assemblies to assemblies. Both these issues were successfully addressed in Peter Johnstone’s paper [7]:

**Theorem 1.6 (Johnstone)** *An applicative morphism  $\gamma : A \rightarrow B$  is computationally dense if and only if there exist an element  $r \in B$  and a function  $g : B \rightarrow A$  satisfying: for all  $b \in B$  and all  $b' \in \gamma(g(b))$ ,  $rb' = b$ .*

We might, extending the notation for inequalities between applicative morphisms, express the last property as:  $\gamma g \leq \text{id}_B$ .

**Theorem 1.7 (Johnstone)** *i) For any geometric morphism  $f : \text{RT}(B) \rightarrow \text{RT}(A)$ , the diagrams*

$$\begin{array}{ccc} \text{Set} & \xrightarrow{\text{id}} & \text{Set} \\ \downarrow & & \downarrow \\ \text{RT}(B) & \xrightarrow{f} & \text{RT}(A) \end{array}$$

*(where the vertical arrows embed  $\text{Set}$  as the category of  $\neg\neg$ -sheaves) is a bipullback in the 2-category of toposes and geometric morphisms.*

*ii) For every geometric morphism  $f : \text{RT}(B) \rightarrow \text{RT}(A)$ , the inverse image functor  $f^*$  preserves assemblies.*

We shall be saying more about this theorem in section 2. For the moment, we continue our treatment of material from the literature, inasmuch it is relevant for our purposes.

**Definition 1.8** A geometric morphism is called *regular* if its direct image functor is a regular functor.

Clearly, by theorems 1.3 and 1.7, a regular geometric morphism  $\text{RT}(B) \rightarrow \text{RT}(A)$  arises from an adjunction in the 2-category of pcas; and therefore Longley studied such adjunctions in his thesis. First, he distinguished a number of types of applicative morphisms:

**Definition 1.9 (Longley)** Let  $\gamma : A \rightarrow B$  be an applicative morphism.

- i)  $\gamma$  is called *decidable* if there is an element  $d \in B$  such that for all  $b \in \gamma(\top_A)$ ,  $db = \top_B$ , and for all  $b \in \mathbf{F}_A$ ,  $db = \mathbf{F}_B$ .
- ii)  $\gamma$  is called *discrete* if  $\gamma(a) \cap \gamma(a') = \emptyset$  whenever  $a \neq a'$ .
- iii)  $\gamma$  is called *projective* if  $\gamma$  is isomorphic to an applicative morphism which is single-valued.

Among other things, Longley proved the statements in the following theorem:

**Theorem 1.10 (Longley)** *Let  $A \xrightleftharpoons[\gamma]{\delta} B$  be a pair of applicative morphisms.*

- i) If  $\gamma\delta \leq \text{id}_B$  then  $\gamma$  is decidable and  $\delta$  is discrete.*
- ii) If  $\gamma \dashv \delta$  then  $\gamma$  is projective.*

- iii) If  $\gamma \dashv \delta$  and  $\delta\gamma \simeq \text{id}_A$  then both  $\delta$  and  $\gamma$  are discrete and decidable.
- iv)  $\gamma$  is decidable if and only if  $\gamma^*$  preserves finite sums, if and only if  $\gamma^*$  preserves the natural numbers object.
- v)  $\gamma$  is projective if and only if  $\gamma^*$  preserves regular projective objects.
- vi)  $\gamma$  is discrete if and only if  $\gamma^*$  preserves discrete objects.
- vii) There exists, up to isomorphism, exactly one decidable applicative morphism  $\mathcal{K}_1 \rightarrow A$ , for any pca  $A$ .

From theorem 1.10 and theorem 1.7 we can draw some immediate inferences:

**Corollary 1.11** *Let  $\gamma : A \rightarrow B$  be an applicative morphism.*

- i) *If  $\gamma$  is computationally dense, then  $\gamma$  is decidable.*
- ii) *If  $\gamma$  is computationally dense and the geometric morphism  $\text{RT}(B) \rightarrow \text{RT}(A)$  induced by  $\gamma$  is regular, then  $\gamma$  is projective.*
- iii) *There exists, up to isomorphism, at most one geometric morphism  $\text{RT}(A) \rightarrow \mathcal{E}\text{ff}$ ; and there is one if and only if the essentially unique decidable morphism from  $\mathcal{K}_1$  to  $A$  is computationally dense.*

Example 2.6 exhibits a computationally dense applicative morphism which is not projective, and therefore cannot have a right adjoint on the level of pcas. Theorem 3.5 gives a criterion for iii) to hold, in terms of local operators on realizability toposes.

Let us draw one more corollary from theorem 1.10:

**Corollary 1.12** *Let  $\gamma$  be computationally dense. Then the geometric morphism induced by  $\gamma$  is regular, if and only if  $\gamma$  has a right adjoint in PCA, if and only if  $\gamma$  is projective.*

**Proof.** The first equivalence was already stated after definition 1.8, and is a direct consequence of the biequivalence expressed by theorem 1.3. For the second equivalence, if  $\gamma \dashv \delta$  then  $\gamma$  is projective by 1.10ii); conversely, if  $\gamma$  is projective then by 1.10v), the functor  $\gamma^*$  preserves regular projective objects, which, given that categories of assemblies always have enough regular projectives, is the case if and only if the right adjoint of  $\gamma^*$  preserves regular epimorphisms, and is therefore induced by some applicative morphism  $\delta$ , which by 1.3 must be right adjoint to  $\gamma$  in PCA. ■

## 1.4 Order-pcas

Although most of our results are about ordinary pcas, the generalization to order-pcas, first defined in [15] and elaborated on in [4], has its advantages for the formulation of some results.

**Definition 1.13** An *order-pca* is a partially ordered set  $A$  with a partial binary application function  $(a, b) \mapsto ab$ ; there are also elements  $k$  and  $s$ , and the axioms are:

- i) If  $ab \downarrow$ ,  $a' \leq a$  and  $b' \leq b$  then  $a'b' \downarrow$  and  $a'b' \leq ab$
- ii)  $kab \leq a$
- iii)  $sab \downarrow$  and whenever  $ac(bc) \downarrow$ ,  $sabc \downarrow$  and  $sabc \leq ac(bc)$

**Definition 1.14** An *applicative morphism* of order-pcas  $A \rightarrow B$  is a function  $f : A \rightarrow B$  satisfying the following requirements:

- i) There is an element  $r \in B$  such that whenever  $aa' \downarrow$  in  $A$ ,  $rf(a)f(a') \downarrow$  in  $B$ , and  $rf(a)f(a') \leq f(aa')$ .
- ii) There is an element  $u \in B$  such that whenever  $a \leq a'$  in  $A$ ,  $uf(a) \downarrow$  and  $uf(a) \leq f(a')$  in  $B$ .

Just as for pcas, we have an order on applicative morphisms, which is analogously defined.

Every order-pca  $A$  determines a category of assemblies: objects are pairs  $(X, E)$  where  $X$  is a set and  $E(x)$  is a nonempty, downward closed subset of  $A$ , for each  $x \in X$ ; morphisms are set-theoretic functions which are tracked just as in the definition for pcas.

On the 2-category of order-pcas there is a 2-monad  $T$ , which at the same time gives the prime examples of interest of genuine order-pcas:  $T(A)$  is the order-pca consisting of nonempty, downward closed subsets of  $A$ , with the inclusion ordering; for  $\alpha, \beta \in T(A)$ , we say  $\alpha\beta \downarrow$  if and only if for all  $a \in \alpha$  and  $b \in \beta$ ,  $ab \downarrow$  in  $A$ ; if that holds,  $\alpha\beta$  is the downward closure of the set  $\{ab \mid a \in \alpha, b \in \beta\}$ .

Note that when we consider applicative morphisms  $f$  to order-pcas of the form  $T(A)$ , we may assume that  $f$  is an order-preserving function; since the element  $u$  of 1.14ii) allows us to find an isomorphism between  $f$  and the map  $x \mapsto \bigcup_{y \leq x} f(y)$ .

The category of assemblies on the order-pca  $T(A)$  has enough regular projectives: a  $T(A)$ -assembly  $(X, E)$  is regular projective if and only if (up to isomorphism)  $E(x)$  is a *principal downset* of  $T(A)$  for each  $x$ ; i.e.,  $E(x) = \{\alpha \subseteq A \mid \alpha \subseteq \beta\}$  for some  $\beta \in T(A)$ . It is now easy to see that the full subcategory of  $\text{Ass}(T(A))$  on the regular projectives is equivalent to  $\text{Ass}(A)$ , and applying a criterion due to Carboni ([2]), one readily verifies

**Theorem 1.15** *The category of assemblies on  $T(A)$  is the regular completion of the category  $\text{Ass}(A)$ .*

## 1.5 Relative recursion

We also need to recall a construction given in [16]. Given a pca  $A$  and a partial function  $f : A \rightarrow A$ , we say that  $f$  is *representable* w.r.t. an applicative

morphism  $\gamma : A \rightarrow B$ , if there is an element  $b \in B$  which satisfies: for each  $a$  in the domain of  $f$  and each  $c \in \gamma(a)$ ,  $bc \downarrow$  and  $bc \in \gamma(f(a))$ . We say that  $f$  is *representable*, or *representable in A*, if  $f$  is representable w.r.t. the identity morphism on  $A$ .

There is a pca  $A[f]$  and a decidable applicative morphism  $\iota_f : A \rightarrow A[f]$  such that  $f$  is representable w.r.t.  $\iota_f$  and  $\iota_f$  is universal with this property: whenever  $\gamma : A \rightarrow B$  is a decidable applicative morphism w.r.t. which  $f$  is representable, then  $\gamma$  factors uniquely through  $\iota_f$ .

It follows that this property determines  $A[f]$  up to isomorphism, and hence, if  $f$  is representable in  $A$  then  $A$  and  $A[f]$  are isomorphic.

The applicative morphism  $\iota_f$  is computationally dense and induces an inclusion of toposes:  $\text{RT}(A[f]) \rightarrow \text{RT}(A)$ . Moreover,  $\iota_f$ , being the identity function on the level of sets, is projective as applicative morphism.

## 2 Geometric morphisms between realizability toposes

We start by formulating a variation on Longley's theorem 1.3. Recall the definition of order-pcas and the monad  $T$  from section 1.4. We wish to characterize finite limit-preserving  $\Gamma$ -functors between categories of assemblies.

**Definition 2.1** Let  $A, B$  be pcas. A *proto-applicative morphism* from  $A$  to  $B$  is an applicative morphism of order-pcas from  $T(A)$  to  $T(B)$ .

Note, that every applicative morphism  $A \rightarrow B$  induces a proto-applicative morphism; and the proto-applicative morphisms which arise in this way are exactly the maps of  $T$ -algebras. In this context,  $\gamma : T(A) \rightarrow T(B)$  is a map of  $T$ -algebras if  $\gamma$  (is isomorphic to a map which) preserves unions of subsets of  $A$ .

**Theorem 2.2** *There is a biequivalence between the following two 2-categories:*

- 1 *The category of pcas, proto-applicative morphisms and inequalities between them*
- 2 *The category of categories of the form  $\text{Ass}(A)$  for a pca  $A$ , finite limit-preserving  $\Gamma$ -functors and natural transformations*

*This biequivalence restricts to the biequivalence of Longley's theorem 1.3: regular  $\Gamma$ -functors correspond to  $T$ -algebra maps.*

**Proof.** Let  $\gamma : T(A) \rightarrow T(B)$  be an applicative morphism, realized by  $r \in B$ . Define  $\gamma^*(X, E) = (X, \gamma \circ E)$ . If  $f : (X, E) \rightarrow (Y, E')$  is tracked by  $t \in A$ , then

$$r\gamma(\{t\})\gamma(E(x)) \subseteq \gamma(E'(f(x)))$$

so whenever  $s \in \gamma(\{t\})$ ,  $rs$  tracks  $f$  as morphism  $(X, \gamma \circ E) \rightarrow (Y, \gamma \circ E')$ . So  $\gamma^*$  is a  $\Gamma$ -functor.

It is immediate that  $\gamma^*$  preserves terminal objects and equalizers; that  $\gamma^*$  preserves finite products is similar to the proof of theorem 1.3 (for which the reader may consult either [12] or [17]).

If  $\gamma \leq \delta : T(A) \rightarrow T(B)$  is realized by  $\beta \in T(B)$  and  $b \in \beta$ , then  $b$  tracks every component of the unique natural transformation  $\gamma^* \Rightarrow \delta^*$ . Conversely, suppose there is a natural transformation  $\gamma^* \Rightarrow \delta^*$ , consider its component at the object  $(T(A), i)$  where  $i$  is the identity function. Any element of  $B$  which tracks this component realizes  $\gamma \leq \delta$ .

Now suppose that  $F : \text{Ass}(A) \rightarrow \text{Ass}(B)$  is a finite-limit preserving  $\Gamma$ -functor. We may well suppose that  $F$  is the identity on the level of sets, as any  $\Gamma$ -functor is isomorphic to a functor having this property. Consider again the object  $(T(A), i)$  of  $\text{Ass}(A)$  and its  $F$ -image  $(T(A), \tilde{F})$  in  $\text{Ass}(B)$ , for some map  $\tilde{F} : T(A) \rightarrow T(B)$ . We wish to show that  $\tilde{F}$  is a proto-applicative morphism  $A \rightarrow B$ .

Let  $P = \{(\alpha, \beta) \in T(A) \times T(A) \mid \alpha\beta\downarrow\}$ . For  $(\alpha, \beta) \in P$  put  $E(\alpha, \beta) = \pi\alpha\beta$  (where  $\pi$  is the pairing combinator in  $A$ ). Then  $(P, E)$  is a regular subobject of  $(T(A), i) \times (T(A), i)$  in  $\text{Ass}(A)$  so by assumption on  $F$ ,  $F(P, E)$  is a regular subobject of  $(T(A), \tilde{F}) \times (T(A), \tilde{F})$ ; we may assume that  $F(P, E) = (P, \hat{E})$  with  $\hat{E}(\alpha, \beta) = \rho\tilde{F}(\alpha)\tilde{F}(\beta)$  (where  $\rho$  is the pairing combinator in  $B$ ). There is an application map  $\text{app} : (P, E) \rightarrow (T(A), i)$ , hence we have a map  $\text{app} : (P, \hat{E}) \rightarrow (T(A), \tilde{F})$ . Modulo a little fiddling with realizers, any element of  $B$  tracking this map realizes  $\tilde{F}$  as applicative morphism  $T(A) \rightarrow T(B)$ .

Furthermore, since any natural transformation between the sort of functors we consider is the identity on the level of sets, if we have a natural transformation  $F \Rightarrow G$  then we have a tracking for the identity function as morphism  $(T(A), \tilde{F}) \rightarrow (T(A), \tilde{G})$ ; such a tracking realizes  $\tilde{F} \leq \tilde{G}$ .

It is immediate that  $\tilde{\gamma}^* = \gamma$ . For the proof that  $(\tilde{F})^* \simeq F$ , we recall two basic facts from Longley's thesis:

- 1) Between any two  $\Gamma$ -functors  $F, G : \text{Ass}(A) \rightarrow \text{Ass}(B)$  there exists at most one natural transformation (whose components are identity arrows modulo the isomorphisms between  $F$  and  $G$  and functors which are the identity on the level of sets)
- 2) Any  $\Gamma$ -functor  $F : \text{Ass}(A) \rightarrow \text{Ass}(B)$  satisfies  $F \circ \nabla_A \simeq \nabla_B$ , where  $\nabla_A, \nabla_B$  are, respectively, right adjoint to  $\Gamma_A : \text{Ass}(A) \rightarrow \text{Set}$ ,  $\Gamma_B : \text{Ass}(B) \rightarrow \text{Set}$ .

Since we assume that  $F$  is the identity on the level of sets, we have  $F\nabla_A = \nabla_B$  and therefore, by the first fact above, for each  $A$ -assembly  $(X, E)$  we have

$$F(\eta_{(X, E)}) = \eta_{F(X, E)} : F(X, E) \rightarrow \nabla_B(X)$$

where  $\eta$  is the unit of the adjunction  $\Gamma \dashv \nabla$ .

Consider an object  $(X, E)$  of  $\text{Ass}(A)$ . There is an obvious map  $E : (X, E) \rightarrow (T(A), i)$  and the diagram

$$\begin{array}{ccc} (X, E) & \xrightarrow{E} & (T(A), i) \\ \eta \downarrow & & \downarrow \eta \\ \nabla_A(X) & \xrightarrow{E} & \nabla_A(T(A)) \end{array}$$

is a pullback. Similarly, for the map  $E : (X, \tilde{F} \circ E) \rightarrow (T(A), \tilde{F})$  the naturality square for  $\eta$  is a pullback diagram. Since  $F$  preserves pullbacks, we have therefore pullback diagrams

$$\begin{array}{ccc} F(X, E) & \xrightarrow{E} & F(T(A), i) & (X, \tilde{F} \circ E) & \xrightarrow{E} & (T(A), \tilde{F}) \\ \eta \downarrow & & \downarrow \eta & \eta \downarrow & & \downarrow \eta \\ \nabla_B(X) & \xrightarrow{E} & \nabla_B(T(A)) & \nabla_B(X) & \xrightarrow{E} & \nabla_B(T(A)) \end{array}$$

Since  $(T(A), \tilde{F}) = F(T(A), i)$  by definition of  $\tilde{F}$ , we see that  $(\tilde{F})^*(X, E) = (X, \tilde{F} \circ E)$  is naturally isomorphic to  $F(X, E)$ , as desired.

In order to see how Longley's biequivalence is a restriction of ours, first note that if  $\gamma : A \rightarrow B$  is an applicative morphism (i.e. the induced proto-applicative morphism  $T(A) \rightarrow T(B)$  is a  $T$ -algebra map) then  $\gamma^*$  is a regular  $\Gamma$ -functor, as Longley showed.

Conversely, suppose  $F$  is a regular  $\Gamma$ -functor. We wish to show that  $\tilde{F} : T(A) \rightarrow T(B)$  is a  $T$ -algebra map, which means that it commutes with unions. Consider the object  $\text{El} = (|\text{El}|, E)$  where

$$|\text{El}| = \{(a, S) \mid a \in S \subseteq A\}$$

and  $E(a, S) = \{a\}$ . We have a second projection  $\pi_2 : \text{El} \rightarrow (T(A), i)$  which is a regular epimorphism. Since by assumption  $F \simeq (\tilde{F})^*$  preserves regular epimorphisms, the map

$$(\tilde{F})^*(\text{El}) \xrightarrow{\pi_2} (\tilde{F})^*(T(A), i) = (T(A), \tilde{F})$$

is regular epi, but this means that, up to isomorphism of proto-applicative morphisms,  $\tilde{F}(\alpha) = \bigcup_{a \in \alpha} \tilde{F}(\{a\})$ , so  $\tilde{F}$  commutes with unions and is induced by an applicative morphism  $A \rightarrow B$ . ■

We can now give another characterization of computationally dense applicative morphisms of pcas. Every applicative morphism  $\gamma : A \rightarrow B$  of pcas is also an applicative morphism  $A \rightarrow T(B)$  of order-pcas and hence induces an applicative morphism  $\tilde{\gamma} : T(A) \rightarrow T(B)$  (and the functors  $\gamma^*$  from 1.3 and  $(\tilde{\gamma})^*$  of 2.2 coincide); by the biequivalence in the latter theorem, we have the following corollary:

**Corollary 2.3** *For an applicative morphism  $\gamma : A \rightarrow B$  the following statements are equivalent:*

- i)  $\gamma$  is computationally dense
- ii)  $\tilde{\gamma}$  has a right adjoint (in the 2-category of order-pcas)
- iii) there is an applicative morphism  $\delta : B \rightarrow A$  such that  $\gamma\delta \leq \text{id}_B$

**Proof.** i) $\Rightarrow$ ii): if  $\gamma$  is computationally dense then it induces a geometric morphism  $\text{RT}(B) \rightarrow \text{RT}(A)$  which, by 1.7, restricts to an adjunction between  $\Gamma$ -functors on the categories of assemblies; by 2.2 this is induced by an adjunction between proto-applicative morphisms.

ii) $\Rightarrow$ iii): let  $\delta : T(B) \rightarrow T(A)$  be right adjoint to  $\tilde{\gamma}$ . Define  $\bar{\delta} : B \rightarrow T(A)$  by

$$\bar{\delta}(b) = \delta(\{b\})$$

Then  $\bar{\delta}$  is an applicative morphism  $A \rightarrow B$  and  $\gamma\bar{\delta} \leq \text{id}_B$  since  $\gamma\bar{\delta}(b) = \tilde{\gamma}\delta(\{b\})$  and  $\tilde{\gamma} \dashv \delta$ .

iii) $\Rightarrow$ i): this is immediate from 1.6. ■

Another corollary is the following:

**Corollary 2.4** *The following data are equivalent:*

- i) a geometric morphism  $\text{RT}(B) \rightarrow \text{RT}(A)$
- ii) an adjunction  $\text{Ass}(B) \begin{matrix} \xleftarrow{f^*} \\ \xrightarrow{f_*} \end{matrix} \text{Ass}(A)$ ,  $f^* \dashv f_*$ , and  $f^*$  preserving finite limits
- iii) an adjunction  $T(B) \begin{matrix} \xleftarrow{\gamma^*} \\ \xrightarrow{\gamma_*} \end{matrix} T(A)$ ,  $\gamma^* \dashv \gamma_*$ , in the 2-category of order-pcas
- iv) a computationally dense applicative morphism  $A \rightarrow B$

**Proof.** By 1.5 and 1.7, i) and iv) are equivalent and imply ii); the equivalence between ii) and iii) is theorem 2.2. Suppose we have an adjunction as in ii). Then  $f_*$  is always a  $\Gamma$ -functor, since  $\Gamma$  is represented by 1 and  $f^*$  preserves 1. So  $f_*$  is, by 2.2, induced by a proto-applicative morphism; and  $f_*$  commutes with  $\nabla$  (as we saw in the proof of 2.2), whence its left adjoint commutes with  $\Gamma$ , and we have an adjunction of  $\Gamma$ -functors, hence an adjunction of proto-applicative morphisms, hence a computationally dense morphism  $A \rightarrow B$ . ■

In the same way we can characterise which computationally dense  $\gamma : A \rightarrow B$  induce geometric *inclusions*:

**Corollary 2.5** *A computationally dense applicative morphism  $\gamma : A \rightarrow B$  induces an inclusion of toposes:  $\text{RT}(B) \rightarrow \text{RT}(A)$  if and only if there is an applicative morphism  $\delta : B \rightarrow A$  such that  $\gamma\delta \simeq \text{id}_B$ .*

We conclude this section with the promised example of a computationally dense applicative morphism which is not projective:

**Example 2.6** Consider the pca  $\mathcal{K}_2^{\text{rec}}$  (see [17], 1.4.9) and the applicative morphism  $\mathcal{K}_2^{\text{rec}} \rightarrow \mathcal{K}_1$  which sends every total recursive function to the set of its indices ([17], p. 95). For recursion-theoretic reasons, this can *not* be isomorphic to a single-valued relation, so this is an example of a geometric morphism  $\mathcal{E}ff \rightarrow \text{RT}(\mathcal{K}_2^{\text{rec}})$  which is not regular.

## 2.1 Intermezzo: total pcas

In this small section we include some material on total pcas; it contains a characterization of the pcas which are isomorphic to a total one.

A pca  $A$  is called *total* if for all  $a$  and  $b$ ,  $ab \downarrow$ . The following results have been established about total pcas:

- The topos  $\mathcal{E}ff$  is not equivalent to a realizability topos on a total pca ([9]). In fact, the proof of that paper shows that if  $A$  is total and  $B$  is decidable (see definition 2.7 below), then  $\text{RT}(A)$  and  $\text{RT}(B)$  cannot be equivalent.
- Every total pca is isomorphic to a nontotal one ([16]).
- Every realizability topos is covered (in the sense of a geometric surjection) by a realizability topos on a total pca ([18]).

**Definition 2.7** Call an element  $a$  of a pca  $A$  *total* if for all  $b \in A$ ,  $ab \downarrow$ . Call a pca  $A$  *almost total* if for every  $a \in A$  there is a total element  $b \in A$  such that for all  $c \in A$ ,  $bc \preceq ac$ .

A pca is called *decidable* if there is an element  $d \in A$  which decides equality in  $A$ , that is: for all  $a, b \in A$ ,

$$dab = \begin{cases} \text{T} & \text{if } a = b \\ \text{F} & \text{if } a \neq b \end{cases}$$

**Proposition 2.8** *A nontrivial decidable pca is never almost total.*

**Proof.** We present a direct, elementary proof of this fact, although it also follows from the next proposition (2.9) by the Johnstone-Robinson result quoted above.

Let  $A$  be nontrivial and decidable. Choose  $e \in A$  such that for all  $x \in A$ ,  $ex \simeq xk$ . Pick elements  $a \neq b \in A$ . Suppose that  $g$  is a total element for  $e$  as in definition 2.7. By the recursion theorem for  $A$  ([17], 1.3.4) there is  $h \in A$  satisfying for all  $y \in A$ :

$$hy \simeq d(gh)aba$$

Then  $hy = b$  if  $gh = a$ , and  $hy = a$  otherwise (recall that  $\text{T}xy = x, \text{F}xy = y$ ). Since  $h$  is total, we have  $eh = hk$ . But now,

$$eh = hk = \begin{cases} b & \text{if } gh = a \\ a & \text{if } gh \neq a \end{cases} = \begin{cases} b & \text{if } eh = a \\ a & \text{if } eh \neq a \end{cases}$$

A clear contradiction. ■

**Proposition 2.9** *Let  $A$  be a pca. The following four conditions are equivalent:*

- i)  $A$  is almost total.*
- ii) There is an element  $g \in A$  such that for all  $e \in A$ ,  $ge$  is total and for all  $x$ ,  $gex \preceq ex$ .*
- iii)  $A$  is isomorphic to a total pca.*

**Proof.** i) $\Rightarrow$ ii): assume  $A$  is almost total. Pick  $f \in A$  such that for all  $y$ ,  $fy \simeq \pi_0 y (\pi_1 y)$  (recall that  $\pi, \pi_0, \pi_1$  are the pairing and unpairing combinators in  $A$ ). By assumption there is a total element  $h$  for  $f$  as in definition 2.7. Let  $g$  be such that  $gxy \simeq h(\pi xy)$ .

Then for every  $e \in A$ ,  $ge$  is a total element and if  $ex \downarrow$  then

$$gex = h(\pi ex) = f(\pi ex) = ex$$

so  $gex \preceq ex$  as required.

ii) $\Rightarrow$ iii): assume  $A$  satisfies condition ii). Define a binary function  $*$  on  $A$  by putting  $a * b = gab$ . We have

$$k * a * b = g(gka)b = kab = a$$

and if  $s' = \langle xyz \rangle g(gxz)(gyz)$  then

$$\begin{aligned} s' * a * b * c &= g(g(s'a)b)c = g(\langle z \rangle g(gaz)(gbz))c \\ &= g(gac)gbc = a * c * (b * c) \end{aligned}$$

So,  $(A, *)$  is a total pca. The identity function  $A \rightarrow A$  is an applicative morphism  $A \rightarrow (A, *)$ , realized by  $s' * k * k$  in  $(A, *)$ , and in the other direction it is realized by  $g \in A$ . So  $A$  is isomorphic to  $(A, *)$ .

iii) $\Rightarrow$ i): suppose  $A$  is isomorphic to  $B$  and  $B$  is total. By 1.10ii) we may assume that the isomorphism is given by functions  $f : A \rightarrow B$  and  $g : B \rightarrow A$  which are each other's inverse; suppose  $r \in B$  realizes  $f$  as applicative morphism, and  $s \in A$  realizes  $g$ .

For  $a \in A$  let  $a' = s(sg(r)gf(a))$ . For any  $x \in A$  we have:

$$\begin{aligned} a'x &= s(sg(r)gf(a))x = s(g(rf(a))x) \\ &= s(g(rf(a)))gf(x) = g(rf(a))f(x) \end{aligned}$$

So,  $a'x \downarrow$ , and if  $ax \downarrow$  then  $a'x = gf(ax) = ax$ . So  $A$  is almost total, as desired. ■

## 2.2 Discrete computationally dense morphisms

We employ the following convention for a parallel pair of geometric morphisms  $\alpha, \beta$  between realizability toposes: we write  $\alpha \leq \beta$  if there is a (necessarily unique) natural transformation  $\alpha^* \Rightarrow \beta^*$ .

**Theorem 2.10** *Let  $\gamma : A \rightarrow B$  be a discrete, computationally dense applicative morphism.*

- i) *There is a pca of the form  $A[f]$  such that the geometric morphism  $\text{RT}(B) \rightarrow \text{RT}(A)$  factors through the inclusion  $\text{RT}(A[f]) \rightarrow \text{RT}(A)$  by a geometric morphism  $\alpha : \text{RT}(B) \rightarrow \text{RT}(A[f])$*
- ii) *Moreover, there is a geometric morphism  $\beta : \text{RT}(A[f]) \rightarrow \text{RT}(B)$  satisfying  $\alpha\beta \leq \text{id}_{\text{RT}(A[f])}$  and  $\text{id}_{\text{RT}(B)} \leq \beta\alpha$ .*
- iii) *If  $\gamma$  induces an inclusion of toposes, then  $\beta\alpha \simeq \text{id}_{\text{RT}(B)}$ , so  $\text{RT}(B)$  is a retract of  $\text{RT}(A[f])$ .*
- iv) *If  $\gamma$  is projective then  $\alpha\beta \simeq \text{id}_{\text{RT}(A[f])}$ , so  $\text{RT}(A[f])$  is a retract of  $\text{RT}(B)$ .*
- v) *Hence, if  $\gamma$  is projective and induces an inclusion,  $\text{RT}(B)$  is equivalent to  $\text{RT}(A[f])$ .*

**Proof.** By 2.3ii),  $\tilde{\gamma} : T(A) \rightarrow T(B)$  has a right adjoint  $\delta$ . Let us write  $\delta'$  for the morphism  $\delta$  from the proof of 2.3:  $\delta'(b) = \delta(\{b\})$ . Assume, as we may, that  $\delta$  preserves inclusions. This means that  $\tilde{\delta}' \leq \delta$  as morphisms  $T(B) \rightarrow T(A)$ . We have  $\gamma\delta' \leq \text{id}_B$ , so  $\delta'$  is discrete by 1.10i).

Since both  $\gamma$  and  $\delta'$  are discrete, so is  $\delta'\gamma$  and we have a partial function  $f : A \rightarrow A$  defined by:  $f(a) = b$  if and only if  $a \in \delta'\gamma(b)$ . The partial function  $f$  is representable w.r.t.  $\gamma$ , for if  $\epsilon \in B$  realizes  $\gamma\delta' \leq \text{id}_B$  and  $f(a) = b$ ,  $c \in \gamma(a)$ , then  $a \in \delta'\gamma(b)$  so  $c \in \gamma\delta'\gamma(b)$  so  $\epsilon c \in \gamma(b)$ ; hence  $\epsilon$  represents  $f$  w.r.t.  $\gamma$ . Since, by section 1.5,  $\gamma$  factors through  $\iota_f$ , the geometric morphism  $\text{RT}(B) \rightarrow \text{RT}(A)$  factors through the inclusion  $\text{RT}(A[f]) \rightarrow \text{RT}(A)$ . This proves i). The geometric morphism  $\alpha$  is induced by  $(\gamma_f)^* \dashv \delta_f$  on the level of assemblies; here  $\gamma_f$  and  $\delta_f$  are the same relations on the level of sets as  $\gamma, \delta$  respectively.

We can regard  $\delta'$  also as applicative morphism  $B \rightarrow A[f]$ . Now in  $A[f]$ ,  $\delta'\gamma_f \leq \text{id}_{A[f]}$ , since if  $u$  represents  $f$  in  $A[f]$  then  $u$  realizes this inequality. This means that  $\delta' : B \rightarrow A[f]$  is computationally dense, by 2.3. So there is a geometric morphism  $\beta : \text{RT}(A[f]) \rightarrow \text{RT}(B)$ ; let  $\zeta : T(A[f]) \rightarrow T(B)$  be the right adjoint to  $\tilde{\delta}'$ . We have the following diagram of order-pcas:

$$T(B) \xleftarrow[\delta]{\tilde{\gamma}} T(A[f]) \xleftarrow[\zeta]{\tilde{\delta}'} T(B)$$

We have  $\tilde{\gamma}\tilde{\delta}' \leq \tilde{\gamma}\delta \leq \text{id}_{T(B)}$  and  $\tilde{\delta}'\tilde{\gamma} \leq \text{id}_{T(A[f])}$ , so this proves ii).

If  $\gamma$  induces an inclusion then  $\gamma\delta' \simeq \text{id}_B$  so  $\beta\alpha \simeq \text{id}_{\text{RT}(B)}$  and  $\text{RT}(B)$  is a retract of  $\text{RT}(A[f])$ .

If  $\gamma$  is projective then  $\tilde{\delta}' \simeq \delta$  so  $\delta\tilde{\gamma} \simeq \tilde{\delta}'\tilde{\gamma} \leq \text{id}_{T(A[f])} \leq \delta\tilde{\gamma}$ , so  $\alpha\beta$  is isomorphic to the identity on  $\text{RT}(A[f])$  and this topos is a retract of  $\text{RT}(B)$ .

v) is obvious. ■

**Corollary 2.11** *If  $A$  is a decidable pca, then every realizability topos which is a subtopos of  $\text{RT}(A)$  is a retract of  $\text{RT}(A[f])$  for some partial function  $f : A \rightarrow A$ .*

*Every realizability topos which is a subtopos of  $\mathcal{E}ff$  is equivalent to one of the form  $\text{RT}(\mathcal{K}_1[f])$  for some partial function on the natural numbers.*

**Proof.** Both statements follow from theorem 2.10, since if  $A$  is decidable, then for every computationally dense applicative morphism  $\gamma : A \rightarrow B$  we have that  $\gamma^*(A, \{\cdot\})$  is decidable in  $\text{Ass}(B)$ , hence discrete; and therefore  $\gamma$  is discrete. For the second statement, note that the essentially unique decidable applicative morphism  $\mathcal{K}_1 \rightarrow B$  is discrete and projective. ■

### 3 Local Operators in Realizability Toposes

Local operators ( $j$ -operators, Lawvere-Tierney topologies) in the Effective topos have been studied in [5, 14, 11, 19]. We quickly recall some basic facts which readily generalize to arbitrary realizability toposes.

Let  $A$  be a pca. For subsets  $U, V$  of  $A$  we denote by  $U \Rightarrow V$  the set of all elements  $a \in A$  which satisfy: for every  $x \in U$ ,  $ax \downarrow$  and  $ax \in V$ . We write  $U \wedge V$  for the set  $\{\pi ab \mid a \in U, b \in V\}$ . The powerset of  $A$  is denoted  $\mathcal{P}(A)$ .

Every local operator in  $\text{RT}(A)$  is represented by a function  $J : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$  for which the sets

- i)  $\bigcap_{U \subseteq A} U \Rightarrow J(U)$
- ii)  $\bigcap_{U \subseteq A} JJ(U) \Rightarrow J(U)$
- iii)  $\bigcap_{U, V \subseteq A} (U \Rightarrow V) \Rightarrow (J(U) \Rightarrow J(V))$

are all nonempty. A map  $J$  for which just the set iii) is nonempty, is said to represent a *monotone map* on  $\Omega$ . Abusing language, we shall just speak of “local operators” and “monotone maps” when we mean the maps representing them.

**Example 3.1** Important examples of local operators are:

- 1) The identity map on  $\mathcal{P}(A)$ ; this is the least local operator, and denoted by  $J_{\perp}$ . Its category of sheaves is just  $\text{RT}(A)$  itself.
- 2) The constant map with value  $A$ . This is the largest local operator, denoted  $J_{\top}$ ; its category of sheaves is the trivial topos.
- 3) The map which sends every nonempty set to  $A$ , and the empty set to itself. This is the  $\neg\neg$ -operator, and we shall also denote it by  $\neg\neg$ . Its category of sheaves is  $\text{Set}$ .
- 4) Suppose  $\gamma : A \rightarrow B$  is a computationally dense applicative morphism, inducing  $\tilde{\gamma} : T(A) \rightarrow T(B)$  and its right adjoint  $\delta$  by the theory of section 2. The map  $J : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$  which sends the empty set to itself, and every nonempty  $U \subseteq A$  to  $\delta\tilde{\gamma}(U)$ , is a local operator; its category of sheaves is the image of the geometric morphism  $\text{RT}(B) \rightarrow \text{RT}(A)$  induced by  $\gamma$ .

There is a partial order on local operators:  $J \leq K$  iff the set

$$\bigcap_{U \subseteq A} J(U) \Rightarrow K(U)$$

is nonempty (strictly speaking this gives a preorder on representatives of local operators). Every local operator is represented by a map  $J$  which preserves inclusions ([11], Remark 2.1). If  $M : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$  is a monotone map, there is a least local operator  $J_M$  such that  $M \leq J_M$ : it is given by

$$J_M(U) = \bigcap \{Q \subseteq A \mid \{\mathbf{T}\} \wedge U \subseteq Q \text{ and } \{\mathbf{F}\} \wedge M(Q) \subseteq Q\}$$

It is a general fact of topos theory that for any monomorphism  $m$  in a topos there is a least local operator which “inverts  $m$ ”, i.e. for which the sheafification of  $m$  is an isomorphism. In  $\text{RT}(A)$ , every object is covered by an  $A$ -assembly, so we need only consider monos into assemblies. Here, we restrict ourselves to two types of monos:

1. Consider an assembly  $(X, E)$  and the mono  $(X, E) \rightarrow \nabla(X)$ . Let  $M$  be the monotone map sending  $U \subseteq A$  to the set

$$\bigcup_{x \in X} E(x) \Rightarrow U$$

Then  $J_M$  is the least local operator inverting the mono  $(X, E) \rightarrow \nabla(X)$ .

2. Consider a partial function  $f : A \rightarrow A$  with domain  $B \subseteq A$ . We have the assemblies  $(B, \{\cdot\})$  and  $(B, E)$  where  $E(b) = \{\pi b f(b)\}$ . The identity on  $B$  is a map of assemblies  $(B, E) \rightarrow (B, \{\cdot\})$ , tracked by  $\pi_0$ . The least local operator inverting this mono (“forcing  $f$  to be realizable”) is  $J_M$ , where  $M$  is the monotone map

$$U \mapsto \{\pi b e \mid e f(b) \in U\}$$

The following theorem generalizes a result by Hyland and Phoa ([5, 13]).

**Theorem 3.2** *The category of sheaves for the local operator of type 2 above, is  $\text{RT}(A[f])$ .*

**Proof.** We refer to [16] for details on  $A[f]$ . The underlying set of  $A[f]$  is  $A$ ; the application map of  $A[f]$  is denoted  $a, b \mapsto a \cdot^f b$ .

It follows from the construction of the elements  $\mathbf{k}$  and  $\mathbf{s}$  in  $A[f]$ , that if  $t(x, a_1, \dots, a_n)$  is a term built from variable  $x$ , parameters  $a_1, \dots, a_n \in A$  and the application of  $A[f]$ , that the element  $\langle x \rangle t(x, \vec{a})$  of  $A[f]$  can be obtained computably in  $A$  from the parameters  $\vec{a}$ .

The computationally dense applicative morphism  $\iota_f : A \rightarrow A[f]$  is just the identity function, and the right adjoint  $\delta : T(A[f]) \rightarrow T(A)$  is given by

$$\delta(U) = \{\pi a e \mid e \cdot^f a \in U\}$$

Indeed,  $\text{id}_{T(A)} \leq \delta \tilde{\iota}_f = \delta$  because we can find,  $A$ -computably in  $a$ , an element  $\xi_a$  satisfying  $\xi_a \cdot^f x = a$  for all  $x$ . Also,  $\delta = \tilde{\iota}_f \delta \leq \text{id}_{T(A[f])}$  by simply evaluating in  $A[f]$ . We need to see that  $\delta$  is applicative; but if  $U \cdot^f V$  is defined in  $T(A[f])$  and  $\pi a e \in \delta(U)$ ,  $\pi b c \in \delta(V)$  then

$$\pi(\pi a b)(\langle x \rangle (e \cdot^f (\pi_0 x)) \cdot^f (e \cdot^f (\pi_1 x)))$$

is an element of  $\delta(U \cdot^f V)$  and we noted that this element can be obtained  $A$ -computably from  $a, e, b, c$ .

So, by Example 3.1, item 4, the local operator on  $\text{RT}(A)$  for which the category of sheaves is  $\text{RT}(A[f])$ , sends  $U$  to  $\{\pi a e \mid e \cdot^f a \in U\}$ . Let us call this map  $J_f$ .

On the other hand, the least local operator which forces the partial function  $f$  to be realizable, is the map  $J_M$  where  $M$  is the monotone map

$$U \mapsto \{\pi b e \mid e f(b) \in U\}$$

We need to prove  $J_M \leq J_f$  and  $J_f \leq J_M$ .

By  $[a_1, \dots, a_n]$  we denote some standard coding in  $A$  of the  $n$ -tuple  $a_1, \dots, a_n$ . We write  $[]$  for the code of the empty tuple. The symbol  $*$  is used for ( $A$ -computable) concatenation of tuples: so

$$[a_1, \dots, a_n] * [b_1, \dots, b_m] = [a_1, \dots, a_n, b_1, \dots, b_m]$$

The definition of  $a \cdot^f b = c$  is as follows:

$a \cdot^f b = c$  if and only if either  $a[b] = \pi \top c$ , or there is a sequence  $a_1, \dots, a_n$  such that  $a[b] = \pi \mathbb{F}d$  for some  $d$  such that  $f(d) = a_1$ , and for all  $1 < k \leq n$ ,  $a[b, a_1, \dots, a_{k-1}] = \pi \mathbb{F}d$  for some  $d$  such that  $f(d) = a_k$ , and moreover,  $a[b, a_1, \dots, a_n] = \pi \top c$

Let us call such a sequence  $a_1, \dots, a_n$  a *computation sequence* for  $a \cdot^f b$ .

Now clearly, if  $a \in A$  satisfies  $a e[b] = \pi \mathbb{F}b$  and  $a e[b, c] \simeq e c$  for all  $e, b, c$ , then we have  $a e \cdot^f b \simeq e f(b)$ . Hence,  $M \leq J_f$  and therefore  $J_M \leq J_f$  by definition of  $J_M$ .

For the converse, let  $\alpha \in \bigcap_{U \subseteq A} U \Rightarrow J_M(U)$  and  $\zeta \in \bigcap_{U \subseteq A} M(J_M(U)) \Rightarrow J_M(U)$ . By the recursion theorem in  $A$ , there is an element  $\gamma \in A$  such that for all  $e, a, \sigma$ :

$$\begin{aligned} \gamma e a \sigma &\leq \text{If } \pi_0(e([a] * \sigma)) \\ &\quad \text{then } \alpha(\pi_1(e([a] * \sigma))) \\ &\quad \text{else } \zeta(\pi(\pi_1(e([a] * \sigma))) \langle x \rangle \gamma e a(\sigma * [x])) \end{aligned}$$

We claim: if  $e \cdot^f a \in U$ , and  $a_1, \dots, a_n$  is a computation sequence for  $e \cdot^f a$ , then for all  $0 \leq k \leq n$ ,

$$\gamma e a[a_1, \dots, a_k] \in J_M(U)$$

For  $k = 0$ ,  $[a_1, \dots, a_k]$  is  $[]$ .

Of course,  $\gamma ea[a_1, \dots, a_n] = e \cdot^f a$  since  $e[a, a_1, \dots, a_n] = \pi T(e \cdot^f a)$ . Hence by assumption that  $e \cdot^f a \in U$ , we have  $\gamma ea[a_1, \dots, a_n] = \alpha(e \cdot^f a) \in J_M(U)$ .

Now suppose  $k < n$  and  $\gamma ea[a_1, \dots, a_{k+1}] \in J_M(U)$ . Let  $e[a, a_1, \dots, a_k] = \pi F u_k$ . We have

$$\gamma ea[a_1, \dots, a_k] = \zeta(\pi u_k \varepsilon)$$

where  $\varepsilon = \langle x \rangle \gamma ea(\sigma * [x])$ . Moreover,  $f(u_k) = a_{k+1}$ . We see that

$$\varepsilon f(u_k) = \gamma ea[a_1, \dots, a_{k+1}] \in J_M(U)$$

so  $\pi u_k \varepsilon \in M(J_M(U))$ , whence  $\zeta(\pi u_k \varepsilon) \in J_M(U)$ . This proves the claim.

We conclude that whenever  $\pi ae \in J_f(U)$ , that is  $e \cdot^f a \in U$ , we have  $\gamma ea[] \in J_M(U)$ ; so  $J_f \leq J_M$  as desired.  $\blacksquare$

As an example of a monomorphism of type 1, we consider the inclusion of assemblies  $2 \rightarrow \nabla(2)$ . It is a result of Hyland, that the least local operator in  $\mathcal{E}ff$  inverting this mono, is  $\neg\neg$ ; we shall see whether this holds for arbitrary realizability toposes. The following lemma is from [5] and generalizes to arbitrary realizability toposes in a straightforward way.

**Lemma 3.3** *Let  $J$  be a local operator. Then  $\neg\neg \leq J$  if and only if the set  $\bigcap_{a \in A} J(\{a\})$  is nonempty.*

We can represent the object 2 in  $\mathbf{RT}(A)$  as the assembly  $(\{0, 1\}, E)$  with  $E(0) = \{\bar{0}\}$  and  $E(1) = \{\bar{1}\}$  (recall that  $\bar{0}, \bar{1}$  are the first two Curry numerals). Therefore the least local operator inverting  $2 \rightarrow \nabla(2)$  is  $J_M$  where

$$M(U) = (\{\bar{0}\} \Rightarrow U) \cup (\{\bar{1}\} \Rightarrow U)$$

Note, that  $M$  is also the least monotone map with the property that  $M(\{\bar{0}\}) \cap M(\{\bar{1}\})$  is nonempty, and therefore  $J_M$  is the least local operator  $J$  for which  $J(\{\bar{0}\}) \cap J(\{\bar{1}\})$  is nonempty.

**Lemma 3.4** *The least local operator which inverts the inclusion  $2 \rightarrow \nabla(2)$  is (up to isomorphism) the map  $J$  which sends  $U \subseteq A$  to  $\bigcup_{n \in \mathbb{N}} (\{\bar{n}\} \Rightarrow U)$ .*

**Proof.** Martin Hyland showed in [5], 16.4, that whenever  $J$  is a local operator in  $\mathcal{E}ff$  such that  $J(\{0\}) \cap J(\{1\})$  is nonempty, then  $\bigcap_{n \in \mathbb{N}} J(\{n\})$  is nonempty. Since the tools for this proof were basic recursion theory, this proof generalizes to an arbitrary pca  $A$  to yield: whenever  $J$  is a local operator in  $\mathbf{RT}(A)$  such that  $J(\{\bar{0}\}) \cap J(\{\bar{1}\})$  is nonempty, then  $\bigcap_{n \in \mathbb{N}} J(\{\bar{n}\})$  is nonempty.

Now the least monotone map  $M$  such that  $\bigcap_{n \in \mathbb{N}} M(\{\bar{n}\})$  is nonempty, is the map  $J$  in the statement of the lemma. So it remains to show that this is a local operator. Clearly, it is a monotone map, and certainly  $\langle xy \rangle x$  is an element of  $U \Rightarrow J(U)$  for all  $U \subseteq A$ . As to  $J(J(U)) \Rightarrow J(U)$ , we note that we have uniform isomorphisms

$$\begin{aligned} J(J(U)) &\cong \bigcup_n (\{n\} \Rightarrow \bigcup_m (\{m\} \Rightarrow U)) \\ &\cong \bigcup_{m,n} (\{m\} \wedge \{n\} \Rightarrow U) \\ &\cong \bigcup_k (\{k\} \Rightarrow U) = J(U) \end{aligned}$$

The last isomorphism is because there exists a recursive pairing on the natural numbers which is a bijection from  $\mathbb{N} \times \mathbb{N}$  to  $\mathbb{N}$ , and which is representable in  $A$ , as well as its unpairing functions. ■

**Theorem 3.5** *For a pca  $A$  the following three statements are equivalent:*

- i) *The least local operator inverting  $2 \rightarrow \nabla(2)$  is  $\neg\neg$ .*
- ii) *There is an element  $h \in A$  such that for every  $a \in A$  there is a natural number  $n$  satisfying  $h\bar{n} = a$ .*
- iii) *There exists a (necessarily essentially unique) geometric morphism  $\text{RT}(A) \rightarrow \mathcal{E}ff$ .*

**Proof.** This is now a trivality: given the characterizations of Lemma 3.3 and Lemma 3.4, we have the equivalence of i) and ii). But clearly, ii) is equivalent to the statement that the essentially unique decidable applicative morphism  $\mathcal{K}_1 \rightarrow A$ , which is the map sending  $n$  to  $\bar{n}$ , is computationally dense. And that is equivalent to iii). ■

**Remark 3.6** We are grateful to Peter Johnstone for the following remark. As pointed out by Olivia Caramello in [1], the least local operator inverting  $2 \rightarrow (2)_{\neg\neg}$  (where  $(2)_{\neg\neg}$  denotes the  $\neg\neg$ -sheafification of 2), is also the least local operator for which the category of sheaves is a *De Morgan topos* (A topos is De Morgan if 2 is a  $\neg\neg$ -sheaf).

This yields another proof of iii) $\Rightarrow$ i) in Theorem 3.5: if  $f : \text{RT}(A) \rightarrow \text{RT}(B)$  is a geometric morphism, then  $f$  restricts to a geometric morphism  $\text{RT}(A)_{\text{dm}} \rightarrow \text{RT}(B)_{\text{dm}}$  (where  $\mathcal{E}_{\text{dm}}$  denotes the largest De Morgan subtopos of  $\mathcal{E}$ ). This is immediate, because  $f^*$  preserves both 2 and  $\nabla(2)$ . This means that if  $\text{RT}(B)_{\text{dm}} = \text{Set}$ , then also  $\text{RT}(A)_{\text{dm}} = \text{Set}$ . This follows from 1.3 of [7].

**Example 3.7** Peter Johnstone has suggested the terminology *effectively numerical* for a pca  $A$  satisfying ii) of 3.5. Clearly, if a pca is effectively numerical, it must be countable. The pca  $\mathcal{K}_2^{\text{rec}}$  is effectively numerical.

In order to see a countable pca which is nevertheless not effectively numerical, consider a nonstandard model of Peano Arithmetic  $A$ .  $A$  is a pca if we define  $ab = c$  to hold precisely if the formula  $\exists x(T(a, b, x) \wedge U(x) = c)$  is true in  $A$  (here  $T$  and  $U$  are Kleene's well-known computation predicate and output function; these things can be expressed in the language of Peano Arithmetic, hence interpreted in  $A$ ).  $A$  will then satisfy the axioms for a pca, since these are consequences of Peano Arithmetic. In  $A$ , the Curry numerals can be identified with the standard part of  $A$ . Now consider, for an arbitrary parameter  $a \in A$ , the following  $\mathbb{N}$ -indexed family of formulas in one variable  $x$ :

$$\Phi_a(x) = \{\forall y(T(a, n, y) \rightarrow U(y) \neq x) \mid n \in \mathbb{N}\}$$

By [10], Theorem 11.5,  $A$  is *saturated* for types like this: there is an element  $\xi \in A$  such that  $\Phi_a(\xi)$  holds in  $A$ . That means, there is no  $n$  such that  $an = \xi$ . Since  $a$  is arbitrary, we see that  $A$  cannot be effectively numerical.

We conclude this paper with a characterization of those local operators in  $\text{RT}(A)$  for which the category of sheaves is  $\text{RT}(A[f])$  for some partial function  $f$  on  $A$ . From Theorem 2.10 we know that if  $\gamma : A \rightarrow B$  is discrete and projective and induces an inclusion, then this inclusion is of the form  $\text{RT}(A[f]) \rightarrow \text{RT}(A)$ . Moreover, we know then that the local operator  $J$  corresponding to this inclusion has the following properties:

- 1)  $J(\{a\}) \cap J(\{b\}) = \emptyset$  whenever  $a \neq b$  (we may call  $J$  *discrete*)
- 2)  $J$  preserves unions.

**Proposition 3.8** *Suppose  $J$  is a discrete local operator which preserves unions. Then there is a partial function  $f$  on  $A$  such that  $J$  is isomorphic to  $J_f$ , the least local operator forcing  $f$  to be realizable.*

**Proof.** Define  $f$  by:  $f(a) = b$  if and only if  $a \in J(\{b\})$ . This is well-defined since  $J$  is discrete. Let  $M$  be the monotone map of the proof of Theorem 3.2, so  $M(U) = \{\pi a e \mid e f(a) \in U\}$  and  $J_f$  is the least local operator majorizing  $M$ . Let  $g$  realize the monotonicity of  $J$ :

$$g \in \bigcap_{U, V \subseteq A} (U \Rightarrow V) \Rightarrow (JU \Rightarrow JV)$$

Now if  $\pi a e \in M(U)$ , then  $e \in \{f(a)\} \Rightarrow U$  so  $ge \in J(\{f(a)\}) \Rightarrow J(U)$ , so  $ge \in \{a\} \Rightarrow J(U)$  (since  $a \in J(\{f(a)\})$ ), so  $gea \in J(U)$ . This shows that  $M \leq J$  and hence  $J_f \leq J$ .

Conversely, if  $a \in J(U)$  then since  $J$  preserves unions, we have  $a \in J(\{x\})$  for some  $x \in U$ , which means  $f(a) \in U$ , which implies that  $\pi a i$  (where  $i$  is such that  $ib = b$  for all  $b \in A$ ) is an element of  $M(U)$ . So  $J \leq M \leq J_f$ . Note that we actually prove that  $M$  is a local operator in this case! ■

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